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Equations paraboliques: comportement qualitatif

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CHAPTER 1

Introduction

The aim of this course is to study the linear and the semilinear heat equations

$$u_t - \Delta u = f$$

and

$$u_t - \Delta u + f(u) = 0$$

on an open subset of $\mathbb{R}^n$. The theory which we will develop in order to study existence and uniqueness of solutions, their regularity, positivity and asymptotic properties (shortly: their qualitative behaviour) serves in fact to study more general parabolic evolution equations. We cite for example

- the semilinear diffusion equation

$$u_t + Lu + f(u) = 0,$$

  where L is an elliptic partial differential operator,
- the Navier-Stokes equation

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f,$$

- free boundary problems, and
- geometric evolution equations.

These partial differential evolution equations can be abstractly rewritten in the form

$$\dot{u} + Au + F(u) = 0,$$

where A and F are linear (resp. nonlinear) operators on a Banach space X which is often a Banach function space. The latter problem is an *ordinary differential equation* with the disadvantage that A and F may be *unbounded* (= not continuous).

In this course, we will study abstract linear Cauchy problems of the form

$$\dot{u} + Au = f, \quad u(0) = u_0,$$

and associated abstract nonlinear Cauchy problems, and we will show that the above partial differential equations are examples to which that abstract theory applies.

CHAPTER 2

Linear parabolic equations

Throughout we denote by X and Y Banach spaces and by H, K, V Hilbert spaces. The norm on a Banach space X is usually denoted by $\|\cdot\|_X$ or $\|\cdot\|$, and the inner product on a Hilbert space H is usually denoted by $(\cdot,\cdot)_H$ or $(\cdot,\cdot)$.

Recall that a linear operator $T : X \to Y$ is continuous if and only if it is bounded, i.e. if and only if $\|T\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|_X \leq 1} \|Tx\|_Y$ is finite. Instead of speaking of continuous linear operators we will in the following speak of bounded linear operators. The space of all bounded linear operators from X into Y is denoted by $\mathcal{L}(X,Y)$. It is a Banach space for the norm $\|\cdot\|_{\mathcal{L}(X,Y)}$.

1. Closed linear operators

For the following, we will have to consider a larger class of linear operators. Whenever X and Y are two Banach spaces, a *linear operator* is a linear mapping $A : D(A) \to Y$ defined on a linear subspace $D(A)$ of X. The space $D(A)$ is called *domain* of A. Note that the domain $D(A)$ need not be a closed linear subspace of X.

DEFINITION 1.1. Let X and Y be two Banach spaces. A linear operator $A : D(A) \to Y$ is called *closed* if its graph

$$G(A) := \{(x, Ax) : x \in D(A)\} \subset X \times Y$$

is closed in the product space $X \times Y$.

LEMMA 1.2. *A linear operator $A : D(A) \to Y$ is closed if and only if the following property holds:*

$$\left. \begin{array}{l} D(A) \ni x_n \to x \ \text{in } X \ \text{and} \\ Ax_n \to y \ \text{in } Y \end{array} \right\} \Rightarrow x \in D(A) \ \text{and } Ax = y.$$

PROOF. It suffices to note that $D(A) \ni x_n \to x$ in X and $Ax_n \to y$ in Y if and only if $G(A) \ni (x_n, Ax_n) \to (x,y)$ in the product space $X \times Y$, by definition of the product topology.

If A is closed and if $G(A) \ni (x_n, Ax_n) \to (x,y)$ then $(x,y) \in G(A)$ by the closedness of A and thus $x \in D(A)$ and $y = Ax$.

Conversely, if $G(A) \ni (x_n, Ax_n) \to (x,y)$ implies necessarily $x \in D(A)$ and $y = Ax$, then $(x,y) \in G(A)$, i.e. $G(A)$ is closed, and thus A is closed. $\qquad\square$

LEMMA 1.3. *A linear operator $A : D(A) \to Y$ is closed if and only if its domain $D(A)$ equipped with the graph norm*

$$\|x\|_{D(A)} := \|x\|_X + \|Ax\|_Y, \quad x \in D(A),$$

is a Banach space.

PROOF. If A is closed, then, by definition, $G(A)$ is a closed subspace of the product space $X \times Y$. Since $X \times Y$ is a Banach space, the graph $G(A)$ is a Banach space. Now note that $D(A)$ equipped with the graph norm and $G(A)$ equipped with the product norm are isometrically isomorphic under the isometry $D(A) \to G(A)$, $x \mapsto (x, Ax)$. Hence $D(A)$ equipped with the graph norm is a Banach space.

Conversely, assume that $D(A)$ equipped with the graph norm is a Banach space. Then $G(A)$ (equipped with the product norm from $X \times Y$) is a Banach space by the same argument as before. In particular, $G(A)$ is a closed subspace of $X \times Y$. Hence, A is closed.   □

LEMMA 1.4. *Every bounded linear operator $T : X \to Y$ (with domain $D(T) = X$) is closed.*

PROOF. Let $T \in \mathcal{L}(X, Y)$. The norms $\| \cdot \|_X$ and $\| \cdot \|_{D(T)}$ are equivalent norms on X which is a Banach space for the norm $\| \cdot \|_X$. Hence $X = D(T)$ is a Banach space for the norm $\| \cdot \|_{D(T)}$. By Lemma 1.3, T is closed.   □

The following theorem is a fundamental theorem in functional analysis. It is a consequence of Baire's theorem, but it will not be proved here.

THEOREM 1.5 (Closed graph theorem). *Let X and Y be Banach spaces and let $T : X \to Y$ (with domain $D(T) = X$) be closed. Then T is bounded.*

EXAMPLE 1.6. Let $X = Y = C([0, 1])$ be the space of continuous functions on $[0, 1]$ with norm $\|f\|_\infty := \sup_{x \in [0,1]} |f(x)|$. Define the *derivation operator D* by

$$D(D) := C^1([0, 1]) \text{ and } Df := f' \text{ for } f \in D(D).$$

Then D is closed. In fact, the space $C^1([0, 1])$ is a Banach space for the graph norm $\|f\|_{D(D)} = \|f\|_\infty + \|f'\|_\infty$ (exercice).

EXAMPLE 1.7. Let $X = Y = L^p(\mathbb{R})$ ($1 \le p \le \infty$) with norm $\| \cdot \|_p$. Define the *multiplication operator M* by

$$D(M) := \{f \in L^p(\mathbb{R}) : xf(x) \in L^p(\mathbb{R})\} \text{ and } (Mf) := xf(x) \text{ for } f \in D(M).$$

Then M is closed. In fact,

$$D(M) = L^p(\mathbb{R}; (1 + |x|^p) \, dx),$$

and the graph norm $\| \cdot \|_{D(M)}$ is equivalent to the norm

$$\|f\|_{L^p(\mathbb{R};(1+|x|^p) \, dx)} := \left(\int_{\mathbb{R}} |f|^p (1 + |x|^p) \, dx \right)^{1/p},$$

which makes $L^p(\mathbb{R}; (1 + |x|^p) \, dx)$ a Banach space.

2. Operators associated with bilinear forms

In this section, V will be a real Hilbert space.

DEFINITION 2.1 (Bilinear form). A function $a : V \times V \to \mathbb{K}$ is called a *bilinear form* if it is linear in each variable, i.e.

$$\begin{aligned} a(\alpha u + \beta v, w) &= \alpha a(u, w) + \beta a(v, w) \text{ and} \\ a(u, \alpha v + \beta w) &= \alpha a(u, v) + \beta a(u, w) \end{aligned}$$

for every u, v, $w \in V$ and every α, $\beta \in \mathbb{K}$.

There are some simple but important examples of bilinear forms.

EXAMPLE 2.2. Every inner product on V is a bilinear form!

EXAMPLE 2.3. Let $V = H_0^1(\Omega)$ ($\Omega \subset \mathbb{R}^n$ open) be the Sobolev space which is obtained by taking the closure of $\mathcal{D}(\Omega)$ (the test functions on Ω) in $H^1(\Omega)$. The space V is equipped with the inner product

$$(u, v)_{H_0^1} := \int_\Omega uv + \int_\Omega \nabla u \nabla v,$$

and the corresponding norm

$$\|u\|_{H_0^1} = \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

On this Sobolev space the equality

$$a(u, v) := \int_\Omega \nabla u \, \nabla v, \quad u, v \in V,$$

defines a bilinear form.

EXAMPLE 2.4. More generally, if $A \in L^\infty(\Omega; \mathbb{R}^{n \times n})$ is a bounded, measurable, matrix valued function, then the equality

$$a(u, v) := \int_\Omega A(x) \nabla u \, \overline{\nabla v}, \quad u, v \in V,$$

defines a bilinear form on the Sobolev space $V = H_0^1(\Omega)$.

DEFINITION 2.5 (Boundedness, coercivity). Let a be a bilinear form on a Hilbert space V.

(a) We say that a is *bounded* if there exists a constant $C \geq 0$ such that

$$|a(u, v)| \leq C \, \|u\|_V \, \|v\|_V \text{ for every } u, v \in V.$$

(b) We say that a is *coercive* if there exists a constant $\eta > 0$ such that

$$\operatorname{Re} a(u, u) \geq \eta \, \|u\|_V^2 \text{ for every } u \in V.$$

In the following, we denote by V' the dual space of V, i.e. $V' = \mathcal{L}(V, \mathbb{R})$. The duality between V' and V is denoted by the bracket $\langle \cdot, \cdot \rangle_{V', V}$.

DEFINITION 2.6 (Operator associated with a bilinear form). Given a bounded, bilinear form a on V, we define the linear operator $A : V \rightarrow V'$ associated with this form by

$$\langle Au, \varphi \rangle_{V',V} := a(u, \varphi), \quad u, \varphi \in V.$$

It follows from the boundedness of a that the operator A is well-defined and bounded. In fact, let $C \geq 0$ be the constant from Definition 2.5 (a). Then

$$
\begin{aligned}
\|Au\|_{V'} &= \sup_{\|\varphi\|_V \leq 1} |\langle Au, \varphi \rangle_{V',V}| \\
&= \sup_{\|\varphi\|_V \leq 1} |a(u, \varphi)| \\
&\leq \sup_{\|\varphi\|_V \leq 1} C \|u\|_V \|\varphi\|_V = C \|u\|_V.
\end{aligned}
$$

The following theorem says something about the solvability of the equation $Au = f$ for given $f \in V'$. As one can see from the statement, coercivity of a implies invertibility of A.

THEOREM 2.7 (Lax-Milgram). *Let a be a bounded, coercive, bilinear form on V. Then for every $f \in V'$ there exists a unique $u \in V$ such that*

$$a(u, \varphi) = \langle f, \varphi \rangle_{V',V} \text{ for every } \varphi \in V.$$

PROOF. We have to prove that the bounded linear operator $A \in \mathcal{L}(V, V')$ associated with a is bijective. By coercivity, for every $u \in V \setminus \{0\}$,

$$
\begin{aligned}
\|Au\|_{V'} &= \sup_{\|v\|_V \leq 1} |\langle Au, v \rangle_{V',V}| \\
&\geq \left|\langle Au, \frac{u}{\|u\|_V} \rangle_{V',V}\right| \\
&= \frac{1}{\|u\|_V} a(u, u) \\
&\geq \eta \|u\|_V.
\end{aligned}
$$

This proves on the one hand injectivity of A, but also that $\operatorname{Rg} A$ is closed.

It is easy to verify that the adjoint $A' \in \mathcal{L}(V'', V') = \mathcal{L}(V, V')$ of the operator A is associated with the bilinear form $a' : V \times V \rightarrow \mathbb{R}$ given by $a'(u, v) = a(v, u)$. Clearly, a' is also bounded and coercive. In particular, by the preceeding argument, A' is injective. This implies that $\operatorname{Rg} A$ is dense in V'. Since $\operatorname{Rg} A$ is also closed, the operator A must be surjective. $\qquad \square$

For the rest of this section, we let H be a second Hilbert space and we suppose that V is *continuously and densely embedded into H*. By this we mean that there exists an injective bounded linear operator $J : V \rightarrow H$ with dense range. Via this injective bounded linear operator, we may actually consider V as a dense subspace of H, and there exists a constant $C \geq 0$ such that

$$\|u\|_H \leq C \|u\|_V \text{ for every } u \in V.$$

We write shortly $V \hookrightarrow H$ if V is densely embedded into H.

Moreover, in the following we will identify the dual H' with the Hilbert space H, using the Riesz-Fréchet representation theorem, i.e. we say that every bounded linear functional comes from a unique element in H. Since V is continuously and densely embedded into H, via the adjoint of this embedding, we see that H' is continuously and densely embedded into V'. Since, moreover, $H = H'$ by convention, we have the following picture

$$(2.1) \qquad\qquad V \hookrightarrow H = H' \hookrightarrow V',$$

and

$$(2.2) \qquad\qquad \langle u, v \rangle_{V',V} = (u, v)_H \text{ for every } u \in H, \, v \in V.$$

Note that in this situation V is also continuously and densely embedded into its dual V', and in particular it is a linear subspace of V'.

Hence, given a bounded, bilinear form a on V, the operator $A : V \to V'$ associated with V may be seen as an unbounded operator on V' with domain $D(A) = V \subset V'$. If we consider A as a linear operator on V' and if a is coercive, then A is in fact a *closed* linear operator on V'. Indeed, by the Lax-Milgram theorem, A is boundedly invertible on V', i.e. A^{-1} is bounded and in particular closed, and therefore A is closed.

We define a second operator associated with the form a.

DEFINITION 2.8 (Operator associated with a bilinear form). Let a be a bounded, bilinear form on V, and let H be a second Hilbert space such that (2.1) holds. We define the operator $A_H : H \supset D(A) \to H$ associated with a by

$$\begin{aligned} D(A_H) &:= \{u \in V : \exists v \in H \, \forall \varphi \in V : a(u, \varphi) = (v, \varphi)_H\}, \\ Au &= v. \end{aligned}$$

The operator A is well-defined in the sense that the element $v \in H$ is uniquely determined if it exists. Indeed, assume that there are two elements $v_1, v_2 \in H$ such that

$$(v_1, \varphi)_H = a(u, \varphi) = (v_2, \varphi)_H \text{ for every } \varphi \in V.$$

Then $(v_1 - v_2, \varphi)_H = 0$ for every $\varphi \in V$, and since V is dense in H (here the density of the embedding is used!), this already implies $v_1 = v_2$.

LEMMA 2.9. *The operator A_H is the restriction of A to the space H, i.e.*

$$D(A_H) = \{u \in V : Au \in H\} \text{ and } A_H u = Au \text{ for } u \in D(A_H).$$

PROOF. Exercise. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Vector-valued L^p spaces

As before X denotes a Banach space. In this section $(\Omega, \mathcal{A}, \mu)$ is a measure space.

DEFINITION 3.1. (a) A function $f : \Omega \to X$ is called *step function*, if there exists a sequence $(A_n) \subset \mathcal{A}$ of mutually disjoint measurable sets and a sequence $(x_n) \subset X$ such that $f = \sum_n 1_{A_n} x_n$.
(b) A function $f : \Omega \to X$ is called *mesurable*, if there exists a sequence (f_n) of step functions $f_n : \Omega \to X$ such that $f_n \to f$ pointwise μ-almost everywhere.

REMARK 3.2. Note that there may be a difference to the definition of mesurability of a scalar valued functions. Measurability of a function is here depending on the measure μ. However, if the measure space $(\Omega, \mathcal{A}, \mu)$ is *complete* in the sense that $\mu(A) = 0$ and $B \subset A$ implies $B \in \mathcal{A}$, then the above definition of measurability and the classical definition of measurability coincide. Note that one may always consider complete measure spaces.

LEMMA 3.3. *If* $f : \Omega \to X$ *is measurable, then* $\|f\| : \Omega \to \mathbb{R}$ *is measurable. More generally, if* $f : \Omega \to X$ *is measurable and if* $g : X \to Y$ *is continuous, then* $g \circ f : \Omega \to Y$ *is measurable.*

PROOF. This is an easy consequence of the definition of measurability and the continuity of g. Note that in particular the norm $\|\cdot\| : X \to \mathbb{R}$ is continous. $\square$

LEMMA 3.4. *If* $f : \Omega \to X$ *and* $g : \Omega \to \mathbb{K}$ *are measurable, then* $fg : \Omega \to X$ *is measurable.*
Similarly, if $f : \Omega \to X$ *and* $g : \Omega \to X'$ *are measurable, then* $\langle g, f \rangle_{X', X} : \Omega \to \mathbb{K}$ *is measurable.*

THEOREM 3.5 (Pettis). *A function* $f : \Omega \to X$ *is measurable if and only if* $\langle x', f \rangle$ *is measurable for every* $x' \in X'$ *(we say that* f *is* weakly measurable*) and if there exists a* μ*-null set* $N \in \mathcal{A}$ *such that* $f(\Omega \setminus N)$ *is separable.*

For a proof of Pettis' theorem, see HILLE & PHILLIPS [**11**].

COROLLARY 3.6. *If* (f_n) *is a sequence of measurable functions* $\Omega \to X$ *such that* $f_n \to f$ *pointwise* μ*-almost everywhere, then* f *is measurable.*

PROOF. We assume that this corollary is known in the scalar case, i.e. when $X = \mathbb{K}$.
By Pettis's theorem, for all n there exists a μ null set $N_n \in \mathcal{A}$ such that $f_n(\Omega \setminus N_n)$ is separable. Moreover there exists a μ null set $N_0 \in \Omega$ such that $f_n(t) \to f(t)$ for all $t \in \Omega \setminus N_0$. Let $N := \bigcup_{n \geq 0} N_n$; as a countable union of μ null sets, N is a μ null set.
Then f (restricted to $\Omega \setminus N$) is the pointwise limit everywhere of the sequence (f_n). In particular f is weakly measurable. Moreover, $f(\Omega \setminus N)$ is separable

since

$$f(\Omega \setminus N) \subset \overline{\bigcup_n f_n(\Omega \setminus N)},$$

and since $f_n(\Omega \setminus N)$ is separable. The claim follows from Pettis' theorem. □

DEFINITION 3.7. A measurable function $f : \Omega \to X$ is called *integrable* if $\int_\Omega \|f\| \, d\mu < \infty$.

LEMMA 3.8. *For every integrable step function $f : \Omega \to X$, $f = \sum_n 1_{A_n} x_n$ the series $\sum_n x_n \mu(A_n)$ converges absolutely and it is independent of the representation of f.*

PROOF. Let $f = \sum_n 1_{A_n} x_n$ be an integrable step function. The sets $(A_n) \subset \mathcal{A}$ are mutually disjoint and $(x_n) \subset X$. Then

$$\sum_n \|x_n\| \, \mu(A_n) = \int_\Omega \|f\| \, d\mu < \infty.$$

□

DEFINITION 3.9 (Bochner integral for integrable step functions). Let $f : \Omega \to X$ be an integrable step function, $f = \sum_n 1_{A_n} x_n$. We define

$$\int_\Omega f \, d\mu := \sum_n x_n \, \mu(A_n).$$

LEMMA 3.10. *(a) For every integrable function $f : \Omega \to X$ there exists a sequence (f_n) of integrable step functions $\Omega \to X$ such that $\|f_n\| \leq \|f\|$ and $f_n \to f$ pointwise μ-almost everywhere.*
(b) Let $f : \Omega \to X$ be integrable. Let (f_n) be a sequence of integrable step functions such that $\|f_n\| \leq \|f\|$ and $f_n \to f$ pointwise μ-almost everywhere. Then

$$x := \lim_{n \to \infty} \int_\Omega f_n \, d\mu \text{ exists}$$

and

$$\|x\| \leq \int_\Omega \|f\| \, d\mu.$$

PROOF. (a) Let $f : \Omega \to X$ be integrable. Then $\|f\| : \Omega \to \mathbb{R}$ is integrable. Therefore there exists a sequence (g_n) of integrable step functions such that $0 \leq g_n \leq \|f\|$ and $g_n \to \|f\|$ pointwise μ-almost everywhere.

Since f is measurable, there exists a sequence $(\tilde{f}_n)$ of step functions $\Omega \to X$ such that $\tilde{f}_n \to f$ pointwise μ-almost everywhere.

Put

$$f_n := \frac{\tilde{f}_n \, g_n}{\|\tilde{f}_n\| + \frac{1}{n}}.$$

(b) For every integrable step function $g : \Omega \to X$ one has

$$\left\| \int_\Omega g \, d\mu \right\| \leq \int_\Omega \|g\| \, d\mu.$$

Hence, for every n, m

$$\left\| \int_\Omega f_n - f_m \, d\mu \right\| \leq \int_\Omega \|f_n - f_m\| \, d\mu,$$

and by Lebesgue's dominated convergence theorem the sequence $(\int_\Omega f_n \, d\mu)$ is a Cauchy sequence. When we put $x = \lim_{n \to \infty} \int_\Omega f_n \, d\mu$ then

$$\|x\| \leq \liminf_{n \to \infty} \int_\Omega \|f_n\| \, d\mu = \int_\Omega \|f\| \, d\mu.$$

$\square$

DEFINITION 3.11 (Bochner integral for integrable functions). Let $f : \Omega \to X$ be integrable. We define

$$\int_\Omega f \, d\mu := \lim_{n \to \infty} \int_\Omega f_n \, d\mu,$$

where (f_n) is a sequence of step functions $\Omega \to X$ such that $\|f_n\| \leq \|f\|$ and $f_n \to f$ pointwise μ-almost everywhere.

REMARK 3.12. The definition of the Bochner integral for integrable functions is independent of the choice of the sequence (f_n) of step functions, by Lemma 3.10.

REMARK 3.13. We will also use the follwing notation for the Bochner integral:

$$\int_\Omega f \quad \text{oder} \quad \int_\Omega f(t) \, d\mu(t),$$

and if $\Omega = (a, b)$ is an interval in $\mathbb{R}$:

$$\int_a^b f \quad \text{oder} \quad \int_a^b f(t) \, d\mu(t).$$

If $\mu = \lambda$ is the Lebesgue measure then we also write

$$\int_a^b f(t) \, dt.$$

LEMMA 3.14. *Let $f : \Omega \to X$ be integrable and $T \in \mathcal{L}(X, Y)$. Then $Tf : \Omega \to Y$ is integrable and*

$$\int_\Omega Tf \, d\mu = T \int_\Omega f \, d\mu.$$

PROOF. Exercise. $\square$

THEOREM 3.15 (Lebesgue, dominates convergence). *Let (f_n) be a sequence of integrable functions. Suppose there exists an integrable function $g : \Omega \to \mathbb{R}$ and an (integrable) measurable function $f : \Omega \to X$ such that $\|f_n\| \le g$ and $f_n \to f$ pointwise μ-almost everywhere. Then*

$$\int_\Omega f \, d\mu = \lim_{n \to \infty} \int_\Omega f_n \, d\mu.$$

PROOF. Exercise.   □

DEFINITION 3.16 ($\mathcal{L}^p$ spaces). For every $1 \le p < \infty$ we define

$$\mathcal{L}^p(\Omega; X) := \{f : \Omega \to X \text{ measurable} : \int_\Omega \|f\|^p \, d\mu < \infty\}.$$

We also define

$$\mathcal{L}^\infty(\Omega; X) := \{f : \Omega \to X \text{ measurable} : \exists C \ge 0 \text{ such that } \mu(\{\|f\| \ge C\}) = 0\}.$$

LEMMA 3.17. *For every $1 \le p < \infty$ we put*

$$\|f\|_p := \left(\int_\Omega \|f\|^p \, d\mu \right)^{1/p}.$$

We also put

$$\|f\|_\infty := \inf\{C \ge 0 : \mu(\{\|f\| \ge C\}) = 0\}.$$

Then $\| \cdot \|_p$ is a seminorm on $\mathcal{L}^p(\Omega; X)$ ($1 \le p \le \infty$).

REMARK 3.18. A function $\| \cdot \| : X \to \mathbb{R}_+$ on a real or complex vector space is called a *seminorm* if

    (i) $x = 0 \Rightarrow \|x\| = 0$,
    (ii) $\|\lambda x\| = |\lambda| \, \|x\|$ for every $\lambda \in \mathbb{K}$ and all $x \in X$,
    (iii) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$.

DEFINITION 3.19 (L^p spaces). For every $1 \le p \le \infty$ we put

$$\begin{aligned} N_p &:= \{f \in \mathcal{L}^p(\Omega; X) : \|f\|_p = 0\} \\ &= \{f \in \mathcal{L}^p(\Omega; X) : f = 0 \, \mu\text{-almost everywhere}\}. \end{aligned}$$

We define the quotient space

$$L^p(\Omega; X) := \mathcal{L}^p(\Omega; X)/N_p,$$

which is the space of all equivalence classes

$$[f] := f + N_p, \quad f \in \mathcal{L}^p(\Omega; X).$$

LEMMA 3.20. *For every $[f] \in L^p(\Omega; X)$ ($f \in \mathcal{L}^p(\Omega; X)$) the value*

$$\|[f]\|_p := \|f\|_p$$

is well defined, i.e. independent of the representant f. The function $\| \cdot \|_p$ is a norm on $L^p(\Omega; X)$. The space $L^p(\Omega; X)$ is a Banach space when equipped with this norm.

REMARK 3.21. As in the scalar case we will in the following identify *functions* $f \in \mathcal{L}^p(\Omega; X)$ with their *equivalence classes* $[f] \in L^p(\Omega; X)$, and we say that L^p is a *function space* although we should be aware that it is only a space of equivalence classes of functions.

REMARK 3.22. For $\Omega = (a, b)$ an interval in $\mathbb{R}$ and for $\mu = \lambda$ the Lebesgue measure we simply write

$$L^p(a, b; X) := L^p((a, b); X).$$

We can do so since the spaces $L^p([a, b]; X)$ and $L^p((a, b); X)$ coincide since the end points $\{a\}$ and $\{b\}$ have Lebesgue measure zero and there is no danger of confusion.

LEMMA 3.23. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then $C(\bar{\Omega}; X) \subset L^p(\Omega; X)$ for every $1 \leq p \leq \infty$.*

PROOF. Actually, for finite measure spaces, we have the more general inclusions

$$L^\infty(\Omega; X) \subset L^p(\Omega; X) \subset L^q(\Omega; X) \subset L^1(\Omega; X)$$

if $1 \leq q \leq p \leq \infty$. $\square$

LEMMA 3.24. *Let the measure space $(\Omega, \mathcal{A}, \mu)$ be such that $L^p(\Omega)$ is separable for $1 \leq p < \infty$ (e.g. $\Omega \subset \mathbb{R}^n$ be an open set with the Lebesgue measure). Let X be separable. Then $L^p(\Omega; X)$ is separable for $1 \leq p < \infty$.*

PROOF. By assumption the spaces $L^p(\Omega)$ and X are separable. Let $(h_n) \subset L^p(\Omega; X)$ and $(x_n) \subset X$ be two dense sequences. Then the set

$$\mathcal{F} := \{f : \Omega \to X : f = h_n \, x_m\}$$

is countable. It suffices to shows that $\mathcal{F} \subset L^p(\Omega; X)$ is total, i.e. $\operatorname{span} \mathcal{F}$ is dense in $L^p(\Omega; X)$. This is an exercise. $\square$

THEOREM 3.25. *Let Ω be as in lemma 3.24. Let $1 < p < \infty$ and assume that X is reflexive. Then the space $L^p(\Omega; X)$ is reflexive and*

$$L^p(\Omega; X)' \cong L^{p'}(\Omega; X').$$

PROOF. Without proof. $\square$

4. Vector-valued Sobolev spaces

DEFINITION 4.1 (Sobolev spaces). Let $-\infty \leq a < b \leq \infty$ and $1 \leq p \leq \infty$. We define

$$W^{1,p}(a, b; X) := \{u \in L^p(a, b; X) \ : \ \exists v \in L^p(a, b; X) \, \forall \varphi \in \mathcal{D}(a, b)$$

$$\int_a^b u\varphi' = - \int_a^b v\varphi\}.$$

Notation: $v =: u'$.

LEMMA 4.2. *For every* $-\infty \leq a < b \leq \infty$ *and every* $1 \leq p \leq \infty$ *one has* $W^{1,p}(a,b;X) \subset C^b(\overline{(a,b)};X)$. *For every* $u \in W^{1,p}(a,b;X)$ *and every* s, $t \in (a,b)$ *one has*

$$u(t) - u(s) = \int_s^t u'(r) \, dr.$$

5. L^p-maximal regularity

Let $A : X \supset D(A) \to X$ be a closed, linear, densely defined operator on X.

In this and the following sections, we consider the abstract linear inhomogeneous Cauchy problem

$$(5.1) \qquad \dot{u}(t) + Au(t) = f(t), \, t \in [0,T], \quad u(0) = 0.$$

Here $A : D(A) \to X$ is a closed, densely defined, linear operator on the Banach space X and $f \in L^p(0,T;X)$ for some $1 \leq p \leq \infty$.

DEFINITION 5.1. (a) A function $u \in C^1([0,T];X) \cap C([0,T];D(A))$ is called a *classical solution* if $u(0) = 0$ and if u satisfies the differential equation (5.1) for every $t \in [0,T]$.

(b) A function $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ is called a *(L^p) strong solution* if $u(0) = 0$ and if u satisfies the differential equation (5.1) for almost every $t \in [0,T]$.

DEFINITION 5.2. We say that A has L^p-*maximal regularity* (on $(0,T)$) if for every $f \in L^p(0,T;X)$ there exists a unique strong solution $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ of the problem (5.1).

By definition, if A has L^p-maximal regularity, then the Cauchy problem (5.1) is uniquely solvable in the space $W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$, for every $f \in L^p(0,T;X)$. It will be convenient to introduce the *maximal regularity space*

$$MR_p(a,b;X,D(A)) := W^{1,p}(a,b;X) \cap L^p(a,b;D(A)) \quad (-\infty \leq a < b \leq \infty)$$

which is naturally endowed with the norm

$$\|u\|_{MR_p} := \|u\|_{W^{1,p}(a,b;X)} + \|u\|_{L^p(a,b;D(A))}.$$

Since $W^{1,p}(a,b;X)$ and $L^p(a,b;D(A))$ are Banach spaces, $MR_p(a,b;X,D(A))$ is also a Banach space. If there is no danger of confusion, we will write $MR_p(a,b)$ instead of $MR_p(a,b;X,D(A))$.

We will first show that the definition of L^p-maximal regularity is independent of $T > 0$, so that it suffices in fact to speak only of L^p-maximal regularity. On the way we will also show that the initial value problem is uniquely solvable in the maximal regularity space, at least for certain initial values. For this, we first need the following locality lemma.

LEMMA 5.3. *Assume that A has L^p-maximal regularity on $(0,T)$. If $f \in L^p(0,T;X)$ is zero on the interval $(0,T')$ (with $0 < T' \leq T$), and if $u \in MR_p(0,T)$ is the corresponding solution of (5.1), then $u = 0$ on $(0,T')$.*

PROOF. Define the function
$$g(t) := \begin{cases} f(t+T') & \text{if } 0 \leq t \leq T-T', \\ 0 & \text{if } T-T' < t \leq T. \end{cases}$$

Then $g \in L^p(0,T;X)$. By definition of L^p-maximal regularity, there exists a unique $v \in MR_p(0,T)$ solution of (5.1).

Now define
$$w(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq T', \\ v(t-T') & \text{if } T' < t \leq T. \end{cases}$$

Then the function w restricted to the two intervals $[0,T']$ and $[T',T]$ belongs to the maximal regularity spaces $MR_p(0,T')$ and $MR_p(T',T)$, respectively. Since w is also continuous in T' (note that $v(0) = 0$!), we actually have $w \in MR_p(0,T)$.

It follows easily from the definition of w (the definition of g and v), that w solves the problem (5.1) for the function f. Since (5.1) is uniquely solvable, $u = w$, and therefore $u = 0$ on $[0,T']$. $\square$

We also have to define the *trace space*
$$Tr_p(X,D(A)) := \{u(0) : u \in MR_p(0,1)\},$$
which is naturally a Banach space for the norm
$$\|u_0\|_{Tr_p} := \inf\{\|u\|_{MR_p(0,1)} : u \in MR_p(0,1) \text{ and } u(0) = u_0\}.$$
If there is no danger of confusion, we simply write Tr_p instead of $Tr_p(X,D(A))$. The space Tr_p is called trace space since it contains all *traces* in $t = 0$ of functions $u \in MR_p(0,1)$. Note that we can evaluate $u(0)$ for every function u in the maximal regularity space $MR_p(0,T)$ since $W^{1,p}(0,T;X)$ is contained in the space of all continuous functions (see vector-valued Sobolev spaces in one dimension). Clearly, by definition, Tr_p is contained in X, and since for every $u_0 \in D(A)$ the constant function $u \equiv u_0$ belongs to $MR_p(0,1)$, one has the inclusions
$$D(A) \hookrightarrow Tr_p \hookrightarrow X.$$
It turns out that Tr_p is a strictly contained between $D(A)$ and X (see below). For the moment, however, we need not to know this.

LEMMA 5.4. *The following are true:*
(a) For every $T > 0$ and every $0 \leq t \leq T$ one has
$$Tr_p = \{u(t) : u \in MR_p(0,T)\}.$$
(b) One has the inclusion
$$MR_p(0,T) \subset C([0,T];Tr_p)$$

and there exists a constant $C \geq 0$ (depending on $T > 0$) such that

$$\|u\|_{C([0,T];Tr_p)} \leq C \|u\|_{MR_p} \text{ for every } u \in MR_p.$$

PROOF. The spaces $MR_p(0,T)$ and $MR_p(0,1)$ are isomorphic via the iso-morphism $u \mapsto u(\cdot T)$. Hence, for every $T > 0$,

$$Tr_p = \{u(0) : u \in MR_p(0,T)\},$$

and

$$\|u_0\|_{Tr_p,T} := \inf\{\|u\|_{MR_p(0,T)} : u \in MR_p(0,T) \text{ and } u(0) = u_0\}$$

defines an equivalent norm on Tr_p.

Given $u \in MR_p(0,T)$ we may define the extension $v \in MR_p(0,2T)$ by

$$v(t) := \begin{cases} u(t) & \text{if } 0 \leq t \leq T, \\ u(2T - t) & \text{if } T < t \leq 2T. \end{cases}$$

We define next the functions $u_t \in MR_p(0,T)$ by

$$u_t(s) := v(t+s), \quad 0 \leq s, \, t \leq T.$$

Then one sees that $u(t) = u_t(0) \in Tr_p$ for every $0 \leq t \leq T$ and since

$$t \mapsto u_t, \quad [0,T] \to MR_p[0,T]$$

is continuous, one obtains from the definition of the norm on Tr_p that

$$t \mapsto u(t), \quad [0,T] \to Tr_p$$

is continuous. Moreover,

$$\sup_{t \in [0,T]} \|u(t)\|_{Tr_p} \leq C \sup_{t \in [0,T]} \|u(t)\|_{Tr_p,2T} \leq C \|v\|_{MR_p(0,2T)} = 2C \|u\|_{MR_p(0,T)}.$$

$\square$

THEOREM 5.5 (Initial value problem). *Assume that A has L^p-maximal reg-ularity on $(0,T)$. Then for every $u_0 \in Tr_p$ there exists a unique $u \in MR_p(0,T)$ solution of the problem*

$$\dot{u}(t) + Au(t) = 0, \, t \in [0,T], \quad u(0) = u_0.$$

PROOF. *Existence:* Let $u_0 \in Tr_p$. By definition of Tr_p and Lemma 5.4, there exists $v \in MR_p(0,T)$ such that $v(0) = u_0$. By definition of L^p-maximal regularity, there exists $w \in MR_p(0,T)$ solution of

$$\dot{w}(t) + Aw(t) = \dot{v}(t) + Av(t), \, t \in [0,T], \quad w(0) = 0.$$

Now put $u := v - w$.

Uniqueness: Let u and v be two solutions of the initial value problem. Then $u - v$ is a solution of the same initial value problem with initial value u_0 replaced by 0. The solution for that problem, however, is unique by definition of L^p-maximal regularity. Hence, $u = v$. $\square$

THEOREM 5.6 (Independence of $T > 0$). *Assume that A has L^p-maximal regularity on $(0, T)$. Then A has L^p-maximal regularity on $(0, T')$ for every $T' > 0$.*

PROOF. Fix $T' \in (0, T]$. Let $f \in L^p(0, T'; X)$ and extend f by zero on $(T', T]$. The resulting function is denoted by $\tilde{f}$. Let $\tilde{u} \in MR_p(0, T)$ be the unique solution of (5.1). Let u be restriction of $\tilde{u}$ to the interval $[0, T']$. Then $u \in MR_p(0, T')$ is a solution of (5.1) with T replaced by T'. Hence, we have proved existence of strong solutions.

In order to prove uniqueness, by linearity, it suffices to show that $u = 0$ is the only solution of (5.1) with T replaced by T' and with $f = 0$. So let u be some solution of the problem

$$\dot{u}(t) + Au(t) = 0, \ t \in [0, T'], \quad u(0) = 0.$$

$\square$

REMARK 5.7. Theorem 5.6 allows us just to speak of L^p-maximal regularity of an operator A or of the Cauchy problem (5.1) without making the $T > 0$ precise.

6. Bilinear forms and L^p-maximal regularity

Throughout this section, V and H are two real Hilbert spaces such that

$$V \hookrightarrow H = H' \hookrightarrow V',$$

with dense injections.

DEFINITION 6.1. We call a form $a : V \times V \to \mathbb{R}$ *elliptic* if there exists $\omega \in \mathbb{R}$ such that the form $a_\omega : V \times V \to \mathbb{R}$ defined by $a_\omega(u, v) := a(u, v) + \omega(u, v)_H$ is coercive, i.e. if there exists $\eta > 0$ such that

$$a(u, u) + \omega \|u\|_H^2 \geq \eta \|u\|_V^2 \text{ for every } u \in V.$$

DEFINITION 6.2. We call a matrix $A \in \mathbb{R}^{n \times n}$ *elliptic* if there exists a constant $\eta > 0$ such that

$$A\xi \bar{\xi} \geq \eta |\xi|^2 \text{ for every } \xi \in \mathbb{C}^n.$$

We call a matrix-valued function $A \in L^\infty(\Omega, \mathbb{R}^{n \times n})$ *uniformly elliptic* if there exists a constant $\eta > 0$ such that

$$\operatorname{Re} A(x)\xi \bar{\xi} \geq \eta |\xi|^2 \text{ for every } \xi \in \mathbb{C}^n, \ x \in \Omega.$$

In the above definition, if the matrix A is symmetric then ellipticity of A is equivalent to saying that A is positive definite.

EXAMPLE 6.3. Take the bilinear form a from Example 2.4 and assume that $A \in L^\infty(\Omega, \mathbb{R}^{n \times n})$ is uniformly elliptic. Then a is bounded and elliptic. Indeed,

$$
\begin{aligned}
a(u, u) + \eta \|u\|_{L^2}^2 &= \int_\Omega A(x) \nabla u \, \nabla u + \eta \|u\|_{L^2}^2 \\
&\geq \eta \int_\Omega |\nabla u|^2 + \eta \|u\|_{L^2}^2 \\
&= \eta \|u\|_{H_0^1}^2.
\end{aligned}
$$

THEOREM 6.4 (J.-L. Lions). *Let $a : V \times V \to \mathbb{R}$ be a bilinear, bounded, elliptic form and let $A : V \to V'$ be the associated operator. Let $T > 0$. Then for every $f \in L^2(0, T; V')$ and every $u_0 \in H$ there exists a unique solution $u \in W^{1,2}(0, T; V') \cap L^2(0, T; V)$ of the problem*

$$
\dot{u}(t) + Au(t) = f(t), \ t \in [0, T], \quad u(0) = u_0.
$$

We will prove this theorem in several steps. First, we study the maximal regularity space $MR_2(a, b; V', V)$.

LEMMA 6.5. *For every $-\infty \leq a < b \leq \infty$ and every $u \in MR_2(a, b; V', V)$ the function $t \mapsto \frac{1}{2} \|u(t)\|_H^2$ is differentiable almost everywhere and*

(6.1) $$\frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 = \langle \dot{u}(t), u(t) \rangle_{V', V}.$$

PROOF. One shows by regularisation that the space $C_c^1(\overline{(a, b)}; V)$ is dense in $MR_2(a, b; V', V)$. Then one verifies that for functions $u \in C_c^1(\overline{(a, b)}; V)$ the equality (6.1) is true, using also the equality

$$
(\dot{u}(t), u(t))_H = \langle \dot{u}(t), u(t) \rangle_{V', V}.
$$

The claim then follows by an approximation argument. □

LEMMA 6.6. *For every $-\infty \leq a < b \leq \infty$ one has*

$$
MR_2(a, b; V', V) \hookrightarrow C_0(\overline{(a, b)}; H).
$$

In particular, $Tr_2(V', V) \subset H$.

PROOF. First let $a = -\infty$ and $b = \infty$. The space $C_c^1(\mathbb{R}; V)$ is dense in $MR_2(-\infty, \infty; V', V)$, by a regularisation argument. For every $u \in C_c^1(\mathbb{R}; V)$

and every $t \in \mathbb{R}$ one has

$$
\begin{aligned}
\|u(t)\|_H^2 &= \int_{-\infty}^{t} \frac{d}{ds} \|u(s)\|_H^2 \, ds \\
&= 2 \int_{-\infty}^{t} (\dot{u}(s), u(s))_H \, ds \\
&= 2 \int_{-\infty}^{t} \langle \dot{u}(s), u(s) \rangle_{V',V} \, ds \\
&\leq 2 \|\dot{u}\|_{L^2(\mathbb{R};V')} \|u\|_{L^2(\mathbb{R};V)} \\
&\leq \|\dot{u}\|_{L^2(\mathbb{R};V')}^2 + \|u\|_{L^2(\mathbb{R};V)}^2 \\
&\leq 2 \|u\|_{MR_2(-\infty,\infty;V',V)}^2 .
\end{aligned}
$$

Hence, the embedding operator

$$
(C_c^1(\mathbb{R};V), \| \cdot \|_{MR_2(-\infty,\infty;V',V)}) \to (C_0(\mathbb{R};H), \| \cdot \|_{C_0(\mathbb{R};H)})
$$

is bounded. Since $C_c^1(\mathbb{R};V)$ is dense in $MR_2(-\infty,\infty;V',V)$, we find that

(6.2) $\qquad\qquad MR_2(-\infty,\infty;V',V) \hookrightarrow C_0(\mathbb{R};H).$

Next, let a, b be arbitrary. There exists a linear bounded extension operator

$$
E : MR_2(a,b;V',V) \to MR_2(-\infty,\infty;V',V)
$$

with the property that Eu restricted to (a,b) equals u (exercice!). Using that the restriction operator

$$
C_0(\mathbb{R};H) \to C_0(\overline{(a,b)};H), \quad u \mapsto u|_{(a,b)}
$$

is linear and bounded, too, the claim follows by considering the composition of the extension operator E, the embedding (6.2), and this restriction operator.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

LEMMA 6.7 (Uniqueness). *Let A be as in Theorem 6.4. Then for every $f \in L^2(0,T;V')$ and every $u_0 \in H$ there exists at most one solution $u \in MR_2(0,T;V',V)$ of the problem*

$$
\dot{u}(t) + Au(t) = f(t), \, t \in [0,T], \quad u(0) = u_0.
$$

PROOF. By linearity, it suffices to prove that if $u \in MR_2(0,T;V',V)$ is a solution of

$$
\dot{u}(t) + Au(t) = 0, \, t \in [0,T], \quad u(0) = 0,
$$

then $u = 0$. So let u be a solution of this problem. Then, by ellipticity of the form a, and by Lemma 6.5,

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 &= \langle \dot{u}(t), u(t) \rangle_{V',V} \\
&\leq \langle \dot{u}(t), u(t) \rangle_{V',V} + \langle Au(t), u(t) \rangle_{V',V} + \omega \|u(t)\|_H^2 \\
&= \omega \|u(t)\|_H^2 .
\end{aligned}
$$

As a consequence,

$$\|u(t)\|_H^2 \leq e^{2\omega t}\|u(0)\|_H^2 = 0 \text{ for every } t \in [0, T].$$

Hence, $u = 0$. $\qquad \square$

PROOF OF THEOREM 6.4. By Lemma 6.7 it remains only to prove existence of a solution. This we will do by a Galerkin approximation.

Let $(w_n) \subset V$ be a linearly independent sequence such that span $\{w_n\}$ is dense in V (here we use that V is separable in order to ensure existence of such a sequence). Let $V_m := \text{span} \{w_n : 1 \leq n \leq m\}$. As a finite dimensional vector space, the space V_m is a closed subspace of V, H and V'. It will be equipped with the norms coming from these three spaces. Note that the three norms are equivalent on V_m.

The restriction of the form a to the space V_m (i.e. the form $a_m : V_m \times V_m \to \mathbb{R}$ defined by $a_m(u, v) := a(u, v)$) is a bilinear, bounded and elliptic form. Hence, there exists an operator $A_m : (V_m, \|\cdot\|_V) \to (V_m, \|\cdot\|_{V'})$ such that

$$\langle A_m u, v \rangle_{V',V} = a_m(u, v) = a(u, v) \text{ for every } u, v \in V_m.$$

Consider the ordinary differential equation

(6.3) $\qquad \dot{u}_m(t) + A_m u_m(t) = f_m(t), \ t \in [0, T], \quad u_m(0) = u_0^m,$

where $u_0^m := P_m u_0$, $P_m : H \to H$ being the orthogonal projection in H onto V_m, and where $f_m(t) = P_m f(t)$ (note that the orthogonal projection P_m extends to a bounded projection $V' \to V'$ and that $\|P_m\|_{V'} = 1$).

The problem (6.3) is a linear inhomogeneous ordinary differential equation in a finite dimensional Hilbert/Banach space and we know from the theory of ordinary differential equations that (6.3) admits a unique solution $u_m \in C^1([0, T]; V_m)$.

Multiplying the equation (6.3) with u_m, we obtain

$$(\dot{u}_m(t), u_m(t))_H + (A_m u_m(t), u_m(t))_H = (f_m(t), u_m(t))_H$$

and hence, by ellipticity of a,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_H^2 + \eta \|u_m(t)\|_V^2 \\
\leq\ & \langle \dot{u}_m(t), u_m(t) \rangle_{V',V} + \langle A_m u_m(t), u_m(t) \rangle_{V',V} + \omega \|u_m(t)\|_H^2 \\
=\ & \langle f_m(t), u_m(t) \rangle_{V',V} + \omega \|u_m(t)\|_H^2 \\
\leq\ & C_\eta \|f_m(t)\|_{V'}^2 + \frac{\eta}{2}\|u_m(t)\|_V^2 + \omega \|u_m(t)\|_H^2.
\end{aligned}
$$

As a first consequence, we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_H^2 \leq C_\eta \|f_m(t)\|_{V'}^2 + \omega \|u_m(t)\|_H^2.$$

By Gronwall's lemma, this implies for every $t \in [0,T]$,

$$\|u_m(t)\|_H^2 \leq e^{2\omega t}\|u_0^m\|_H^2 + C_\eta \int_0^t e^{2\omega(t-s)}\|f_m(s)\|_{V'}^2 \, ds$$

$$\leq C\big(\|u_0\|_H^2 + \int_0^T \|f(s)\|_{V'}^2 \, ds\big),$$

where $C \geq e^{2\omega T}(C_\eta + 1)$. When we plug this inequality into the above inequality, then we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_H^2 + \frac{\eta}{2}\|u_m(t)\|_V^2 \leq C_\eta\|f_m(t)\|_{V'}^2 + C\big(\|u_0\|_H^2 + \int_0^T \|f(s)\|_{V'}^2 \, ds\big).$$

This implies, when integrating over $(0,T)$,

$$\frac{\eta}{2}\int_0^T \|u_m(t)\|_V^2 \, dt + \frac{1}{2}\|u_m(T)\|_H^2 \leq C\big(\|u_0\|_H^2 + \int_0^T \|f(s)\|_{V'}^2 \, ds\big).$$

This and the equation (6.3) imply

$$\int_0^T \|\dot{u}_m(t)\|_{V'}^2 \, dt \leq \int_0^T \|A_m u_m(t)\|_{V'}^2 \, dt + \int_0^T \|f_m(t)\|_{V'}^2 \, dt$$

$$\leq M\int_0^T \|u_m(t)\|_V^2 \, dt + \int_0^T \|f(t)\|_{V'}^2 \, dt$$

$$\leq \frac{2(MC+1)}{\eta}\big(\int_0^T \|f(t)\|_{V'}^2 \, dt + \|u_0\|_H^2\big).$$

Summing up, we see that there exists a constant $C \geq 0$ such that for every $m \geq 1$

$$\int_0^T \|u_m(t)\|_V^2 \, dt + \int_0^T \|\dot{u}_m(t)\|_{V'}^2 \, dt \leq C\left(\int_0^T \|f(t)\|_{V'}^2 \, dt + \|u_0\|_H^2\right).$$

The right-hand side is finite by assumption and does not depend on $m \geq 1$.

As a consequence, (u_m) is bounded in $L^2(0,T;V)$ and $W^{1,2}(0,T;V')$. By reflexivity, we can thus extract a subsequence (which we denote again by (u_m)) such that

$$u_m \rightharpoonup u \text{ in } L^2(0,T;V) \text{ and}$$
$$\dot{u}_m \rightharpoonup v \text{ in } L^2(0,T;V').$$

This means that for every $\varphi \in L^2(0,T;V')$

$$\lim_{m\to\infty} \int_0^T \langle \varphi(t), u_m(t)\rangle_{V',V} = \int_0^T \langle \varphi(t), u(t)\rangle_{V',V}$$

and for every $\varphi \in L^2(0,T;V)$

$$\lim_{m\to\infty} \int_0^T \langle \dot{u}_m(t), \varphi(t)\rangle_{V',V} = \int_0^T \langle v(t), \varphi(t)\rangle_{V',V}.$$

Let $w \in V$ be any fixed vector and let $\varphi \in \mathcal{D}(0,T)$ be a scalar test function. Then an integration by parts yields

$$
\begin{aligned}
\langle \int_0^T u(t)\dot{\varphi}(t)\, dt, w \rangle_{V',V} &= \int_0^T \langle u(t), \dot{\varphi}(t)w \rangle_{V',V} \\
&= \lim_{m \to \infty} \int_0^T \langle u_m(t), \dot{\varphi}(t)w \rangle_{V',V} \\
&= -\lim_{m \to \infty} \int_0^T \langle \dot{u}_m(t), \varphi(t)w \rangle_{V',V} \\
&= -\int_0^T \langle v(t), \varphi(t)w \rangle_{V',V} \\
&= -\langle \int_0^T v(t)\varphi(t), w \rangle_{V',V}.
\end{aligned}
$$

Since this equality is true for every $w \in V$, we find that

$$
\int_0^T u\dot{\varphi} = -\int_0^T v\varphi \text{ in } V',
$$

for every test function $\varphi \in \mathcal{D}(0,T)$. Hence, by definition of the Sobolev space, the function u belongs to $W^{1,2}(0,T;V')$ and $\dot{u} = v$.

Since $A : V \to V'$ is a bounded linear operator, we find that

$$
Au_m \rightharpoonup Au \text{ in } L^2(0,T;V'),
$$

i.e. for every $\varphi \in L^2(0,T;V)$

$$
\lim_{m \to \infty} \int_0^T \langle Au_m(t), \varphi(t) \rangle_{V',V} = \int_0^T \langle Au(t), \varphi(t) \rangle_{V',V}.
$$

Note that also

$$
A_m u_m \rightharpoonup Au \text{ in } L^2(0,T;V').
$$

In order to see this, let $w \in V_n$ for some $n \geq 1$ and let $\varphi \in L^2(0,T)$. Then, for every $m \geq n$,

$$
\begin{aligned}
\int_0^T \langle A_m u_m(t), \varphi(t)w \rangle_{V',V} &= \int_0^T a_m(u_m(t), \varphi(t)w) \\
&= \int_0^T a(u_m(t), \varphi(t)w) \\
&= \int_0^T \langle Au_m(t), \varphi(t)w \rangle_{V',V} \\
&\to \int_0^T \langle Au(t), \varphi(t)w \rangle_{V',V} \quad (m \to \infty).
\end{aligned}
$$

Since $\bigcup_n V_n$ is dense in V, and since therefore the set $\{\varphi(\cdot)v : \varphi \in L^2(0,T), v \in \bigcup_n V_n\}$ is total in $L^2(0,T;V)$, the last claim follows.

Note also that $f_m \to f$ in $L^2(0, T; V')$. We thus obtain for every $\varphi \in L^2(0, T; V)$

$$
\begin{aligned}
\int_0^T \langle \dot{u}(t), \varphi(t) \rangle_{V', V} &= \lim_{m \to \infty} \int_0^T \langle \dot{u}_m(t), \varphi(t) \rangle_{V', V} \\
&= \lim_{m \to \infty} \int_0^T \langle f_m(t) - A_m u_m(t), \varphi(t) \rangle_{V', V} \\
&= \int_0^T \langle f(t) - Au(t), \varphi(t) \rangle_{V', V}.
\end{aligned}
$$

Since this equality holds for every $\varphi \in L^2(0, T; V)$, we find that

$$
\dot{u}(t) + Au(t) = f(t) \text{ for a.e. } t \in [0, T],
$$

i.e. u is a solution of our differential equation.

It remains to show that u verifies also the initial condition. Let $w \in V$ and let $\varphi \in C^1([0, T])$ be such that $\varphi(0) = 1$ and $\varphi(T) = 0$. Then an integration by parts yields on the one hand

$$
\int_0^T \langle u, \dot{\varphi} w \rangle_{V', V} = -\langle u(0), w \rangle_{V', V} - \int_0^T \langle \dot{u}, \varphi w \rangle_{V', V}.
$$

On the other hand, since $u_0^m \to u_0$ in H,

$$
\begin{aligned}
\int_0^T \langle u, \dot{\varphi} w \rangle_{V', V} &= \lim_{m \to \infty} \int_0^T \langle u_m, \dot{\varphi} w \rangle_{V', V} \\
&= \lim_{m \to \infty} \left(-\langle u_m(0), w \rangle_{V', V} - \int_0^T \langle \dot{u}_m, \varphi w \rangle_{V', V} \right. \\
&= -\lim_{m \to \infty} \langle u_0^m, w \rangle_{V', V} - \int_0^T \langle \dot{u}, \varphi w \rangle_{V', V} \\
&= -\langle u_0, w \rangle_{V', V} - \int_0^T \langle \dot{u}, \varphi w \rangle_{V', V}.
\end{aligned}
$$

Comparing both equalities, we obtain

$$
\langle u(0), w \rangle_{V', V} = \langle u_0, w \rangle_{V', V}
$$

for every $w \in V$. Hence, $u(0) = u_0$. $\qquad \square$

REMARK 6.8. Lions' Theorem says that the operator $A : V \to V'$, considered as a closed, unbounded operator on V' with domain $D(A) = V$, has L^2-maximal regularity. This follows when regarding the inhomogeneous problem with initial value $u(0) = 0$.

Moreover, it follows from Lions' Theorem, especially the solvability of the initial value problem, that $H \subset Tr_2(V', V)$. Together with Lemma 6.6 this implies the identity

$$
Tr_2(V', V) = H,
$$

i.e. a complete description of the trace space in this special situation.

EXAMPLE 6.9. We consider the linear heat equation with Dirichlet boundary conditions and initial condition

(6.4)
$$u_t(t,x) - \Delta u(t,x) = f(t,x) \quad (t,x) \in \Omega_T,$$
$$u(t,x) = 0 \qquad\qquad x \in \partial\Omega,$$
$$u(0,x) = u_0(x) \qquad\qquad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is any open set and $\Omega_T = (0,T) \times \Omega$. This heat equation can be abstractly rewritten as a linear Cauchy problem

$$\dot{u}(t) + Au(t), \ t \in [0,T], \quad u(0) = u_0,$$

where $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Dirichlet-Laplace operator associated with the form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$a(u,v) = \int_\Omega \nabla u \nabla v.$$

It follows from Lions' Theorem that for every $f \in L^2(0,T;H^{-1}(\Omega))$ and every $u_0 \in L^2(\Omega)$ there exists a unique solution

$$u \in W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$

of this problem.

A particular situation arises when $\Omega = \mathbb{R}^n$ (in this case the boundary conditions are obsolete) and when $f = 0$, because in this case we have an explicit formula for the solution. Using the heat kernel, one has for every $u_0 \in L^2(\mathbb{R}^n)$ the solution u of the heat equation is given by

$$u(t,x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{|x-y|^2/(4t)} u_0(y) \, dy.$$

Lions' Theorem implies that this solution belongs to the space

$$u \in W^{1,2}(0,T;H^{-1}(\mathbb{R}^n)) \cap L^2(0,T;H^1(\mathbb{R}^n)) \cap C([0,T];L^2(\mathbb{R}^n)).$$

7. Interpolation and L^p-maximal regularity

The aim of this section is to study interpolation results for maximal regularity. In particular, as a corollary, we will prove that the operator $A_H : D(A_H) \to H$ associated with a bounded, elliptic bilinear form $a : V \times V \to \mathbb{R}$ has L^2-maximal regularity.

Given two Banach spaces X, Y such that $Y \hookrightarrow X$, and given $T > 0$, $p \in [1,\infty]$, we define the *maximal regularity space*

$$MR_p(0,T;X,Y) := W^{1,p}(0,T;X) \cap L^p(0,T;Y)$$

and the *trace space*

$$Tr_p(X,Y) := \{u(0) : u \in MR_p(0,T;X,Y)\}$$

with usual norms. The maximal regularity space and the trace space used up to now was obtained for $Y = D(A)$. The definition of $Tr_p(X, Y)$ is independent of $T > 0$.

LEMMA 7.1 (Interpolation of a bounded linear operator). *Let X_1, X_2, Y_1, Y_2 be four Banach spaces such that $Y_i \hookrightarrow X_i$ for $i = 1$, 2. Let $S : X_1 \to X_2$ be a bounded linear operator such that its restriction to Y_1 is a bounded linear operator $S : Y_1 \to Y_2$. Then, for every $p \in [1, \infty]$, the restriction of S to $Tr_p(X_1, Y_1)$ is a bounded linear operator $S : Tr_p(X_1, Y_1) \to Tr_p(X_2, Y_2)$ and*

$$\|S\|_{\mathcal{L}(Tr_p(X_1,Y_1), Tr_p(X_2,Y_2))} \leq \max\{\|S\|_{\mathcal{L}(X_1,X_2)}, \|S\|_{\mathcal{L}(Y_1,Y_2)}\}.$$

Moreover, if $S : X_1 \to X_2$ and $S : Y_1 \to Y_2$ are invertible, then $S : Tr_p(X_1, Y_1) \to Tr_p(X_2, Y_2)$ is invertible, too.

PROOF. Let $u_0 \in Tr_p(X_1, Y_1)$. By definition of the trace space, and by definition of the norm on the trace space, for every $\varepsilon > 0$ there exists $u \in MR_p(0, T; X_1, Y_1)$ such that $\|u\|_{MR_p} \leq (1 + \varepsilon)\|u_0\|_{Tr_p}$. Put $v(t) := Su(t)$. Then $v \in MR_p(0, T; X_2, Y_2)$ and

$$
\begin{aligned}
\|v\|_{MR_p} &= \|v\|_{W^{1,p}(0,T;X_2)} + \|v\|_{L^p(0,T;Y_2)} \\
&\leq \|S\|_{\mathcal{L}(X_1,X_2)} \|u\|_{W^{1,p}(0,T;X_1)} + \|S\|_{\mathcal{L}(Y_1,Y_2)} \|u\|_{L^p(0,T;Y_1)} \\
&\leq \max\{\|S\|_{\mathcal{L}(X_1,X_2)}, \|S\|_{\mathcal{L}(Y_1,Y_2)}\} \|u\|_{MR_p} \quad < \quad \infty.
\end{aligned}
$$

In particular, $v(0) = Su(0) = Su_0 \in Tr_p(X_2, Y_2)$ and

$$
\begin{aligned}
\|Su_0\|_{Tr_p(X_2,Y_2)} &\leq \|v\|_{MR_p} \leq \max\{\|S\|_{\mathcal{L}(X_1,X_2)}, \|S\|_{\mathcal{L}(Y_1,Y_2)}\} \|u\|_{MR_p} \leq \\
&\leq (1 + \varepsilon) \max\{\|S\|_{\mathcal{L}(X_1,X_2)}, \|S\|_{\mathcal{L}(Y_1,Y_2)}\} \|u_0\|_{Tr_p(X_1,Y_1)}.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, the first claim follows.

If $S : X_1 \to X_2$ and $S : Y_1 \to Y_2$ are invertible, then one applies the above argument to the operator $S^{-1} : X_2 \to X_1$ whose restriction to Y_2 is a bounded linear operator $S^{-1} : Y_2 \to Y_1$. $\square$

REMARK 7.2. The situation in the interpolation lemma. The boundedness of $S : X_1 \to X_2$ and $S : Y_1 \to Y_2$ is assumed, the boundedness of S in the interpolation spaces is a consequence:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{S} & X_2 \\
\uparrow & & \uparrow \\
Tr_p(X_1, Y_1) & \xrightarrow{S} & Tr_p(X_2, Y_2) \\
\uparrow & & \uparrow \\
Y_1 & \xrightarrow{S} & Y_2
\end{array}
$$

The following lemma will not be proved.

LEMMA 7.3. *Let X, Y be two Banach spaces such that $Y \hookrightarrow X$. Then, for every $p \in [1, \infty]$,*

$$Tr_p(L^p(0, T; X), L^p(0, T; Y)) = L^p(0, T; Tr_p(X, Y)).$$

Let $A : D(A) \to X$ be a closed linear operator on X. This implies that the domain $D(A)$ equipped with the graph norm is a Banach space. We can define the restriction of A to the space $D(A)$ by

$$
\begin{aligned}
D(A_1) &:= \{x \in D(A) : Ax \in D(A)\}, \\
A_1 x &:= Ax.
\end{aligned}
$$

This restriction is again a closed linear operator (exercice!).

LEMMA 7.4. *Let $A : D(A) \to X$ be a closed linear operator on X and define $A_1 : D(A_1) \to D(A)$ as above. Assume that $A + \omega I$ is invertible and that A has L^p-maximal regularity. Then A_1 has L^p-maximal regularity.*

PROOF. The operator $A + \omega I$ is an isomorphism between the Banach spaces $D(A)$ and X, and also between the Banach spaces $D(A_1)$ and $D(A)$.

Let $f \in L^p(0, T; D(A))$. Then $(A + \omega I)f \in L^p(0, T; X)$ and by L^p-maximal regularity there exists a unique solution $u \in MR_p(0, T; X, D(A))$ of the problem

$$\dot{u} + Au = (A + \omega I)f, \, t \in [0, T], \quad u(0) = 0.$$

Multiply this differential equation by $(A + \omega I)^{-1}$ and put $v(t) := (A + \omega I)^{-1} u(t)$. Then $v \in MR_p(0, T; D(A), D(A_1))$ is solution of the problem

$$\dot{v} + Av = f, \, t \in [0, T], \quad v(0) = 0.$$

This solution is unique since every solution in $MR_p(0, T; D(A), D(A_1))$ is also a solution in $MR_p(0, T; X, D(A))$ of the same problem and the solution in the latter space is unique by L^p-maximal regularity.

As a consequence, $A_1 : D(A_1) \to D(A)$ has L^p-maximal regularity. $\quad\square$

REMARK 7.5. One can repeat the above argument and restrict the operator A to the space $D(A_1)$ which is also a Banach space. This restriction is given by

$$
\begin{aligned}
D(A_2) &:= \{x \in D(A_1) : Ax \in D(A_1)\}, \\
A_2 x &:= Ax,
\end{aligned}
$$

and it is also a closed linear operator. By iteration, one can define closed linear operators

$$
\begin{aligned}
D(A_k) &:= \{x \in D(A_{k-1}) : Ax \in D(A_{k-1})\}, \\
A_k x &:= Ax,
\end{aligned}
$$

and one obtains the following picture:

$$
\begin{array}{ccccc}
D(A) & \xrightarrow{A} & X & =: X_0 \\
\uparrow & & \uparrow & \\
D(A_1) & \xrightarrow{A} & D(A) & =: X_1 \\
\uparrow & & \uparrow & \\
D(A_2) & \xrightarrow{A} & D(A_1) & =: X_2 \\
\vdots & & \vdots &
\end{array}
$$

If $A + \omega I$ is invertible and if A has L^p-maximal regularity, then each operator A_k has L^p-maximal regularity.

Even more is true: we know from the interpolation lemma (Lemma 7.1) that A is also a closed linear operator on the interpolation spaces between X and $D(A)$. In the following theorem we prove that if $A + \omega I$ is invertible and if A has L^p-maximal regularity, then also the restriction of A to $Tr_p(X, D(A))$ has L^p-maximal regularity.

THEOREM 7.6. *Let $A : D(A) \to X$ be a closed linear operator on X and define its restriction to $Tr_p(X, D(A))$ by*

$$
\begin{aligned}
D(A_{Tr_p}) &:= & Tr_p(D(A), D(A_1)), \\
A_{Tr_p} x &:= & Ax.
\end{aligned}
$$

Assume that $A + \omega I$ is invertible and that A has L^p-maximal regularity. Then A_{Tr_p} has L^p-maximal regularity.

PROOF. Define

$$
MR_p^0(0, T; X, D(A)) := \{u \in MR_p(0, T; X, D(A)) : u(0) = 0\}
$$

and define the operator

$$
\begin{aligned}
S : MR_p^0(0, T; X, D(A)) &\to & L^p(0, T; X), \\
u &\mapsto & \dot{u} + Au.
\end{aligned}
$$

The operator S is clearly bounded. Moreover, the operator A has L^p-maximal regularity if and only if the operator S is invertible. Hence, by assumption, S is invertible.

The restriction of S to the space $MR_p(0, T; D(A), D(A_1))$ is a bounded operator with values in $L^p(0, T; D(A))$, and, by Lemma 7.4, this restriction is also invertible.

By Lemma 7.3, we have

$$
Tr_p(L^p(0, T; D(A)), L^p(0, T; D(A_1))) = L^p(0, T; Tr_p(D(A), D(A_1))),
$$

and

$$
Tr_p(W^{1,p}(0, T; X); W^{1,p}(0, T; D(A))) = W^{1,p}(0, T; Tr_p(X, D(A))).
$$

It then follows that

$$Tr_p(MR_p(0,T;X,D(A)), MR_p(0,T;D(A),D(A_1))) =$$
$$= MR_p(0,T;Tr_p(X,D(A)), Tr_p(D(A),D(A_1))).$$

By the Interpolation Lemma (Lemma 7.1), the restriction

$$S : MR_p^0(0,T;Tr_p(X,D(A)), Tr_p(D(A),D(A_1))) \to L^p(0,T;Tr_p(X,D(A))),$$
$$u \mapsto \dot{u} + Au.$$

is bounded and invertible. This means that the operator A_{Tr_p} has L^p-maximal regularity. □

REMARK 7.7. One has the equality

$$Tr_p(D(A),D(A_1)) = \{x \in D(A) : Ax \in Tr_p(X,D(A))\},$$

so that A_{Tr_p} is really the restriction of A to the space $Tr_p(X,D(A))$.

In order to prove this equality, let $u_0 \in Tr_p(D(A),D(A_1)) \subset D(A)$. Then there exists $u \in MR_p(0,T;D(A),D(A_1))$ such that $u(0) = u_0$. Put $v(t) := (A+\omega I)u(t)$. Then $v \in MR_p(0,T;X,D(A))$ and thus $(A+\omega I)u_0 \in Tr_p(X,D(A))$. Hence, $u_0 \in D(A_{Tr_p})$. The other inclusion is proved similarly, using the invertibility of $A+\omega I$.

REMARK 7.8. As before, the procedure of considering restrictions to intermediate spaces can be repeated on the smaller spaces $X_1 = D(A)$, $X_2 = D(A_1)$, etc.. One thus obtains the following picture:

$$
\begin{array}{ccccc}
D(A) & \xrightarrow{A} & X & =: X_0 \\
\uparrow & & \uparrow & \\
Tr_p(D(A),D(A_1)) & \xrightarrow{A} & Tr_p(X,D(A)) & =: X_{1/p'} \\
\uparrow & & \uparrow & \\
D(A_1) & \xrightarrow{A} & D(A) & =: X_1 \\
\uparrow & & \uparrow & \\
Tr_p(D(A_1),D(A_2)) & \xrightarrow{A} & Tr_p(D(A),D(A_1)) & =: X_{1+1/p'} \\
\uparrow & & \uparrow & \\
D(A_2) & \xrightarrow{A} & D(A_1) & =: X_2 \\
\vdots & & \vdots &
\end{array}
$$

If $A+\omega I$ is invertible and if A has L^p-maximal regularity, then each operator in this picture has L^p-maximal regularity.

COROLLARY 7.9. *Let $a : V \times V \to \mathbb{R}$ be a bilinear, bounded, elliptic form and let $A_H : D(A_H) \to H$ be the associated operator on H. Then A_H has L^2-maximal regularity. In particular, for every $f \in L^2(0,T;H)$ and every*

$u_0 \in V$ *there exists a unique solution* $u \in W^{1,2}(0,T;H) \cap L^2(0,T;D(A_H))$ *of the problem*

$$\dot{u}(t) + Au(t) = f(t),\ t \in [0,T], \quad u(0) = u_0.$$

PROOF. By Lions' Theorem (Theorem 6.4, see also Remark 6.8), the operator $A : V \to V'$ associated with the form a has L^2-maximal regularity.

By ellipticity of the form a and by the theorem of Lax-Milgram, the operator $A + \omega I$ is invertible. Hence, by Theorem 7.6, the restriction of A to the trace space $Tr_2(V',V)$ has L^2-maximal regularity, too.

But by Remark 6.8, this trace space is equal to H and the restriction of A to the space H is nothing else than A_H. Hence, A_H has L^2-maximal regularity.

For the second statement, one has to prove that $Tr_2(H,D(A_H)) = V$. □

EXAMPLE 7.10. We consider again the linear heat equation (6.4) with Dirichlet boundary conditions and initial condition from Example 6.9. From the results in this section follows that for every $u_0 \in H_0^1(\Omega)$ and every $f \in L^2(0,T;L^2(\Omega))$ the heat equation (6.4) admits a unique solution

$$u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;D(\Delta_{L_2})) \cap C([0,T];H_0^1(\Omega)),$$

where $D(\Delta_{L^2})$ is the domain in $L^2(\Omega)$ of the Laplace operator with Dirichlet boundary conditions. If $\Omega = (a,b)$ is a bounded interval, then $D(\Delta_{L^2}) = H^2(a,b) \cap H_0^1(a,b)$ (exercice). One also has $D(\Delta_{L^2}) = H^2(\Omega) \cap H_0^1(\Omega)$ if $\Omega \subset \mathbb{R}^N$ has smooth boundary, but this result is more difficult to prove and will be omitted.

Note that one can identify the spaces $L^2(0,T;L^2(\Omega))$ and $L^2((0,T) \times \Omega)$ in a natural way so that the inhomogeneity f is actually a real valued function on the product $(0,T) \times \Omega$ as suggested in the heat equation (6.4).

CHAPTER 3

Nonlinear parabolic equations

1. Existence and uniqueness of local solutions

Let X and D be two Banach spaces such that D is densely and continuously embedded into X. Let $1 < p < \infty$ and $u_0 \in Tr_p(X, D)$ be fixed, and let $A : Tr_p(X, D) \to \mathcal{L}(D, X)$ and $F : Tr_p(X, D) \to X$ be locally lipschitz continuous functions.

In this section, we consider the quasilinear problem

$$(1.1) \qquad \dot{u} + A(u)u + F(u) = 0, \, t \geq 0, \quad u(0) = u_0.$$

Note that for every $u \in Tr_p(X, D)$ the operator $A(u)$ is a linear operator $D \to X$ which may be considered as an unbounded operator on X with domain D. The problem (1.1), however, is not linear even if $F = 0$, since the linear operator $A(u)$ depends on u. We call the problem (1.1) *quasilinear*.

THEOREM 1.1 (Existence and uniqueness). *Under the above assumptions on u_0, A and F, and if $A(u_0)$ has L^p-maximal regularity, the quasilinear problem (1.1) admits a unique local solution $u \in MR_p(0, T; X, D)$.*

PROOF. We put $A_0 := A(u_0)$. By assumption, the operator $A_0 : D \to X$ has L^p-maximal regularity. Let $T > 0$ and consider the set

$$\tilde{M} := \{u \in MR_p(0, T; X, D) : u(0) = u_0\}$$

with the induced metric (induced from the norm in MR_p). Functions in $\tilde{M}$ already satisfy the initial condition from (1.1).

Consider the nonlinear map

$$\begin{aligned} R : \tilde{M} &\to L^p(0, T; X), \\ u &\mapsto (A_0 - A(u))u - F(u). \end{aligned}$$

It follows from the embedding $MR_p(0, T; X, D) \hookrightarrow C([0, T]; Tr_p(X, D))$ and the assumptions on A and F that the map R is well defined.

Consider also the solution map

$$\begin{aligned} S : L^p(0, T; X) &\to \tilde{M}, \\ f &\mapsto Sf := u, \end{aligned}$$

which assigns to every function $f \in L^p(0, T; X)$ the unique solution in $\tilde{M}$ of the problem

$$\dot{u}(t) + A_0 u(t) = f(t), \, t \in [0, T], \quad u(0) = u_0.$$

By L^p-maximal regularity of A_0, this solution map is well defined, too.

By definition of the two maps above, a function $u \in MR_p(0, T; X, D)$ is a solution of the quasilinear problem (1.1) if and only if $u \in \tilde{M}$ and $SRu = u$, i.e. if u is a fixed point of SR. We have thus reduced the problem of existence to a fixed point problem which we will solve by using Banach's fixed point theorem.

Let $S_0 : L^p(0, T; X) \to MR_p^0(0, T; X, D)$ be the solution operator which assigns to every function $f \in L^p(0, T; X)$ the unique solution $u := Sf$ of the problem

$$\dot{u}(t) + A_0 u(t) = f(t), \ t \in [0, T], \quad u(0) = 0.$$

There exists a constant $C_S \geq 0$ independent of $T \in (0, 1]$ such that $\|S_0\| \leq C_S$ for every $T \in (0, T]$.

Moreover, for every $T > 0$ and every $u \in MR_p^0(0, T; X, D)$ one has

(1.2) $$\|u\|_{C([0,T];Tr_p)} \leq 2 \|u\|_{MR_p^0(0,T;X,D)}.$$

In fact, every function $u \in MR_p^0(0, T; X, D)$ can be extended to a function $\bar{u} \in MR_p^0(0, \infty; X, D)$ by setting

$$\bar{u}(t) := \begin{cases} u(t) & \text{if } 0 \leq t \leq T, \\ u(2T - t) & \text{if } T \leq t \leq 2T, \\ 0 & \text{if } 2T \leq t, \end{cases}$$

and for this particular extension one has

$$\|\bar{u}\|_{MR_p(0,\infty;X,D)} \leq 2 \|u\|_{MR_p(0,T;X,D)}.$$

Note that in this reasoning it is important that $u(0) = 0$! As a consequence, by definition of the norm in the trace space, for every $t \geq 0$,

$$\|u(t)\|_{Tr_p} \leq \|\bar{u}(t + \cdot)\|_{MR_p(0,1;X,D)} \leq 2 \|u\|_{MR_p(0,T;X,D)},$$

and the inequality (1.2) follows.

By local Lipschitz continuity of A and F, there exist $r > 0$, $L \geq 0$ such that for every v_0, w_0 in the closed ball $\bar{B}(u_0, r) \subset Tr_p(X, D)$ one has

$$\|A(v_0) - A(w_0)\|_{\mathcal{L}(D,X)} \leq L \|v_0 - w_0\|_{Tr_p} \text{ and}$$
$$\|F(v_0) - F(w_0)\|_X \leq L \|v_0 - w_0\|_{Tr_p}.$$

Let

$$r' := \min\{r, \frac{1}{4C_S L}\}.$$

Let $\tilde{u} := S0$ be the solution of the problem

$$\dot{u}(t) + A_0 u(t) = 0, \ t \in [0, T], \quad u(0) = u_0.$$

Choose first $\varepsilon > 0$ such that

$$\varepsilon < \min\{1, \frac{r'}{6}\} = \min\{1, \frac{r}{6}, \frac{1}{12C_S L}\}.$$

and choose then $T \in (0, 1]$ sufficiently small so that

$$\sup_{t \in [0,T]} \|\tilde{u}(t) - u_0\|_{Tr_p} \leq \frac{r'}{3},$$

$$\|\tilde{u}\|_{L^p(0,T;D)} \leq \varepsilon \text{ and}$$

$$T^{\frac{1}{p}} \leq \min\{\frac{\varepsilon}{8C_S L}, \frac{\varepsilon}{4C_S \|F(\tilde{u})\|_{C([0,1];X)}}\}.$$

For this fixed $T > 0$, let $\tilde{M}$ be defined as above, and define the closed subset

$$M := \{u \in \tilde{M} : \|u - \tilde{u}\|_{MR_p(0,T;X,D)} \leq \varepsilon\}.$$

For every $u \in M$ one has

$$\sup_{t \in [0,T]} \|u(t) - u_0\|_{Tr_p} \leq \frac{2r'}{3}$$

by the triangle inequality, the inequality (1.2), and the choice of ε. Moreover, by the triangle inequality, one has

$$\|u\|_{L^p(0,T;D)} \leq 2\varepsilon.$$

We first prove that SR maps the set M into itself. In order to see this, let $u \in M$. Then

$$
\begin{aligned}
\|SRu - \tilde{u}\|_{MR_p} &= \|SRu - S0\|_{MR_p} \\
&= \|S_0 Ru\|_{MR_p} \\
&\leq \|S_0\| \left(\|(A_0 - A(u))u\|_{L^p(0,T;X)} + \|F(u)\|_{L^p(0,T;X)} \right) \\
&\leq C_S \sup_{t \in [0,T]} \|A_0 - A(u(t))\|_{\mathcal{L}(D,X)} \|u\|_{L^p(0,T;D)} + \\
&\quad + C_S T^{\frac{1}{p}} \left(\|F(u) - F(\tilde{u})\|_{C([0,T];X)} + \|F(\tilde{u})\|_{C([0,T];X)} \right) \\
&\leq C_S L \sup_{t \in [0,T]} \|u(t) - u_0\|_{Tr_p} \|u\|_{L^p(0,T;D)} + \\
&\quad + C_S T^{\frac{1}{p}} L \|u - \tilde{u}\|_{C([0,T];Tr_p)} + \frac{\varepsilon}{4} \\
&\leq C_S L r' 2\varepsilon + C_S T^{\frac{1}{p}} L 2\varepsilon + \frac{\varepsilon}{4} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \quad = \quad \varepsilon.
\end{aligned}
$$

This proves that $SRu \in M$.

We next prove that SR is a strict contraction. In order to see this, let u, $v \in M$. Then

$$
\begin{aligned}
\|SRu - SRv\|_{MR_p} &= \|S_0(Ru - Rv)\|_{MR_p} \\
&\leq \|S_0\| \left(\|(A_0 - A(u))u - (A_0 - A(v))v\|_{L^p(0,T;X)} + \right. \\
&\qquad + \|F(u) - F(v)\|_{L^p(0,T;X)} \Big) \\
&\leq C_S \left(\|(A(u) - A(v))u\|_{L^p(0,T;X)} + \right. \\
&\qquad + \|(A_0 - A(v))(u - v)\|_{L^p(0,T;X)} + \\
&\qquad + T^{\frac{1}{p}} \|F(u) - F(v)\|_{C([0,T];X)} \Big).
\end{aligned}
$$

For the first term on the right-hand side we have

$$
\begin{aligned}
\|(A(u) - A(v))u\|_{L^p(0,T;X)} &\leq \sup_{t \in [0,T]} \|A(u(t)) - A(v(t))\|_{\mathcal{L}(D,X)} \|u\|_{L^p(0,T;D)} \\
&\leq L\|u - v\|_{C([0,T];Tr_p)} \, 2\varepsilon \\
&\leq 4L\varepsilon \, \|u - v\|_{MR_p}.
\end{aligned}
$$

For the second term on the right-hand side we have

$$
\begin{aligned}
\|(A_0 - A(v))(u - v)\|_{L^p(0,T;X)} &\leq \sup_{t \in [0,T]} \|A_0 - A(v(t))\|_{\mathcal{L}(D,X)} \|u - v\|_{L^p(0,T;D)} \\
&\leq Lr' \, \|u - v\|_{MR_p} \\
&\leq \frac{1}{4C_S} \|u - v\|_{MR_p}.
\end{aligned}
$$

And for the third term on the right-hand side we have

$$
\begin{aligned}
\|F(u) - F(v)\|_{C([0,T];X)} &\leq L \, \|u - v\|_{C([0,T];Tr_p)} \\
&\leq 2L \, \|u - v\|_{MR_p}.
\end{aligned}
$$

Putting the above estimates together we find that

$$
\begin{aligned}
\|SRu - SRv\|_{MR_p} &\leq C_S\left(4L\varepsilon + \frac{1}{4C_S} + 2T^{\frac{1}{p}}L\right)\|u - v\|_{MR_p} \\
&\leq \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right)\|u - v\|_{MR_p} = \frac{3}{4}\|u - v\|_{MR_p}.
\end{aligned}
$$

Hence, $SR : M \to M$ is a strict contraction. By Banach's fixed point theorem, there exists a unique fixed point $u \in M$ which by construction of S and R is a solution of the quasilinear problem (1.1). $\square$

EXAMPLE 1.2 (Semilinear heat equation). Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be open, $T > 0$, $\Omega_T = (0,T) \times \Omega$, $f \in C^1(\mathbb{R})$, and consider the semilinear heat equation with Dirichlet boundary conditions and initial condition:

$$
(1.3) \quad
\begin{cases}
u_t(t,x) - \Delta u(t,x) + f(u(t,x)) = 0 & (t,x) \in \Omega_T, \\
u(t,x) = 0 & x \in \partial\Omega, \\
u(0,x) = u_0(x) & x \in \Omega,
\end{cases}
$$

Assume in addition that there exists some constant $C \geq 0$ such that

$$|f'(s)| \leq |s|^{\frac{2}{N-2}} \text{ for every } s \in \mathbb{R}.$$

COROLLARY 1.3. *Then for every $u_0 \in H_0^1(\Omega)$ there exists a unique local solution $u \in W^{1,2}(0, T'; L^2(\Omega)) \cap L^2(0, T'; D(\Delta_{L^2}))$ of the problem (6.4).*

PROOF. In fact, we may apply Theorem 1.1, where $A \equiv -\Delta_{L^2}$ is the constant Dirichlet-Laplace operator associated with the form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ given by $a(u,v) = \int_\Omega \nabla u \nabla v$ (which has L^2-maximal regularity on $L^2(\Omega)$ by Corollary 7.9) and where $F : H_0^1(\Omega) \to L^2(\Omega)$ is the *Nemytski operator* associated with the function f: $F(u)(x) := f(u(x))$. One only has to show that this Nemytski operator is locally Lipschitz continuous. For this, we need the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega),$$

which holds for $q = \frac{2N}{N-2}$. From this embedding, the growth condition on f, the mean value theorem and Hölder's inequality we deduce that for every u, $v \in H_0^1(\Omega)$

$$
\begin{aligned}
\|F(u) - F(v)\|_{L^2}^2 &= \int_\Omega |f(u) - f(v)|^2 \\
&= \int_\Omega |f'(\xi(x))(u(x) - v(x))|^2 \\
&\leq \int_\Omega |\xi(x)|^{\frac{4}{N-2}} |u - v|^2 \\
&\leq \int_\Omega ||u| \vee |v||^{\frac{4}{N-2}} |u - v|^2 \\
&\leq \left(\int_\Omega ||u| \vee |v||^{\frac{2N}{N-2}} \right)^{\frac{2}{N}} \left(\int_\Omega |u - v|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} \\
&\leq \max\{\|u\|_{H_0^1}, \|v\|_{H_0^1}\}^{\frac{4}{N-2}} \|u - v\|_{H_0^1}^2.
\end{aligned}
$$

Hence, for every $R > 0$ there exists a Lipschitz constant $L \geq 0$ such that for every u, $v \in H_0^1(\Omega)$ with norms less than R one has

$$\|F(u) - F(v)\|_{L^2} \leq L \|u - v\|_{H_0^1}.$$

In fact, one may take $L := R^{\frac{2}{N-2}}$. In other words, F is Lipschitz continuous on bounded subsets of $H_0^1(\Omega)$. $\square$

EXAMPLE 1.4 (Cahn-Hilliard equation). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain which is regular in the send that the domain of the Dirichlet-Laplace operator $A = A_{L^2}$ which is associated with the form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, $a(u,v) = \int_\Omega \nabla u \nabla v$, is given by

$$D(A) = H^2 \cap H_0^1(\Omega).$$

In this example, we consider the Cahn-Hilliard equation

(1.4)
$$\begin{cases} u_t(t,x) + \Delta(\Delta u(t,x) - f(u(t,x))) = 0 & (t,x) \in \Omega_T, \\ u(t,x) = \Delta u(t,x) = 0 & x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases}$$

where as before $\Omega_T = (0,T) \times \Omega$, and where the nonlinearity f belongs to $C^3(\mathbb{R})$. No growth restrictions on f are imposed. We will apply Theorem 1.1 in order to prove existence and uniqueness of solutions of the Cahn-Hilliard equation, at least for initial values $u_0 \in H^2 \cap H_0^1(\Omega)$. For this, we start by the following lemma.

LEMMA 1.5. *The bilinear form* $b : H^2 \cap H_0^1(\Omega) \times H^2 \cap H_0^1(\Omega) \to \mathbb{R}$ *defined by*

$$b(u,v) = \int_\Omega \Delta u \Delta v$$

is bounded and elliptic.

PROOF. The Dirichlet-Laplace operator on $L^2(\Omega)$ is a closed linear operator. In particular, its domain $D(A)$ is a Banach space for the graph norm

$$\|u\|_{D(A)} = \|u\|_{L^2} + \|\Delta u\|_{L^2}.$$

By assumption, the domain $D(A)$ coincides with $H^2 \cap H_0^1(\Omega)$. Hence, by the bounded inverse theorem, the graph norm is equivalent to the usual norm in $H^2 \cap H_0^1$, which is the norm induced from H^2. This implies that there exists $\eta > 0$ such that

$$b(u,u) + \|u\|_{L^2}^2 = \|\Delta u\|_{L^2}^2 + \|u\|_{L^2}^2 \geq \eta \|u\|_{H^2 \cap H_0^1}^2.$$

Hence, the form b is elliptic. Boundedness of b is straightforward. $\qquad\square$

LEMMA 1.6. *Let* $B = B_{L^2}$ *be the operator on* L^2 *which is associated with the form* b *from Lemma 1.5. Then*

$$D(B) = \{u \in H^2 \cap H_0^1(\Omega) : \Delta u \in H^2 \cap H_0^1(\Omega)\}$$

and $Bu = \Delta^2 u$.

PROOF. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

LEMMA 1.7. *Let* $f \in C^3(\mathbb{R})$ *be such that* $f(0) = 0$. *Then the Nemytski operator* $F : H^2(\Omega) \to H^2(\Omega)$ *given by* $F(u)(x) := f(u(x))$ *is well-defined and Lipschitz continuous on bounded subsets of* $H^2(\Omega)$.

PROOF. We use the Sobolev embeddings

$$H^2(\Omega) \hookrightarrow L^\infty(\Omega) \text{ and } H^1(\Omega) \hookrightarrow L^4(\Omega),$$

which are valid since Ω is a subset of $\mathbb{R}^3$ (in fact, the above embeddings are also true for open subsets of $\mathbb{R}$ and $\mathbb{R}^2$, but they are not true in dimension 4 and higher dimensions).

Let $u \in H^2(\Omega)$. Then, by the Sobolev embeddings and by the fact that f and its derivatives are bounded on bounded intervals one obtains

$$f(u) \in L^2(\Omega),$$

$$\frac{\partial}{\partial x_i} f(u) = f'(u) \frac{\partial u}{\partial x_i} \in L^\infty \cdot \mathcal{L}^2 \subset L^2(\Omega)$$

and

$$\frac{\partial^2}{\partial x_j \partial x_i} f(u) = f''(u) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} + f'(u) \frac{\partial^2 u}{\partial x_j \partial x_i} \in L^\infty \cdot L^4 \cdot L^4 + L^\infty \cdot \mathcal{L}^2 \subset L^2(\Omega),$$

where in the last inclusion we also used Hölder's inequality. This proves that F is well-defined from $H^2(\Omega)$ into $H^2(\Omega)$.

We next show that F is Lipschitz continuous on bounded sets of $H^2(\Omega)$. Let $R > 0$ and let $u, v \in H^2(\Omega)$ such that $\|u\|_{H^2}, \|v\|_{H^2} \leq R$. By the Sobolev embeddings, there exists a constant $C \geq 0$ such that

$$\|u\|_{L^\infty}, \|v\|_{L^\infty}, \|\nabla u\|_{L^4}, \|\nabla v\|_{L^4} \leq CR.$$

Let $M = M(R) \geq 0$ be a constant such that

$$\|f^{(k)}\|_{L^\infty(-CR,CR)} \leq M \text{ for every } k \in \{0, 1, 2, 3\}.$$

Then

$$
\begin{aligned}
\|f(u) - f(v)\|_{L^2}^2 &= \int_\Omega |f(u) - f(v)|^2 \\
&\leq \|f'\|_{L^\infty(-CR,CR)}^2 \|u - v\|_{L^2}^2 \\
&\leq M^2 \|u - v\|_{H^2}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|\frac{\partial}{\partial x_i}(f(u) - f(v))\|_{L^2}^2 &\leq \int_\Omega |f'(u) - f'(v)|^2 |\frac{\partial u}{\partial x_i}|^2 + \int_\Omega |f'(v)|^2 |\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}|^2 \\
&\leq \|f''\|_{L^\infty(-CR,CR)}^2 \|u - v\|_{L^\infty}^2 \|u\|_{H^2}^2 + \\
&\qquad\qquad + \|f'\|_{L^\infty(-CR,CR)}^2 \|u - v\|_{H^2}^2 \\
&\leq (M^2 C^2 R^2 + M^2) \|u - v\|_{H^2}^2 \\
&= L_2(R)^2 \|u - v\|_{H^2}^2.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\|\frac{\partial^2}{\partial x_j \partial x_i}(f(u)-f(v))\|_{L^2}^2 \;\leq\; & \int_\Omega |f''(u)-f''(v)|^2 |\frac{\partial u}{\partial x_j}|^2 |\frac{\partial u}{\partial x_i}|^2 + \\
& + \int_\Omega |f''(v)|^2 |\frac{\partial u}{\partial x_j}-\frac{\partial v}{\partial x_j}|^2 |\frac{\partial u}{\partial x_i}|^2 + \\
& + \int_\Omega |f''(v)|^2 |\frac{\partial v}{\partial x_j}|^2 |\frac{\partial u}{\partial x_i}-\frac{\partial v}{\partial x_i}|^2 + \\
& + \int_\Omega |f'(u)-f'(v)|^2 |\frac{\partial^2 u}{\partial x_j \partial x_i}|^2 + \\
& + \int_\Omega |f'(v)|^2 |\frac{\partial^2 u}{\partial x_j \partial x_i}-\frac{\partial^2 v}{\partial x_j \partial x_i}|^2 \\
\leq\; & M^2 \|u-v\|_{L^\infty}^2 \|\frac{\partial u}{\partial x_j}\|_{L^4}^2 \|\frac{\partial u}{\partial x_i}\|_{L^4}^2 + \\
& + M^2 \|\frac{\partial u}{\partial x_j}-\frac{\partial v}{\partial x_j}\|_{L^4}^2 \|\frac{\partial u}{\partial x_i}\|_{L^4}^2 + \\
& + M^2 \|\frac{\partial v}{\partial x_j}\|_{L^4}^2 \|\frac{\partial u}{\partial x_i}-\frac{\partial v}{\partial x_i}\|_{L^4}^2 + \\
& + M^2 \|u-v\|_{L^\infty}^2 \|\frac{\partial^2 u}{\partial x_j \partial x_i}\|_{L^2}^2 + \\
& + M^2 \|\frac{\partial^2 u}{\partial x_j \partial x_i}-\frac{\partial^2 v}{\partial x_j \partial x_i}\|_{L^2}^2 \\
\leq\; & L_3(R)^2 \|u-v\|_{H^2}^2.
\end{aligned}
$$

Putting the last three estimates together we have thus proved that F is Lipschitz continuous on bounded subsets of $H^2(\Omega)$. □

THEOREM 1.8. *Assume that $f \in C^3(\mathbb{R})$ satisfies $f(0) = 0$ and that Ω is regular in the sense described above. Then for every $u_0 \in H^2 \cap H_0^1(\Omega)$ there exists a unique local solution*

$$ u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; D(B)) $$

of the Cahn-Hilliard equation (1.4). *Here, $D(B)$ is as in Lemma 1.6.*

EXAMPLE 1.9 (Quasilinear diffusion equation). More generally, if $\Omega \subset \mathbb{R}^N$ is open and bounded (!), we may also solve the following quasilinear problem:

$$
(1.5) \quad
\begin{cases}
u_t - \operatorname{div}(a(x, u)\nabla u) + f(x, u) = 0 & (t, x) \in \Omega_T, \\
u(t, x) = 0 & x \in \partial\Omega, \\
u(0, x) = u_0(x) & x \in \Omega,
\end{cases}
$$

where a, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are of class C^1. We assume that there exist constants C, L, $\eta > 0$ such that for every $x \in \Omega$, u, $v \in \mathbb{R}$

$$(H1) \quad C \geq a(x,u) \geq \eta > 0,$$

$$(H2) \quad |a(x,u) - a(x,v)| \leq L|u-v|,$$

$$(H3) \quad |f'(u)| \leq C\,|u|^{\frac{2}{N-2}}.$$

COROLLARY 1.10. *Then for every $u_0 \in L^2(\Omega)$ there exists a unique local solution $u \in W^{1,2}(0,T'; H^{-1}(\Omega)) \cap L^2(0,T'; H^1_0(\Omega))$ of the problem (6.4).*

PROOF. We apply again Theorem 1.1, but this time we work in the Hilbert space $H^{-1}(\Omega)$. The Nemytski operator $F : L^2(\Omega) \to H^{-1}(\Omega)$ defined by $F(u)(x) := f(x, u(x))$ is locally Lipschitz continuous by hypothesis (H3); the proof is very similar to that in the preceeding example, but we now use that $L^{\frac{2N}{N+2}}(\Omega)$ embeds continuously into $H^{-1}(\Omega)$ and we actually show that the Nemytski operator F is locally Lipschitz continuous from $L^2(\Omega)$ into $L^{\frac{2N}{N+2}}(\Omega)$.

Next, for every $u \in H^1_0(\Omega)$ we define the bilinear form $a(u) : H^1_0 \times H^1_0 \to \mathbb{R}$ by

$$a(u)(w_1, w_2) = \int_\Omega a(x, u(x)) \nabla w_1 \nabla w_2.$$

By hypothesis (H1), for every $u \in H^1_0(\Omega)$. $\qquad\square$

REMARK 1.11. It should be clear from the proof of Theorem 1.1 that one could actually also study non-autonomous (i.e. time-dependent) quasi-linear problems of the form

$$(1.6) \qquad \dot{u} + A(t,u)u + F(t,u) = 0,\ t \geq 0, \quad u(0) = u_0.$$

Here $A : [0,T] \times Tr_p(X,D) \to \mathcal{L}(D,X)$ and $F : [0,T] \times Tr_p(X,D) \to X$ are continuous functions which are locally Lipschitz continuous with respect to the second variable, uniformly with respect to the first one, and $u_0 \in Tr_p(X,D)$. The proof of the following theorem is an adaptation of the proof of Theorem 1.1.

THEOREM 1.12. *Under the above assumptions on u_0, A and F, and if $A(0, u_0)$ has L^p-maximal regularity, the quasilinear problem (1.6) admits a unique local solution $u \in MR_p(0, T'; X, D)$ $(T' \leq T)$.*

2. Regularity of solutions

In this section we want to study the regularity of solutions of the semilinear equation

$$(2.1) \qquad \dot{u} + Au + F(u) = 0,\ t \geq 0, \quad u(0) = u_0.$$

Here $A : D(A) \to X$ is a closed linear operator on the Banach space X, $F : Tr_p(X, D(A)) \to X$ is a locally Lipschitz continuous function, and $u_0 \in Tr_p(X, D(A))$. In fact, the proof of the regularity theorem will show that we could actually also study the quasilinear problem (1.1), but for the

sake of simplicity we restrict ourselves to the problem (2.1).

For the proof of our regularity theorem we need the following classical theorem from calculus (which is in fact also proved by using Banach's fixed point theorem, like our theorem of existence and uniqueness for the quasilinear problem).

THEOREM 2.1 (Implicit function theorem). *Let X, Y, Z be three Banach spaces and let $G : X \times Y \to Z$ be of class C^k for some $k \geq 1$. Assume that $G(\bar{x}, \bar{y}) = 0$ and assume that the partial derivative $\frac{\partial G}{\partial y}(\bar{x}, \bar{y}) : Y \to Z$ is boundedly invertible.*

Then there exists a neighbourhood $U \subset X$ of $\bar{x}$, a neighbourhood $V \subset Y$ of $\bar{y}$, and a function $g : U \to Y$ of class C^k such that

$$\{(x, y) \in U \times V : G(x, y) = 0\} = \{(x, g(x)) : x \in U\}.$$

If, in addition, the function G is analytic, then the implicit function g is analytic, too.

THEOREM 2.2 (Regularity for the quasilinear problem). *In the semilinear problem (2.1) assume in addition that A has L^p-maximal regularity and that F is of class C^k for some $k \geq 1$. Then the unique local solution $u \in MR_p(0, T; X, D(A))$ has the following regularity properties: for every $\tau > 0$*

$$\begin{aligned} u &\in W^{k+1,p}(\tau, T; X) \cap W^{k,p}(\tau, T; D(A)) \text{ and} \\ u &\in C^k(]0, T]; X) \cap C^{k-1}(]0, T]; D(A)). \end{aligned}$$

If F is of class C^∞, then in fact $u \in C^\infty(]0, T]; D(A))$, and if F is analytic, then u is analytic.

In fact, for the proof of Theorem 2.2 we will also need the following perturbation result.

THEOREM 2.3. *Let $A : D(A) \to X$ be a closed, linear operator on a Banach space X, and let $B \in \mathcal{L}(Tr_p, X)$. Assume that A has L^p-maximal regularity. Then $A + B$ has L^p-maximal regularity.*

PROOF. Saying that the operator A has L^p-maximal regularity is equivalent to saying that the operator

$$\begin{aligned} S_0 : MR_p^0(0, T; X, D(A)) &\to L^p(0, T; X), \\ u &\mapsto \dot{u} + Au, \end{aligned}$$

is invertible for some (for all) $T > 0$. Note that the norm of the inverse $\|S_0^{-1}\|$ is uniformly bounded in $T \in (0, 1]$.

We have to prove that the operator

$$\begin{aligned} S : MR_p^0(0, T; X, D(A)) &\to L^p(0, T; X), \\ u &\mapsto \dot{u} + Au + Bu, \end{aligned}$$

is invertible for some (for all) $T > 0$. There exists a constant $C \geq 0$ such that for every $T \in (0, 1]$

$$
\begin{aligned}
\|S_0^{-1}B\|_{\mathcal{L}(MR_p^0)} &\leq\ C\,\|B\|_{\mathcal{L}(MR_p^0, L^p)} \\
&\leq\ C \sup_{\|u\|_{MR_p^0} \leq 1} \|Bu\|_{L^p(0,T;X)} \\
&\leq\ CT^{1/p} \sup_{\|u\|_{MR_p^0} \leq 1} \|Bu\|_{C([0,T];X)} \\
&\leq\ CT^{1/p}\|B\|_{\mathcal{L}(Tr_p, X)} \sup_{\|u\|_{MR_p^0} \leq 1} \|u\|_{C([0,T];Tr_p)} \\
&\leq\ 2CT^{1/p}\|B\|_{\mathcal{L}(Tr_p, X)}.
\end{aligned}
$$

Hence, if $T > 0$ is small enough, then, by the Neumann series, $I + S_0^{-1}B$ is invertible in $MR_p^0(0, T; X, D)$. As a consequence, the operator $S = S_0(I + S_0^{-1}B)$ is invertible for $T > 0$ small enough. Hence, $A + B$ has L^p-maximal regularity. $\qquad\square$

PROOF OF THEOREM 2.2. Let $\varepsilon > 0$ be sufficiently small so that for every $\lambda \in (-\varepsilon, \varepsilon)$ the function

$$
u_\lambda(t) := u((1 + \lambda)t), \quad t \in [0, T],
$$

is well-defined. For every $\lambda \in (-\varepsilon, \varepsilon)$ the function $u_\lambda \in MR_p(0, T; X, D(A))$ is the unique solution of the semilinear problem

$$
\dot{u} + (1 + \lambda)Au + (1 + \lambda)F(u) = 0,\ t \geq 0, \quad u(0) = u_0.
$$

Consider the nonlinear operator

$$
\begin{aligned}
G : (-\varepsilon, \varepsilon) \times MR_p(0, T; X, D(A)) &\ \rightarrow\ L^p(0, T; X) \times Tr_p(X, D), \\
(\lambda, v) &\ \mapsto\ (\dot{v} + (1 + \lambda)Av + (1 + \lambda)F(v), v(0) - u_0).
\end{aligned}
$$

Since F is of class C^k, the operator G is also of class C^k as one easily verifies.

Moreover, by definition of G and the functions u_λ, one has

$$
G(\lambda, u_\lambda) = (0, 0) \text{ for every } \lambda \in (-\varepsilon, \varepsilon).
$$

We show that G actually satisfies the assumptions of the implicit function theorem in $(0, u)$. For this, we have to consider the partial derivative $\frac{\partial G}{\partial u}(0, u)$ which is the linear operator given by

$$
\begin{aligned}
\frac{\partial G}{\partial u}(0, u) : MR_p(0, T; X, D(A)) &\ \rightarrow\ L^p(0, T; X) \times Tr_p(X, D), \\
v &\ \mapsto\ (\dot{v} + Av + F'(u(t))v, v(0)).
\end{aligned}
$$

Since A has L^p-maximal regularity, and by a fixed point argument as in the proof of Theorem 1.1, for every $f \in L^p(0, T; X)$ and every $v_0 \in Tr_p(X, D(A))$ the non-autonomous, linear problem

$$
\dot{v} + Av + F'(u(t))v = f,\ t \geq 0, \quad v(0) = v_0,
$$

admits a unique solution $v \in MR_p(0, T; X, D(A))$. But this actually means nothing else than that the partial derivative $\frac{\partial G}{\partial u}(0, u)$ is boundedly invertible.

By the implicit function theorem, there exists $\varepsilon' \in (0, \varepsilon)$, a neighbourhood $U \subset MR_p(0, T; X, D(A))$ and an implicit function $g : (-\varepsilon', \varepsilon') \to U$ of class C^k such that

$$G(\lambda, g(\lambda)) = (0, 0),$$

and all solutions in $(-\varepsilon', \varepsilon') \times U$ of the equation $G(\lambda, v) = (0, 0)$ are of the form $(\lambda, g(\lambda))$. Since the elements (λ, u_λ) are solutions of this equation, we obtain $u_\lambda = g(\lambda)$.

Moreover, the function

$$\begin{aligned} g : (-\varepsilon', \varepsilon') &\to MR_p(0, T; X, D(A)), \\ \lambda &\mapsto u_\lambda = u((1 + \lambda)\cdot), \end{aligned}$$

is of class C^k. When calculating the consecutive derivates of g in $\lambda = 0$, we see that

$$t \mapsto t\dot{u}(t) \in MR_p,$$

$$\vdots$$

$$t \mapsto t^k u^{(k)}(t) \in MR_p,$$

and this yields the stated regularity of the solution u. $\qquad\square$

COROLLARY 2.4. *Assume that $A : D(A) \to X$ is a closed linear operator on a Banach space X and assume that A has L^p-maximal regularity. Then for every $u_0 \in Tr_p(X, D(A))$ the unique solution u of the linear problem*

$$\dot{u} + Au = 0, \, t \geq 0, \quad u(0) = u_0,$$

satisfies for every $k \geq 1$

$$u \in MR_p(0, T; X, D(A)) \cap C^\infty(]0, T]; D(A^k)).$$

In fact, the solution u is analytic on $(0, \infty)$.

PROOF. Applying Theorem 2.2, one obtains $u \in C^\infty([\tau, T; D(A))$ for every $0 < \tau < T$. From this and the Cauchy problem one obtains then $u \in L^p(\tau, T; D(A^2))$ and derivating in the Cauchy problem one successively obtains first $u \in C^\infty([\tau, T]; D(A^2))$ and then by induction $u \in C^\infty([\tau, T]; D(A^k))$ for every $k \geq 1$. $\qquad\square$

REMARK 2.5 (C_0-semigroups). A family $(S(t))_{t \geq 0}$ of bounded linear operators on a Banach space X is called a C_0-*semigroup* if

  (i) $S(0) = I$,
  (ii) $S(t + s) = S(t)S(s)$ for every $t, s \geq 0$, and
  (iii) for every $x \in X$ the function $t \mapsto S(t)x$ is continuous.

A closed linear operator $A : D(A) \to X$ is called the *generator* of a C_0-semigroup $(S(t))_{t\geq0}$ if for every $x \in X$ and $t \geq 0$ one has $\int_0^t S(s)x \; ds \in D(A)$ and $A \int_0^t S(s)x \; ds = S(t)x - x$.

Let $A : D(A) \to X$ be a closed linear operator on a Banach space X and assume that A has L^p-maximal regularity. Then for every $x \in Tr_p = Tr_p(X, D(A))$ there exists a unique solution $u \in MR_p$ of the problem

$$\dot{u} + Au = 0, \; t \geq 0, \quad u(0) = x.$$

If one puts $S(t)x := u(t)$ (u being the unique solution for the initial value $x \in Tr_p$), then $(S(t))_{t\geq0}$ is a C_0-semigroup on the trace space Tr_p. The orbits $S(\cdot)x$ being by definition the solutions of the above Cauchy problem, this semigroup is even *analytic* by Corollary 2.4.

In the special situation when $A : V \to V'$ is the operator associated with a bilinear, bounded and coercive form $a : V \times V \to \mathbb{R}$ ($V \hookrightarrow H = H' \hookrightarrow V$), then we obtain an analytic semigroup $(S(t))_{t\geq0}$ on the space H whose generator is actually the operator A_H.

EXAMPLE 2.6. Let $\Omega = (0,1)$, $f \in C^\infty(\mathbb{R})$, and consider the semilinear heat equation

(2.2)
$$\begin{cases} u_t - u_{xx} + f(u) = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases}$$

COROLLARY 2.7. *For every $u_0 \in H_0^1(0,1)$ there exists a unique local solution $u \in W^{1,2}(0,T; L^2(0,1)) \cap L^2(0,T; H^2 \cap H_0^1(0,1))$ of the semilinear heat equation (2.2) satisying*

$$u \in C^\infty((0,T) \times (0,1)).$$

PROOF. The negative Dirichlet-Laplace operator on $L^2(0,1)$ given by

$$-\Delta_{L^2} : H^2 \cap H_0^1(0,1) \to L^2(0,1), \quad u \mapsto -u_{xx},$$

has L^2-maximal regularity by Lions' theorem and by interpolation.

The Nemytski operator

$$F : H_0^1(0,1) \to L^2(0,1), \quad u \mapsto f(u),$$

is of class C^∞. In order to see this, one may use the embedding of $H_0^1(0,1)$ into $C_0(0,1)$. In particular, F is locally Lipschitz continuous.

By Theorem 1.1, the heat equation (2.2) admits a unique local solution in the maximal regularity space. By Theorem 2.2, this solution even satisfies

(2.3) $$u \in C^\infty(]0,T]; H^2(0,1)).$$

Note that for every $k \in \mathbb{N}$ one has

$$F(H^k(0,1)) \subset H^k(0,1),$$

and the restriction of the Nemytski operator F to $H^k(0,1)$ is again of class C^∞. In particular, by (2.3),

$$\partial_t u, \; f(u) \in C^\infty(]0,T]; H^2(0,1)),$$

which by the heat equation implies

$$u_{xx} \in C^\infty(]0,T]; H^2(0,1))$$

or

$$u \in C^\infty(]0,T]; H^4(0,1))$$

By induction, one shows that for every $k \in \mathbb{N}$

$$u \in C^\infty(]0,T]; H^k(0,1)) \hookrightarrow C^\infty(]0,T]; C^{k-1}([0,1])).$$

In particular, all the partial derivatives of u exist and are continuous. The claim follows. $\qquad\square$

3. Navier-Stokes equations: local existence of regular solutions

In this section we are looking for (regular) solutions

$$u : [0,T] \times \mathbb{R}^n \;\to\; \mathbb{R}^n,$$
$$p : [0,T] \times \mathbb{R}^n \;\to\; \mathbb{R}$$

of the Navier-Stokes equation

(3.1) $$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ \operatorname{div} u = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

The Navier-Stokes equation is actually a system of n equations which we may also write in the form

$$\begin{cases} \partial_t u_i - \Delta u_i + \sum_{j=1}^n u_j \partial_{x_j} u_i + \partial_{x_i} p = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ \operatorname{div} u = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

with $u = (u_1, \ldots, u_n)$.

The first step in solving the Navier-Stokes equation will be to rewrite it in an abstract functional analytic setting and to obtain an abstract nonlinear evolution equation of parabolic type.

We introduce the following spaces. First the Sobolev space of all *solenoidal* (i.e. divergence free) vector fields

$$H^1_\sigma := H^1_\sigma(\mathbb{R}^n; \mathbb{R}^n) := \{u \in H^1(\mathbb{R}^n; \mathbb{R}^n) : \operatorname{div} u = 0\},$$

and also the Lebesgue space of all solenoidal vector fields

$$L^2_\sigma := L^2_\sigma(\mathbb{R}^n; \mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n; \mathbb{R}^n) : \forall \varphi \in H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} u \cdot \nabla \varphi = 0\}.$$

LEMMA 3.1. *The spaces H_σ^1 and L_σ^2 are closed subspaces of H^1 and L^2, respectively. The space H_σ^1 is a dense subspace of L_σ^2.*

PROOF. The closedness of the two spaces is straightforward. Let $u \in H_\sigma^1$ and let $\varphi \in H^1(\mathbb{R}^n)$. Then an integration by parts yields

$$0 = \int_{\mathbb{R}^n} \operatorname{div} u \varphi = \int_{\mathbb{R}^n} u \cdot \nabla \varphi,$$

and hence $u \in L_\sigma^2$. The density of H_σ^1 in L_σ^2 follows from a usual regularization argument. $\square$

LEMMA 3.2. *Let*

$$L_\nabla^2 := L_\nabla^2(\mathbb{R}^n; \mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n; \mathbb{R}^n) : \exists v \in H^1(\mathbb{R}^n) \ s.t. \ u = \nabla v\}$$

be the space of all gradient vector fields. Then

$$L_\sigma^2 \perp L_\nabla^2,$$

i.e. the two spaces are orthogonal in $L^2(\mathbb{R}^n; \mathbb{R}^n)$.

PROOF. This follows from the very definition of L_σ^2. $\square$

Since L_σ^2 is a closed subspace of L^2, there exists the orthogonal projection $P : L^2 \to L^2$ onto the space L_σ^2. This projection is called the *Helmholtz projection*. By Lemma 3.2, $PL_\nabla^2 = \{0\}$.

Next, we need the following form $a : H_\sigma^1 \times H_\sigma^1 \to \mathbb{R}$ which is defined by

$$a(u, v) := \int_{\mathbb{R}^n} \nabla u \nabla v = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_{x_j} u_i \partial_{x_j} v_i.$$

The following lemma is straightforward.

LEMMA 3.3. *The form a defined above is bilinear, bounded and elliptic.*

Let $A = A_{L_\sigma^2}$ be the operator on L_σ^2 associated with the form a. The operator A is called the *Stokes operator*.

For every $u \in H_\sigma^2 = H^2 \cap H_\sigma^1$ and every $v \in H_\sigma^1$ an integration by parts yields

$$
\begin{aligned}
a(u, v) &= \int_{\mathbb{R}^n} \nabla u \nabla v \\
&= -\int_{\mathbb{R}^n} \Delta u v \\
&= -\int_{\mathbb{R}^n} \Delta u P v \\
&= \int_{\mathbb{R}^n} (-P\Delta u) v,
\end{aligned}
$$

where we have used that $Pv = v$ since P is the identity on L^2_σ. Hence, for every $u \in H^2_\sigma$ one has $u \in D(A)$ and

$$Au = -P\Delta u.$$

We will use in the following (without proof) that $D(A) = H^2_\sigma$. This can be shown by using the Fourier transform on $L^2(\mathbb{R}^n)$ and the Plancherel theorem.

LEMMA 3.4. *The Stokes operator $A : H^2_\sigma \to L^2_\sigma$ has L^2-maximal regularity.*

PROOF. This lemma is a direct consequence of Lions' theorem (Theorem 6.4) and interpolation. See in particular Corollary 7.9.   □

Assume for the moment, that the Navier-Stokes equation admits a solution (u, p) such that $u(t, \cdot) \in L^2$, $p(t, \cdot) \in H^1(\mathbb{R}^n)$ and such that the members $\partial_t u(t, \cdot)$ and $\Delta u(t, \cdot)$ belong to L^2 (so that necessarily also the nonlinear term belongs to L^2). Then for each t we can apply the Helmholtz projection to each member and we obtain the following equation

$$\begin{cases} \partial_t Pu - P\Delta u + P((u \cdot \nabla)u) = 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ \operatorname{div} u = 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

Here we have used that the Helmholtz projection applied to gradient vector fields gives 0. The resulting equation is only an equation in the unknown function u. Since $\operatorname{div} u = 0$, one has $u \in L^2_\sigma$, and the above equation can abstractly be rewritten in the space L^2_σ:

(3.2) $$\begin{cases} \dot{u} - Au + P((u \cdot \nabla)u) = 0, & t \in [0, T], \\ u(0) = u_0. \end{cases}$$

This is the equation which we will solve by abstract methods. Let

$$MR_2 = W^{1,2}(0, T; L^2_\sigma) \cap L^2(0, T; H^2_\sigma).$$

LEMMA 3.5. *Assume that $n = 2$ or $n = 3$. Then the operator*

$$\begin{aligned} B : MR_2 \times MR_2 & \to & L^2(0, T; L^2_\sigma), \\ (u, v) & \mapsto & P((u \cdot \nabla)v) \end{aligned}$$

is well-defined, bilinear and bounded (i.e. continuous).

PROOF. We will use the Sobolev embeddings

$$H^1_\sigma \hookrightarrow L^4 \text{ and } H^2_\sigma \hookrightarrow L^\infty$$

which hold true if $n = 2$ or $n = 3$.

These Sobolev embeddings imply the embeddings

$$MR_2 \hookrightarrow C([0, T]; H^1_\sigma) \hookrightarrow L^\infty(0, T; L^4)$$

and for every $u \in MR_2$ one has

$$\nabla u \in L^2(0, T; H^1_\sigma) \hookrightarrow L^2(0, T; L^4).$$

By Hölder's inequality, this implies for every u, $v \in MR_2$

$$(u \cdot \nabla)v \in L^2(0, T; L^2),$$

and

$$\|(u \cdot \nabla)v\|_{L^2(0,T;L^2)} \leq C \|u\|_{MR_2} \|v\|_{MR_2},$$

for some constant $C \geq 0$ independent of u and v. Hence, $P((u \cdot \nabla)v) \in L^2(0, T; L^2_\sigma)$, i.e. B is well-defined. Moreover, by the preceeding inequality, B is also bounded. $\qquad\square$

THEOREM 3.6. *Assume that $n = 2$ or $n = 3$. For every $u_0 \in H^1_\sigma$ the equation (3.2) admits a unique local solution*

$$u \in W^{1,2}(0, T; L^2_\sigma) \cap L^2(0, T; H^2_\sigma).$$

SKETCH OF THE PROOF. Let

$$\tilde{M} := \{u \in MR_2 : u(0) = u_0\}.$$

Define the nonlinear operator

$$\begin{aligned} R : \tilde{M} &\rightarrow L^2(0, T; L^2_\sigma), \\ u &\mapsto -P((u \cdot \nabla)u). \end{aligned}$$

This operator is well-defined by Lemma 3.5. Actually, for every $u \in \tilde{M}$ one has $Ru = -B(u, u)$.

Define in addition the operator

$$\begin{aligned} S : L^2(0, T; L^2_\sigma) &\rightarrow \tilde{M}, \\ f &\mapsto Sf, \end{aligned}$$

which assigns to every f the unique solution $u = Sf$ of the problem

$$\dot{u} + Au = f,\ t \in [0, T], \quad u(0) = u_0,$$

where A is the Stokes operator. The operator S is well-defined by L^2-maximal regularity of the Stokes operator (Lemma 3.4).

Then $u \in MR_2$ is a solution of the abstract Navier-Stokes equation (3.2) if and only if $u \in \tilde{M}$ is a fixed point of $SR : \tilde{M} \rightarrow \tilde{M}$.

The rest of the proof is very similar to the proof of Theorem 1.1 on existence and uniqueness of local solutions of the quasilinear problem. In particular, existence and uniqueness for the abstract Navier-Stokes equation follows from Banach's fixed point theorem. We omit this part of the proof. $\qquad\square$

COROLLARY 3.7. *Assume that $n = 2$ or $n = 3$. Then for every $u_0 \in H^1_\sigma$ there exists a unique local solution of the Navier-Stokes equation:*

$$\begin{aligned} u &\in W^{1,2}(0, T; L^2_\sigma) \cap L^2(0, T; H^2_\sigma) \cap C([0, T]; H^1_\sigma) \text{ and} \\ u, \nabla p &\in C^\infty(]0, T] \times \mathbb{R}^n; \mathbb{R}^n). \end{aligned}$$

PROOF. $\qquad\square$

4. Diffusion equations: comparison principle

In this section, we want to study order preservingness of semilinear diffusion equations of the form

(4.1)
$$\begin{cases} u_t - Lu + f(u) = 0, & (t,x) \in \mathbb{R}_+ \times \Omega, \\ u(t,x) = 0, & t \in \mathbb{R}_+ \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

where L is a second order elliptic operator of the form

$$Lu = \sum_{i,j} \partial_i a_{ij}(x) \partial_j u + \sum_i \left(b_i(x) \partial_i u + \partial_i(c_i(x) u) \right) + d(x) u.$$

We will work on the Hilbert space $H = L^2(\Omega)$. We say that a function $u \in L^2(\Omega)$ is *positive* (and we write $u \geq 0$) if $u(x) \geq 0$ almost everywhere. We also write $u \geq v$ if $u - v \geq 0$. For every pair of functions $u, v \in L^2(\Omega)$ we define the *supremum* $u \vee v$ and the *infimum* $u \wedge v$ respectively by

$$u \vee v(x) := \sup\{u(x), v(x)\} \text{ and } u \wedge v(x) := \inf\{u(x), v(x)\}..$$

Note that $u \vee v$ and $u \wedge v$ belong to $L^2(\Omega)$. For a function $u \in L^2(\Omega)$ we define the *positive part* u^+, the *negative part* u^- and the *absolute value* $|u|$ respectively by

$$u^+ := u \vee 0, \ u^- := (-u) \vee 0 \text{ and } |u| = u^+ + u^-.$$

Note that u^+, u^- and $|u|$ are positive, and $u = u^+ - u^-$. Note also that

$$\| \, |u| \, \|_{L^2} = \|u\|_{L^2}.$$

LEMMA 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. For every $u \in H_0^1(\Omega)$ one has u^+, $u^- \in H_0^1(\Omega)$ and*

$$\nabla u^+ = 1_{u \geq 0} \nabla u \text{ and } \nabla u^- = 1_{u \leq 0} \nabla u.$$

PROOF. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(t) := \begin{cases} 0, & t \leq 0, \\ t, & t \geq 0. \end{cases}$$

Moreover, for every $\varepsilon > 0$ we put

$$g_\varepsilon(t) := \begin{cases} 0, & t \leq 0, \\ \frac{t^2}{2\varepsilon}, & 0 < t < \varepsilon, \\ t - \frac{\varepsilon}{2}, & t \geq \varepsilon. \end{cases}$$

Note that $g_\varepsilon \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ with $\|g_\varepsilon'\|_\infty \leq 1$.

We first show that for every $u \in H_0^1(\Omega)$ one has $g_\varepsilon \circ u \in H_0^1(\Omega)$ and $\nabla g_\varepsilon \circ u = g_\varepsilon'(u) \nabla u$. In fact, given $u \in H_0^1(\Omega)$, there exists a sequence $(u_n) \in \mathcal{D}(\Omega)$ such that $u_n \to u$ in $H_0^1(\Omega)$. By the classical chain rule, for every $n \in \mathbb{N}$,

$g_\varepsilon \circ u_n \in H_0^1(\Omega)$ (even $C_c^1(\Omega)$) and $\nabla(g_\varepsilon \circ u_n) = g_\varepsilon'(u_n)\nabla u_n$. In particular, if $\varphi \in \mathcal{D}(\Omega)$, then

$$\int_\Omega g_\varepsilon \circ u_n \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_\varepsilon'(u_n)\frac{\partial u_n}{\partial x_i}\varphi.$$

Letting $n \to \infty$ and using Lebesgue's dominated convergence theorem, we obtain

$$\int_\Omega g_\varepsilon \circ u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_\varepsilon'(u)\frac{\partial u}{\partial x_i}\varphi.$$

This implies the first claim about $g_\varepsilon \circ u$. But in this equation we may now let tend $\varepsilon \to 0$ and use that

$$g_\varepsilon(u) \to u^+ \text{ and } g_\varepsilon'(u) \to 1_{u>0}.$$

By Lebesgue's dominated convergence theorem again, we find that

$$\int_\Omega u^+ \frac{\partial \varphi}{\partial x_i} = -\int_\Omega 1_{u>0}\frac{\partial u}{\partial x_i}\varphi.$$

The claim is proved. $\qquad\square$

REMARK 4.2. Lemma 4.1 remains true for functions in $H^1(\Omega)$, but then one should use the fact that $C^1(\Omega) \cap H^1(\Omega)$ is dense in $H^1(\Omega)$, at least if one wants to copy the proof above.

LEMMA 4.3. *For every u, $v \in H_0^1(\Omega)$ one has $u \vee v$, $u \wedge v \in H_0^1(\Omega)$.*

PROOF. Note that

$$u \vee v = v + (u - v)^+ \text{ and } u \wedge v = v - (u - v)^-,$$

and use Lemma 4.1. $\qquad\square$

Let V be a Hilbert space which is densely and continuously embedded into $H = L^2(\Omega)$. Let $a : V \times V \to \mathbb{R}$ be a bounded, bilinear, elliptic form. Let $A = A_{L^2}$ be the operator on $L^2(\Omega)$ which is associated with the form a. Let $F : V \to L^2(\Omega)$ be a nonlinear locally Lipschitz continuous operator. Consider the semilinear evolution problem

(4.2) $$\dot{u} + Au + F(u) = 0, \, t \geq 0, \quad u(0) = u_0.$$

By Theorem 1.1 we know that for every $u_0 \in V$ there exists a unique local solution $u \in MR_2(0, T; L^2(\Omega), V)$ of this problem. Let u and v be two solutions of (4.2) and assume that $u_0 \leq v_0$. We prove the following comparison principle under additional assumptions on V, a and F.

THEOREM 4.4 (Comparison principle). *Assume that there exists $\omega \in \mathbb{R}$, $L \geq 0$ such that for every $u \in V$ one has $u^+ \in V$ and*

$$a(u, u^+) + \omega\|u^+\|_{L^2}^2 \geq 0$$

and for every u, $v \in V$,

$$\left|(F(u) - F(v))(u - v)^+\right| \leq L\left((u - v)^+\right)^2.$$

Let u_0, $v_0 \in V$, and let u and v be two local solutions of (4.2) (both existing on $[0, T]$) corresponding to the initial condition u_0 and v_0, respectively. Assume that $u_0 \leq v_0$.

Then $u(t) \leq v(t)$ for every $t \in [0, T]$.

PROOF. Since u and v are solutions of (4.2), we have

$$(\dot{u} - \dot{v}) + A(u - v) + F(u) - F(v) = 0.$$

Multiplying this equation scalarly in $H = L^2(\Omega)$ by $(u - v)^+ \in V$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left((u - v)^+\right)^2 + a(u - v, (u - v)^+) + \int_\Omega (F(u) - F(v))(u - v)^+ = 0.$$

By hypothesis on a and F, this implies

$$\frac{1}{2}\frac{d}{dt}\|(u - v)^+\|_{L^2}^2 \leq (\omega + L)\|(u - v)^+\|_{L^2}^2.$$

By assumption on u_0 and v_0 one has $(u - v)^+(0) = (u_0 - v_0)^+ = 0$. Hence, by Gronwall's lemma,

$$\|(u(t) - v(t))^+\|_{L^2}^2 = 0 \text{ for every } t \in [0, T],$$

i.e. $(u(t) - v(t))^+ = 0$ or $u(t) \leq v(t)$ for every $t \in [0, T]$. □

REMARK 4.5. The condition on a from Theorem 4.4 is satisfied if for every $u \in V$ one has u^+, $u^- \in V$ and

$$a(u^-, u^+) \geq 0.$$

EXAMPLE 4.6. Let $V = H_0^1(\Omega)$ so that V is densely and continuously embedded into $H = L^2(\Omega)$. Let $a : V \times V \to \mathbb{R}$ be the bounded, bilinear, elliptic form given by

$$a(u, v) = \int_\Omega A(x)\nabla u \nabla v + \int_\Omega \left(b(x)\nabla u v + c(x)u\nabla v\right) + \int_\Omega d(x)uv.$$

Here, the coefficient matrix $A \in L^\infty(\Omega; \mathbb{R}^{N \times N})$ is uniformly elliptic, and b_i, c_i, $d \in L^\infty(\Omega)$, $b = (b_i)$, $c = (c_i)$.

Let $A = A_{L^2}$ be the operator on $L^2(\Omega)$ which is associated with the form a. Formally, A is a realization of the elliptic operator from the diffusion equation (4.1).

Let $f \in C^1(\mathbb{R})$ be globally Lipschitz continuous and let $F : L^2(\Omega) \to L^2(\Omega)$ be the Nemytski operator associated with f (note that F is also Lipschitz continuous). Consider the semilinear evolution problem (4.1) from the beginning of this section. We know that for every $u_0 \in H_0^1(\Omega)$ there exists a unique solution $u \in MR_2(0, T; L^2(\Omega), D(A))$ of (4.1). By Theorem 4.4, if u and v are two solutions of (4.1), and if $u(0) \leq v(0)$, then $u(t) \leq v(t)$ for every t in the common interval of existence. In fact, we have to prove the two conditions from Theorem 4.4.

First, if $u \in H_0^1(\Omega)$, then u^+, $u^- \in H_0^1(\Omega)$ by Lemma 4.1 and by Lemma 4.1 one also obtains

$$
\begin{aligned}
a(u^-, u^+) &= \int_\Omega A(x)\nabla u^- \nabla u^+ + \int_\Omega \left(b(x)\nabla u^- u^+ + c(x)u^- \nabla u^+ \right) + \int_\Omega d(x)u^- u^+ \\
&= \int_\Omega A(x)|\nabla u|^2 1_{\{u<0\}} 1_{\{u>0\}} + \int_\Omega b(x)\nabla u u 1_{\{u<0\}} 1_{\{u>0\}} + \\
&\quad + \int_\Omega c(x)u\nabla u 1_{\{u<0\}} 1_{\{u>0\}} + \int_\Omega d(x)uu 1_{\{u<0\}} 1_{\{u>0\}} \\
&= 0.
\end{aligned}
$$

Hence, by Remark 4.5, the condition on a is satisfied.

Next, since f is globally Lipschitz continuous, for every u, $v \in H_0^1(\Omega)$,

$$
\begin{aligned}
\int_\Omega (f(u) - f(v))(u - v)^+ &= \int_\Omega (f(u) - f(v))(u - v)\, 1_{\{u>v\}} \\
&\geq -L \int_\Omega (u - v)^2 1_{\{u>v\}} \\
&= -L \int_\Omega \left((u - v)^+ \right)^2,
\end{aligned}
$$

where $L \geq 0$ is the Lipschitz constant of f.

If one assumes in addition that $f(0) = 0$, so that the constant function $u \equiv 0$ is a solution of the diffusion equation (4.1), then the comparison principle yields the following form of the maximum principle. If u is a solution of the diffusion equation (4.1) such that $u(0) \geq 0$, then $u \geq 0$.

EXAMPLE 4.7. One may also consider the heat equation with Neumann boundary conditions, i.e. the problem

(4.3)
$$
\begin{cases}
u_t - \Delta u + f(u) = 0, & (t, x) \in \mathbb{R}_+ \times \Omega, \\
\frac{\partial u}{\partial \nu}(t, x) = 0, & t \in \mathbb{R}_+ \times \partial\Omega, \\
u(0, x) = u_0(x), & x \in \Omega.
\end{cases}
$$

Assume that f is globally Lipschitz continuous. The negative Laplacian with Neumann boundary conditions is realized (at least for regular Ω such as intervals or smooth domains) on $L^2(\Omega)$ by the operator $A = A_{L^2}$ which is associated with the form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$
a(u, v) = \int_\Omega \nabla u \nabla v.
$$

It follows from the previous results that for every $u_0 \in H^1(\Omega)$ there exists a unique local solution $u \in MR_2(0, T; L^2(\Omega), D(A))$ of (4.3). Moreover, the comparison principle can be applied.

Assume that $f(0) = 0$ and $f(\bar{u}) = 0$ for some $\bar{u} > 0$. Then the constant functions $u \equiv 0$ and $u \equiv \bar{u}$ are global solutions of the heat equation (4.3). Hence, if an initial value u_0 satisfies $0 \leq u_0 \leq \bar{u}$, and if $u \in MR_2(0, T; L^2(\Omega), D(A))$ is

a local solution of (4.3), then the comparison principle implues $0 \leq u \leq \bar{u}$. In particular, the solution u is bounded uniformly in time and space. We will see in the next section that the solution can be extended to a global solution (i.e. existing for all $t \geq 0$).

EXAMPLE 4.8. Also in this example we show how the comparison principle may be applied in order to prove global existence of solutions. We consider the semilinear heat equation

$$
(4.4) \quad
\begin{cases}
u_t - u_{xx} - |u|^{p-2}u = 0, & (t, x) \in \mathbb{R}_+ \times (0, 1), \\
u(t, 0) = u(t, 1) = 0, & t \in \mathbb{R}_+, \\
u(0, x) = u_0(x), & x \in (0, 1),
\end{cases}
$$

where $p \geq 2$ is a real parameter. Again, we know that for every $u_0 \in H_0^1(0, 1)$ there exists a unique local solution

$$
u \in MR_2(0, T; L^2(0, 1), H^2 \cap H_0^1(0, 1))
$$

of this heat equation.

We calculate the *stationary solutions* of (4.4), i.e. the solutions $\varphi \in H^2 \cap H_0^1(0, 1)$ of the ordinary differential boundary value problem

$$
(4.5) \quad
\begin{cases}
-\varphi_{xx} - |\varphi|^{p-2}\varphi = 0, & x \in (0, 1), \\
\varphi(0) = \varphi(1) = 0.
\end{cases}
$$

Assume that φ is a solution of this stationary problem. Then a multiplcation of (4.5) by φ_x implies that

$$
\frac{d}{dx}\left(\varphi_x^2 + \frac{1}{p}|\varphi|^p\right) = 0,
$$

or

$$
(4.6) \quad \varphi_x^2 + \frac{1}{p}|\varphi|^p = C
$$

for a constant C which is necessarily positive. The constant C represents an energy of the solution.

Clearly, every solution φ of the stationary problem (4.5) is necessarily also a solution of the initial value problem

$$
(4.7) \quad
\begin{cases}
-\varphi_{xx} - |\varphi|^{p-2}\varphi = 0, & x \in (0, 1), \\
\varphi(0) = 0, \\
\varphi_x(0) = c.
\end{cases}
$$

The theory of ordinary differential equations implies that for every $c \in \mathbb{R}$ the initial value problem (4.7) admits a unique solution φ existing for all $x \in \mathbb{R}$ and in particular on the interval $[0, 1]$. For every such solution φ, the identity (4.6) holds, but $\varphi(1)$ is not necessarily equal to 0.

By identity (4.6), for every solution φ of the initial value problem (4.7) one has

$$\varphi_x(0) = \pm\sqrt{C} = c,$$

and if $x_0 \in \mathbb{R}$ is a local extremum, then

$$\varphi(x_0) = \pm(pC)^{\frac{1}{p}}.$$

Assume that $\varphi_x(0) > 0$ (so that φ is positive on some interval $[0, \varepsilon]$) and let $x_0 \in (0, \infty)$ be the first maximum of φ. The function φ is thus positive and increasing on $[0, x_0]$. By the identity (4.6),

$$\varphi_x = \sqrt{C - \frac{1}{p}\varphi^p},$$

which implies

$$\int_0^{x_0} \frac{\varphi_x}{\sqrt{C - \frac{1}{p}\varphi^p}}\, dx = x_0.$$

After substitution, one obtains

$$\begin{aligned}
x_0 &= \int_0^{(pC)^{\frac{1}{p}}} \frac{1}{\sqrt{C - \frac{1}{p}s^p}}\, ds \\
&= \frac{1}{\sqrt{C}} \int_0^{p^{\frac{1}{p}}} \frac{1}{\sqrt{1 - \frac{1}{p}s^p}}\, ds.
\end{aligned}$$

The above calculation shows that if we define x_0 as in this last equality, then x_0 is a maximum of φ. Solutions of the initial value problem are then obtained by taking a solution on $[0, x_0]$, extending it by reflection to $[0, 2x_0]$ and $[0, 4x_0]$, and then to extend the thus obtained function $4x_0$-periodically.

Hence, if φ is a solution of the boundary value problem (4.5), then $x_0 = \frac{1}{2n}$ for some $n \in \mathbb{N}$. This shows that solutions of (4.5) exist for a discrete set of energies C_n. For every $n \in \mathbb{N}$, there exist two stationary solutions φ_n and φ_{-n} which have their first extremum in $x_0 = \frac{1}{2n}$. The function φ_n is positive in x_0, the function φ_{-n} is negative in x_0. To this set of stationary solutions one has to add the solution $\varphi_0 \equiv 0$. We see that the stationary problem (4.5) admits a countable number of solutions which form a discrete subset of $H^2 \cap H_0^1(0, 1)$.

Let φ_1 and φ_{-1} be the solutions of (4.5) which have exactly one maximum (resp. minimum) in $(0, 1)$. Then φ_1 is positive on $(0, 1)$ and φ_{-1} is negative.

The functions u_1 and u_{-1} defined by

$$u_1(t, x) = \varphi_1(x) \text{ and } u_{-1}(t, x) = \varphi_{-1}(x)$$

are global solutions of the heat equation (4.4). If $u_0 \in H_0^1(0, 1)$ is an initial value such that $\varphi_{-1} \le u_0 \le \varphi_1$, and if u is the corresponding solution of (4.4),

then the comparison principle implies

$$\varphi_{-1}(x) \le u(t,x) \le \varphi_1(x).$$

In particular, the solution u remains uniformly bounded in time and space, as long as it exists. We will see in the next section that this implies that for every initial value as above the corresponding solution can be extended to a global solution (i.e. existing for every $t \ge 0$).

5. Energy methods and stability

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