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Skript zur Vorlesung

# Funktionalanalysis

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**Vorwort:**

Ich habe dieses Skript zur Vorlesung Funktionalanalysis an der Universität Ulm und an der TU Dresden nach meinem besten Wissen und Gewissen geschrieben. Mit Sicherheit schlichen sich jedoch Druckfehler oder gar mathematische Ungenauigkeiten ein, die man beim ersten Schreiben eines Skripts nicht vermeiden kann. Möge man mir diese Fehler verzeihen.

Obwohl die Vorlesung auf Deutsch gehalten wird, habe ich mich entschieden, dieses Skript auf Englisch zu verfassen. Auf diese Weise wird eine Brücke zwischen der Vorlesung und der (meist englischsprachigen) Literatur geschlagen. Mathematik sollte jedenfalls unabhängig von der Sprache sein in der sie präsentiert wird.

Ich danke Johannes Ruf und Manfred Sauter für ihre Kommentare zu einer früheren Version dieses Skripts. Für weitere Kommentare, die zur Verbesserungen beitragen, bin ich sehr dankbar.

# Contents

# Chapter 0
# Primer on metric spaces

It is the purpose of this introductory chapter to recall some basic facts about metric spaces, sequences in metric spaces, compact metric spaces, and continuous functions between metric spaces. Most of the material should be known, and if it is not known in the context of metric spaces, it has certainly been introduced on $\mathbb{R}^d$. The generalization to metric spaces should be straightforward, but it is nevertheless worthwhile to spend some time on the examples.

We also introduce some further notions from topology which may be new; see for example the definitions of density or of completion of a metric space.

## 0.1 Metric spaces

Let $M$ be a set. We call a function $d : M \times M \to \mathbb{R}_+$ a **metric** or a **distance** on $M$ if for every $x$, $y$, $z \in M$

(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ (symmetry), and

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality).

A pair $(M,d)$ of a set $M$ and a metric $d$ on $M$ is called a **metric space**.

It will be convenient to write only $M$ instead of $(M,d)$ if the metric $d$ on $M$ is known from the context, and to speak of a metric space $M$.

**Example 0.1.** 1. Let $M \subseteq \mathbb{R}^d$ and

$$d(x,y) := \sum_{i=1}^{d} |x_i - y_i|$$

or

$$d(x,y) := \left( \sum_{i=1}^{d} |x_i - y_i|^2 \right)^{\frac{1}{2}} .$$

Then $(M,d)$ is a metric space. The second metric is called the **Euclidean metric**. Often, if the metric on $\mathbb{R}^d$ is not explicitly given, we mean the Euclidean metric.

2. Let $M \subseteq C([0,1])$, the space of all continuous functions on the interval $[0,1]$, and

$$d(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Then $(M,d)$ is a metric space.

3. Let $M$ be any set and

$$d(x,y) := \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Then $(M,d)$ is a metric space. The metric $d$ is called the **discrete metric**.

4. Let $(M,d)$ be a metric space. Then

$$d_1(x,y) := \frac{d(x,y)}{1 + d(x,y)}$$

and

$$d_2(x,y) := \min\{d(x,y), 1\}$$

are metrics on $M$, too.

5. Let $M = C(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$, and let

$$d_n(f,g) := \sup_{x\in[-n,n]} |f(x) - g(x)| \quad (n \in \mathbb{N})$$

and

$$d(f,g) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{d_n(f,g)}{1 + d_n(f,g)}.$$

Then $(M,d)$ is a metric space. Note that the functions $d_n$ are not metrics for any $n \in \mathbb{N}$!

6. Let $(M,d)$ be a metric space. Then any subset $\tilde{M} \subseteq M$ is a metric space for the **induced metric**

$$\tilde{d}(x,y) = d(x,y), \quad x, y \in \tilde{M}.$$

We may sometimes say that $\tilde{M}$ is a **subspace** of $M$, that is, a subset and a metric space, but certainly this is not to be understood in the sense of linear subspaces of vector spaces ($M$ need not be a vector space).

7. Let $(M_n, d_n)$ be metric spaces ($n \in \mathbb{N}$). Then the cartesian product $M := \bigotimes_{n\in\mathbb{N}} M_n$ is a metric space for the metric

$$d(x,y) := \sum_{n\in\mathbb{N}} 2^{-n} \min\{d_n(x_n, y_n), 1\}.$$

Clearly, in a similar way, every finite cartesian product of metric spaces is a metric space.

Let $(M,d)$ be a metric space. For every $x \in M$ and every $r > 0$ we define the **open ball** $B(x,r) := \{y \in M : d(x,y) < r\}$ with center $x$ and radius $r$. A set $O \subseteq M$ is called **open** if for every $x \in O$ there exists some $r > 0$ such that $B(x,r) \subseteq O$. A set $A \subseteq M$ is called **closed** if its complement $A^c = M \setminus A$ is open. A set $U \subseteq M$ is called a **neighbourhood** of $x \in M$ if there exists $r > 0$ such that $B(x,r) \subseteq U$.

**Remark 0.2.** (a) The notions *open*, *closed*, *neighbourhood* depend on the set $M$!! For example, $M$ is always closed and open in $M$. The set $\mathbb{Q}$ is not closed in $\mathbb{R}$ (for the Euclidean metric), but it is closed in $\mathbb{Q}$ for the induced metric! Therefore, one should always say in which metric space some given set is open or closed.
(b) Clearly, a set $O \subseteq M$ is open (in $M$) if and only if it is a neighbourhood of every of its elements.

**Lemma 0.3.** *Let $(M,d)$ be a metric space. The following are true:*

(a) *Arbitrary unions of open sets are open. That means: if $(O_i)_{i\in I}$ is an arbitrary family of open sets (no restrictions on the index set $I$), then $\bigcup_{i\in I} O_i$ is open.*

(b) *Arbitrary intersections of closed sets are closed. That means: if $(A_i)_{i\in I}$ is an arbitrary family of closed sets, then $\bigcap_{i\in I} A_i$ is closed.*

(c) *Finite intersections of open sets are open.*

(d) *Finite unions of closed sets are closed.*

*Proof.* (a) Let $(O_i)_{i\in I}$ be an arbitrary family of open sets and let $O := \bigcup_{i\in I} O_i$. If $x \in O$, then $x \in O_i$ for some $i \in I$, and since $O_i$ is open, $B(x,r) \subseteq O_i$ for some $r > 0$. This implies that $B(x,r) \subseteq O$, and therefore $O$ is open.

(c) Next let $(O_i)_{i\in I}$ be a finite family of open sets and let $O := \bigcap_{i\in I} O_i$. If $x \in O$, then $x \in O_i$ for every $i \in I$. Since the $O_i$ are open, there exist $r_i$ such that $B(x,r_i) \subseteq O_i$. Let $r := \min_{i\in I} r_i$ which is positive since $I$ is finite. By construction, $B(x,r) \subseteq O_i$ for every $i \in I$, and therefore $B(x,r) \subseteq O$, that is, $O$ is open.

The proofs for closed sets are similar or follow just from the definition of closed sets and the above two assertions.

**Exercise 0.4** *Determine all open sets (respectively, all closed sets) of a metric space $(M,d)$, where $d$ is the discrete metric.*

**Exercise 0.5** *Show that a ball $B(x,r)$ in a metric space $M$ is always open. Show also that*

$$\bar{B}(x,r) := \{y \in M : d(x,y) \leq r\}$$

*is always closed.*

Let $(M,d)$ be a metric space and let $S \subseteq M$ be a subset. Then the set $\bar{S} := \bigcap\{A : A \subseteq M \text{ is closed and } S \subseteq A\}$ is called the **closure** of $S$. The set $S^\circ := \bigcup\{O : O \subseteq M \text{ is open and } O \subseteq S\}$ is called the **interior** of $S$. Finally, we call $\partial S := \{x \in M : \forall \varepsilon > 0\, B(x,\varepsilon) \cap S \neq \emptyset \text{ and } B(x,\varepsilon) \cap S^c \neq \emptyset\}$ the **boundary** of $S$.

By Lemma 0.3, the closure of a set $S$ is always closed (arbitrary intersections of closed sets are closed). By definition, $\bar{S}$ is the smallest closed set which contains $S$.

Similarly, the interior of a set $S$ is always open, and by definition it is the largest open set which is contained in $S$. Note that the interior might be empty.

**Exercise 0.6** *Give an example of a metric space $M$ and some $x \in M$, $r > 0$, to show that $\bar{B}(x,r)$ need not coincide with the closure of $B(x,r)$.*

**Exercise 0.7** *Let $(M,d)$ be a metric space and consider the metrics $d_1$ and $d_2$ from Example 0.1* (4)*. Show that the set of all open subsets, closed subsets or neighbourhoods of $M$ is the same for the three given metrics.*

*The set of all open subsets is also called the* **topology** *of $M$. The three metrics $d$, $d_1$ and $d_2$ thus induce the same topology. Sometimes it is good to know that one can pass from a given metric $d$ to a finite metric ($d_1$ and $d_2$ take only values between* 0 *and* 1*) without changing the topology.*

## 0.2 Sequences, convergence

Throughout the following, sequences will be denoted by $(x_n)$. Only when it is necessary, we make precise the index $n$; usually, $n \geq 0$ or $n \geq 1$, but sometimes we will also consider finite sequences or sequences indexed by $\mathbb{Z}$.

Let $(M,d)$ be a metric space. We call a sequence $(x_n) \subseteq M$ a **Cauchy sequence** if

$$\forall \varepsilon > 0 \, \exists n_0 \in \mathbb{N} \, \forall n, m \in \mathbb{N}_{\geq n_0} : d(x_n, x_m) < \varepsilon.$$

We say that a sequence $(x_n) \subseteq M$ **converges** to some element $x \in M$ if

$$\forall \varepsilon > 0 \, \exists n_0 \in \mathbb{N} \, \forall n \in \mathbb{N}_{\geq n_0} : d(x_n, x) < \varepsilon.$$

If $(x_n)$ converges to $x$, we also write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

**Exercise 0.8** *Let $C([0,1])$ be the metric space from Example 0.1* (2)*. Show that a sequence $(f_n) \subseteq C([0,1])$ converges to some $f$ for the metric $d$ if and only if it converges uniformly. We say that the metric $d$ induces the topology of* **uniform convergence**.

*Show also that a sequence $(f_n) \subseteq C(\mathbb{R})$ (Example 0.1* (5)*) converges to some $f$ for the metric $d$ if and only if it converges uniformly on compact subsets of $\mathbb{R}$. In this example, we say that the metric $d$ induces the topology of* **local uniform convergence**.

**Exercise 0.9** *Determine all Cauchy sequences and all convergent sequences in a discrete metric space.*

**Lemma 0.10.** *Let $M$ be a metric space and $(x_n) \subseteq M$ be a sequence. Then:*

(a) $\lim_{n\to\infty} x_n = x$ *for some element $x \in M$ if and only if for every neighbourhood $U$ of $x$ there exists $n_0$ such that for every $n \geq n_0$ one has $x_n \in U$.*

(b) (Uniqueness of the limit) *If* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} x_n = y$*, then $x = y$.*

**Lemma 0.11.** *A set $A \subseteq M$ is closed if and only it is* **sequentially closed***, that is, if for every sequence $(x_n) \subseteq A$ which converges to some $x \in M$ one has $x \in A$.*

*Proof.* Assume first that $A$ is closed and let $(x_n) \subseteq A$ be convergent to $x \in M$. If $x$ does not belong to $A$, then it belongs to $A^c$ which is open. By definition, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq A^c$. Given this $\varepsilon$, there exists $n_0$ such that $x_n \in B(x,\varepsilon)$ for every $n \geq n_0$, a contradiction to the assumption that $x_n \in A$. Hence, $x \in A$.

On the other hand, assume that $\lim_{n\to\infty} x_n = x \in A$ for every convergent $(x_n) \subseteq A$ and assume in addition that $A$ is not closed or, equivalently, that $A^c$ is not open. Then there exists $x \in A^c$ such that for every $n \in \mathbb{N}$ the set $B(x, \frac{1}{n}) \cap A$ is nonempty. From this one can construct a sequence $(x_n) \subseteq A$ which converges to $x$, which is a contradiction because $x \in A^c$.

**Lemma 0.12.** *Let $(M,d)$ be a metric space, and let $S \subseteq M$ be a subset. Then*

$$\begin{aligned}\bar{S} &= \{x \in M : \exists (x_n) \subseteq S \text{ s.t. } \lim_{n\to\infty} x_n = x\} \\ &= \{x \in M : d(x,S) := \inf_{y\in S} d(x,y) = 0\}.\end{aligned}$$

*Proof.* Let

$$A := \{x \in M : \exists (x_n) \subseteq S \text{ s.t. } \lim_{n\to\infty} x_n = x\}$$

and

$$B := \{x \in M : d(x,S) := \inf_{y\in S} d(x,y) = 0\}.$$

These two sets are clearly equal by the definition of the inf and the definition of convergence. Moreover, the set $B$ is closed by the following argument. Assume that $(x_n) \subseteq B$ is convergent to $x \in M$. By definition of $B$, for every $n$ there exists $y \in S$ such that $d(x_n, y_n) \leq 1/n$. Hence,

$$\limsup_{n\to\infty} d(x,y_n) \leq \limsup_{n\to\infty} d(x,x_n) + \limsup_{n\to\infty} d(x_n,y_n) = 0,$$

so that $x \in B$.

Clearly, $B$ contains $S$, and since $B$ is closed, $B$ contains $\bar{S}$. It remains to show that $B \subseteq \bar{S}$. If this is not true, then there exists $x \in B \setminus \bar{S}$. Since the complement of $\bar{S}$ is open in $M$, there exists $r > 0$ such that $B(x,r) \cap \bar{S} = \emptyset$, a contradiction to the definition of $B$.

A metric space $(M,d)$ is called **complete** if every Cauchy sequence converges.

**Exercise 0.13** *Show that the spaces $\mathbb{R}^d$, $C([0,1])$ and $C(\mathbb{R})$ are complete. Show also that any discrete metric space is complete.*

**Lemma 0.14.** *A subspace $N \subseteq M$ of a complete metric space is complete if and only if it is closed in $M$.*

*Proof.* Assume that $N \subseteq M$ is closed, and let $(x_n)$ be a Cauchy sequence in $N$. By the assumption that $M$ is complete, $(x_n)$ is convergent to some element $x \in M$. Since $N$ is closed, $x \in N$.

Assume on the other hand that $N$ is complete, and let $(x_n) \subseteq N$ be convergent to some element $x \in M$. Clearly, every convergent sequence is also a Cauchy sequence, and since $N$ is complete, $(x_n)$ converges to some element $y \in N$. By uniqueness of the limit, $x = y \in N$. Hence, $N$ is closed.

## 0.3 Compact spaces

We say that a metric space $(M,d)$ is **compact** if for every open covering there exists a finite subcovering, that is, whenever $(O_i)_{i\in I}$ is a family of open sets (no restrictions on the index set $I$) such that $M = \bigcup_{i\in I} O_i$, then there exists a *finite* subset $I_0 \subseteq I$ such that $M = \bigcup_{i\in I_0} O_i$.

**Lemma 0.15.** *A metric space $(M,d)$ is compact if and only if it is* **sequentially compact***, that is, if and only if every sequence $(x_n) \subseteq M$ has a convergent subsequence.*

*Proof.* Assume that $M$ is compact and let $(x_n) \subseteq M$. Assume that $(x_n)$ does not have a convergent subsequence. Then for every $x \in M$ there exists $\varepsilon_x > 0$ such that $B(x,\varepsilon_x)$ contains only finitely many elements of $\{x_n\}$. Note that $(B(x,\varepsilon_x))_{x\in M}$ is an open covering of $M$ so that by the compactness of $M$ there exists a finite subset $N \subseteq M$ such that $M = \bigcup_{x\in N} B(x,\varepsilon_x)$. But this means that $(x_n)$ takes only finitely many values, and hence there exists even a constant subsequence which is in particular also convergent; a contradiction to the assumption on $(x_n)$.

On the other hand, assume that $M$ is sequentially compact and let $(O_i)_{i\in I}$ be an open covering of $M$. We first show that there exists $\varepsilon > 0$ such that for every $x \in M$ there exists $i_x \in I$ with $B(x,\varepsilon) \subseteq O_{i_x}$. If this were not true, then for every $n \in \mathbb{N}$ there exists $x_n$ such that $B(x_n, \frac{1}{n}) \not\subseteq O_i$ for every $i \in I$. Passing to a subsequence, we may assume that $(x_n)$ is convergent to some $x \in M$. There exists some $i_0 \in I$ such that $x \in O_{i_0}$, and since $O_{i_0}$ is open, we find some $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq O_{i_0}$. Let $n_0$ be such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$. By the triangle inequality, for every $n \geq n_0$ we have $B(x_n, \frac{1}{n}) \subseteq B(x,\varepsilon) \subseteq O_{i_0}$, a contradiction to the construction of the sequence $(x_n)$.

Next we show that $M = \bigcup_{j=1}^{n} B(x_j,\varepsilon)$ for a finite family of $x_j \in M$. Choose any $x_1 \in M$. If $B(x_1,\varepsilon) = M$, then we are already done. Otherwise we find $x_2 \in M \setminus B(x_1,\varepsilon)$. If $B(x_1,\varepsilon) \cup B(x_2,\varepsilon) \neq M$, then we even find $x_3 \in M$ which does not belong to $B(x_1,\varepsilon) \cup B(x_2,\varepsilon)$, and so on. If $\bigcup_{j=1}^{n} B(x_j,\varepsilon)$ is never all of $M$, then we find actually a sequence $(x_j)$ such that $d(x_j,x_k) \geq \varepsilon$ for all $j \neq k$. This sequence can not have a convergent subsequence, a contradiction to sequential compactness.

Since every of the $B(x_j,\varepsilon)$ is a subset of $O_{i_{x_j}}$ for some $i_{x_j} \in I$, we have proved that $M = \bigcup_{j=1}^{n} O_{i_{x_j}}$, i.e. the open covering $(O_i)$ admits a finite subcovering. The proof is complete.

**Lemma 0.16.** *Any compact metric space is complete.*

*Proof.* Let $(x_n)$ be a Cauchy sequence in $M$. By the preceeding lemma, there exists a subsequence which converges to some $x \in M$. If a subsequence of a Cauchy sequence converges, then the sequence itself converges, too.

## 0.4 Continuity

Let $(M_1,d_1)$, $(M_2,d_2)$ be two metric spaces, and let $f: M_1 \to M_2$ be a function. We say that $f$ is **continuous at some point** $x \in M_1$ if

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall y \in B(x,\delta) : d_2(f(x), f(y)) < \varepsilon.$$

We say that $f$ is **continuous** if it is continuous at every point. We say that $f$ is **uniformly continuous** if

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x,y \in M_1 : d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon.$$

We say that $f$ is **Lipschitz continuous** if

$$\exists L \geq 0\, \forall x,y \in M_1 : d_2(f(x), f(y)) \leq L d_1(x,y).$$

**Lemma 0.17.** *A function $f: M_1 \to M_2$ between two metric spaces is continuous at some point $x \in M_1$ if and only if it is* **sequentially continuous** *at $x$, that is, if and only if for every sequence $(x_n) \subseteq M_1$ which converges to $x$ one has $\lim_{n\to\infty} f(x_n) = f(x)$.*

*Proof.* Assume that $f$ is continuous at $x \in M_1$ and let $(x_n)$ be convergent to $x$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that for every $y \in B(x,\delta)$ one has $f(y) \in B(f(x),\varepsilon)$. By definition of convergence, there exists $n_0$ such that for every $n \geq n_0$ one has $x_n \in B(x,\delta)$. For this $n_0$ and every $n \geq n_0$ one has $f(x_n) \in B(f(x),\varepsilon)$. Hence, $\lim_{n\to\infty} f(x_n) = f(x)$.

Assume on the other hand that $f$ is sequentially continuous at $x$. If $f$ was not continuous in $x$ then there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $x_n \in B(x, \frac{1}{n})$ with $f(x_n) \notin B(f(x),\varepsilon)$. By construction, $\lim_{n\to\infty} x_n = x$. Since $f$ is sequentially continuous, $\lim_{n\to\infty} f(x_n) = f(x)$. But this is a contradiction to $f(x_n) \notin B(f(x),\varepsilon)$, and therefore $f$ is continuous.

**Lemma 0.18.** *A function $f: M_1 \to M_2$ between two metric spaces is continuous if and only if preimages of open sets are open, that is, if and only if for every open set $O \subseteq M_2$ the preimage $f^{-1}(O)$ is open in $M_1$.*

*Proof.* Let $f: M_1 \to M_2$ be continuous and let $O \subseteq M_2$ be open. Let $x \in f^{-1}(O)$. Since $O$ is open, there exists $\varepsilon > 0$ such that $B(f(x),\varepsilon) \subseteq O$. Since $f$ is continuous, there exists $\delta > 0$ such that for every $y \in B(x,\delta)$ one has $f(y) \in B(f(x),\varepsilon)$. Hence, $B(x,\delta) \subseteq f^{-1}(O)$ so that $f^{-1}(O)$ is open.

On the other hand, if the preimage of every open set is open, then for every $x \in M_1$ and every $\varepsilon > 0$ the preimage $f^{-1}(B(f(x),\varepsilon))$ is open. Clearly, $x$ belongs to this preimage, and therefore there exists $\delta > 0$ such that $B(x,\delta) \subseteq f^{-1}(B(f(x),\varepsilon))$. This proves continuity.

**Lemma 0.19.** *Let $f: K \to M$ be a continuous function from a compact metric space $K$ into a metric space $M$. Then:*

(a) *The image $f(K)$ is compact.*

(b) *The function $f$ is uniformly continuous.*

*Proof.* (a) Let $(O_i)_{i\in I}$ be an open covering of $f(K)$. Since $f$ is continuous, $f^{-1}(O_i)$ is open in $K$. Moreover, $(f^{-1}(O_i))_{i\in I}$ is an open covering of $K$. Since $K$ is compact, there exists a finite subcovering: $K = \bigcup_{i\in I_0} f^{-1}(O_i)$ for some finite $I_0 \subseteq I$. Hence, $(O_i)_{i\in I_0}$ is a finite subcovering of $f(K)$.

(b) Let $\varepsilon > 0$. Since $f$ is continuous, for every $x \in K$ there exists $\delta_x > 0$ such that for all $y \in B(x,\delta_x)$ one has $f(y) \in B(f(x),\varepsilon)$. By compactness, there exists a finite family $(x_i)_{1\leq i\leq n} \subseteq K$ such that $K = \bigcup_{i=1}^n B(x_i,\delta_{x_i}/2)$. Let $\delta = \min\{\delta_{x_i}/2 : 1 \leq i \leq n\}$ and let $x$, $y \in K$ such that $d(x,y) < \delta$. Since $x \in B(x_i,\delta_{x_i}/2)$ for some $1 \leq i \leq n$, we find that $y \in B(x_i,\delta_{x_i})$. By construction, $f(x)$, $f(y) \in B(f(x_i),\varepsilon)$ so that the triangle inequality implies $d(f(x), f(y)) < 2\varepsilon$.

**Lemma 0.20.** *Any Lipschitz continuous function $f : M_1 \to M_2$ between two metric spaces is uniformly continuous.*

*Proof.* Let $L > 0$ be a Lipschitz constant for $f$ and let $\varepsilon > 0$. Define $\delta := \varepsilon/L$. Then, for every $x$, $y \in M$ such that $d_1(x,y) \leq \delta$ one has

$$d_2(f(x), f(y)) \leq Ld_1(x,y) \leq \varepsilon,$$

and therefore $f$ is uniformly continuous.

## 0.5 Completion of a metric space

We say that a subset $D \subseteq M$ of a metric space $(M,d)$ is **dense in** $M$ if $\bar{D} = M$. Equivalently, $D$ is dense in $M$ if for every $x \in M$ there exists $(x_n) \subseteq D$ such that $\lim_{n\to\infty} x_n = x$.

**Lemma 0.21 (Completion).** *Let $(M,d)$ be a metric space. Then there exists a complete metric space $(\hat{M},\hat{d})$ and a continuous, injective $j : M \to \hat{M}$ such that*

$$d(x,y) = \hat{d}(j(x), j(y)), \quad x, y \in M,$$

*and such that the image $j(M)$ is dense in $\hat{M}$.*

Let $(M,d)$ be a metric space. A complete metric space $(\hat{M},\hat{d})$ fulfilling the properties from Lemma 0.21 is called a **completion** of $M$.

*Proof (Proof of Lemma 0.21).* Let

$$\bar{M} := \{(x_n) \subseteq M : (x_n) \text{ is a Cauchy sequence}\}.$$

We say that two Cauchy sequences $(x_n)$, $(y_n) \subseteq \bar{M}$ are equivalent (and we write $(x_n) \sim (y_n)$) if $\lim_{n\to\infty} d(x_n,y_n) = 0$. Clearly, $\sim$ is an equivalence relation on $\bar{M}$.

We denote by $[(x_n)]$ the equivalence class in $\bar{M}$ of a Cauchy sequence $(x_n)$, and we let

$$\hat{M} := \bar{M}/\sim = \{[(x_n)] : (x_n) \in \bar{M}\}$$

be the set of all equivalence classes. If we define

$$\hat{d}([(x_n)],[(y_n)]) := \lim_{n\to\infty} d(x_n,y_n),$$

then $\hat{d}$ is well defined (the definition is independent of the choice of representatives) and it is a metric on $\hat{M}$. The fact that $\hat{d}$ is a metric and also that $(\hat{M},\hat{d})$ is a complete metric space are left as exercises.

One also easily verifies that $j : M \to \hat{M}$ defined by $j(x) = [(x)]$ (the equivalence class of the constant sequence $(x)$) is continuous, injective and in fact isometric, i.e.

$$d(x,y) = \hat{d}(j(x), j(y))$$

for every $x$, $y \in M$. The proof is here complete.

**Lemma 0.22.** *Let $(\hat{M}_i,\hat{d}_i)$ ($i = 1, 2$) be two completions of a metric space $(M,d)$. Then there exists a bijection $b : \hat{M}_1 \to \hat{M}_2$ such that for every $x$, $y \in \hat{M}_1$*

$$\hat{d}_1(x,y) = \hat{d}_2(b(x),b(y)).$$

Lemma 0.22 shows that up to isometric bijections there exists only one completion of a given metric space and it allows us to speak of *the* completion of a metric space.

**Lemma 0.23.** *Let $f : M_1 \to M_2$ be a uniformly (!) continuous function between two metric spaces. Let $\hat{M}_1$ and $\hat{M}_2$ be the completions of $M_1$ and $M_2$, respectively. Then there exists a unique continuous extension $\hat{f} : \hat{M}_1 \to \hat{M}_2$ of $f$.*

*Proof.* Since $f$ is uniformly continuous, it maps equivalent Cauchy sequences into equivalent Cauchy sequences (equivalence of Cauchy sequences is defined as in the proof of Lemma 0.21). Hence, the function $\hat{f}([(x_n)]) := [(f(x_n))]$ is well defined. It is easy to check that $\hat{f}$ is an extension of $f$ and that $\hat{f}$ is continuous (even uniformly continuous).

The assumption of uniform continuity in Lemma 0.23 is necessary in general. The functions $f(x) = \sin(1/x)$ and $f(x) = 1/x$ on the open interval $(0,1)$ do not admit continuous extensions to the closed interval $[0,1]$ (which is the completion of $(0,1)$).

# Chapter 1
# Banach spaces and bounded linear operators

Throughout, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. In this chapter we introduce normed spaces, Banach spaces and bounded linear operators between normed spaces. Besides defining these notions, we give some standard examples of normed spaces, Banach spaces and of bounded linear operators. There are two main theoretical results of this chapter, and they have to do with characteristic properties of dimensional normed spaces. The first result states that any two norms on a finite dimensional real or complex vector space are equivalent (this is not true in infinite dimensional spaces), while the second result states that the closed unit ball of a normed space is compact if and only if the space is finite dimensional.

## 1.1 Normed spaces

Let $X$ be a vector space over $\mathbb{K}$. A function $\|\cdot\| : X \to \mathbb{R}_+$ is called a **norm** if for every $x$, $y \in X$ and every $\lambda \in \mathbb{K}$

(i) $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|\lambda x\| = |\lambda| \, \|x\|$, and

(iii) $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality).

A pair $(X, \|\cdot\|)$ of a vector space $X$ and a norm $\|\cdot\|$ is called a **normed space**.

Often, we will speak of a normed space $X$ if it is clear which norm is given on $X$.

**Example 1.1.** 1. (Finite dimensional spaces) Let $X = \mathbb{K}^d$. Then

$$\|x\|_p := \left( \sum_{i=1}^{d} |x_i|^p \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|x\|_\infty := \sup_{1 \leq i \leq d} |x_i|$$

are norms on $X$.

2. (Sequence spaces) Let $1 \leq p < \infty$, and let
$$l^p := \{(x_n) \subseteq \mathbb{K} : \sum_n |x_n|^p < \infty\}$$
with norm
$$\|x\|_p := \left(\sum_n |x_n|^p\right)^{1/p}.$$
Then $(l^p, \|\cdot\|_p)$ is a normed space.

3. (Sequence spaces) Let $X$ be one of the spaces
$$\begin{aligned} l^\infty &:= \{(x_n) \subseteq \mathbb{K} : \sup_n |x_n| < \infty\}, \\ c &:= \{(x_n) \subseteq \mathbb{K} : \lim_{n\to\infty} x_n \text{ exists}\}, \text{ or} \\ c_0 &:= \{(x_n) \subseteq \mathbb{K} : \lim_{n\to\infty} x_n = 0\}, \text{ or} \\ c_{00} &:= \{(x_n) \subseteq \mathbb{K} : \text{the set } \{n : x_n \neq 0\} \text{ is finite}\}, \end{aligned}$$
and let
$$\|x\|_\infty := \sup_n |x_n|.$$
Then $(X, \|\cdot\|_\infty)$ is a normed space.

4. (Function spaces: continuous functions) Let $C([a,b])$ be the space of all continuous, $\mathbb{K}$-valued functions on a compact interval $[a,b] \subset \mathbb{R}$. Then
$$\|f\|_p := \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p}, \quad 1 \leq p < \infty,$$
and
$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|$$
are norms on $C([a,b])$.

5. (Function spaces: continuous functions) Let $K$ be a compact metric space and let $C(K)$ be the space of all continuous, $\mathbb{K}$-valued functions on $K$. Then
$$\|f\|_\infty := \sup_{x\in K} |f(x)|$$
is a norm on $C(K)$.

6. (Function spaces: integrable functions) Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and let $X_p = L^p(\Omega)$ $(1 \leq p \leq \infty)$. Let
$$\|f\|_p := \left(\int_\Omega |f|^p \,\mathrm{d}\mu\right)^{1/p}, \quad 1 \leq p < \infty,$$

or

$$\|f\|_\infty := \text{ess sup}|f(x)| := \inf\{c \in \mathbb{R}_+ : \mu(\{|f| > c\}) = 0\}.$$

Then $(X_p, \|\cdot\|_p)$ is a normed space.

7. (Function spaces: differentiable functions) Let

$$C^1([a,b]) := \{f \in C([a,b]) : f \text{ is continuously differentiable}\}.$$

Then $\|\cdot\|_\infty$ and

$$\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$$

are norms on $C^1([a,b])$.

We will see more examples in the sequel.

**Lemma 1.2.** *Every normed space $(X, \|\cdot\|)$ is a metric space for the metric*

$$d(x,y) := \|x-y\|, \quad x, y \in X.$$

By the above lemma, also every subset of a normed space becomes a metric space in a natural way. Moreover, it is natural to speak of closed or open subsets (or linear subspaces!) of normed spaces, or of closures and interiors of subsets.

**Exercise 1.3** *Show that in a normed space $X$, for every $x \in X$ and every $r > 0$ the closed ball $\bar{B}(x,r)$ coincides with closure $\overline{B(x,r)}$ of the open ball.*

Also the notion of continuity of functions between normed spaces (or between a metric space and a normed space) makes sense. The following is a first example of a continuous function.

**Lemma 1.4.** *Given a normed space, the norm is a continuous function.*

This lemma is a consequence of the following lemma.

**Lemma 1.5 (Triangle inequality from below).** *Let $X$ be a normed space. Then, for every $x$, $y \in X$,*

$$\|x-y\| \geq \big|\|x\| - \|y\|\big|.$$

*Proof.* The triangle inequality implies

$$\begin{aligned}\|x\| &= \|x-y+y\| \\ &\leq \|x-y\| + \|y\|,\end{aligned}$$

so that

$$\|x\| - \|y\| \leq \|x-y\|.$$

Changing the role of $x$ and $y$ implies

$$\|y\| - \|x\| \leq \|y-x\| = \|x-y\|,$$

and the claim follows.

A notion which can not really be defined in metric spaces but in normed spaces is the following. A subset $B$ of a normed space $X$ is called **bounded** if

$$\sup\{\|x\| : x \in B\} < \infty.$$

It is easy to check that if $X$ is a normed space, and $M$ is a metric space, then the set $C(M;X)$ of all continuous functions from $M$ into $X$ is a vector space for the obvious addition and scalar multiplication. If $M$ is in addition compact, then $f(M) \subseteq X$ is also compact for every such function, and hence $f(M)$ is necessarily bounded (every compact subset of a normed space is bounded!). So we can give a new example of a normed space.

**Example 1.6.** 8. (Function spaces: vector-valued continuous functions) Let $(X, \|\cdot\|)$ be a normed space and let $K$ be a compact metric space. Let $E = C(K;X)$ be the space of all $X$-valued continuous functions on $K$. Then

$$\|f\|_\infty := \sup_{x\in K} \|f(x)\|$$

is a norm on $C(K;X)$.

Also the notions of Cauchy sequences and convergent sequences make sense in normed spaces. In particular, one can speak of a **complete normed space**, that is, a normed space in which every Cauchy sequence converges. A complete normed space is called a **Banach space**.

**Example 1.7.** The finite dimensional spaces, the sequence spaces $l^p$ $(1 \le p \le \infty)$, $c$, and $c_0$, and the function spaces $(C([a,b]), \|\cdot\|_\infty)$, $(L^p(\Omega), \|\cdot\|_p)$ are Banach spaces.

The spaces $(c_{00}, \|\cdot\|_\infty)$, $(C([a,b]), \|\cdot\|_p)$ $(1 \le p < \infty)$ are not Banach spaces.

If $X$ is a Banach space, then also $(C(K;X), \|\cdot\|_\infty)$ is a Banach space.

We say that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a real or complex vector space $X$ are **equivalent** if there exist two constants $c, C > 0$ such that for every $x \in X$

$$c\,\|x\|_1 \le \|x\|_2 \le C\,\|x\|_1.$$

**Lemma 1.8.** *Let $\|\cdot\|_1$, $\|\cdot\|_2$ be two norms on a vector space $X$ (over $\mathbb{K}$). The following are equivalent:*

(i) *The norms $\|\cdot\|_1$, $\|\cdot\|_2$ are equivalent.*

(ii) *A set $O \subseteq X$ is open for the norm $\|\cdot\|_1$ if and only if it is open for the norm $\|\cdot\|_2$ (and similarly for closed sets).*

(iii) *A sequence $(x_n) \subseteq X$ converges to $0$ for the norm $\|\cdot\|_1$ if and only if it converges to $0$ for the norm $\|\cdot\|_2$.*

In other words, if two norms $\|\cdot\|_1$, $\|\cdot\|_2$ on a vector space $X$ are equivalent, then the open sets, the closed sets and the null sequences are the same. We also say that the two norms define the same *topology*. In particular, if $X$ is a Banach space for one norm then it is also a Banach space for the other (equivalent) norm.

**Exercise 1.9** *The norms* $\|\cdot\|_\infty$ *and* $\|\cdot\|_p$ *are not equivalent on* $C([0,1])$.

**Theorem 1.10.** *Any two norms on a finite dimensional real or complex vector space are equivalent.*

*Proof.* We may without loss of generality consider $\mathbb{K}^d$. Let $\|\cdot\|$ be a norm on $\mathbb{K}^d$ and let $(e_i)_{1\leq i\leq d}$ be the canonical basis of $\mathbb{K}^d$. For every $x \in \mathbb{K}^d$

$$\begin{aligned}\|x\| &= \|\sum_{i=1}^d x_i e_i\| \\ &\leq \sum_{i=1}^d |x_i| \, \|e_i\| \\ &\leq C\,\|x\|_1,\end{aligned}$$

where $C := \sup_{1\leq i\leq d} \|e_i\| < \infty$ and $\|\cdot\|_1$ is the norm from Example 1.1.1. By the triangle inequality from below, for every $x$, $y \in \mathbb{K}^d$,

$$|\|x\| - \|y\|| \leq \|x-y\| \leq C\,\|x-y\|_1.$$

Hence, the norm $\|\cdot\| : (\mathbb{K}^d, \|\cdot\|_1) \to \mathbb{R}_+$ is continuous (on $\mathbb{K}^d$ equipped with the norm $\|\cdot\|_1$). If $S := \{x \in \mathbb{K}^d : \|x\|_1 = 1\}$ denotes the unit sphere for the norm $\|\cdot\|_1$, then $S$ is compact. As a consequence

$$c := \inf\{\|x\| : x \in S\} > 0,$$

since the infimum is attained by the continuity of $\|\cdot\|$. This implies

$$c\,\|x\|_1 \leq \|x\| \quad \text{for every } x \in \mathbb{K}^d.$$

We have proved that every norm on $\mathbb{K}^d$ is equivalent to the norm $\|\cdot\|_1$. Hence, any two norms on $\mathbb{K}^d$ are equivalent.

**Corollary 1.11.** *Any finite dimensional normed space is complete. Any finite dimensional subspace of a normed space is closed.*

*Proof.* The space $(\mathbb{K}^d, \|\cdot\|_1)$ is complete (exercise!). If $\|\cdot\|$ is a second norm on $\mathbb{K}^d$ and if $(x_n)$ is a Cauchy sequence for that norm, then it is also a Cauchy sequence in $(\mathbb{K}^d, \|\cdot\|_1)$ (use that the norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent), and therefore convergent in $(\mathbb{K}^d, \|\cdot\|_1)$. By equivalence of norms again, the sequence $(x_n)$ is also convergent in $(\mathbb{K}^d, \|\cdot\|)$, and therefore $(\mathbb{K}^d, \|\cdot\|)$ is complete.

Let $Y$ be a finite dimensional subspace of a normed space $X$, and let $(x_n) \subseteq Y$ be a convergent sequence with $x = \lim_{n\to\infty} x_n \in X$. Since $(x_n)$ is also a Cauchy sequence, and since $Y$ is complete, we find (by uniqueness of the limit) that $x \in Y$, and therefore $Y$ is closed (Lemma 0.11).

Let $(x_n)$ be a sequence in a normed space $X$. We say that the series $\sum_n x_n$ is **convergent** if the sequence $(\sum_{j\leq n} x_j)$ of partial sums is convergent. We say that the series $\sum_n x_n$ is **absolutely convergent** if $\sum_n \|x_n\| < \infty$.

**Lemma 1.12.** *Let $(x_n)$ be a sequence in a normed space $X$. If the series $\sum_n x_n$ is convergent, then necessarily* $\lim_{n\to\infty} x_n = 0$.

Note that in a normed space not every absolutely convergent series is convergent. In fact, the following is true.

**Lemma 1.13.** *A normed space $X$ is a Banach space if and only if every absolutely convergent series converges.*

*Proof.* Assume that $X$ is a Banach space, and let $\sum_n x_n$ be absolutely convergent. It follows easily from the triangle inequality that the corresponding sequence of partial sums is a Cauchy sequence, and since $X$ is complete, the series $\sum_n x_n$ is convergent.

On the other hand, assume that every absolutely convergent series is convergent. Let $(x_n)_{n\geq 1} \subseteq X$ be a Cauchy sequence. From this Cauchy sequence, one can extract a subsequence $(x_{n_k})_{k\geq 1}$ such that $\|x_{n_{k+1}} - x_{n_k}\| \leq 2^{-k}$, $k \geq 1$. Let $y_0 = x_{n_1}$ and $y_k = x_{n_{k+1}} - x_{n_k}$, $k \geq 1$. Then the series $\sum_{k\geq 0} y_k$ is absolutely convergent. By assumption, it is also convergent. But by construction, $(\sum_{l=0}^{k} y_l) = (x_{n_k})$, so that $(x_{n_k})$ is convergent. Hence, we have extracted a subsequence of the Cauchy sequence $(x_n)$ which converges. As a consequence, $(x_n)$ is convergent, and since $(x_n)$ was an arbitrary Cauchy sequence, $X$ is complete.

**Lemma 1.14 (Riesz).** *Let $X$ be a normed space and let $Y \subseteq X$ be a closed linear subspace. If $Y \neq X$, then for every $\delta > 0$ there exists $x \in X \setminus Y$ such that $\|x\| = 1$ and*

$$\operatorname{dist}(x,Y) = \inf\{\|x-y\| : y \in Y\} \geq 1-\delta.$$

*Proof.* Let $z \in X \setminus Y$. Since $Y$ is closed,

$$d := \operatorname{dist}(z,Y) > 0.$$

Let $\delta > 0$. By definition of the infimum, there exists $y \in Y$ such that

$$\|z-y\| \leq \frac{d}{1-\delta}.$$

Let $x := \frac{z-y}{\|z-y\|}$. Then $x \in X \setminus Y$, $\|x\| = 1$, and for every $u \in Y$

$$\begin{aligned} \|x-u\| &= \|z-y\|^{-1}\,\|z-(y+\|z-y\|u)\| \\ &\geq \|z-y\|^{-1}\,d \quad \geq \quad 1-\delta, \end{aligned}$$

since $(y+\|z-y\|u) \in Y$.

**Theorem 1.15.** *A normed space is finite dimensional if and only if every closed bounded set is compact.*

*Proof.* If the normed space is finite dimensional, then every closed bounded set is compact by the Theorem of Heine-Borel. Note that by Theorem 1.10 it is not important which norm on the finite dimensional space is considered. By Lemma 1.8, the closed and bounded sets do not change.

On the other hand, if the normed space is infinite dimensional, then, by the Lemma of Riesz, one can construct inductively a sequence $(x_n) \subseteq X$ such that $\|x_n\| = 1$ and $\operatorname{dist}(x_{n+1}, X_n) \geq \frac{1}{2}$ for every $n \in \mathbb{N}$, where $X_n = \operatorname{span}\{x_i : 1 \leq i \leq n\}$ (note that $X_n$ is closed by Corollary 1.11). By construction, $(x_n)$ belongs to the closed unit ball, but it can not have a convergent subsequence (even not a Cauchy subsequence). Hence, the closed unit ball is not compact. We state this result separately.

**Theorem 1.16.** *In an infinite dimensional Banach space the closed unit ball is not compact.*

**Lemma 1.17 (Completion of a normed space).** *For every normed space $X$ there exists a Banach space $\hat{X}$ and a linear injective $j : X \to \hat{X}$ such that $\|j(x)\| = \|x\|$ ($x \in X$) and $j(X)$ is dense in $\hat{X}$.* Up to isometry, *the Banach space $\hat{X}$ is unique (up to isomorphism). It is called the* **completion** *of $X$.*

*Proof.* It suffices to repeat the proof of Lemma 0.21 and to note that the completion $\hat{X}$ of $X$ (considered as a metric space) carries in a natural way a linear structure: addition of - equivalence classes of - Cauchy sequences is their componentwise addition, and also multiplication of - an equivalence class - of a Cauchy sequence and a scalar is done componentwise. Moreover, for every $[(x_n)]$, one defines the norm

$$\|[(x_n)]\| := \lim_{n\to\infty} \|x_n\|.$$

Uniqueness of $\hat{X}$ follows from Lemma 0.22.

## 1.2 Product spaces and quotient spaces

**Lemma 1.18 (Product spaces).** *Let $(X_i)_{i\in I}$ be a finite (!) family of normed spaces, and let $\mathscr{X} := \bigotimes_{i\in I} X_i$ be the cartesian product. Then*

$$\|x\|_p := \left( \sum_{i\in I} \|x_i\|_{X_i}^p \right)^{1/p} \qquad (1 \leq p < \infty),$$

*and*

$$\|x\|_\infty := \sup_{i\in I} \|x_i\|_{X_i}$$

*define equivalent norms on $\mathscr{X}$. In particular, the cartesian product is a normed space.*

*Proof.* The easy proof is left to the reader.

**Lemma 1.19.** *Let $(X_i)_{i\in I}$ be a finite family of normed spaces, and let $\mathscr{X} := \bigotimes_{i\in I} X_i$ be the cartesian product equipped with one of the equivalent norms $\|\cdot\|_p$ from*

*Lemma 1.18. Then a sequence* $(x^n) = ((x_i^n)_i) \subseteq \mathscr{X}$ *converges (is a Cauchy sequence) if and only if* $(x_i^n) \subseteq X_i$ *is convergent (is a Cauchy sequence) for every* $i \in I$.

*As a consequence,* $\mathscr{X}$ *is a Banach space if and only if all the* $X_i$ *are Banach spaces.*

**Proposition 1.20 (Quotient space).** *Let* $X$ *be a vector space (!) over* $\mathbb{K}$*, and let* $Y \subseteq X$ *be a linear subspace. Define, for every* $x \in X$*, the affine subspace*

$$x + Y := \{x + y : y \in Y\},$$

*and define the* **quotient space** *or* **factor space**

$$X/Y := \{x + Y : x \in X\}.$$

*Then* $X/Y$ *is a vector space for the addition*

$$(x+Y) + (z+Y) := (x+z+Y),$$

*and the scalar multiplication*

$$\lambda(x+Y) := (\lambda x + Y).$$

*The neutral element is* $Y$.

For the definition of quotient spaces, it is not important that we consider real or complex vector spaces.

Examples of quotient spaces are already known. In fact, $L^p$ is such an example. Usually, one defines

$$\mathscr{L}^p(\Omega, \mathscr{A}, \mu)$$

to be the space of *all* measurable functions $f : \Omega \to \mathbb{K}$ such that $\int_\Omega |f|^p \, \mathrm{d}\mu < \infty$. Moreover,

$$N := \{f \in \mathscr{L}^p(\Omega, \mathscr{A}, \mu) : \int_\Omega |f|^p = 0\}.$$

Note that $N$ is a linear subspace of $\mathscr{L}^p(\Omega, \mathscr{A}, \mu)$, and that $N$ is the space of all functions $f \in \mathscr{L}^p$ which vanish almost everywhere. Then

$$L^p(\Omega, \mathscr{A}, \mu) := \mathscr{L}^p(\Omega, \mathscr{A}, \mu)/N.$$

**Proposition 1.21.** *Let* $X$ *be a normed space and let* $Y \subseteq X$ *be a linear subspace. Then*

$$\|x + Y\| := \inf\{\|x - y\| : y \in Y\}$$

*defines a norm on* $X/Y$ *if and only if* $Y$ *is closed in* $X$*. If* $X$ *is a Banach space and* $Y \subseteq X$ *closed, then* $X/Y$ *is also a Banach space.*

*Proof.* We have to check that $\|\cdot\|$ satisfies all properties of a norm. Recall that $0_{X/Y} = Y$, and that for all $x \in X$

$$\begin{aligned}
&\|x+Y\| = 0 \\
&\Leftrightarrow \inf\{\|x-y\| : y \in Y\} = 0 \\
&\Leftrightarrow \exists (y_n) \subseteq Y : \lim_{n\to\infty} y_n = x \\
&\Leftrightarrow (\Rightarrow \text{ if } Y \text{ closed}) : x \in Y \\
&\Leftrightarrow x+Y = Y.
\end{aligned}$$

Second, for every $x \in X$ and every $\lambda \in \mathbb{K} \setminus \{0\}$,

$$\begin{aligned}
\|\lambda(x+Y)\| &= \|\lambda x + Y\| \\
&= \inf\{\|\lambda x - y\| : y \in Y\} \\
&= \inf\{\|\lambda(x-y)\| : y \in Y\} \\
&= |\lambda| \inf\{\|x-y\| : y \in Y\} \\
&= |\lambda| \, \|x+Y\|.
\end{aligned}$$

Third, for every $x, z \in X$,

$$\begin{aligned}
\|(x+Y)+(z+Y)\| &= \|(x+z)+Y\| \\
&= \inf\{\|x+z-y\| : y \in Y\} \\
&= \inf\{\|x+z-y_1-y_2\| : y_1, y_2 \in Y\} \\
&\leq \inf\{\|x-y_1\| + \|z-y_2\| : y_1, y_2 \in Y\} \\
&\leq \inf\{\|x-y\| : y \in Y\} + \inf\{\|z-y\| : y \in Y\} \\
&= \|x+Y\| + \|z+Y\|.
\end{aligned}$$

Hence, $X/Y$ is a normed space if $Y$ is closed.

Assume next that $X$ is a Banach space. Let $(x_n) \subseteq X$ be such that the series $\sum_{n\geq 1} x_n + Y$ converges absolutely, that is, $\sum_{n\geq 1} \|x_n + Y\| < \infty$. By definition of the norm in $X/Y$, we find $(y_n) \subseteq Y$ such that $\|x_n - y_n\| \leq \|x_n + Y\| + 2^{-n}$. Replacing $(x_n)$ by $(\hat{x}_n) = (x_n - y_n)$, we find that $x_n + Y = \hat{x}_n + Y$ and that the series $\sum_{n\geq 0} \hat{x}_n$ is absolutely convergent. Since $X$ is complete, by Lemma 1.13, the limit $\sum_{n\geq 1} \hat{x}_n = x \in X$ exists. As a consequence,

$$\begin{aligned}
\|(x+Y) - \sum_{k=1}^{n} (\hat{x}_k + Y)\| &= \|(x - \sum_{k=1}^{n} \hat{x}_k) + Y\| \\
&\leq \|x - \sum_{k=1}^{n} \hat{x}_k\| \quad \to \quad 0,
\end{aligned}$$

that is, the series $\sum_{n\geq 1} x_n + Y$ converges. By Lemma 1.13, $X/Y$ is complete.

## 1.3 Bounded linear operators

In the following a linear mapping between two normed spaces $X$ and $Y$ will also be called a **linear operator** or just **operator**. If $Y = \mathbb{K}$, then we call linear operators also **linear functionals**. If $T : X \to Y$ is a linear operator between two normed spaces, then we denote by

$$\ker T := \{x \in X : Tx = 0\}$$

its **kernel** or **null space**, and by

$$\operatorname{ran} T := \{Tx : x \in X\}$$

its **range** or **image**. Observe that we simply write $Tx$ instead of $T(x)$, meaning that $T$ is applied to $x \in X$. The identity operator $X \to X$, $x \mapsto x$ is denoted by $I$.

**Lemma 1.22.** *Let $T : X \to Y$ be a linear operator between two normed spaces $X$ and $Y$. Then the following are equivalent*

(i) *$T$ is continuous.*

(ii) *$T$ is continuous at* $0$.

(iii) *$TB$ is bounded in $Y$, where $B = B(0,1)$ denotes the unit ball in $X$.*

(iv) *There exists a constant $C \geq 0$ such that for every $x \in X$*

$$\|Tx\| \leq C\,\|x\|.$$

*Proof.* The implication (i)⇒(ii) is trivial.

(ii)⇒(iii). If $T$ is continuous at 0, then there exists some $\delta > 0$ such that for every $x \in B(0,\delta)$ one has $Tx \in B(0,1)$ (so the $\varepsilon$ from the $\varepsilon$-$\delta$ definition of continuity is chosen to be 1 here). By linearity, for every $x \in B = B(0,1)$

$$\|Tx\| = \frac{1}{\delta}\,\|T(\delta x)\| \leq \frac{1}{\delta},$$

and this means that $TB$ is bounded.

(iii)⇒(iv). The set $TB$ being bounded in $Y$ means that there exists some constant $C \geq 0$ such that for every $x \in B$ one has $\|Tx\| \leq C$. By linearity, for every $x \in X \setminus \{0\}$,

$$\|Tx\| = \|T\frac{x}{\|x\|}\|\,\|x\| \leq C\,\|x\|.$$

(iv)⇒(i). Let $x \in X$, and assume that $\lim_{n\to\infty} x_n = x$. Then

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \leq C\|x_n - x\| \to 0 \quad \text{as } n \to \infty,$$

so that $\lim_{n\to\infty} Tx_n = Tx$.

We call a continuous linear operator $T : X \to Y$ between two normed spaces $X$ and $Y$ also a **bounded operator** (since it maps the unit ball of $X$ to a bounded subset of $Y$). The set of all bounded linear operators is denoted by $\mathscr{L}(X,Y)$. Special cases: If $X = Y$, then we write $\mathscr{L}(X,X) =: \mathscr{L}(X)$. If $Y = \mathbb{K}$, then we write $\mathscr{L}(X,\mathbb{K}) =: X'$.

**Lemma 1.23.** *The set $\mathscr{L}(X,Y)$ is a vector space and*

$$\begin{aligned} \|T\| &:= \sup\{\|Tx\| : \|x\| \le 1\} \\ &= \sup\{\|Tx\| : \|x\| = 1\} \\ &= \sup\{\frac{\|Tx\|}{\|x\|} : x \neq 0\} \\ &= \inf\{C \ge 0 : \|Tx\| \le C\|x\| \textit{ for all } x \in X\} \\ &= \min\{C \ge 0 : \|Tx\| \le C\|x\| \textit{ for all } x \in X\} \end{aligned} \tag{1.1}$$

*is a norm on $\mathscr{L}(X,Y)$.*

*Proof.* We first show that the three quantities on the right-hand side of (1.1) are equal. In fact, the equality

$$\sup\{\|Tx\| : \|x\| \le 1\} = \sup\{\|Tx\| : \|x\| = 1\}$$

is easy to check so that it remains only to show that

$$A := \inf\{C \ge 0 : \|Tx\| \le C\|x\| \text{ for all } x \in X\} = \sup\{\|Tx\| : \|x\| = 1\} =: B.$$

If $C > A$, then for every $x \in X \setminus \{0\}$, $\|Tx\| \le C\|x\|$ or $\|T\frac{x}{\|x\|}\| \le C$. Hence, $C \ge B$ which implies that $A \ge B$. If $C > B$, then for every $x \in X \setminus \{0\}$, $\|T\frac{x}{\|x\|}\| \le C$, and therefore $\|Tx\| \le C\|x\|$. Hence, $C \ge A$ which implies that $A \le B$.

Now we check that $\|\cdot\|$ is a norm on $\mathscr{L}(X,Y)$. First, for every $T \in \mathscr{L}(X,Y)$,

$$\begin{aligned} \|T\| = 0 &\Leftrightarrow \sup\{\|Tx\| : \|x\| \le 1\} = 0 \\ &\Leftrightarrow \forall x \in X, \|x\| \le 1 : \|Tx\| = 0 \\ &\Leftrightarrow (\|\cdot\| \text{ is a norm on } Y)\, \forall x \in X, \|x\| \le 1 : Tx = 0 \\ &\Leftrightarrow (\Rightarrow \text{ linearity of } T)\, \forall x \in X : Tx = 0 \\ &\Leftrightarrow T = 0. \end{aligned}$$

Second, for every $T \in \mathscr{L}(X,Y)$ and every $\lambda \in \mathbb{K}$

$$\begin{aligned} \|\lambda T\| &= \sup\{\|(\lambda T)x\| : \|x\| \le 1\} \\ &= \sup\{|\lambda|\,\|Tx\| : \|x\| \le 1\} \\ &= |\lambda|\,\|T\|. \end{aligned}$$

Finally, for every $T, S \in \mathscr{L}(X,Y)$,

$$\begin{aligned}\|T+S\| &= \sup\{\|(T+S)x\| : \|x\| \le 1\}\\ &\le \sup\{\|Tx\| + \|Sx\| : \|x\| \le 1\}\\ &\le \|T\| + \|S\|.\end{aligned}$$

The proof is complete.

**Remark 1.24.** (a) Note that the infimum on the right-hand side of (1.1) in Lemma 1.23 is always attained. Thus, for every operator $T \in \mathscr{L}(X,Y)$ and every $x \in X$,

$$\|Tx\| \le \|T\| \, \|x\|.$$

This inequality shall be frequently used in the sequel! Note that on the other hand the suprema on the right-hand side of (1.1) are not always attained. (b) From Lemma 1.23 we can learn how to show that some operator $T : X \to Y$ is bounded and how to calculate the norm $\|T\|$. Usually (in most cases), one should prove in the *first step* some inequality of the form

$$\|Tx\| \le C\,\|x\|, \quad x \in X,$$

because this inequality shows on the one hand that $T$ *is* bounded, and on the other hand it shows the estimate $\|T\| \le C$. In the *second step* one should prove that the estimate $C$ was optimal by finding some $x \in X$ of norm $\|x\| = 1$ such that $\|Tx\| = C$, or by finding some sequence $(x_n) \subseteq X$ of norms $\|x_n\| \le 1$ such that $\lim_{n\to\infty} \|Tx_n\| = C$, because this shows that $\|T\| = C$. Of course, the second step only works if one has not lost anything in the estimate of the first step. There are in fact many examples of bounded operators for which it is difficult to estimate their norm.

**Example 1.25.** 1. (Shift-operator). On $l^p(\mathbb{N})$ consider the *left-shift operator*

$$Lx = L(x_n) = (x_{n+1}).$$

Then

$$\|L(x_n)\|_p = \Big(\sum_n |x_{n+1}|^p\Big)^{1/p} \le \Big(\sum_n |x_n|^p\Big)^{1/p},$$

so that $L$ is bounded and $\|L\| \le 1$. On the other hand, for $x = (0,1,0,0,\dots)$ one computes that $\|x\|_p = 1$ and $\|Lx\|_p = \|(1,0,0,\dots)\|_p = 1$, and one concludes that $\|L\| = 1$.

2. (Shift-operator). Similarly, one shows that the *right-shift operator* $R$ on $l^p(\mathbb{N})$ defined by

$$Rx = R(x_n) = (0, x_0, x_1, \dots)$$

is bounded and $\|R\| = 1$. Note that actually $\|Rx\|_p = \|x\|_p$ for every $x \in l^p$.

3. (Multiplication operator). Let $m \in l^\infty$ and consider on $l^p$ the *multiplication operator*

$$Mx = M(x_n) = (m_n x_n).$$

4. (Functionals on $C$). Consider the linear functional $\varphi : C([0,1]) \to \mathbb{K}$ defined by

$$\varphi(f) := \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x.$$

Then

$$|\varphi(f)| \leq \int_0^{\frac{1}{2}} |f(x)|\,\mathrm{d}x \leq \frac{1}{2}\|f\|_\infty,$$

so that $\varphi$ is bounded and $\|\varphi\| \leq \frac{1}{2}$. On the other hand, for the constant function $f = 1$ one has $\|f\|_\infty = 1$ and $|\varphi(f)| = \frac{1}{2}$, so that $\|\varphi\| = \frac{1}{2}$.

**Lemma 1.26.** *Let $X$, $Y$, $Z$ be three Banach spaces, and let $T \in \mathscr{L}(X,Y)$ and $S \in \mathscr{L}(Y,Z)$. Then $ST \in \mathscr{L}(X,Z)$ and*

$$\|ST\| \leq \|S\|\,\|T\|.$$

*Proof.* The boundedness of $ST$ is clear since compositions of continuous functions are again continuous. To obtain the bound on $ST$, we calculate

$$\begin{aligned}\|ST\| &= \sup_{\|x\|\leq 1} \|STx\| \\ &\leq \sup_{\|x\|\leq 1} \|S\|\,\|Tx\| \\ &= \|S\|\,\|T\|.\end{aligned}$$

**Lemma 1.27.** *If $Y$ is a Banach space then $\mathscr{L}(X,Y)$ is a Banach space.*

*Proof.* Assume that $Y$ is a Banach space and let $(T_n)$ be a Cauchy sequence in $\mathscr{L}(X,Y)$. By the estimate

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\|\,\|x\|,$$

the sequence $(T_n x)$ is a Cauchy sequence in $Y$ for every $x \in X$. Since $Y$ is complete, the limit $\lim_{n\to\infty} T_n x$ exists for every $x \in X$. Define $Tx := \lim_{n\to\infty} T_n x$. Clearly, $T : X \to Y$ is linear. Moreover, since any Cauchy sequence is bounded, we find that

$$\|Tx\| \leq \sup_n \|T_n x\| \leq C\,\|x\|$$

for some constant $C \geq 0$, that is, $T$ is bounded. Moreover, for every $n \in \mathbb{N}$ we have the estimate

$$\begin{aligned}\|T - T_n\| &= \sup_{\|x\|\leq 1} \|Tx - T_n x\| \\ &\leq \sup_{\|x\|\leq 1} \sup_{m\geq n} \|T_m x - T_n x\| \\ &\leq \sup_{m\geq n} \|T_m - T_n\|.\end{aligned}$$

Since that right-hand side of this inequality becomes arbitrarily small for large $n$, we see that $\lim_{n\to\infty} T_n = T$ exists, and so we have proved that $\mathscr{L}(X,Y)$ is a Banach space.

**Remark 1.28.** The converse of the statement in Lemma 1.27 is also true, that is, if $\mathscr{L}(X,Y)$ is a Banach space then necessarily $Y$ is a Banach space. For the proof, however, one has to know that there are nontrivial operators in $\mathscr{L}(X,Y)$ as soon as $Y$ is nontrivial (that is, $Y \neq \{0\}$). For this, we need the Theorem of Hahn-Banach and its consequences discussed in Chapter 3.

**Corollary 1.29.** *The space $X' = \mathscr{L}(X,\mathbb{K})$ of all bounded linear functionals on $X$ is always a Banach space. The space $X'$ is called the* **dual space** *of $X$.*

Let $X$, $Y$ be two normed spaces. We call $T \in \mathscr{L}(X,Y)$ an **isomorphism** if $T$ is bijective and $T^{-1} \in \mathscr{L}(Y,X)$. We call $T \in \mathscr{L}(X,Y)$ an **isometry** if $\|Tx\| = \|x\|$ for every $x \in X$. We say that space $X$ and $Y$ are **isomorphic** (and we write $X \cong Y$) if there exists an isomorphism $T \in \mathscr{L}(X,Y)$. We say that $X$ and $Y$ are **isometrically isomorphic** if there exists an isometric isomorphism $T \in \mathscr{L}(X,Y)$.

**Remark 1.30.** 1. Two norms $\|\cdot\|_1$, $\|\cdot\|_2$ on a $\mathbb{K}$ vector space $X$ are equivalent if and only if the identity operator $I : (X,\|\cdot\|_1) \to (X,\|\cdot\|_2)$ is an isomorphism.
2. Saying that two *normed* spaces $X$ and $Y$ are isomorphic means that they are not only 'equal' as vector spaces (in the sense that we find a bijective linear operator) but also as normed spaces (that is, the bijection is continuous as well as its inverse).
3. If $T \in \mathscr{L}(X,Y)$ and $S \in \mathscr{L}(Y,Z)$ are isomorphisms, then $ST \in \mathscr{L}(X,Z)$ is an isomorphism and $(ST)^{-1} = T^{-1}S^{-1}$.
4. Every isometry $T \in \mathscr{L}(X,Y)$ is clearly injective. If it is also surjective, then $T$ is an isometric isomorphism, that is, the inverse $T^{-1}$ is also bounded (even isometric).
5. Clearly, if $T \in \mathscr{L}(X,Y)$ is isometric, then it is an isometric isomorphism from $X$ onto $\operatorname{ran} T$, and we may say that $X$ is **isometrically embedded** into $Y$ (via $T$).

**Example 1.31.** The right-shift operator from Example 1.25 (2) is isometric, but not surjective. In particular, $l^p$ is isometrically isomorphic to a proper subspace of $l^p$.

**Exercise 1.32** *Show that the spaces $(c,\|\cdot\|_\infty)$ of all convergent sequences and $(c_0,\|\cdot\|_\infty)$ of all null sequences are isomorphic.*

**Exercise 1.33** *Show that $(c_0,\|\cdot\|_\infty)$ is (isometrically) isomorphic to a linear subspace of $(C([0,1]),\|\cdot\|_\infty)$, that is, find an isometry $T : c_0 \to C([0,1])$.*

**Lemma 1.34 (Neumann series).** *Let $X$ be a Banach space and let $T \in \mathscr{L}(X)$ be such that $\|T\| < 1$. Then $I - T$ is boundedly invertible, that is, it is an isomorphism. Moreover, $(I-T)^{-1} = \sum_{n\geq 0} T^n$.*

*Proof.* Since $X$ is a Banach space, $\mathscr{L}(X)$ is also a Banach space by Lemma 1.27. By assumption on $\|T\|$, the series $\sum_{n\geq 0} T^n$ is absolutely convergent, and hence, by Lemma 1.13, it is convergent to some element $S \in \mathscr{L}(X)$. Moreover,

$$(I-T)S = \lim_{n\to\infty}(I-T)\sum_{k=0}^{n} T^k = \lim_{n\to\infty}(I-T^{k+1}) = I,$$

and similarly, $S(I-T) = I$.

**Corollary 1.35.** *Let $X$ and $Y$ be two Banach spaces. Then the set $\mathscr{I}(X,Y)$ of all isomorphisms in $\mathscr{L}(X,Y)$ is open, and the mapping $T \mapsto T^{-1}$ is continuous from $\mathscr{I}(X,Y)$ onto $\mathscr{I}(Y,X)$.*

*Proof.* Let $\mathscr{I} \subseteq \mathscr{L}(X,Y)$ be the set of all isomorphisms, and assume that $\mathscr{I}$ is not empty (if it is empty, then it is also open). Let $T \in \mathscr{I}$. Then for every $S \in B(T, \frac{1}{\|T^{-1}\|})$ we have

$$S = T + S - T = T(I + T^{-1}(S-T)),$$

and since $\|T^{-1}(S-T)\| \leq \|T^{-1}\| \, \|S-T\| < 1$, the operator $I + T^{-1}(S-T) \in \mathscr{L}(X)$ is an isomorphism by Lemma 1.34. As a composition of two isomorphisms, $S \in \mathscr{I}$, and hence $\mathscr{I}$ is open. The continuity is also a direct consequence of the above representation of $S$ (and thus of its inverse), using the Neumann series.

## 1.4 The Arzelà-Ascoli theorem

It is a consequence of Riesz' Lemma (Lemma 1.14) that the unit ball in an infinite dimensional Banach space is not compact; see also Theorem 1.16. But compact sets play an important role in many theorems from analysis, in particular when one wants to prove the existence of some fixed point, the existence of a solution to an algebraic equation, the existence of a solution of a differential equation, the existence of a solution of a partial differential equation etc. It is therefore important to identify the compact sets in Banach spaces, in particular in the classical Banach spaces. The Arzalà-Ascoli theorem characterizes the compact subsets of $C(K;X)$, where $(K,d)$ is a compact metric space and $X$ is a Banach space.

We say that a subset $B \subseteq C(K;X)$ is **equicontinuous at some point** $x \in K$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in K$ and every $f \in B$ the implication

$$d(x,y) < \delta \quad \Rightarrow \quad \|f(x) - f(y)\| < \varepsilon$$

holds.

**Theorem 1.36 (Arzelà-Ascoli).** *Let $(K,d)$ be a compact metric space, $X$ be a Banach space and consider the Banach space $C(K;X)$ of all continuous functions $K \to X$ equipped with the supremum norm $\|f\|_\infty = \sup_{x\in K}\|f(x)\|$. For a subset $B \subseteq C(K;X)$, the following assertions are equivalent:*

(i) *The set $B$ is relatively compact.*

(ii) *The set $B$ is equicontinuous at every $x \in K$ and there exists a dense set $D \subseteq K$ such that for every $x \in D$ the set $B_x = \{f(x) : f \in B\}$ is relatively compact.*

We point out that, by the Heine-Borel theorem, the condition of pointwise relative compactness of $B$ can be replaced by mere pointwise or global boundedness as soon as the space $X$ is finite dimensional.

**Corollary 1.37 (Arzelà-Ascoli).** *Let $(K,d)$ be a compact metric space, and consider the Banach space $C(K;\mathbb{R}^d)$ of all continuous functions $K \to \mathbb{R}^d$ equipped with the supremum norm $\|f\|_\infty = \sup_{x\in K} \|f(x)\|$. For a subset $B \subseteq C(K;\mathbb{R}^d)$, the following assertions are equivalent:*

(i) *The set $B$ is compact.*

(ii) *The set $B$ is closed, equicontinuous at every $x \in K$ and pointwise bounded in the sense that for every $x \in K$ the set $B_x = \{f(x) : f \in B\}$ is bounded.*

*Proof (of Theorem 1.36).* The proof of the Arzelà-Ascoli theorem is a nice application of Cantor's diagonal sequence argument which we see here for the first time, but which we will see again below when we prove that every bounded sequence in a reflexive Banach space admits a weakly convergent subsequence. Given a sequence, Cantor's argument allows us to construct a subsequence which satisfies a countable number of properties. It is instructive to learn the idea of Cantor's argument since it can be help in various situations.

We first assume that $B \subseteq C(K;X)$ is relatively compact. Any relatively compact subset of a Banach space is bounded, and therefore $B$ is bounded, too. For every $x \in K$, the point evaluation $C(K;X) \to X$, $f \mapsto f(x)$ is linear and continuous. Since continuous images of relatively compact sets are relatively compact, the image of $B$ under the point evaluation, that is the set $B_x = \{f(x) : f \in B\}$, is relatively compact.

We show that $B$ is equicontinuous at every $x$. Assume that this was not the case. Then there exist $x \in K$ and $\varepsilon > 0$ such that for every $n \geq 1$ there exist $y_n \in K$ and $f_n \in B$ such that $d(x,y_n) < \frac{1}{n}$ and $\|f_n(x) - f_n(y_n)\| \geq \varepsilon$. Since $B$ is relatively compact, there exists a subsequence of $(f_n)$ (which we denote for simplicity again by $(f_n)$) such that $\lim_{n\to\infty} f_n = f$ in $C(K;X)$. Then, by the triangle inequality from below,

$$\begin{aligned}
\liminf_{n\to\infty} \|f(x) - f(y_n)\| &= \liminf_{n\to\infty} \|f(x) - f_n(x) + f_n(x) - f_n(y_n) + f_n(y_n) - f(y_n)\| \\
&\geq \liminf_{n\to\infty} \big(\|f_n(x) - f_n(y_n)\| - 2\|f - f_n\|_\infty\big) \\
&\geq \varepsilon.
\end{aligned}$$

This inequality, however, contradicts to the continuity of $f$ (note that $\lim_{n\to\infty} y_n = x$), and therefore, $B$ is equicontinuous at every $x \in K$.

Assume now that $B$ satisfies the properties from assertion (ii). In order to show that $B$ is relatively compact, it suffices to show that every sequence $(f_n) \subseteq B$ admits a convergent subsequence, that is, $B$ is relatively sequentially compact. So let $(f_n) \subseteq B$ be an arbitrary sequence.

Recall that every compact metric space is separable. Moreover, every subset of a separable space is separable. Hence, there exists a sequence $(x_m)_{m\geq 1} \subseteq K$ which is dense in $K$.

Consider the sequence $(f_n(x_1)) \subseteq B_{x_1} \subseteq X$. Since $B_{x_1}$ is relatively compact by assumption, there exists a subsequence $(f_{\varphi_1(n)})$ of $(f_n)$ such that $\lim_{n\to\infty} f_{\varphi_1(n)}(x_1)$ exists.

Consider next the sequence $(f_{\varphi_1(n)}(x_2)) \subseteq B_{x_2} \subseteq X$. Since $B_{x_2}$ is relatively compact by assumption, there exists a subsequence $(f_{\varphi_2(n)})$ of $(f_{\varphi_1(n)})$ such that $\lim_{n\to\infty} f_{\varphi_2(n)}(x_2)$ exists. Note that we have also the existence of the limit $\lim_{n\to\infty} f_{\varphi_2(n)}(x_1)$.

Iterating this argument, we obtain for every $m \geq 2$ a subsequence $(f_{\varphi_m(n)})$ of $(f_{\varphi_{m-1}(n)})$ such that $\lim_{n\to\infty} f_{\varphi_m(n)}(x_i)$ exists for every $1 \leq i \leq m$. These subsequences converge therefore pointwise at a finite number of elements of $K$.

We now consider the *diagonal subsequence* $(f_{\varphi(n)}) = (f_{\varphi_n(n)})$. This diagonal subsequence has the property of being a subsequence of $(f_{\varphi_m(n)})$ for every $m \geq 1$, up to a finite number of initial elements perhaps. It enjoys therefore the property that $\lim_{n\to\infty} f_{\varphi(n)}(x_m)$ exists for every $m \geq 1$, that is, it converges pointwise on a dense subset of $K$. We will show that $(f_{\varphi(n)})$ converges everywhere and uniformly on $K$. Since $C(K;X)$ is complete, it suffices to show that $(f_{\varphi(n)})$ is a Cauchy sequence in $C(K;X)$.

Let $\varepsilon > 0$. Since $B$ is equicontinuous at every $x \in K$, for every $x \in K$ there exists $\delta_x > 0$ such that for every $y \in K$ and every $f \in B$ the implication

$$d(x,y) < \delta \quad \Rightarrow \quad \|f(x) - f(y)\| < \varepsilon \tag{1.2}$$

is true. We clearly have $K = \bigcup_{x\in K} B(x,\delta_x)$, and since $K$ is compact, we find finitely many points $x'_1, \ldots, x'_k$ such that $K = \bigcup_{i=1}^{k} B(x'_i,\delta_i)$ (with $\delta_i = \delta_{x'_i}$). Since the sequence $(x_m)$ is dense in $K$, for every $1 \leq i \leq k$ there exists $m_i \geq 1$ such that $x_{m_i} \in B(x'_i,\delta_i)$. Since the sequence $(f_{\varphi(n)})$ converges pointwise on $(x_m)$, there exists $n_0 \geq 0$ such that

$$\text{for every } n, n' \geq n_0 \text{ and every } 1 \leq i \leq k \qquad \|f_{\varphi(n)}(x_{m_i}) - f_{\varphi(n')}(x_{m_i})\| < \varepsilon.$$

Let now $x \in K$ be arbitrary. Then $x \in B(x'_i,\delta_i)$ for some $1 \leq i \leq k$. Hence, for every $n, n' \geq n_0$, by the preceding estimate and by the implication (1.2),

$$\begin{aligned}
\|f_{\varphi(n)}(x) - f_{\varphi(n')}(x)\| &\leq \|f_{\varphi(n)}(x) - f_{\varphi(n)}(x'_i)\| + \\
&\quad + \|f_{\varphi(n)}(x'_i) - f_{\varphi(n)}(x_{m_i})\| + \\
&\quad + \|f_{\varphi(n)}(x_{m_i}) - f_{\varphi(n')}(x_{m_i})\| + \\
&\quad + \|f_{\varphi(n')}(x_{m_i}) - f_{\varphi(n')}(x'_i)\| + \\
&\quad + \|f_{\varphi(n')}(x'_i) - f_{\varphi(n')}(x)\| \\
&\leq 5\varepsilon.
\end{aligned}$$

Since $n_0 \geq 0$ did not depend on $x \in K$, and since $\varepsilon > 0$ was arbitrary, this proves that $(f_{\varphi(n)})$ is a Cauchy sequence in $C(K;X)$. We have therefore proved that every sequence in $B$ admits a convergent subsequence.

# Chapter 2
# Hilbert spaces

Let $H$ be a vector space over $\mathbb{K}$.

## 2.1 Inner product spaces

A function $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{K}$ is called an **inner product** if for every $x$, $y$, $z \in H$ and every $\lambda \in \mathbb{K}$

(i) $\langle x,x\rangle \geq 0$ for every $x \in H$ and $\langle x,x\rangle = 0$ if and only if $x = 0$,

(ii) $\langle x,y\rangle = \overline{\langle y,x\rangle}$,

(iii) $\langle \lambda x + y, z\rangle = \lambda\langle x,z\rangle + \langle y,z\rangle$.

A pair $(H, \langle\cdot,\cdot\rangle)$ of a vector space over $\mathbb{K}$ and a scalar product is called an **inner product space**.

**Example 2.1.** 1. On the space $H = \mathbb{K}^d$,

$$\langle x,y\rangle := \sum_{i=1}^{d} x_i \bar{y}_i$$

defines an inner product.

2. On the space $H = l^2 := \{(x_n) \subseteq \mathbb{K} : \sum |x_n|^2 < \infty\}$,

$$\langle x,y\rangle := \sum_{n} x_n \bar{y}_n$$

defines an inner product.

3. On the space $H = C([0,1])$, the Riemann integral

$$\langle f,g\rangle := \int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x$$

defines an inner product.

4. On the space $H = L^2(\Omega)$, the integral

$$\langle f, g\rangle := \int_\Omega f\bar{g}\,\mathrm{d}\mu$$

defines an inner product.

**Lemma 2.2.** *Let $\langle\cdot,\cdot\rangle$ be an inner product on a vector space $H$. Then, for every $x$, $y$, $z \in H$ and $\lambda \in \mathbb{K}$*

*(iv)* $\langle x, \lambda y + z\rangle = \bar{\lambda}\langle x, y\rangle + \langle x, z\rangle$.

*Proof.*

$$\langle x, \lambda y + z\rangle = \overline{\langle \lambda y + z, x\rangle} = \bar{\lambda}\,\overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} = \bar{\lambda}\langle x, y\rangle + \langle x, z\rangle.$$

In the following, if $H$ is an inner product space, then we put

$$\|x\| := \sqrt{\langle x, x\rangle}, \quad x \in H.$$

**Lemma 2.3 (Cauchy-Schwarz inequality).** *Let $H$ be an inner product space. Then, for every $x$, $y \in H$,*

$$|\langle x, y\rangle| \le \|x\|\,\|y\|,$$

*and equality holds if and only if $x$ and $y$ are colinear.*

*Proof.* Let $\lambda \in \mathbb{K}$. Then

$$\begin{aligned} 0 &\le \langle x + \lambda y, x + \lambda y\rangle \\ &= \langle x, x\rangle + \langle \lambda y, x\rangle + \langle x, \lambda y\rangle + |\lambda|^2\langle y, y\rangle \\ &= \langle x, x\rangle + \lambda\overline{\langle x, y\rangle} + \bar{\lambda}\langle x, y\rangle + |\lambda|^2\langle y, y\rangle, \end{aligned}$$

that is,

$$0 \le \|x + \lambda y\|^2 = \|x\|^2 + 2\mathrm{Re}\,\bar{\lambda}\langle x, y\rangle + |\lambda|^2\,\|y\|^2. \tag{2.1}$$

Assuming that $y \neq 0$ (for $y = 0$ the Cauchy-Schwarz inequality is trivial), we may put $\lambda := -\langle x, y\rangle/\|y\|^2$. Then

$$\begin{aligned} 0 &\le \langle x - \frac{\langle x, y\rangle}{\|y\|^2}y, x - \frac{\langle x, y\rangle}{\|y\|^2}y\rangle \\ &= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}, \end{aligned}$$

which is the Cauchy-Schwarz inequality. The calculation also shows that equality holds if and only if $x = \lambda y$, that is, if $x$ and $y$ are colinear.

**Lemma 2.4.** *Every inner product space $H$ is a normed linear space for the norm*

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad x \in H.$$

*Proof.* Properties (i) and (ii) in the definition of a norm follow from the properties (i) and (iii) (together with Lemma 2.2) in the definition of an inner product. The only difficulty is to show that $\|\cdot\|$ satisfies the triangle inequality. This, however, follows from putting $\lambda = 1$ in (2.1) and estimating with the Cauchy-Schwarz inequality:

$$\|x+y\|^2 \leq (\|x\|+\|y\|)^2.$$

A complete inner product space is called a **Hilbert space**.

**Example 2.5.** The spaces $\mathbb{K}^d$ (with Euclidean inner product), $l^2$ and $L^2(\Omega)$ are Hilbert spaces. More examples are given by the Sobolev spaces defined below.

**Lemma 2.6 (Completion of an inner product space).** *Let $H$ be an inner product space. Then there exists a Hilbert space $K$ and a bounded linear operator $j : H \to K$ such that for every $x, y \in H$*

$$\langle x,y\rangle_H = \langle j(x), j(y)\rangle_K,$$

*and such that $j(H)$ is dense in $K$. The Hilbert space $K$ is unique* up to isometry. *It is called the* **completion** *of $H$.*

**Lemma 2.7 (Parallelogram identity).** *Let $H$ be an inner product space. Then for every $x, y \in H$*

$$\|x+y\|^2 + \|x-y\|^2 = 2\,(\|x\|^2 + \|y\|^2).$$

*Proof.* The parallelogram identity follows immediately from (2.1) by putting $\lambda = \pm 1$ and adding up.

**Exercise 2.8 (von Neumann)** *Show that a norm satisfying the parallelogram identity comes from a scalar product. That means, the parallelogram identity characterises inner product spaces.*

A subset $K$ of a real or complex vector space $X$ is **convex** if for every $x, y \in K$ and every $t \in [0,1]$ one has $tx + (1-t)y \in K$.

**Theorem 2.9 (Projection onto closed, convex sets).** *Given a nonempty closed, convex subset $K$ of a Hilbert space $H$, and given a point $x \in H$, there exists a unique $y \in K$ such that*

$$\|x-y\| = \inf\{\|x-z\| : z \in K\}.$$

*Proof.* Let $d := \inf\{\|x-z\| : z \in K\}$, and choose $(y_n) \in K$ such that

$$\lim_{n\to\infty} \|x-y_n\| = d. \tag{2.2}$$

Applying the parallelogram identity to $(x-y_n)/2$ and $(x-y_m)/2$, we obtain

$$\|x - \frac{y_n+y_m}{2}\|^2 + \frac{1}{4}\|y_n - y_m\|^2 = \frac{1}{2}(\|x-y_n\|^2 + \|x-y_m\|^2).$$

Since $K$ is convex, $\frac{y_n+y_m}{2} \in K$ and hence $\|x - \frac{y_n+y_m}{2}\|^2 \geq d^2$. Using this and (2.2), the last identity implies that $(y_n)$ is a Cauchy sequence. Since $H$ is complete, $y := \lim_{n\to\infty} y_n$ exists. Since $K$ is closed, $y \in K$. Moreover, $\|x-y\| = \lim_{n\to\infty} \|x-y_n\| = d$, so that $y$ is a minimizer for the distance to $x$. To see that there is only one such minimizer, suppose that $y' \in K$ is a second one, and apply the parallelogram identity to $x-y$ and $x-y'$.

Let $H$ be an inner product space. We say that two vectors $x$, $y \in H$ are **orthogonal** (and we write $x \perp y$), if $\langle x,y\rangle = 0$. Given a subset $S \subseteq H$, we define the **orthogonal space** $S^\perp := \{y \in H : x \perp y \text{ for all } x \in S\}$. If $S = K$ is a linear subspace of $H$, then we call $K^\perp$ also the **orthogonal complement** of $K$.

**Theorem 2.10.** *Let $H$ be a Hilbert space, $S \subseteq H$ be a subset and $K$ a closed linear subspace. Then:*

(a) *$S^\perp$ is a closed linear subspace of $H$,*

(b) *$K$ and $K^\perp$ are complementary subspaces, i.e. every $x \in H$ can be decomposed uniquely as a sum of an $x_0 \in K$ and an $x_1 \in K^\perp$,*

(c) *$(K^\perp)^\perp = K$ and $(S^\perp)^\perp = \overline{\operatorname{span}} S$.*

(d) *$\operatorname{span} S$ is dense in $H$ if and only if $S^\perp = \{0\}$.*

*Proof.* (a) It follows from the bilinearity of the inner product that $S^\perp$ is a linear subspace of $H$. Let $(y_n) \subseteq S^\perp$ be convergent to some $y \in H$. Then, for every $x \in S$, by the Cauchy-Schwarz inequality,

$$\langle x,y\rangle = \lim_{n\to\infty} \langle x,y_n\rangle = 0,$$

that is, $y \in S^\perp$ and therefore $S^\perp$ is closed.

(b) For every $x \in H$ we let $x_0 \in K$ be the unique element (Theorem 2.9) such that

$$\|x-x_0\| = \inf\{\|x-y\| : y \in K\}.$$

Put $x_1 = x - x_0$. For every $y \in K$ and every $\lambda \in \mathbb{K}$, by the minimum property of $x_0$,

$$\begin{aligned}\|x_1\|^2 &\leq \|x_1 - \lambda y\|^2 \\ &= \|x_1\|^2 - 2\mathrm{Re}\,\bar{\lambda}\langle x_1,y\rangle + |\lambda|^2\|y\|^2.\end{aligned}$$

This implies that $\langle x_1,y\rangle = 0$, that is, $x_1 \in K^\perp$. Every decomposition $x = x_0 + x_1$ with $x_0 \in K$ and $x_1 \in K^\perp$ is unique since $x \in K \cap K^\perp$ implies $\langle x,x\rangle = 0$, that is, $x = 0$.

(c) and (d) follow immediately from (a) and (b).

**Lemma 2.11 (Pythagoras).** *Let $H$ be an inner product space. Whenever $x$, $y \in H$ are orthogonal, then*

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

*Proof.* The claim follows from (2.1) and putting $\lambda = 1$.

We call an operator $P : X \to X$ on a linear space $X$ a **projection** if $P^2 = P$.

**Lemma 2.12.** *Let $X$ be a normed space and let $P \in \mathscr{L}(X)$ be a bounded projection. Then the following are true:*

(a) *$Q = I - P$ is a projection.*

(b) *Either $P = 0$ or $\|P\| \geq 1$.*

(c) *The kernel $\ker P$ and the range $\operatorname{ran} P$ are closed in $X$.*

(d) *Every $x \in X$ can be decomposed uniquely as a sum of an $x_0 \in \ker P$ and an $x_1 \in \operatorname{ran} P$, and $X \cong \ker P \oplus \operatorname{ran} P$.*

*Proof.* (a) $Q^2 = (I-P)^2 = I - 2P + P^2 = I - P = Q$.

(b) follows from $\|P\| = \|P^2\| \leq \|P\|^2$.

(c) Since $\{0\}$ is closed in $X$ and since $P$ is continuous, $\ker P = P^{-1}(\{0\})$ is closed. Similarly, $\operatorname{ran} P = \ker(I-P)$ is closed.

(d) For every $x \in X$ we can write $x = Px + (I-P)x = x_1 + x_2$ with $x_1 \in \operatorname{ran} P$ and $x_2 \in \ker P$. The decomposition is unique since if $x \in \ker P \cap \operatorname{ran} P$, then $x = Px = 0$. This proves that the *vector* spaces $X$ and $\ker P \oplus \operatorname{ran} P$ are isomorphic. That they are also isomorphic as normed spaces follows from the continuity of $P$.

**Lemma 2.13.** *Let $H$ be a Hilbert space and $K \subseteq H$ be a closed linear subspace. For every $x \in H$ we let $x_1 = Px$ be the unique element in $K$ which minimizes the distance to $x$ (Theorem 2.9). Then $P : H \to H$ is a bounded projection satisfying $\operatorname{ran} P = K$. Moreover, $\ker P = K^{\perp}$. We call $P$ the* orthogonal projection *onto $K$.*

## 2.2 Orthogonal decomposition

We call a metric space **separable** if there exists a countable dense subset.

**Example 2.14.** The space $\mathbb{R}^d$ (or $\mathbb{C}^d$) is separable: one may take $\mathbb{Q}^d$ as an example of a dense countable subset. It is not too difficult to see that subsets of separable metric spaces are separable (note, however, that in general the dense subset has to be constructed carefully), and that finite products of separable metric spaces are separable.

**Lemma 2.15.** *A normed space $X$ is separable if and only if there exists a sequence $(x_n) \subseteq X$ such that $\operatorname{span}\{x_n : n \in \mathbb{N}\}$ is dense in $X$ (such a sequence is in general called a* total *sequence).*

*Proof.* If $X$ is separable, then there exists a sequence $(x_n) \subseteq X$ such that $\{x_n : n \in \mathbb{N}\}$ is dense. In particular, the larger set $\operatorname{span}\{x_n : n \in \mathbb{N}\}$ is dense.

If, on the other hand, there exists a total sequence $(x_n) \subseteq X$, and if we put $D = \mathbb{Q}$ in the case $\mathbb{K} = \mathbb{R}$ and $D = \mathbb{Q} + i\mathbb{Q}$ in the case $\mathbb{K} = \mathbb{C}$, then the set

$$\{\sum_{i=1}^{m} \lambda_i x_{n_i} : m \in \mathbb{N},\, \lambda_i \in D,\, n_i \in \mathbb{N}\}$$

is dense in $X$ (in fact, the closure contains all finite linear combinations of the $x_n$, that is, it contains span$\{x_n : n \in \mathbb{N}\}$). It is an exercise to show that this set is countable. The claim follows.

**Corollary 2.16.** *The space* $(C([0,1]), \|\cdot\|_\infty)$ *is separable.*

*Proof.* By Weierstrass' theorem, the subspace of all polynomials is dense in $C([0,1])$ (Weierstrass' theorem says that every continuous function $f : [0,1] \to \mathbb{R}$ can be uniformly approximated by polynomials). The polynomials, however, are the linear span of the monomials $f_n(t) = t^n$. The claim therefore follows from Lemma 2.15.

**Corollary 2.17.** *The space* $l^p$ *is separable if* $1 \leq p < \infty$. *The space* $c_0$ *is separable.*

*Proof.* Let $e_n = (\delta_{nk})_k \in l^p$ be the $n$-th unit vector in $l^p$ (here $\delta_{nk}$ denotes the Kronecker symbol: $\delta_{nk} = 1$ if $n = k$ and $\delta_{nk} = 0$ otherwise). Then $span\{e_n : n \in \mathbb{N}\} = c_{00}$ (the space of all finite sequences) is dense in $l^p$ if $1 \leq p < \infty$. The claim for $l^p$ follows from Lemma 2.15. The argument for $c_0$ is similar.

**Lemma 2.18.** *The space* $l^\infty$ *is not separable.*

*Proof.* The set $\{0,1\}^{\mathbb{N}} \subseteq l^\infty$ of all sequences taking only values 0 or 1 is uncountable. Moreover, whenever $x, y \in \{0,1\}^{\mathbb{N}}$, $x \neq y$, then

$$\|x - y\|_\infty = 1.$$

Hence, the balls $B(x, \frac{1}{2})$ with centers $x \in \{0,1\}^{\mathbb{N}}$ and radius $\frac{1}{2}$ are mutually disjoint. If $l^\infty$ was separable, that is, if there exists a dense countable set $D \subseteq l^\infty$, then in each $B(x, \frac{1}{2})$ there exists at least one element $y \in D$, a contradiction.

**Definition 2.19.** Let $H$ be an inner product space. A family $(e_l)_{l \in I} \subseteq H$ is called

(a) an *orthogonal system* if $(e_l, e_k) = 0$ whenever $l \neq k$,

(b) an *orthonormal system* if it is an orthogonal system and $\|e_l\| = 1$ for every $l \in I$, and

(c) an *orthonormal basis* if it is an orthonormal system and span$\{e_l : l \in I\}$ is dense in $H$.

**Lemma 2.20 (Gram-Schmidt process).** *Let* $(x_n)$ *be a sequence in an inner product space* $H$. *Then there exists an orthonormal system* $(e_n)$ *such that* span$\{x_n\} =$ span$\{e_n\}$.

*Proof.* Passing to a subsequence, if necessary, we may assume that the $(x_n)$ are linearly independent.

Let $e_1 := x_1/\|x_1\|$. Then $e_1$ and $x_1$ span the same linear subspace. Next, assume that we have constructed an orthonormal system $(e_k)_{1 \leq k \leq n}$ such that

$$\text{span}\{x_k : 1 \leq k \leq n\} = \text{span}\{e_k : 1 \leq k \leq n\}.$$

Let $e'_{n+1} := x_{n+1} - \sum_{k=1}^{n} \langle x_{n+1}, e_k \rangle e_k$. Since the $x_n$ are linearly independent, we find $e'_{n+1} \neq 0$. Let $e_{n+1} := e'_{n+1}/\|e'_{n+1}\|$. By construction, for every $1 \leq k \leq n$, $\langle e_{n+1}, e_k \rangle = 0$, and

$$\operatorname{span}\{x_k : 1 \leq k \leq n+1\} = \operatorname{span}\{e_k : 1 \leq k \leq n+1\}.$$

Proceeding inductively, the claim follows.

**Corollary 2.21.** *Every separable inner product space admits an orthonormal basis.*

**Example 2.22.** Consider the inner product space $C([-1,1])$ equiped with the scalar product $\langle f, g \rangle = \int_{-1}^{1} f(t)\overline{g(t)}\,\mathrm{d}t$ and resulting norm $\|\cdot\|_2$. Let $f_n(t) := t^n$ ($n \geq 0$), so that $\operatorname{span}\{f_n\}$ is the space of all polynomials on the interval $[-1,1]$. Applying the Gram-Schmidt process to the sequence $(f_n)$ yields a orthonormal sequence $(p_n)$ of polynomials. The $p_n$ are called *Legendre polynomials*.

Recall that the space of all polynomials is dense in $C([-1,1])$ by Weierstrass' theorem (even for the uniform norm; *a fortiori* also for the norm $\|\cdot\|_2$). Hence, the Legendre polynomials form an orthonormal basis in $C([-1,1])$.

**Lemma 2.23 (Bessel's inequality).** *Let $H$ be an inner product space, $(e_n)_{n\in\mathbb{N}} \subseteq H$ an orthonormal system. Then, for every $x \in H$,*

$$\sum_{n\in\mathbb{N}} |\langle x, e_n \rangle|^2 \leq \|x\|^2.$$

*Proof.* Let $N \in \mathbb{N}$. Put $x_N = x - \sum_{n=1}^{N} \langle x, e_n \rangle e_n$ so that $x_N \perp e_n$ for every $1 \leq n \leq N$. By Pythagoras (Lemma 2.11),

$$\begin{aligned}
\|x\|^2 &= \|x_N\|^2 + \Big\|\sum_{n=1}^{N} \langle x, e_n \rangle e_n\Big\|^2 \\
&= \|x_N\|^2 + \sum_{n=1}^{N} |\langle x, e_n \rangle|^2 \\
&\geq \sum_{n=1}^{N} |\langle x, e_n \rangle|^2.
\end{aligned}$$

Since $N$ was arbitrary, the claim follows.

**Lemma 2.24.** *Let $H$ be a (separable) Hilbert space, $(e_n)_{n\in\mathbb{N}} \subseteq H$ an orthonormal system. Then:*

(a) *For every $x \in H$, the series $\sum_{n\in\mathbb{N}} \langle x, e_n \rangle e_n$ converges.*

(b) *$P : H \to H$, $x \mapsto \sum_{n\in\mathbb{N}} \langle x, e_n \rangle e_n$ is the orthogonal projection onto $\overline{\operatorname{span}}\{e_n : n \in \mathbb{N}\}$.*

*Proof.* (a) Let $x \in H$. Since $(e_n)$ is an orthonormal system, by Pythagoras (Lemma 2.11), for every $l > k \geq 1$,

$$\| \sum_{n=1}^{l} \langle x, e_n \rangle e_n - \sum_{n=1}^{k} \langle x, e_n \rangle e_n \|^2 = \| \sum_{n=k+1}^{l} \langle x, e_n \rangle e_n \|^2$$
$$= \sum_{n=k+1}^{l} |\langle x, e_n \rangle|^2.$$

Hence, by Bessel's inequality, the sequence $(\sum_{n=1}^{l} \langle x, e_n \rangle e_n)$ of partial sums forms a Cauchy sequence. Since $H$ is complete, the series $\sum_{n\in\mathbb{N}} \langle x, e_n \rangle e_n$ converges.

(b) is an exercise.

**Theorem 2.25.** *Let $H$ be a (separable) Hilbert space, $(e_n)_{n\in\mathbb{N}}$ an orthonormal system. Then the following are equivalent:*

(i) *$(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis.*

(ii) *If $x \perp e_n$ for every $n \in \mathbb{N}$, then $x = 0$.*

(iii) *$x = \sum_{n\in\mathbb{N}} \langle x, e_n \rangle e_n$ for every $x \in H$.*

(iv) *$\langle x, y \rangle = \sum_{n\in\mathbb{N}} \langle x, e_n \rangle \langle e_n, y \rangle$ for every $x$, $y \in H$.*

(v) *(Parseval's identity) For every $x \in H$,*

$$\|x\|^2 = \sum_{n\in\mathbb{N}} |\langle x, e_n \rangle|^2.$$

*Proof.* (i)⇒(ii) follows from Theorem 2.10.

(ii)⇒(iii) follows from Lemma 2.24 (i). In fact, let $x_0 = \sum_{n\in\mathbb{N}} \langle x, e_n \rangle e_n$ (which exists by Lemma 2.24 (i)). Then $\langle x - x_0, e_n \rangle = 0$ for every $n \in \mathbb{N}$, and by assumption (ii), this implies $x = x_0$.

(iii)⇒(iv) follows when multiplying $x$ scalarly with $y$, applying also the Cauchy-Schwarz inequality for the sequences $(\langle x, e_l \rangle)$, $(\langle e_l, y \rangle) \in l^2$.

(iv)⇒(v) follows from putting $x = y$.

(v)⇒(i). Let $x \in \operatorname{span}\{e_n : n \in \mathbb{N}\}^\perp$. Then Parseval's identity implies $\|x\|^2 = 0$, that is, $x = 0$. By Theorem 2.10, $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ is dense in $H$, that is, $(e_n)$ is an orthonormal basis.

A bounded linear operator $U \in \mathscr{L}(H, K)$ between two Hilbert spaces is called a **unitary operator** if it is invertible and for every $x$, $y \in H$,

$$\langle x, y \rangle_H = \langle Ux, Uy \rangle_K.$$

Two Hilbert spaces $H$ and $K$ are **unitarily equivalent** if there exists a unitary operator $U \in \mathscr{L}(H, K)$.

**Corollary 2.26.** *Every infinite dimensional separable Hilbert space $H$ is unitarily equivalent to $l^2$.*

*Proof.* Choose an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $H$ (which exists by Corollary 2.21), and define $U : H \to l^2$ by $U(x) = (\langle x, e_n \rangle)_{n\in\mathbb{N}}$. Then $\langle x, y \rangle_H = \langle U(x), U(y) \rangle_{l^2}$ by Theorem 2.25; in particular, $U$ is bounded, isometric and injective. The fact that $U$

is surjective, that is, that $\sum_n c_n e_n$ converges for every $c = (c_n) \in l^2$, follows as in the proof of Lemma 2.24 (i).

Clearly, if a sequence $(e_n)$ in a Hilbert space $H$ is an orthonormal basis, then necessarily $H$ is separable by Lemma 2.15. Hence, the equivalent statements of Theorem 2.25 are only satisfied in separable Hilbert spaces. In most of the applications (if not all!), we will only deal with separable Hilbert spaces so that Theorem 2.25 is sufficient for our purposes.

However, what is true in general Hilbert spaces? The following sequence of results generalizes the preceeding results to arbitrary Hilbert spaces.

Let $X$ be a normed space, $(x_i)_{i\in I}$ be a family. We say that the series $\sum_{i\in I} x_i$ converges **unconditionally** if the set $I_0 := \{i \in I : x_i \neq 0\}$ is countable, and for every bijective $\varphi : \mathbb{N} \to I_0$ the series $\sum_{n=1}^{\infty} x_{\varphi(n)}$ converges.

**Corollary 2.27 (Bessel's inequality, general case).** *Let $H$ be an inner product space, $(e_l)_{l\in I} \subseteq H$ an orthonormal system. Then, for every $x \in H$, the set $\{l \in I : \langle x, e_l\rangle \neq 0\}$ is countable and*

$$\sum_{l\in I} |\langle x, e_l\rangle|^2 \leq \|x\|^2. \tag{2.3}$$

*Proof.* By Bessel's inequality, the sets $\{l \in I : |\langle x, e_l\rangle| \geq 1/n\}$ must be finite for every $n \in \mathbb{N}$. The countability of $\{l \in I : \langle x, e_l\rangle \neq 0\}$ follows. The inequality (2.3) is then a direct consequence of Bessel's inequality.

**Lemma 2.28.** *Let $H$ be a Hilbert space, $(e_l)_{l\in I} \subseteq H$ an orthonormal system. Then:*

(a) *For every $x \in H$, the series $\sum_{l\in I} \langle x, e_l\rangle e_l$ converges unconditionally.*

(b) *$P : H \to H$, $x \mapsto \sum_{l\in I} \langle x, e_l\rangle e_l$ is the orthogonal projection onto $\overline{\operatorname{span}}\{e_l : l \in I\}$.*

**Corollary 2.29.** *Every Hilbert space admits an orthonormal basis.*

*Proof.* If $H$ is separable, the claim follows directly from the Gram-Schmidt process and has already been stated in Corollary 2.21. In general, one may argue as follows:

The set of all orthonormal systems in $H$ forms a partially ordered set by inclusion. Given a totally ordered collection of orthonormal systems, the union of all vectors contained in all systems in this collection forms a supremum. By Zorn's lemma, there exists an orthonormal system $(e_l)_{l\in I}$ which is maximal. It follows from Bessel's inequality (2.3) that this system is actually an orthonormal basis.

Theorem 2.25 remains true for arbitrary Hilbert spaces when replacing the countable orthonormal system $(e_n)_{n\in\mathbb{N}}$ by an arbitrary orthonormal system $(e_l)_{l\in I}$.

## 2.3 * Fourier series

In the following we will identify the space $L^1(0, 2\pi)$ with

$$L^1_{2\pi}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \text{ measurable, } 2\pi\text{-periodic} : \int_0^{2\pi} |f| \, \mathrm{d}\lambda < \infty\}.$$

Similarly, we identify $L^2(0,2\pi)$ with $L^2_{2\pi}(\mathbb{R})$, and we define

$$C_{2\pi}(\mathbb{R}) := \{f \in C(\mathbb{R}) : f \text{ is } 2\pi\text{-periodic}\}.$$

For every $f \in L^1(0,2\pi) = L^1_{2\pi}(\mathbb{R})$ and every $n \in \mathbb{Z}$ we call

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, \mathrm{d}t$$

the ***n*-th Fourier coefficient** of $f$. The sequence $\hat{f} = (\hat{f}(n))$ is called the **Fourier transform** of $f$. The formal series $\frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{in\cdot}$ is called the **Fourier series** of $f$.

**Lemma 2.30.** *For every $f \in L^1(0,2\pi) = L^1_{2\pi}(\mathbb{R})$ we have $\hat{f} \in l^\infty(\mathbb{Z})$ and the* Fourier transform$\hat{}$: $L^1(0,2\pi) \to l^\infty$ *is a bounded, linear operator. More precisely,*

$$\|\hat{f}\|_\infty \le \frac{1}{2\pi} \|f\|_1, \quad f \in L^1(0,2\pi).$$

*Proof.* For every $f \in L^1(0,2\pi)$ and every $n \in \mathbb{Z}$,

$$|\hat{f}(n)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(t) e^{-int} \, \mathrm{d}t \right| \le \frac{1}{2\pi} \int_0^{2\pi} |f(t)| dt.$$

This proves that $\hat{f} \in l^\infty$ and the required bound on $\|\hat{f}\|_\infty$. Linearity of$\hat{}$is clear.

**Lemma 2.31 (Riemann-Lebesgue).** *For every $f \in L^1(0,2\pi) = L^1_{2\pi}(\mathbb{R})$ we have $\hat{f} \in c_0(\mathbb{Z})$, i.e.*

$$\lim_{|n|\to\infty} |\hat{f}(n)| = 0.$$

*Proof.* Let $f \in L^1(0,2\pi) = L^1_{2\pi}(\mathbb{R})$ and $n \in \mathbb{Z}$, $n \ne 0$. Then

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, \mathrm{d}t \\
&= \frac{1}{4\pi} \int_0^{2\pi} f(t) e^{-int} (1 - e^{i\pi\frac{n}{n}}) \, \mathrm{d}t \\
&= \frac{1}{4\pi} \int_0^{2\pi} f(t) (e^{-int} - e^{-in(t-\frac{\pi}{n})}) \, \mathrm{d}t \\
&= \frac{1}{4\pi} \int_0^{2\pi} (f(t) - f(t + \frac{\pi}{n})) e^{-int} \, \mathrm{d}t,
\end{aligned}$$

so that

$$|\hat{f}(n)| \le \frac{1}{4\pi} \int_0^{2\pi} |f(t) - f(t + \frac{\pi}{n})| \, \mathrm{d}t.$$

Hence, if $f = 1_O \in L^1(0,2\pi)$ for some open set $O \subseteq [0,2\pi]$, then $\hat{f} \in c_0(\mathbb{Z})$ by Lebesgue dominated convergence theorem. On the other hand, since $\operatorname{span}\{1_O : O \subseteq [0,2\pi] \text{ open}\}$ is dense in $L^1(0,2\pi)$, since the Fourier transform is bounded with values in $l^\infty(\mathbb{Z})$ (Lemma 2.30), and since $c_0(\mathbb{Z})$ is a closed subspace of $l^\infty(\mathbb{Z})$, we find that $\hat{f} \in c_0(\mathbb{Z})$ for every $f \in L^1(0,2\pi)$.

**Remark 2.32.** At the end of the proof of the Lemma of Riemann-Lebesgue, we used the following general principle: if $T \in \mathscr{L}(X,Y)$ is a bounded linear operator between two normed linear spaces $X$, $Y$, and if $M \subseteq X$ is dense, then $\operatorname{ran} T \subseteq \overline{T(M)}$. We used in addition that $c_0(\mathbb{Z})$ is closed in $l^\infty(\mathbb{Z})$.

**Theorem 2.33.** *Let $f \in C_{2\pi}(\mathbb{R})$ be differentiable in some point $s \in \mathbb{R}$. Then*

$$f(s) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{ins}.$$

*Proof.* Note that for $f_s(t) := f(s+t)$,

$$\hat{f_s}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(s+t)e^{-int}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-in(t-s)}\,\mathrm{d}t = e^{ins}\hat{f}(n).$$

Hence, replacing $f$ by $f_s$, if necessary, we may without loss of generality assume that $s = 0$. Moreover, replacing $f$ by $f - f(0)$, if necessary, we may without loss of generality assume that $f(0) = 0$. We hence have to show that if $f$ is differentiable in 0 and if $f(0) = 0$, then $\sum_{n\in\mathbb{Z}} \hat{f}(n) = 0$.

Let $g(t) := \frac{f(t)}{1-e^{it}}$. Since $f$ is differentiable in 0, $f(0) = 0$, and since $f$ is $2\pi$-periodic, the function $g$ belongs to $C_{2\pi}(\mathbb{R})$. By the Lemma of Riemann-Lebesgue, $\hat{g} \in c_0(\mathbb{Z})$. Note that

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} g(t)(1-e^{it})e^{-int}\,\mathrm{d}t = \hat{g}(n) - \hat{g}(n-1).$$

Hence,

$$\begin{aligned}\sum_{k=-n}^{n} \hat{f}(k) &= \sum_{k=-n}^{n} \hat{g}(k) - \hat{g}(k-1) \\ &= \hat{g}(n) - \hat{g}(-n-1) \to 0 \qquad (n\to\infty).\end{aligned}$$

This is the claim.

**Corollary 2.34.** *For every $f \in C^1_{2\pi}(\mathbb{R}) := C_{2\pi}(\mathbb{R}) \cap C^1(\mathbb{R})$ and every $t \in \mathbb{R}$*

$$f(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{int}.$$

**Remark 2.35.** We will see that the convergence in the preceeding corollary is even uniform in $t \in \mathbb{R}$.

Throughout the following, we equip the space $L^2(0,2\pi) = L^2_{2\pi}(\mathbb{R})$ with the scalar product given by

$$\langle f,g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,\mathrm{d}t,$$

which differs from the usual scalar product by the factor $\frac{1}{2\pi}$.

**Lemma 2.36.** *The space $C^1_{2\pi}(\mathbb{R})$ is dense in $L^2_{2\pi}(\mathbb{R})$.*

*Proof.* We first prove that $C([0,2\pi])$ is dense in $L^2(0,2\pi) = L^2_{2\pi}(\mathbb{R})$. For this, consider first a characteristic function $f = 1_{(a,b)} \in L^2(0,2\pi)$. Let $(g_n) \subseteq C([0,2\pi])$ be defined by

$$g_n(t) := \begin{cases} 1, & t\in[a,b],\\ 1+n(t-a), & t\in[a-1/n,a),\\ 1-n(t-b), & t\in(b,b+1/n],\\ 0, & \text{else.}\end{cases}$$

It is then easy to see that $\lim_{n\to\infty}\|f-g_n\|_{L^2} = 0$, so that $f = 1_{(a,b)} \in \overline{C([0,2\pi])}^{\|\cdot\|_{L^2}}$.

In the second step, consider a characteristic function $f = 1_A$ of an arbitrary Borel set $A \in \mathscr{B}([0,2\pi])$, and let $\varepsilon > 0$. By outer regularity of the Lebesgue measure, there exists an open set $O \supset A$ such that $\lambda(O\setminus A) < \varepsilon^2$. Recall that $O$ is the countable union of mutually disjoint intervals. Since $O$ has finite measure, there exist finitely many (mutually disjoint) intervals $(a_n,b_n) \subseteq O$ $(1 \le n \le N)$ such that $\lambda(O\setminus\bigcup_{n=1}^N(a_n,b_n)) \le \varepsilon^2$. By the preceeding step, for every $1\le n\le N$ there exists $g_n \in C([0,2\pi])$ such that $\|1_{(a_n,b_n)} - g_n\|_2 \le \frac{\varepsilon}{N}$. Let $g := \sum_{n=1}^N g_n \in C([0,2\pi])$. Then

$$\begin{aligned}\|f-g\|_2 &\le \|1_A - 1_O\|_2 + \|1_O - 1_{\bigcup_{n=1}^N(a_n,b_n)}\|_2 + \|1_{\bigcup_{n=1}^N(a_n,b_n)} - g\|_2\\ &\le \varepsilon+\varepsilon+\|\sum_{n=1}^N(1_{(a_n,b_n)} - g_n)\|_2\\ &\le 3\varepsilon.\end{aligned}$$

This proves $1_A \in \overline{C([0,2\pi])}^{\|\cdot\|_{L^2}}$ for every Borel set $A\in\mathscr{B}([0,2\pi])$. Since $\overline{\operatorname{span}}\{1_A : A\in\mathscr{B}([0,2\pi])\} = L^2(0,2\pi)$, we find that $C([0,2\pi])$ is dense in $L^2(0,2\pi)$.

It remains to show that $C^1_{2\pi}(\mathbb{R})$ is dense in $C([0,2\pi])$ for the norm $\|\cdot\|_2$. So let $f\in C([0,2\pi])$ and let $\varepsilon>0$. By Weierstrass' theorem, there exists a function $g_0 \in C^\infty([0,2\pi])$ (even a polynomial!) such that $\|f-g_0\|_\infty \le \varepsilon$. Let $g_1 \in C^1([0,2\pi])$ be such that $g_1(2\pi) = g_1'(2\pi) = 0$, $g_1(0) = g_0(2\pi) - g_0(0)$ and $g_1'(0) = g_0'(2\pi) - g_0'(0)$ and $\|g_1\|_2 \le \varepsilon$. Such a function $g_1$ exists: it suffices for example to consider functions for which the derivative is of the form

$$g_1'(t) = \begin{cases} g_0(2\pi) - g_0(0) + ct, & t\in[0,h_1],\\ g_0(2\pi)-g_0(0)+ch_1+d(t-h_1), & t\in(h_1,h_2),\\ 0, & t\in[h_2,2\pi],\end{cases}$$

with appropriate constants $0 \le h_1 \le h_2$ and $c$, $d \in \mathbb{C}$. Having chosen $g_1$, we let $g = g_0 + g_1$ and we calculate that

$$\|f-g\|_2 \le \|f-g_0\|_2 + \|g_1\|_2 \le 2\varepsilon.$$

Since $g$ extends to a function in $C^1_{2\pi}(\mathbb{R})$, we have thus proved that $C^1_{2\pi}(\mathbb{R})$ is dense in $L^2_{2\pi}(\mathbb{R})$.

**Remark 2.37.** An adaptation of the above proof actually shows that for every $1 \le p < \infty$ and every compact interval $[a,b] \subseteq \mathbb{R}$, the space $C([a,b])$ is dense in $L^p(a,b)$. A further application of Weierstrass' theorem actually shows that the space of all polynomials is dense in $L^p(a,b)$. In particular, we may obtain the following result.

**Corollary 2.38.** *The space $L^p(a,b)$ is separable if $1 \le p < \infty$. The space $L^\infty(a,b)$ is* not *separable.*

**Corollary 2.39.** *Let $e_n(t) := e^{int}$, $n \in \mathbb{Z}$, $t \in \mathbb{R}$. Then $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2_{2\pi}(\mathbb{R})$.*

*Proof.* The fact that $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal system in $L^2_{2\pi}(\mathbb{R})$ is an easy calculation. We only have to prove that $\operatorname{span}\{e_n : n \in \mathbb{Z}\}$ is dense in $L^2_{2\pi}(\mathbb{R})$. Note that $\hat{f}(n) = \langle f, e_n\rangle$ for every $f \in L^2_{2\pi}(\mathbb{R})$ and every $n \in \mathbb{Z}$. By Lemma 2.24, we know that for every $f \in L^2_{2\pi}(\mathbb{R})$

$$g := \sum_{n\in\mathbb{Z}} \hat{f}(n)e_n \text{ exists in } L^2_{2\pi}(\mathbb{R}).$$

In particular, a subsequence of $(\sum_{n=-k}^{k} \hat{f}(n)e_n)$ converges almost everywhere to $g$. But by Corollary 2.34 we know that $(\sum_{n=-k}^{k} \hat{f}(n)e_n)$ converges pointwise everywhere to $f$ if $f \in C^1_{2\pi}(\mathbb{R})$. As a consequence, for every $f \in C^1_{2\pi}(\mathbb{R})$,

$$\lim_{k\to\infty} \sum_{n=-k}^{k} \hat{f}(n)e_n = f \text{ in } L^2_{2\pi}(\mathbb{R}),$$

so that $\operatorname{span}\{e_n : n \in \mathbb{Z}\}$ is dense in $(C^1_{2\pi}(\mathbb{R}), \|\cdot\|_{L^2_{2\pi}})$. Since $C^1_{2\pi}(\mathbb{R})$ is dense in $L^2_{2\pi}(\mathbb{R})$ by Lemma 2.36, we find that $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2_{2\pi}(\mathbb{R})$.

**Theorem 2.40 (Plancherel).** *For every $f \in L^2_{2\pi}(\mathbb{R})$ we have $\hat{f} \in l^2(\mathbb{Z})$ and the Fourier transform $\hat{}: L^2_{2\pi}(\mathbb{R}) \to l^2(\mathbb{Z})$ is an isometric isomorphism. Moreover, for every $f \in L^2_{2\pi}(\mathbb{R})$,*

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n = f \text{ in } L^2_{2\pi}(\mathbb{R}),$$

*that is, the Fourier series of $f$ converges to $f$ in the $L^2$ sense.*

*Proof.* By Corollary 2.39, the sequence $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2_{2\pi}(\mathbb{R})$. Moreover, recall that for every $f \in L^2_{2\pi}(\mathbb{R})$ and every $n \in \mathbb{Z}$, $\hat{f}(n) = \langle f, e_n\rangle$. Hence, by Theorem 2.25, $\hat{f} \in l^2(\mathbb{Z})$, $f = \sum_{n\in\mathbb{Z}} \hat{f}(n)e_n$, and $\|f\|_{L^2_{2\pi}} = \|\hat{f}\|_{l^2}$ (the last property being Parseval's identity).

**Corollary 2.41.** *Let $f \in C_{2\pi}(\mathbb{R})$ be such that $\hat{f} \in l^1(\mathbb{Z})$. Then*

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n = f \text{ in } C_{2\pi}(\mathbb{R}),$$

*that is, the Fourier series of $f$ converges uniformly to $f$.*

*Proof.* Note that for every $n \in \mathbb{Z}$, $\|e_n\|_\infty = 1$. The assumption $\hat{f} \in l^1(\mathbb{Z})$ therefore implies that the series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n$ converges absolutely in $C_{2\pi}(\mathbb{R})$, i.e. for the uniform norm $\|\cdot\|_\infty$. Since $(C_{2\pi}(\mathbb{R}), \|\cdot\|_\infty)$ is complete, the series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n$ converges uniformly to some element $g \in C_{2\pi}(\mathbb{R})$. By Plancherel, $g = f$.

**Remark 2.42.** The assumption $\hat{f} \in l^1(\mathbb{Z})$ in Corollary 2.41 is essential. For general $f \in C_{2\pi}(\mathbb{R})$, the Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n$ need not not converge uniformly. Questions regarding the convergence of Fourier series (which type of convergence? for which function?) can go deeply into the theory of harmonic analysis and answers are sometimes quite involved. The $L^2$ theory gives in this context satisfactory answers with relatively easy proofs (see Plancherel's theorem). For continuous functions we state the following result without giving a proof.

**Theorem 2.43 (Féjer).** *For every $f \in C_{2\pi}(\mathbb{R})$ one has*

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \sum_{n=-k}^{k} \hat{f}(n)e_n = f \text{ in } C_{2\pi}(\mathbb{R}),$$

*that is, the Fourier series of $f$ converges in the* **Cesàro mean** *uniformly to $f$.*

## 2.4 Linear functionals on Hilbert spaces

In this section, we discuss bounded functionals on Hilbert spaces. Compared to the case of bounded linear functionals on general Banach spaces, the case of bounded linear functionals on Hilbert spaces is considerably easy but it has far reaching consequences.

**Theorem 2.44 (Riesz-Fréchet).** *Let $H$ be a Hilbert space. Then for every bounded linear functional $\varphi \in H'$ there exists a unique $y \in H$ such that*

$$\varphi(x) = \langle x, y \rangle \quad \text{for every } x \in H.$$

*Proof.* *Uniqueness.* Let $y_1, y_2 \in H$ be two elements such that

$$\varphi(x) = \langle x, y_1 \rangle = \langle x, y_2 \rangle \quad \text{for every } x \in H.$$

Then $\langle x, y_1 - y_2 \rangle = 0$ for every $x \in H$, in particular also for $x = y_1 - y_2$. This implies $\|y_1 - y_2\|^2 = 0$, that is, $y_1 = y_2$.

*Existence.* We may assume that $\varphi \neq 0$ since the case $\varphi = 0$ is trivial. Let $\tilde{y} \in (\ker\varphi)^{\perp} \setminus \{0\}$. Since $H \neq \ker\varphi$ and since $\ker\varphi$ is closed, such a $\tilde{y}$ exists. Next, let

$$y := \overline{\varphi(\tilde{y})}/\|\tilde{y}\|^2\,\tilde{y}.$$

Note that $\varphi(y) = \|y\|^2 = \langle y,y\rangle$. Recall that every $x \in H$ can be uniquely written as $x = x_0 + \lambda y$ with $x_0 \in \ker\varphi$ and $\lambda \in \mathbb{K}$ so that $\lambda y \in (\ker\varphi)^{\perp}$. Note that $(\ker\varphi)^{\perp}$ is one-dimensional. Hence, for every $x \in H$,

$$\begin{aligned}
\varphi(x) &= \varphi(x_0 + \lambda y) \\
&= \varphi(x_0) + \lambda\,\varphi(y) \\
&= \lambda\,\varphi(y) \\
&= \lambda\,\langle y,y\rangle \\
&= \langle \lambda y, y\rangle \\
&= \langle x_0, y\rangle + \langle \lambda y, y\rangle \\
&= \langle x, y\rangle .
\end{aligned}$$

The claim is proved.

**Corollary 2.45.** *Let $J : H \to H'$ be the mapping which maps to every $y \in H$ the functional $Jy \in H'$ given by $Jy(x) = \langle x,y\rangle$. Then $J$ is antilinear if $\mathbb{K} = \mathbb{C}$ and linear if $\mathbb{K} = \mathbb{R}$. Moreover, $J$ is isometric and bijective.*

*Proof.* The fact that $J$ is isometric follows from the Cauchy-Schwarz inequality. Antilinearity (or linearity in case $\mathbb{K} = \mathbb{R}$) follows from the sesquilinearity (resp. bilinearity) of the scalar product on $H$. Since $J$ is isometric, it is injective. The surjectivity of $J$ follows from Theorem 2.44.

**Remark 2.46.** The theorem of Riesz-Fréchet allows us to identify any (real) Hilbert space $H$ with its dual space $H'$. Note, however, that there are situations in which one does not identify $H'$ with $H$. This is for example the case when $V$ is a second Hilbert space which embeds continuously and densely into $H$, that is, for which there exists a bounded, injective $J : V \to H$ with dense range.

## 2.5 Weak convergence in Hilbert spaces

Let $H$ be a Hilbert space. We say that a sequence $(x_n) \subseteq H$ **converges weakly** to some element $x \in H$ if for every $y \in H$ one has $\lim_{n\to\infty}\langle x_n, y\rangle = \langle x, y\rangle$. We write $x_n \rightharpoonup x$ or $x_n \overset{\text{weak}}{\to} x$ if $(x_n)$ converges weakly to $x$.

**Theorem 2.47.** *Every bounded sequence $(x_n)$ in a Hilbert space $H$ admits a weakly convergent subsequence, that is, there exists $x \in H$ and there exists a subsequence $(x_{n_k})$ of $(x_n)$ such that $x_{n_k} \overset{\text{weak}}{\to} x$.*

In the proof of this theorem, we will use the following general result.

**Lemma 2.48.** *Let $X$ and $Y$ be two normed spaces, let $(T_n) \in \mathscr{L}(X,Y)$ be a bounded sequence of bounded operators. Assume that there exists a dense set $M \subseteq X$ such that $\lim_{n\to\infty} T_n x$ exists for every $x \in M$. Then $\lim_{n\to\infty} T_n x =: Tx$ exists for every $x \in X$ and $T \in \mathscr{L}(X,Y)$.*

*Proof.* Define $Tx := \lim_{n\to\infty} T_n x$ for every $x \in \operatorname{span} M$. Then

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \sup_n \|T_n\| \, \|x\|,$$

that is. $T : \operatorname{span} M \to Y$ is a bounded linear operator. Since $M$ is dense in $X$, $T$ admits a unique bounded extension $T : X \to Y$.

Let $x \in X$ and $\varepsilon > 0$. Since $M$ is dense in $X$, there exists $y \in M$ such that $\|x - y\| \leq \varepsilon$. By assumption, there exists $n_0$ such that for every $n \geq n_0$ we have $\|T_n y - Ty\| \leq \varepsilon$. Hence, for every $n \geq n_0$,

$$\begin{aligned}
\|T_n x - Tx\| &\leq \|T_n x - T_n y\| + \|T_n y - Ty\| + \|Ty - Tx\| \\
&\leq \sup_n \|T_n\| \, \|x - y\| + \varepsilon + \|T\| \, \|x - y\| \\
&\leq \varepsilon (\sup_n \|T_n\| + 1 + \|T\|),
\end{aligned}$$

and therefore $\lim_{n\to\infty} T_n x = Tx$.

*Proof (of Theorem 2.47).* As in the proof of the Arzela-Ascoli theorem (Theorem 1.36), we use Cantor's diagonal sequence argument. Let $(x_n)$ be a bounded sequence in $H$. We first assume that $H$ is separable, and we let $(y_m) \subseteq H$ be a dense sequence.

Since $(\langle x_n, y_1 \rangle)$ is bounded by the boundedness of $(x_n)$, there exists a subsequence $(x_{\varphi_1(n)})$ of $(x_n)$ ($\varphi_1 : \mathbb{N} \to \mathbb{N}$ is increasing, unbounded) such that

$$\lim_{n\to\infty} \langle x_{\varphi_1(n)}, y_1 \rangle \text{ exists.}$$

Similarly, there exists a subsequence $(x_{\varphi_2(n)})$ of $(x_{\varphi_1(n)})$ such that

$$\lim_{n\to\infty} \langle x_{\varphi_2(n)}, y_2 \rangle \text{ exists.}$$

Note that for this subsequence, we also have that

$$\lim_{n\to\infty} \langle x_{\varphi_2(n)}, y_1 \rangle \text{ exists.}$$

Iterating this argument, we find a subsequence $(x_{\varphi_3(n)})$ of $(x_{\varphi_2(n)})$ and finally for every $m \in \mathbb{N}$, $m \geq 2$, a subsequence $(x_{\varphi_m(n)})$ of $(x_{\varphi_{m-1}(n)})$ such that

$$\lim_{n\to\infty} \langle x_{\varphi_m(n)}, y_j \rangle \text{ exists for every } 1 \leq j \leq m.$$

Let $(x'_n) := (x_{\varphi_n(n)})$ be the 'diagonal sequence'. Then $(x'_n)$ is a subsequence of $(x_n)$ and

$$\lim_{n\to\infty} \langle x_n', y_m \rangle \text{ exists for every } m \in \mathbb{N}.$$

By Lemma 2.48 and the Riesz-Fréchet representation theorem (Theorem 2.44), there exists $x \in H$ such that

$$\lim_{n\to\infty} \langle x_n', y \rangle = \langle x, y \rangle \text{ for every } y \in H,$$

and the claim is proved in the case when $H$ is separable.

If $H$ is not separable as we first assumed, then one may replace $H$ by $\tilde{H} := \overline{\text{span}}\,\{x_n : n \in \mathbb{N}\}$ which is separable. By the above, there exists $x \in \tilde{H}$ and a subsequence of $(x_n)$ (which we denote again by $(x_n)$) such that for every $y \in \tilde{H}$,

$$\lim_{n\to\infty} \langle x_n, y \rangle = \langle x, y \rangle,$$

that is, $(x_n)$ converges weakly in $\tilde{H}$. On the other hand, for every $y \in \tilde{H}^\perp$ and every $n$,

$$\langle x_n, y \rangle = \langle x, y \rangle = 0.$$

The decomposition $H = \tilde{H} \oplus \tilde{H}^\perp$ therefore yields that $(x_n)$ converges weakly in $H$.

# Chapter 3
# Dual spaces and weak convergence

## 3.1 The theorem of Hahn-Banach

Given a normed space $X$, we denote by $X' := \mathscr{L}(X, \mathbb{K})$ the space of all bounded linear functionals on $X$. It is called the **dual space** of $X$. Recall that $X'$ is always a Banach space by Corollary 1.29 of Chapter 1.

However, *a priori* it is not clear whether there exists any bounded linear functional on a normed space $X$ (apart from the zero functional). This fundamental question and the analysis of dual spaces (analysis of functionals) shall be developed in this chapter.

The existence of nontrivial bounded functionals is guaranteed by the Hahn-Banach theorem which actually admits several versions. However, before stating the first version, we need the following definition.

Let $X$ be a real or complex vector space. A function $p : X \to \mathbb{R}$ is called **sublinear** if

(i) $p(\lambda x) = \lambda p(x)$ for every $\lambda > 0$, $x \in X$, and

(ii) $p(x+y) \leq p(x) + p(y)$ for every $x, y \in X$.

**Example 3.1.** On a normed space $X$, the norm $\|\cdot\|$ is sublinear. Every linear $p : X \to \mathbb{R}$ is sublinear.

**Theorem 3.2 (Hahn-Banach; version of linear algebra, real case).** *Let $X$ be a real vector space, $U \subseteq X$ a linear subspace, and $p : X \to \mathbb{R}$ sublinear. Let $\varphi : U \to \mathbb{R}$ be linear such that*

$$\varphi(x) \leq p(x) \text{ for all } x \in U.$$

*Then there exists a linear $\tilde{\varphi} : X \to \mathbb{R}$ such that $\tilde{\varphi}(x) = \varphi(x)$ for every $x \in U$ (that is, $\tilde{\varphi}$ is an extension of $\varphi$) and*

$$\tilde{\varphi}(x) \leq p(x) \text{ for all } x \in X. \tag{3.1}$$

The following lemma asserts that this version of Hahn-Banach is true in the special case when $X/U$ has dimension 1. It is an essential step in the proof of Theorem 3.2.

**Lemma 3.3.** *Take the assumptions of Theorem 3.2 and assume in addition that* $\dim X/U = 1$. *Then the assertion of Theorem 3.2 is true.*

*Proof.* If $\dim X/U = 1$, then there exists $x_0 \in X \setminus U$ such that every $x \in X$ can be uniquely written in the form $x = u + \lambda x_0$ with $u \in U$ and $\lambda \in \mathbb{R}$. So we define $\tilde{\varphi} : X \to \mathbb{R}$ by

$$\tilde{\varphi}(x) := \tilde{\varphi}(u + \lambda x_0) := \varphi(u) + \lambda r,$$

where $r \in \mathbb{R}$ is a parameter which has to be chosen such that (3.1) holds, that is, such that for every $u \in U$, $\lambda \in \mathbb{R}$,

$$\varphi(u) + \lambda r \leq p(u + \lambda x_0). \tag{3.2}$$

If $\lambda = 0$, then this condition clearly holds for every $u \in U$ by the assumption on $\varphi$. If $\lambda > 0$, then (3.2) holds for every $u \in U$ if and only if

$$\begin{aligned} &\lambda r \leq p(u + \lambda x_0) - \varphi(u) \text{ for every } u \in U \\ &\Leftrightarrow r \leq p(\frac{u}{\lambda} + x_0) - \varphi(\frac{u}{\lambda}) \text{ for every } u \in U \\ &\Leftrightarrow r \leq \inf_{v \in U} p(v + x_0) - \varphi(v). \end{aligned}$$

Similarly, if $\lambda < 0$, then (3.2) holds for every $u \in U$ if and only if

$$\begin{aligned} &\lambda r \leq p(u + \lambda x_0) - \varphi(u) \text{ for every } u \in U \\ &\Leftrightarrow -r \leq p(\frac{u}{-\lambda} - x_0) - \varphi(\frac{u}{-\lambda}) \text{ for every } u \in U \\ &\Leftrightarrow r \geq \sup_{w \in U} \varphi(w) - p(w - x_0). \end{aligned}$$

So it is possible to find an appropriate $r \in \mathbb{R}$ in the definition of $\tilde{\varphi}$ if and only if

$$\varphi(w) - p(w - x_0) \leq p(v + x_0) - \varphi(v) \text{ for all } v, w \in U,$$

or, equivalently, if

$$\varphi(w) + \varphi(v) \leq p(v + x_0) + p(w - x_0) \text{ for all } v, w \in U.$$

However, by the assumptions on $\varphi$ and $p$, for every $v$, $w \in U$,

$$\varphi(w) + \varphi(v) = \varphi(w + v) \leq p(w + v) = p(v + x_0 + w - x_0) \leq p(v + x_0) + p(w - x_0).$$

For the second step in the proof of Theorem 3.2, we need the Lemma of Zorn.

**Lemma 3.4 (Zorn).** *Let $(M, \leq)$ be a ordered set. Assume that every totally ordered subset $T \subseteq M$ (i.e. for every $x$, $y \in T$ one either has $x \leq y$ or $y \leq x$) admits an* *upper bound. Then for every $x \in M$ there exists a maximal element $m \geq x$ (that is, an element $m$ such that $m \leq \tilde{m}$ implies $m = \tilde{m}$ for every $\tilde{m} \in M$).*

*Proof (of Theorem 3.2).* Define the following set

$$M := \{(V, \varphi_V) : V \subseteq X \text{ linear subspace}, U \subseteq V,\ \varphi_V : V \to \mathbb{R} \text{ linear, s.t.} \\ \varphi(x) = \varphi_V(x)\,(x \in U) \text{ and } \varphi_V(x) \leq p(x)\,(x \in V)\},$$

and equip it with the order relation $\leq$ defined by

$$(V_1, \varphi_{V_1}) \leq (V_2, \varphi_{V_2}) :\Leftrightarrow V_1 \subseteq V_2 \text{ and } \varphi_{V_1}(x) = \varphi_{V_2}(x) \text{ for all } x \in V_1.$$

Then $(M, \leq)$ is an ordered set. Let $T = ((V_i, \varphi_{V_i}))_{i \in I} \subseteq M$ be a totally ordered subset. Then the element $(V, \varphi_V) \in M$ defined by

$$V := \bigcup_{i \in I} V_i \text{ and } \varphi_V(x) = \varphi_{V_i}(x) \text{ for } x \in V_i$$

is an upper bound of $T$. By the Lemma of Zorn, the set $M$ admits a maximal element $(X_0, \varphi_{X_0})$. Assume that $X_0 \neq X$. Then, by Lemma 3.3, we could construct an element which is strictly larger than $(X_0, \varphi_{X_0})$, a contradiction to the maximality of $(X_0, \varphi_{X_0})$. Hence, $X = X_0$, and $\tilde{\varphi} := \varphi_{X_0}$ is an element we are looking for.

The complex version of the Hahn-Banach theorem reads as follows.

**Theorem 3.5 (Hahn-Banach; version of linear algebra, complex case).** *Let $X$ be a complex vector space, $U \subseteq X$ a linear subspace, and $p : X \to \mathbb{R}$ sublinear. Let $\varphi : U \to \mathbb{C}$ be linear such that*

$$\operatorname{Re} \varphi(x) \leq p(x) \text{ for all } x \in U.$$

*Then there exists a linear $\tilde{\varphi} : X \to \mathbb{C}$ such that $\tilde{\varphi}(x) = \varphi(x)$ for every $x \in U$ (that is $\tilde{\varphi}$ is an extension of $\varphi$) and*

$$\operatorname{Re} \tilde{\varphi}(x) \leq p(x) \text{ for all } x \in X. \tag{3.3}$$

*Proof.* We may consider $X$ also as a real vector space. Note that $\psi(x) := \operatorname{Re} \varphi(x)$ is an $\mathbb{R}$-linear functional on $X$. By Theorem 3.2, there exists an extension $\tilde{\psi} : X \to \mathbb{R}$ of $\psi$ satisfying

$$\tilde{\psi}(x) \leq p(x) \text{ for every } x \in X.$$

Let

$$\tilde{\varphi}(x) := \tilde{\psi}(x) - i\tilde{\psi}(ix), \quad x \in X.$$

It is an exercise to show that $\tilde{\varphi}$ is $\mathbb{C}$-linear, that $\varphi(x) = \tilde{\varphi}(x)$ for every $x \in U$ and it is clear from the definition that $\operatorname{Re} \tilde{\varphi}(x) = \tilde{\psi}(x)$. Thus, $\tilde{\varphi}$ is a possible element we are looking for.

**Theorem 3.6 (Hahn-Banach; extension of bounded linear functionals).** *Let $X$ be a normed space and $U \subseteq X$ a linear subspace. Then for every bounded linear*

*$u' : U \to \mathbb{K}$ there exists a bounded linear extension $x' : X \to \mathbb{K}$ (that is, $x'|_U = u'$) such that $\|x'\| = \|u'\|$.*

*Proof.* We first assume that $X$ is a real normed space. The function $p : X \to \mathbb{R}$ defined by $p(x) := \|u'\|\,\|x\|$ is sublinear and

$$u'(x) \leq p(x) \text{ for every } x \in U.$$

By the first Hahn-Banach theorem (Theorem 3.2), there exists a linear $x' : X \to \mathbb{R}$ extending $u'$ such that

$$x'(x) \leq p(x) = \|u'\|\,\|x\| \text{ for every } x \in X.$$

Replacing $x$ by $-x$, this implies that

$$|x'(x)| \leq \|u'\|\,\|x\| \text{ for every } x \in X.$$

Hence, $x'$ is bounded and $\|x'\| \leq \|u'\|$. On the other hand, one trivially has

$$\|x'\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |x'(x)| \geq \sup_{\substack{x \in U \\ \|x\| \leq 1}} |x'(x)| = \sup_{\substack{x \in U \\ \|x\| \leq 1}} |u'(x)| = \|u'\|.$$

If $X$ is a complex normed space, then the second Hahn-Banach theorem (Theorem 3.5) implies that there exists a linear $x' : X \to \mathbb{C}$ such that

$$\operatorname{Re} x'(x) \leq p(x) = \|u'\|\,\|x\| \text{ for every } x \in X.$$

In particular,

$$|x'(x)| = \sup_{\theta \in [0,2\pi]} \operatorname{Re} x'(e^{i\theta}x) \leq \|u'\|\,\|x\| \text{ for every } x \in X,$$

so that again $x'$ is bounded and $\|x'\| \leq \|u'\|$. The inequality $\|x'\| \geq \|u'\|$ follows as above.

**Corollary 3.7.** *If $X$ is a normed space, then for every $x \in X \setminus \{0\}$ there exists $x' \in X'$ such that*

$$\|x'\| = 1 \text{ and } x'(x) = \|x\|.$$

*In particular, $X'$* separates the points *of $X$, i.e. for every $x_1$, $x_2 \in X$, $x_1 \neq x_2$, there exists $x' \in X'$ such that $x'(x_1) \neq x'(x_2)$.*

*Proof.* By the Hahn-Banach theorem (Theorem 3.6), there exists an extension $x' \in X'$ of the functional $u' : \operatorname{span}\{x\} \to \mathbb{K}$ defined by $u'(\lambda x) = \lambda \|x\|$ such that $\|x'\| = \|u'\| = 1$.

For the proof of the second assertion, set $x := x_1 - x_2$.

**Corollary 3.8.** *If $X$ is a normed space, then for every $x \in X$*

$$\|x\| = \sup_{\substack{x' \in X' \\ \|x'\| \leq 1}} |x'(x)|. \tag{3.4}$$

*Proof.* For every $x' \in X'$ with $\|x'\| \leq 1$ one has

$$|x'(x)| \leq \|x'\| \, \|x\| \leq \|x\|,$$

which proves one of the required inequalities. The other inequality follows from Corollary 3.7.

**Remark 3.9.** The equality (3.4) should be compared to the definition of the norm of an element $x' \in X'$:

$$\|x'\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |x'(x)|.$$

From now on, it will be convenient to use the following notation. Given a normed space $X$ and elements $x \in X$, $x' \in X'$, we write

$$\langle x', x \rangle := \langle x', x \rangle_{X' \times X} := x'(x).$$

For the bracket $\langle \cdot, \cdot \rangle$, we note the following properties. The function

$$\begin{aligned} \langle \cdot, \cdot \rangle : X' \times X &\to \mathbb{K}, \\ (x', x) &\mapsto \langle x', x \rangle = x'(x) \end{aligned}$$

is bilinear and for every $x' \in X'$, $x \in X$,

$$|\langle x', x \rangle| \leq \|x'\| \, \|x\|.$$

The bracket $\langle \cdot, \cdot \rangle$ thus appeals to the notion of the scalar product on inner product spaces, and the last inequality appeals to the Cauchy-Schwarz inequality, but note, however, that the bracket is *not* a scalar product since it is defined on a pair of two different spaces. Moreover, even if $X = H$ is a complex Hilbert space, then the bracket differs from the scalar product in that it is bilinear instead of sesquilinear.

**Corollary 3.10.** *Let $X$ be a normed space, $U \subseteq X$ a closed linear subspace and $x \in X \setminus U$. Then there exists $x' \in X'$ such that*

$$x'(x) \neq 0 \text{ and } x'(u) = 0 \text{ for every } u \in U.$$

*Proof.* Let $\pi : X \to X/U$ be the quotient map ($\pi(x) = x + U$). Since $x \notin U$, we have $\pi(x) \neq 0$. By Corollary 3.7, there exists $\varphi \in (X/U)'$ such that $\varphi(\pi(x)) \neq 0$. Then $x' := \varphi \circ \pi \in X'$ is a functional we are looking for.

A linear operator $P : X \to X$ on a linear space $X$ is called a **projection** if $P^2 = P$. A linear subspace $U$ of a normed space $X$ is called **complemented** if there exists a projection $P \in \mathscr{L}(X)$ such that $\operatorname{ran} P = U$.

**Remark 3.11.** If $P$ is a projection, then $Q = I - P$ is also a projection and $\operatorname{ran} P = \ker Q$. Hence, if $P$ is a bounded projection on a normed space, then $\operatorname{ran} P$ is necessarily closed. Thus, a necessary condition for $U$ to be complemented is that $U$ is closed.

**Corollary 3.12.** *Every finite dimensional subspace of a normed space is complemented.*

*Proof.* Let $U$ be a finite dimensional subspace of a normed space $X$. Let $(b_i)_{1\leq i\leq N}$ be a basis of $U$. By Corollary 3.10, there exist functionals $x_i'\in X'$ such that

$$\langle x_i', b_j\rangle = \begin{cases} 1 \text{ if } i=j, \\ 0 \text{ otherwise.}\end{cases}$$

Let $P: X\to X$ be defined by

$$Px := \sum_{i=1}^N \langle x_i', x\rangle b_i, \quad x\in X.$$

Then $Pb_i = b_i$ for every $1\leq i\leq N$, and thus $P^2=P$, that is, $P$ is a projection. Moreover, $\operatorname{ran} P = U$ by construction. By the estimate

$$\begin{aligned}\|Px\| &\leq \sum_{i=1}^N |\langle x_i', x\rangle|\,\|b_i\| \\ &\leq \big(\sum_{i=1}^N \|x_i'\|\,\|b_i\|\big)\,\|x\|,\end{aligned}$$

the projection $P$ is bounded.

The following lemma which does not depend on the Hahn-Banach theorem is stated for completeness.

**Lemma 3.13.** *In a Hilbert space every closed linear subspace is complemented.*

*Proof.* Take the orthogonal projection onto the closed subspace as a possible projection.

**Corollary 3.14.** *If $X$ is a normed space such that $X'$ is separable, then $X$ is separable, too.*

*Proof.* Let $D' = \{x_n' : n\in\mathbb{N}\}$ be a dense subset of the unit sphere of $X'$. For every $n\in\mathbb{N}$ we choose an element $x_n\in X$ such that $\|x_n\|\leq 1$ and $|\langle x_n', x_n\rangle|\geq \frac{1}{2}$. We claim that $D := \operatorname{span}\{x_n : n\in\mathbb{N}\}$ is dense in $X$. If this was not true, i.e. if $\bar{D}\neq X$, then, by Corollary 3.10, we find an element $x'\in X'\setminus\{0\}$ such that $x'(x_n)=0$ for every $n\in\mathbb{N}$. We may without loss of generality assume that $\|x'\|=1$. Since $D'$ is dense in the unit sphere of $X'$, we find $n_0\in\mathbb{N}$ such that $\|x'-x_{n_0}'\|\leq\frac{1}{4}$. But then

$$\frac{1}{2}\leq |\langle x_{n_0}', x_{n_0}\rangle| = |\langle x_{n_0}'-x', x_{n_0}\rangle| \leq \|x_{n_0}'-x'\|\,\|x_{n_0}\|\leq\frac{1}{4},$$

which is a contradiction. Hence, $\bar{D}=X$ and $X$ is separable by Lemma 2.15 of Chapter 2.

## 3.2 Weak$^*$ convergence and the theorem of Banach-Alaoglu

Let $X$ be a Banach space. We say that a sequence $(x'_n) \subseteq X'$ **converges weak**$^*$ to some element $x' \in X'$ if for every $x \in X$ one has $\lim_{n\to\infty}\langle x'_n, x\rangle = \langle x', x\rangle$. We write $x'_n \overset{\text{weak}*}{\to} x'$ if $(x'_n)$ converges weak$^*$ to $x'$.

**Theorem 3.15 (Banach-Alaoglu).** *Let $X$ be a separable Banach space. Then every bounded sequence $(x'_n) \subseteq X'$ admits a weak$^*$ convergent subsequence, that is, there exists $x' \in X'$ and there exists a subsequence $(x'_{n_k})$ of $(x'_n)$ such that $x'_{n_k} \overset{\text{weak}*}{\to} x'$.*

*Proof.* As in the proof of the Arzelá-Ascoli theorem (Theorem 1.36) and the theorem about weak sequential compactness of the unit ball in Hilbert spaces (Theorem 2.47), we use Cantor's diagonal sequence argument. Let $(x'_n)$ be a bounded sequence in $X'$.

Since $X$ is separable by assumption, we can choose a dense sequence $(x_m) \subseteq X$. Since $(\langle x'_n, x_1\rangle)$ is bounded by the boundedness of $(x'_n)$, there exists a subsequence $(x'_{\varphi_1(n)})$ of $(x'_n)$ ($\varphi_1 : \mathbb{N} \to \mathbb{N}$ is increasing, unbounded) such that

$$\lim_{n\to\infty}\langle x'_{\varphi_1(n)}, x_1\rangle \text{ exists.}$$

Similarly, there exists a subsequence $(x'_{\varphi_2(n)})$ of $(x'_{\varphi_1(n)})$ such that

$$\lim_{n\to\infty}\langle x'_{\varphi_2(n)}, x_2\rangle \text{ exists.}$$

Note that for this subsequence, we also have that

$$\lim_{n\to\infty}\langle x'_{\varphi_2(n)}, x_1\rangle \text{ exists.}$$

Iterating this argument, we find a subsequence $(x'_{\varphi_3(n)})$ of $(x'_{\varphi_2(n)})$ and finally for every $m \in \mathbb{N}$, $m \geq 2$, a subsequence $(x'_{\varphi_m(n)})$ of $(x'_{\varphi_{m-1}(n)})$ such that

$$\lim_{n\to\infty}\langle x'_{\varphi_m(n)}, x_j\rangle \text{ exists for every } 1 \leq j \leq m.$$

Let $(y'_n) := (x'_{\varphi_n(n)})$ be the 'diagonal sequence'. Then $(y'_n)$ is a subsequence of $(x'_n)$ and

$$\lim_{n\to\infty}\langle y'_n, x_m\rangle \text{ exists for every } m \in \mathbb{N}.$$

By Lemma 2.48 of Chapter 2, there exists $x' \in X'$ such that

$$\lim_{n\to\infty}\langle y'_n, x\rangle = \langle x', x\rangle \text{ for every } x \in X.$$

This is the claim.

## 3.3 Weak convergence and reflexivity

Given a normed space $X$, we call $X'' := (X')' = \mathscr{L}(X', \mathbb{K})$ the **bidual** of $X$.

**Lemma 3.16.** *Let $X$ be a normed space. Then the mapping*

$$\begin{aligned} J : X &\to X'', \\ x &\mapsto (x' \mapsto \langle x', x \rangle), \end{aligned}$$

*is well defined and isometric.*

*Proof.* The linearity of $x' \mapsto \langle x', x \rangle$ is clear, and from the inequality

$$|Jx(x')| = |\langle x', x \rangle| \leq \|x'\| \, \|x\|,$$

follows that $Jx \in X''$ (that is, $J$ is well defined) and $\|Jx\| \leq \|x\|$. The fact that $J$ is isometric follows from Corollary 3.7.

A normed space $X$ is called **reflexive** if the isometry $J$ from Lemma 3.16 is surjective, i.e. if $JX = X''$. In other words: a normed space $X$ is reflexive if for every $x'' \in X''$ there exists $x \in X$ such that

$$\langle x'', x' \rangle = \langle x', x \rangle \text{ for all } x' \in X'.$$

**Remark 3.17.** If a normed space is reflexive then $X$ and $X''$ are isometrically isomorphic (via the operator $J$). Since $X''$ is always complete, a reflexive space is necessarily a Banach space.

Note that it can happen that $X$ and $X''$ are isomorphic without $X$ being reflexive (the example of such a Banach space is however quite involved). We point out that reflexivity means that the special operator $J$ is an isomorphism.

**Lemma 3.18.** *Every Hilbert space is reflexive.*

*Proof.* By the Theorem of Riesz-Fréchet, we may identify $H$ with its dual $H'$ and thus also $H$ with its bidual $H''$. The identification is done via the scalar product. It is an exercise to show that this identification of $H$ with $H''$ coincides with the mapping $J$ from Lemma 3.16.

**Remark 3.19.** It should be noted that for complex Hilbert spaces, the identification of $H$ with its dual $H'$ is only antilinear, but after the second identification ($H'$ with $H''$) it turns out that the identification of $H$ with $H''$ is linear.

**Lemma 3.20.** *Every finite dimensional Banach space is reflexive.*

*Proof.* It suffices to remark that if $X$ is finite dimensional, then

$$\dim X = \dim X' = \dim X'' < \infty.$$

Surjectivity of the mapping $J$ (which is always injective) thus follows from linear algebra.

**Theorem 3.21.** *The space $L^p(\Omega)$ is reflexive if $1 < p < \infty$ ($(\Omega, \mathscr{A}, \mu)$ being an arbitrary measure space).*

We will actually only prove the following special case.

**Theorem 3.22.** *The spaces $l^p$ are reflexive if $1 < p < \infty$.*

The proof of Theorem 3.22 is based on the following lemma.

**Lemma 3.23.** *Let $1 \le p < \infty$ and let $q := \frac{p}{p-1}$ be the conjugate exponent so that $\frac{1}{p} + \frac{1}{q} = 1$. Then the operator*

$$T : l^q \to (l^p)',$$
$$(a_n) \mapsto ((x_n) \mapsto \sum_n a_n x_n),$$

*is an isometric isomorphism, that is, $(l^p)' = l^q$.*

*Proof.* Linearity of $T$ is obvious. Assume first $p > 1$, so that $q < \infty$. Note that for every $a := (a_n) \in l^q \setminus \{0\}$ the sequence $(x_n) := (c\bar{a}_n |a_n|^{q-2})$ $(c = \|a\|_q^{-q/p})$ belongs to $l^p$ and

$$\|x\|_p^p = \|a\|_q^{-q} \sum_n |a_n|^{(q-1)p} = 1.$$

This particular $x \in l^p$ shows that

$$\|Ta\|_{(l^p)'} \ge \sum_n a_n x_n = \|a\|_q^{-q/p} \sum_n |a_n|^q = \|a\|_q^{q(p-1)/p} = \|a\|_q.$$

On the other hand, by Hölder's inequality,

$$\|Ta\|_{(l^p)'} = \sup_{\|x\|_p \le 1} |\sum_n a_n x_n| \le \|a\|_q,$$

so that $T$ is isometric in the case $p \in (1, \infty)$. The case $p = 1$ is very similar and will be omitted.

In order to show that $T$ is surjective, let $\varphi \in (l^p)'$. Denote by $e_n$ the $n$-th unit vector in $l^p$, and let $a_n := \varphi(e_n)$. If $p = 1$, then $(a_n) \in l^\infty = l^q$ by the trivial estimate

$$|a_n| = |\varphi(e_n)| \le \|\varphi\| \, \|e_n\|_1 = \|\varphi\|.$$

If $p > 1$, then we may argue as follows. For every $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^N |a_n|^q &= \sum_{n=1}^N a_n \bar{a_n}\, |a_n|^{q-2} \\ &= \varphi(\sum_{n=1}^N \bar{a_n}\, |a_n|^{q-2}\, e_n) \\ &\le \|\varphi\|\, \big(\sum_{n=1}^N |a_n|^{(q-1)p}\big)^{\frac{1}{p}} \\ &= \|\varphi\|\, \big(\sum_{n=1}^N |a_n|^q\big)^{\frac{1}{p}},\end{aligned}$$

which is equivalent to

$$\big(\sum_{n=1}^N |a_n|^q\big)^{1-\frac{1}{p}} = \big(\sum_{n=1}^N |a_n|^q\big)^{\frac{1}{q}} \le \|\varphi\|.$$

Since the right-hand side of this inequality does not depend on $N \in \mathbb{N}$, we obtain that $a := (a_n) \in l^q$ and $\|a\|_q \le \|\varphi\|$.

Next, observe that for every $x \in l^p$ one has

$$x = \sum_n x_n e_n = \lim_{N\to\infty} \sum_{n=1}^N x_n e_n,$$

the series converging in $l^p$ (here we need the restriction $p < \infty$!). Hence, for every $x \in l^p$, by the boundedness of $\varphi$,

$$\begin{aligned}\varphi(x) &= \lim_{N\to\infty} \varphi(\sum_{n=1}^N x_n e_n) \\ &= \lim_{N\to\infty} \sum_{n=1}^N x_n a_n \\ &= \sum_n x_n a_n \\ &= Ta(x).\end{aligned}$$

Hence, $T$ is surjective.

*Proof (of Theorem 3.22).* By Lemma 3.23, we may identify $(l^p)'$ with $l^q$ and, if $1 < p < \infty$ (!), also $(l^p)'' = (l^q)'$ with $l^p$. One just has to notice that this identification of $l^p$ with $(l^p)'' = l^p$ (the identity map on $l^p$) coincides with the operator $J$ from Lemma 3.16, so that $l^p$ is reflexive if $1 < p < \infty$.

**Lemma 3.24.** *The spaces $l^1$, $L^1(\Omega)$ ($\Omega \subseteq \mathbb{R}^N$) and $C([0,1])$ are not reflexive.*

*Proof.* For every $t \in [0,1]$, let $\delta_t \in C([0,1])'$ be defined by

$$\langle \delta_t, f\rangle := f(t), \quad f \in C([0,1]).$$

Then $\|\delta_t\| = 1$ and whenever $t \neq s$, then

$$\|\delta_t - \delta_s\| = 2.$$

In particular, the uncountably many balls $B(\delta_t, \frac{1}{2})$ ($t \in [0,1]$) are mutually disjoint so that $C([0,1])'$ is not separable.

Now, if $C([0,1])$ were reflexive, then $C([0,1])'' = C([0,1])$ would be separable (since $C([0,1])$ is separable), and then, by Corollary 3.14, $C([0,1])'$ would be separable; a contradiction to what has been said before. This proves that $C([0,1])$ is not reflexive.

The cases of $l^1$ and $L^1(\Omega)$ are proved similarly. They are separable Banach spaces with nonseparable dual.

**Theorem 3.25.** *Every closed subspace of a reflexive Banach space is reflexive.*

*Proof.* Let $X$ be a reflexive Banach space, and let $U \subseteq X$ be a closed subspace. Let $u'' \in U''$. Then the mapping $x'' : X' \to \mathbb{K}$ defined by

$$\langle x'', x' \rangle = \langle u'', x'|_U \rangle, \quad x' \in X',$$

is linear and bounded, i.e. $x'' \in X''$. By reflexivity of $X$, there exists $x \in X$ such that

$$\langle x', x \rangle = \langle u'', x'|_U \rangle, \quad x' \in X'. \tag{3.5}$$

Assume that $x \notin U$. Then, by Corollary 3.10, there exists $x' \in X'$ such that $x'|_U = 0$ and $\langle x', x \rangle \neq 0$; a contradiction to the last equality. Hence, $x \in U$. We need to show that

$$\langle u'', u' \rangle = \langle u', x \rangle, \forall u' \in U'. \tag{3.6}$$

However, if $u' \in U'$, then, by Hahn-Banach we can choose an extension $x' \in X'$, i.e. $x'|_U = u'$. The equation (3.6) thus follows from (3.5).

**Corollary 3.26.** *The Sobolev spaces $W^{k,p}(\Omega)$ ($\Omega \subseteq \mathbb{R}^N$ open) are reflexive if $1 < p < \infty$, $k \in \mathbb{N}$.*

*Proof.* For example, for $k = 1$, the operator

$$\begin{aligned} T : W^{1,p}(\Omega) \to & \quad L^p(\Omega)^{1+N}, \\ u \mapsto & \quad (u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}), \end{aligned}$$

is isometric, so that we may consider $W^{1,p}(\Omega)$ as a closed subspace of $L^p(\Omega)^{1+N}$ which is reflexive by Theorem 3.21. The claim follows from Theorem 3.25.

**Corollary 3.27.** *A Banach space is reflexive if and only if its dual is reflexive.*

*Proof.* Assume that the Banach space $X$ is reflexive. Let $x''' \in X'''$ (the tridual!). Then the mapping $x' : X \to \mathbb{K}$ defined by

$$\langle x',x\rangle := \langle x''',J_X(x)\rangle, \quad x\in X,$$

is linear and bounded, i.e. $x'\in X'$ (here $J_X$ denotes the isometry $X\to X''$). Let $x''\in X''$ be arbitrary. Since $X$ is reflexive, there exists $x\in X$ such that $J_X x = x''$. Hence,

$$\langle x''',x''\rangle = \langle x''',J_X x\rangle = \langle x',x\rangle = \langle x'',x'\rangle,$$

which proves that $J_{X'}x' = x'''$, i.e. the isometry $J_{X'}: X'\to X'''$ is surjective. Hence, $X'$ is reflexive.

On the other hand, assume that $X'$ is reflexive. Then $X''$ is reflexive by the preceeding argument, and therefore $X$ (considered as a closed subspace of $X''$ via the isometry $J$) is reflexive by Theorem 3.25.

Let $X$ be a normed space. We say that a sequence $(x_n)\subseteq X$ *converges weakly* to some $x\in X$ if

$$\lim_{n\to\infty}\langle x',x_n\rangle = \langle x',x\rangle \text{ for every } x'\in X'.$$

Notations: if $(x_n)$ converges weakly to $x$, then we write $x_n \rightharpoonup x$, $w-\lim_{n\to\infty} x_n = x$, $x_n\to x$ in $\sigma(X,X')$, or $x_n\to x$ weakly.

**Theorem 3.28.** *In a reflexive Banach space every bounded sequence admits a weakly convergent subsequence.*

*Proof.* Let $(x_n)$ be a bounded sequence in a reflexive Banach space $X$. We first assume that $X$ is separable. Then $X''$ is separable by reflexivity, and $X'$ is separable by Corollary 3.14. Let $(x'_m)\subseteq X'$ be a dense sequence.

Since $(\langle x'_1,x_n\rangle)$ is bounded by the boundedness of $(x_n)$, there exists a subsequence $(x_{\varphi_1(n)})$ of $(x_n)$ ($\varphi_1:\mathbb{N}\to\mathbb{N}$ is increasing, unbounded) such that

$$\lim_{n\to\infty}\langle x'_1,x_{\varphi_1(n)}\rangle \text{ exists.}$$

Similarly, there exists a subsequence $(x_{\varphi_2(n)})$ of $(x_{\varphi_1(n)})$ such that

$$\lim_{n\to\infty}\langle x'_2,x_{\varphi_2(n)}\rangle \text{ exists.}$$

Note that for this subsequence, we also have that

$$\lim_{n\to\infty}\langle x'_1,x_{\varphi_2(n)}\rangle \text{ exists.}$$

Iterating this argument, we find a subsequence $(x_{\varphi_3(n)})$ of $(x_{\varphi_2(n)})$ and finally for every $m\in\mathbb{N}$, $m\geq 2$, a subsequence $(x_{\varphi_m(n)})$ of $(x_{\varphi_{m-1}(n)})$ such that

$$\lim_{n\to\infty}\langle x'_j,x_{\varphi_m(n)}\rangle \text{ exists for every } 1\leq j\leq m.$$

Let $(y_n) := (x_{\varphi_n(n)})$ be the 'diagonal sequence'. Then $(y_n)$ is a subsequence of $(x_n)$ and

$$\lim_{n\to\infty}\langle x'_m,y_n\rangle \text{ exists for every } m\in\mathbb{N}.$$

By Lemma 2.48 of Chapter 2, there exists $x'' \in X''$ such that

$$\lim_{n\to\infty} \langle x', y_n \rangle = \langle x', x'' \rangle \text{ for every } x' \in X'.$$

Since $X$ is reflexive, there exists $x \in X$ such that $Jx = x''$. For this $x$, we have by definition of $J$

$$\lim_{n\to\infty} \langle x', y_n \rangle = \langle x', x \rangle \text{ exists for every } x' \in X',$$

that is, $(y_n)$ converges weakly to $x$.

If $X$ is not separable as we first assumed, then one may replace $X$ by $\tilde{X} := \overline{\text{span}}\,\{x_n : n \in \mathbb{N}\}$ which is separable. By the above, there exists $x \in \tilde{X}$ and a subsequence of $(x_n)$ (which we denote again by $(x_n)$) such that for every $\tilde{x}' \in \tilde{X}'$,

$$\lim_{n\to\infty} \langle \tilde{x}', x_n \rangle = \langle \tilde{x}', x \rangle,$$

that is, $(x_n)$ converges weakly in $\tilde{X}$. If $x' \in X'$, then $x'|_{\tilde{X}} \in \tilde{X}'$, and it follows easily that the sequence $(x_n)$ also converges weakly in $X$ to the element $x$.

## 3.4 * Minimization of convex functionals

Recall from page 31 that subset $K$ of a real or complex vector space is **convex** if for every $x$, $y \in K$ and every $t \in [0,1]$ one has $tx + (1-t)y \in K$.

**Theorem 3.29 (Hahn-Banach; separation of convex sets).** *Let $X$ be a Banach space, $K \subseteq X$ a closed, nonempty, convex subset, and $x_0 \in X \setminus K$. Then there exists $x' \in X'$ and $\varepsilon > 0$ such that*

$$\operatorname{Re} \langle x', x \rangle + \varepsilon \le \operatorname{Re} \langle x', x_0 \rangle, \quad x \in K.$$

**Lemma 3.30.** *Let $K$ be an open, nonempty, convex subset of a Banach space $X$ such that $0 \in K$. Define the* **Minkowski functional** *$p : X \to \mathbb{R}$ by*

$$p(x) = \inf\{\lambda > 0 : \frac{x}{\lambda} \in K\}.$$

*Then $p$ is sublinear, there exists $M \ge 0$ such that*

$$p(x) \le M \|x\|, \quad x \in X,$$

*and $K = \{x \in X : p(x) < 1\}$.*

*Proof.* Since $B(0,r) \subseteq K$ for some $r > 0$, we find that

$$p(x) \le \frac{1}{r} \|x\| \text{ for every } x \in X.$$

The property $p(\alpha x) = \alpha p(x)$ for every $\alpha > 0$ and every $x \in X$ is obvious.

Next, if $p(x) < 1$, then there exists $\lambda \in (0,1)$ such that $x/\lambda \in K$. Hence, by convexity, $x = \lambda \frac{x}{\lambda} = \lambda \frac{x}{\lambda} + (1-\lambda)0 \in K$. On the other hand, if $x \in K$, then $(1+\varepsilon)x \in K$, since $K$ is open. Hence, $p(x) \leq (1+\varepsilon)^{-1} < 1$, so that $K = \{x \in X : p(x) < 1\}$.

Let finally $x, y \in X$. Then for every $\varepsilon > 0$, $x/(p(x)+\varepsilon) \in K$ and $y/(p(y)+\varepsilon) \in K$. In particular, for every $t \in [0,1]$,

$$\frac{t}{p(x)+\varepsilon}x + \frac{1-t}{p(y)+\varepsilon}y \in K.$$

Setting $t = (p(x)+\varepsilon)/(p(x)+p(y)+2\varepsilon)$, one finds that

$$\frac{x+y}{p(x)+p(y)+2\varepsilon} \in K,$$

so that $p(x+y) \leq p(x)+p(y)+2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we find $p(x+y) \leq p(x)+p(y)$. The claim is proved.

*Proof (of Theorem 3.29).* We prove the theorem for the case when $X$ is a real Banach space. The complex case is proved similarly.

We may without loss of generality assume that $0 \in K$; it suffices to translate $K$ and $x_0$ for this. Since $x_0 \notin K$ and since $K$ is closed, we find that $d := \operatorname{dist}(x_0, K) > 0$. Put

$$K_d := \{x \in X : \operatorname{dist}(x, K) < d/2\},$$

so that $K_d$ is an open, convex subset such that $0 \in K_d$. Let $p$ be the corresponding Minkowski functional (see Lemma 3.30).

Define on the one-dimensional subspace $U := \{\lambda x_0 : \lambda \in \mathbb{R}\}$ the functional $u' : U \to \mathbb{R}$ by $\langle u', \lambda x_0 \rangle = \lambda$. Then

$$\langle u', u \rangle \leq p(u), \quad u \in U.$$

By the Hahn-Banach theorem 3.2, there exists a linear extension $x' : X \to \mathbb{R}$ such that

$$\langle x', x \rangle \leq p(x), \quad x \in X. \tag{3.7}$$

In particular, by Lemma 3.30,

$$|\langle x', x \rangle| \leq M \|x\|,$$

so that $x' \in X'$ and $\|x'\| \leq M$. By construction, $\langle x', x_0 \rangle = 1$. Moreover, by (3.7) and Lemma 3.30, $\langle x', x \rangle < 1$ for every $x \in K \subseteq K_d$, so that

$$\langle x', x \rangle \leq \langle x', x_0 \rangle \, (= 1), \quad x \in K_d.$$

Replacing the above argument with $(1-\varepsilon')x_0$ instead of $x_0$ (where $\varepsilon' > 0$ is chosen so small that $(1-\varepsilon')x_0 \notin K_d$), we find that

$$\langle x', x \rangle + \varepsilon' \langle x', x_0 \rangle \leq \langle x', x_0 \rangle, \quad x \in K \subseteq K_d,$$

and putting $\varepsilon := \varepsilon' = \varepsilon' \langle x', x_0 \rangle > 0$ yields the claim.

**Corollary 3.31.** *Let $X$ be a Banach space and $K \subseteq X$ a closed, convex subset (closed for the norm topology). If $(x_n) \subseteq K$ converges weakly to some $x \in X$, then $x \in K$.*

*Proof.* Assume the contrary, that is, $x \notin K$. By the Hahn-Banach theorem (Theorem 3.29), there exist $x' \in X'$ and $\varepsilon > 0$ such that

$$\mathrm{Re}\,\langle x', x_n \rangle + \varepsilon \leq \mathrm{Re}\,\langle x', x \rangle \text{ for every } n \in \mathbb{N},$$

a contradiction to the assumption that $x_n \rightharpoonup x$.

A function $f : K \to \mathbb{R}$ on a convex subset $K$ of a Banach space $X$ is called *convex* if for every $x, y \in K$, and every $t \in [0,1]$,

$$f(tx + (1-t)y) \leq t\, f(x) + (1-t)\, f(y). \tag{3.8}$$

**Corollary 3.32.** *Let $X$ be a Banach space, $K \subseteq X$ a closed, convex subset, and $f : K \to \mathbb{R}$ a continuous, convex function. If $(x_n) \subseteq K$ converges weakly to $x \in K$, then*

$$f(x) \leq \liminf_{n \to \infty} f(x_n).$$

*Proof.* For every $l \in \mathbb{R}$, the set $K_l := \{x \in K : f(x) \leq l\}$ is closed (by continuity of $f$) and convex (by convexity of $f$). After extracting a subsequence, if necessary, we may assume that $l := \liminf_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x_n)$. Then for every $\varepsilon > 0$ the sequence $(x_n)$ is eventually in $K_{l+\varepsilon}$, i.e. except for finitely many $x_n$, the sequence $(x_n)$ lies in $K_{l+\varepsilon}$. Hence, by Corollary 3.31, $x \in K_{l+\varepsilon}$, which means that $f(x) \leq l + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the claim follows.

Let $K \subseteq X$ be a convex subset of a real or complex vector space. A function $f : K \to \mathbb{R}$ is called **convex** if for every $x, y \in K$ and every $t \in [0,1]$ one has

$$f(tx + (1-t)y) \leq t\, f(x) + (1-t)\, f(y).$$

It is called **strictly convex** if for every $x, y \in K$, $x \neq y$ and every $t \in (0,1)$ the above inequality is strict.

**Theorem 3.33.** *Let $X$ be a reflexive Banach space, $K \subseteq X$ a closed, convex, nonempty subset, and $f : K \to \mathbb{R}$ a continuous, convex function such that*

$$\lim_{\substack{\|x\|\to\infty \\ x \in K}} f(x) = +\infty \textit{ (coercivity).}$$

*Then there exists $x_0 \in K$ such that*

$$f(x_0) = \inf\{f(x) : x \in K\} > -\infty.$$

*Proof.* Let $(x_n) \subseteq K$ be such that $\lim_{n\to\infty} f(x_n) = \inf\{f(x) : x \in K\}$. By the coercivity assumption on $f$, the sequence $(x_n)$ is bounded. Since $X$ is reflexive, there exists a weakly convergent subsequence (Theorem 3.28); we denote by $x_0$ the limit. By Corollary 3.31, $x_0 \in K$. By Corollary 3.32,

$$f(x_0) \leq \lim_{n\to\infty} f(x_n) = \inf\{f(x) : x \in K\}.$$

The claim is proved.

**Remark 3.34.** Theorem 3.33 remains true if $f$ is only lower semicontinuous, i.e. if

$$\liminf_{n\to\infty} f(x_n) \geq f(x)$$

for every convergent $(x_n) \subseteq K$ with $x = \lim x_n$. In fact, already Corollary 3.32 remains true if $f$ is only lower semicontinuous (and then Corollary 3.32 says that lower semicontinuity of a convex function in the norm topology implies lower semicontinuity in the weak topology). It suffices for example to remark that the sets $K_l := \{f \leq l\}$ ($l \in \mathbb{R}$) are closed as soon as $f$ is lower semicontinuous.

## 3.5 * The von Neumann minimax theorem

In the following theorem, we call a function $f : K \to \mathbb{R}$ on a convex subset $K$ of a Banach space $X$ **concave** if $-f$ is convex, or, equivalently, if for every $x, y \in K$ and every $t \in [0,1]$,

$$f(tx + (1-t)y) \geq t\,f(x) + (1-t)\,f(y). \tag{3.9}$$

A function $f : K \to \mathbb{R}$ is called *strictly convex* (resp. *strictly concave*) if for every $x$, $y \in K$, $x \neq y$, $f(x) = f(y)$ the inequality in (3.8) (resp. (3.9)) is strict for $t \in (0,1)$.

**Theorem 3.35 (von Neumann minimax theorem).** *Let $K$ and $L$ be two closed, bounded, nonempty, convex subsets of reflexive Banach spaces $X$ and $Y$, respectively. Let $f : K \times L \to \mathbb{R}$ be a continuous function such that*

$$x \mapsto f(x,y) \text{ is strictly convex for every } y \in L, \text{ and}$$
$$y \mapsto f(x,y) \text{ is concave for every } x \in K.$$

*Then there exists $(\bar{x},\bar{y}) \in K \times L$ such that*

$$f(\bar{x},y) \leq f(\bar{x},\bar{y}) \leq f(x,\bar{y}) \text{ for every } x \in K, y \in L. \tag{3.10}$$

**Remark 3.36.** A point $(\bar{x},\bar{y}) \in K \times L$ satisfying (3.10) is called a **saddle point** of $f$.

A saddle point is a point of **equilibrium** in a two-person zero-sum game in the following sense: If the player controlling the strategy $x$ modifies his strategy when the second player plays $\bar{y}$, he increases his loss; hence, it is his interest to play $\bar{x}$. Similarly, if the player controlling the strategy $y$ modifies his strategy when the first player plays $\bar{x}$, he diminishes his gain; thus it is in his interest to play $\bar{y}$. This property of equilibrium of saddle points justifies their use as a (reasonable) solution in a two-person zero-sum game ([**?**]).

*Proof.* Define the function $F : L \to \mathbb{R}$ by $F(y) := \inf_{x\in K} f(x,y)$ $(y \in L)$. By Theorem 3.33, for every $y \in L$ there exists $x \in K$ such that $F(y) = f(x,y)$. By strict convexity, this element $x$ is uniquely determined. We denote $x := \Phi(y)$ and thus obtain

$$F(y) = \inf_{x\in K} f(x,y) = f(\Phi(y),y), \quad y \in L. \tag{3.11}$$

By concavity of the function $y \mapsto f(x,y)$ and by the definition of $F$, for every $y_1$, $y_2 \in L$ and every $t \in [0,1]$,

$$\begin{aligned} F(ty_1 + (1-t)y_2) &= f(\Phi(ty_1 + (1-t)y_2), ty_1 + (1-t)y_2) \\ &\geq t\, f(\Phi(ty_1 + (1-t)y_2), y_1) + (1-t)\, f(\Phi(ty_1 + (1-t)y_2), y_2) \\ &\geq t\, F(y_1) + (1-t)\, F(y_2), \end{aligned}$$

so that $F$ is concave. Moreover, $F$ is upper semicontinuous: let $(y_n) \subseteq L$ be convergent to $y \in L$. For every $x \in K$ and every $n \in \mathbb{N}$ one has $F(y_n) \leq f(x,y_n)$, and taking the limes superior on both sides, we obtain, by continuity of $f$,

$$\limsup_{n\to\infty} F(y_n) \leq \limsup_{n\to\infty} f(x,y_n) = f(x,y).$$

Since $x \in K$ was arbitrary, this inequality implies $\limsup_{n\to\infty} F(y_n) \leq F(y)$, i.e. $F$ is upper semicontinuous.

By Theorem 3.33 (applied to $-F$; use also Remark 3.34), there exists $\bar{y} \in L$ such that

$$f(\Phi(\bar{y}),\bar{y}) = F(\bar{y}) = \sup_{y\in L} F(y).$$

We put $\bar{x} = \Phi(\bar{y})$ and show that $(\bar{x},\bar{y})$ is a saddle point. Clearly, for every $x \in K$,

$$f(\bar{x},\bar{y}) \leq f(x,\bar{y}). \tag{3.12}$$

Therefore it remains to show that for every $y \in L$,

$$f(\bar{x},\bar{y}) \geq f(\bar{x},y). \tag{3.13}$$

Let $y \in L$ be arbitrary and put $y_n := (1-\frac{1}{n})\bar{y} + \frac{1}{n}y$ and $x_n = \Phi(y_n)$. Then, by concavity,

$$\begin{aligned} F(\bar{y}) \geq F(y_n) \quad &= \quad f(x_n,y_n) \\ &\geq (1-\frac{1}{n}) f(x_n,\bar{y}) + \frac{1}{n} f(x_n,y) \\ &\geq (1-\frac{1}{n}) F(\bar{y}) + \frac{1}{n} f(x_n,y), \end{aligned}$$

or

$$F(\bar{y}) \geq f(x_n, y) \text{ for every } n \in \mathbb{N}.$$

Since $K$ is bounded and closed, the sequence $(x_n) \subseteq K$ has a weakly convergent subsequence which converges to some element $x_0 \in K$ (Theorem 3.28 and Corollary 3.31). By the preceeding inequality and Corollary 3.32,

$$F(\bar{y}) \geq f(x_0, y).$$

This is just the remaining inequality (3.13) if we can prove that $x_0 = \bar{x}$. By concavity, for every $x \in K$ and every $n \in \mathbb{N}$,

$$\begin{aligned} f(x, y_n) &\geq f(x_n, y_n) \\ &\geq (1 - \frac{1}{n}) f(x_n, \bar{y}) + \frac{1}{n} f(x_n, y) \\ &\geq (1 - \frac{1}{n}) f(x_n, \bar{y}) + \frac{1}{n} F(y). \end{aligned}$$

Letting $n \to \infty$ in this inequality and using Corollary 3.32 again, we obtain that for every $x \in K$,

$$f(x, \bar{y}) \geq f(x_0, \bar{y}).$$

Hence, $x_0 = \Phi(\bar{y}) = \bar{x}$ and the theorem is proved.

# Chapter 4
# Uniform boundedness, bounded inverse and closed graph

This chapter is devoted to the other fundamental theorems in functional analysis; other than the Hahn-Banach theorem which has been discussed in the previous chapter. These fundamental results are

- the uniform boundedness principle or the Banach-Steinhaus theorem,
- the bounded inverse theorem (and the related open mapping theorem), and
- the closed graph theorem.

All these fundamental results rely on an abstract lemma for metric spaces.

## 4.1 The lemma of Baire

**Lemma 4.1 (Baire).** *Let $(M,d)$ be a complete metric space, and let $(O_n)$ be a sequence of open and dense subsets of $M$. Then $\bigcap_n O_n$ is dense in $M$.*

*Proof.* We can assume that $M$ is not empty since the statement is trivial otherwise. Let $x_0 \in M$ and $\varepsilon > 0$ be arbitrary. We have to prove that $\bigcap_n O_n \cap B(x_0,\varepsilon)$ is not empty.

Since $O_1$ is dense and open in $M$, the intersection $B(x_0,\varepsilon) \cap O_1$ is open and nonempty. Hence, there exists $\varepsilon_1 > 0$ (w.l.o.g. $\varepsilon_1 \leq \varepsilon/2$) and $x_1 \in B(x_0,\varepsilon) \cap O_1$ such that

$$B(x_1,\varepsilon_1) \subseteq B(x_0,\varepsilon) \cap O_1.$$

Choosing $\varepsilon_1$ a little bit smaller, if necessary, we can even assume that

$$\overline{B(x_1,\varepsilon_1)} \subseteq B(x_0,\varepsilon) \cap O_1.$$

Since $O_2$ is dense and open in $M$, the intersection $B(x_1,\varepsilon_1) \cap O_2$ is open and nonempty. Hence, there exists $\varepsilon_2 > 0$ (w.l.o.g. $\varepsilon_2 \leq \varepsilon_1/2$) and $x_2 \in B(x_1,\varepsilon_1) \cap O_2$ such that

$$\overline{B(x_2,\varepsilon_2)} \subseteq B(x_1,\varepsilon_1) \cap O_2 \subseteq B(x_0,\varepsilon) \cap O_1 \cap O_2.$$

Proceeding inductively, we can construct sequences $(\varepsilon_n) \subseteq (0,\infty)$ and $(x_n) \subseteq M$ such that

(i) $\varepsilon_n \leq \varepsilon_{n-1}/2$ and

(ii) for every $n \in \mathbb{N}$

$$\overline{B(x_n,\varepsilon_n)} \subseteq B(x_{n-1},\varepsilon_{n-1}) \cap O_n \subseteq B(x_0,\varepsilon) \cap \bigcap_{j=1}^{n} O_j.$$

In particular, $x_m \in B(x_n,\varepsilon_n)$ for every $m \geq n$ and $\lim_{n\to\infty} \varepsilon_n = 0$. Hence, the sequence $(x_n)$ is a Cauchy sequence in $M$. Since $M$ is complete, there exists $x := \lim_{n\to\infty} x_n \in M$. By the above,

$$x \in \overline{B(x_n,\varepsilon_n)} \text{ for every } n \in \mathbb{N},$$

or

$$x \in \bigcap_n \overline{B(x_n,\varepsilon_n)} \subseteq B(x_0,\varepsilon) \cap \bigcap_n O_n.$$

The claim is proved.

**Lemma 4.2 (Baire).** *Let $(M,d)$ be a complete, nonempty, metric space, and let $(A_n)$ be a sequence of closed subsets in $M$ such that $M = \bigcup_n A_n$. Then there exists $n_0 \in \mathbb{N}$ such that $A_{n_0}$ has nonempty interior.*

*Proof.* Assume the contrary, i.e. that every $A_n$ has empty interior. In this case, the sets $O_n := M \setminus A_n$ are open and dense. By assumption,

$$\emptyset = M \setminus \bigcup_n A_n = \bigcap_n O_n,$$

a contradiction to Lemma 4.1 and the assumption that $M$ is nonempty.

**Remark 4.3.** The assumption in Lemma 4.1 or Lemma 4.2 that $M$ is complete is necessary in general. For example,

$$\mathbb{Q} = \bigcup_{x\in\mathbb{Q}} \{x\},$$

and this union is countable. Each one point set $\{x\}$ is closed but in this example, none of these sets has nonempty interior.

**Remark 4.4.** As a corollary to the lemma of Baire one obtains for example that there exists a continuous function $f \in C([0,1])$ which is nowhere differentiable. In fact, the set of such functions is dense in $C([0,1])$; see [**?**].

## 4.2 The uniform boundedness principle

**Theorem 4.5 (Uniform boundedness principle; Banach-Steinhaus).** *Let $X$, $Y$ be Banach spaces and let $(T_i)_{i\in I} \subseteq \mathscr{L}(X,Y)$ be a family of bounded linear operators such that*

$$\sup_{i\in I} \|T_i x\| < \infty \textit{ for every } x \in X.$$

*Then*

$$\sup_{i\in I} \|T_i\| < \infty.$$

**Remark 4.6.** Theorem 4.5 is in general not true if $X$ is only a normed space. For example, let $X = c_{00}(= Y)$ be the space of all finite sequences equipped with the supremum norm (or any other reasonable norm). Let

$$T_n x = T_n(x_m) = (a_{nm} x_m)$$

with

$$a_{nm} = \begin{cases} m \text{ if } m \le n, \\ 0 \;\text{ if } m > n. \end{cases}$$

Then $\sup_n \|T_n x\|$ is finite for every $x \in X$, but $\|T_n\| = n$ is unbounded.

**Remark 4.7.** The fact that in Theorem 4.5 we suppose also $Y$ to be a Banach space is not important. In fact, if $Y$ is not complete, then we may embed $Y$ into its completion $\tilde{Y}$ and consider every operator $T_i \in \mathscr{L}(X,Y)$ also as an operator in $\mathscr{L}(X,\tilde{Y})$.

*Proof (Proof of Theorem 4.5).* Let $A_n := \{x \in X : \sup_{i\in I} \|T_i x\| \le n\}$. Since arbitrary intersections of closed sets are closed, and by the boundedness of the $T_i$, the sets $A_n$ are closed for every $n \in \mathbb{N}$. By assumption, $X = \bigcup_n A_n$.

Hence, by the lemma of Baire (Lemma 4.2), there exists $n_0 \in \mathbb{N}$ such that $A_{n_0}$ has nonempty interior, i.e. there exist $n_0 \in \mathbb{N}$, $x_0 \in X$ and $\varepsilon > 0$ such that

$$\sup_{i\in I} \|T_i x\| \le n_0 \text{ for every } x \in B(x_0,\varepsilon),$$

or, in other words, there exists $n_0 \in \mathbb{N}$, $x_0 \in X$ and $\varepsilon > 0$ such that

$$\|T_i(x_0 + \varepsilon x)\| \le n_0 \text{ for every } x \in B(0,1),\, i \in I.$$

This implies, by the triangle inequality,

$$\varepsilon \|T_i x\| \le n_0 + \|T_i x_0\| \le 2n_0 \text{ for every } x \in B(0,1),\, i \in I.$$

The claim is proved.

**Corollary 4.8.** *Let $X$, $Y$ be Banach spaces and let $(T_n) \subseteq \mathscr{L}(X,Y)$ be a* strongly convergent *sequence of bounded linear operators, i.e.*

$$Tx := \lim_{n\to\infty} T_n x \text{ exists for every } x \in X.$$

*Then* $\sup_{n\in\mathbb{N}} \|T_n\| =: M < \infty$ *and* $T \in \mathscr{L}(X,Y)$.

*Proof.* Linearity of $T$ is clear. Since $(T_n)$ is strongly convergent, the sequence $(T_n x)$ is bounded for every $x \in X$. By the uniform bounded principle (Theorem 4.5), $\sup_{n\in\mathbb{N}} \|T_n\| =: M < \infty$. As a consequence, for every $x \in X$,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq M \|x\|,$$

so that $T$ is bounded.

**Corollary 4.9.** *Every weakly convergent sequence in a Banach space is bounded.*

*Proof.* Let $X$ be a Banach space and $(x_n) \subseteq X$ be weakly convergent. Considering the $x_n$ as elements in $X'' = \mathscr{L}(X',\mathbb{K})$ by the embedding $J : X \to X''$, the claim follows from Corollary 4.8.

## 4.3 Open mapping theorem, bounded inverse theorem

**Theorem 4.10 (Open mapping theorem).** *Let $X$, $Y$ be two Banach spaces and let $T \in \mathscr{L}(X,Y)$ be surjective. Then there exists $r > 0$ such that*

$$TB_X(0,1) \supseteq B_Y(0,r). \tag{4.1}$$

*Proof. First step:* We show that there exists $r > 0$ such that

$$B(0,2r) \subseteq \overline{TB(0,1)}. \tag{4.2}$$

For this, we remark first that by surjectivity,

$$Y = TX = \bigcup_n TB(0,n) = \bigcup_n \overline{TB(0,n)}.$$

By the Lemma of Baire, there exists $n_0$ such that $\overline{TB(0,n_0)}$ has nonempty interior, i.e. there exist $x \in \overline{TB(0,n_0)}$ and $\varepsilon > 0$ such that

$$B(x,\varepsilon) \subseteq \overline{TB(0,n_0)}.$$

By symmetry,

$$B(-x,\varepsilon) \subseteq \overline{TB(0,n_0)},$$

and adding both 'inequalities' together, we obtain

$$B(0,\varepsilon) \subseteq \overline{TB(0,n_0)},$$

which implies the required inclusion (4.2) if we put $r = \frac{\varepsilon}{2n_0}$.

*Second step:* We prove (4.1). Let $y \in B(0,r)$, where $r > 0$ is as in (4.2) from the first step. Then, by (4.2), for every $\varepsilon > 0$ there exists $x \in B(0,\frac{1}{2})$ such that $\|y - Tx\| < \varepsilon$. In particular, if we choose $\varepsilon = \frac{r}{2}$, then there exists $x_1 \in B(0,\frac{1}{2})$ such that $\|y - Tx_1\| < \frac{r}{2}$.

Similarly, since $y - Tx_1 \in B(0,\frac{r}{2})$, there exists $x_2 \in B(0,\frac{1}{4})$ such that $\|(y - Tx_1) - Tx_2\| \leq \frac{r}{4}$. Iterating this construction, we find a sequence $(x_n)$ such that $x_n \in B(0,2^{-n})$ and such that $\|y - \sum_{j=1}^n Tx_n\| \leq 2^{-n}r$. Since $X$ is complete and since $\sum_n x_n$ is absolutely convergent with $\sum_n \|x_n\| < 1$, the limit $x = \sum_n x_n$ exists and $x \in B(0,1)$. By the preceeding estimates, $\|y - Tx\| = 0$ or $Tx = y$. Thus we have proved (4.1).

**Remark 4.11.** It is not difficult to prove that if an operator $T \in \mathscr{L}(X,Y)$ satisfies (4.1), then $TO$ is open for every open $O \subseteq X$. A function which maps open sets into open sets is called *open*; whence the name of the open mapping theorem.

**Corollary 4.12 (Bounded inverse theorem).** *Let $X$, $Y$ be two Banach spaces and let $T \in \mathscr{L}(X,Y)$ be bijective. Then $T^{-1} \in \mathscr{L}(Y,X)$.*

*Proof.* Linearity of $T^{-1}$ is clear. By the open mapping theorem (Theorem 4.10), we have

$$T^{-1}B_Y(0,1) \subseteq B_X(0,\frac{1}{r})$$

for some $r > 0$. Hence, $T^{-1}$ is bounded.

**Corollary 4.13.** *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space $X$ such that $(X,\|\cdot\|_1)$ and $(X,\|\cdot\|_2)$ are complete. If there exists a constant $C > 0$ such that*

$$\|x\|_2 \leq C\,\|x\|_1 \text{ for every } x \in X,$$

*then the two norms are equivalent.*

*Proof.* It suffices to consider the identity $I : (X,\|\cdot\|_1) \to (X,\|\cdot\|_2)$. It is bounded by assumption, and clearly it is bijective. By the bounded inverse theorem (Corollary 4.12), the inverse $I^{-1} : (X,\|\cdot\|_2) \to (X,\|\cdot\|_1)$ is bounded, i.e. there exists $c > 0$ such that

$$\|x\|_1 \leq c\,\|x\|_2 \text{ for every } x \in X.$$

## 4.4 Closed graph theorem

Let $X$, $Y$ be two Banach spaces, and let $\operatorname{dom} T \subseteq X$ be a linear subspace. A linear operator $T : \operatorname{dom} T \to Y$ is called a **closed operator** if the graph

$$\operatorname{Graph} T := \{(x,Tx) : x \in \operatorname{dom} T\}$$

is closed in $X \times Y$.

**Lemma 4.14.** *A linear operator $T : X \supseteq \operatorname{dom} T \to Y$ is closed if and only if*

$$\left.\begin{array}{l}\operatorname{dom} T \ni x_n \to x \text{ in } X \text{ and} \\ Tx_n \to y \text{ in } Y\end{array}\right\} \Rightarrow \quad x \in \operatorname{dom} T \text{ and } Tx = y. \tag{4.3}$$

*Proof.* Exercise.

**Lemma 4.15.** *Every bounded linear operator $T \in \mathscr{L}(X,Y)$ ($X$, $Y$ Banach spaces) is closed.*

*Proof.* This is an immediate consequence of Lemma 4.14.

**Lemma 4.16.** *A linear operator $T : X \supseteq \operatorname{dom} T \to Y$ is closed if and only if the space* $\operatorname{dom} T$ *equipped with the* **graph norm**

$$\|x\|_{\operatorname{dom} T} := \|x\|_X + \|Tx\|_Y, \quad x \in X,$$

*is complete.*

*Proof.* $\Rightarrow$ Assume that $T$ is closed. Let $(x_n)$ be a Cauchy sequence in $(\operatorname{dom} T, \|\cdot\|_{\operatorname{dom} T})$. Then $(x_n)$ is a Cauchy sequence in $X$ and $(Tx_n)$ is a Cauchy sequence in $Y$. Since $X$ and $Y$ are complete, there exist $x \in X$ and $y \in Y$ such that $x_n \to x$ and $Tx_n \to y$. Since $T$ is closed, and by Lemma 4.14, this implies $x \in \operatorname{dom} T$ and $Tx = y$. Moreover,

$$\|x_n - x\|_{\operatorname{dom} T} = \|x_n - x\|_X + \|Tx_n - Tx\|_Y \to 0,$$

so that $(x_n)$ converges in $(\operatorname{dom} T, \|\cdot\|_{\operatorname{dom} T})$. Hence, $\operatorname{dom} T$ equipped with the graph norm is complete.

$\Leftarrow$ Assume that $(\operatorname{dom} T, \|\cdot\|_{\operatorname{dom} T})$ is complete. Assume that $\operatorname{dom} T \ni x_n \to x \in X$ and $Tx_n \to y \in Y$. Then $(x_n)$ and $(Tx_n)$ are Cauchy sequences in $X$ and $Y$, respectively. By the definition of $\|\cdot\|_{\operatorname{dom} T}$, this implies that $(x_n)$ is a Cauchy sequence in $(\operatorname{dom} T, \|\cdot\|_{\operatorname{dom} T})$. By completeness, there exists $\bar{x} \in \operatorname{dom} T$ such that $x_n \to \bar{x}$ in $\operatorname{dom} T$ (with respect to the graph norm). Since convergence of $(x_n)$ in $\operatorname{dom} T$ implies the convergence of $(x_n)$ in $X$, and since $(x_n)$ converges to $x$ in $X$, we find $x = \bar{x} \in \operatorname{dom} T$ by the uniqueness of the limit. Moreover, since $T$ is always bounded from $\operatorname{dom} T$ (when equipped with the graph norm) into $Y$, we have $Tx = \lim_{n\to\infty} Tx_n = y$. Hence, by Lemma 4.14, $T$ is closed.

**Example 4.17.** Let $X = Y = C([0,1])$ be equipped with the supremum norm, and let $\operatorname{dom} T := C^1([0,1]) \subseteq X$. Let $Tf := f'$ for $f \in \operatorname{dom} T$. Then $T$ is a closed operator. In fact, the graph norm $\|\cdot\|_{\operatorname{dom} T}$ coincides with the canonical norm on $C^1([0,1])$, i.e.

$$\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty,$$

and $(C^1([0,1]), \|\cdot\|_{C^1})$ is complete.

**Theorem 4.18 (Closed graph theorem).** *Let $X$, $Y$ be two Banach spaces and let $T : X \to Y$ be a closed operator. Then $T$ is bounded.*

**Remark 4.19.** The important assumption in Theorem 4.18, besides the assumption that $T$ is closed, is the assumption that $\operatorname{dom} T = X$! The Example 4.17 shows that closed operators need not be bounded in general; when considered from $(\operatorname{dom} T, \|\cdot\|_X)$ with values in $Y$. Note that in Example 4.17, $\operatorname{dom} T$ is not complete when equipped with the norm coming from $X$.

*Proof (Proof of Theorem 4.18).* By assumption $(X, \|\cdot\|_X)$ is a Banach space, and by closedness of $T$ and Lemma 4.16, also $(X, \|\cdot\|_{\operatorname{dom} T})$ is a Banach space, where $\|\cdot\|_{\operatorname{dom} T}$ denotes the graph norm. Moreover, trivially,

$$\|x\|_X \leq \|x\|_{\operatorname{dom} T} \text{ for every } x \in X.$$

By Corollary 4.13, the two norms $\|\cdot\|_X$ and $\|\cdot\|_{\operatorname{dom} T}$ are equivalent, that is, there exists a constant $C \geq 0$ such that

$$\|x\|_X + \|Tx\|_Y \leq C\,\|x\|_X \text{ for every } x \in X.$$

As a consequence, $T$ is bounded.

**Example 4.20 (Sobolev embedding).** Let $-\infty < a < b < \infty$. Then the embedding

$$J : W^{1,p}(a,b) \to C([a,b]),$$
$$u \mapsto u$$

is well defined and bounded, that is, there exists a constant $C \geq 0$ such that

$$\|u\|_\infty \leq C\,\|u\|_{W^{1,p}} \text{ for every } u \in W^{1,p}(a,b).$$

Recall that this embedding is well defined since every function $u \in W^{1,p}(a,b)$ is continuous on $[a,b]$ by Theorem 10.8 of Chapter 10.

In order to see that $J$ is also bounded, we apply the closed graph theorem together with the characterization in Lemma 4.14: let $(u_n) \subseteq W^{1,p}(a,b)$ be such that $u = \lim_{n\to\infty} u_n$ exists in $W^{1,p}(a,b)$ and such that $v = \lim_{n\to\infty} u_n$ exists in $C([a,b])$. The convergence in $W^{1,p} \subseteq L^p$ implies that $u_n \to u$ almost everywhere if we extract a subsequence. The convergence in $C$ implies that $u_n \to v$ everywhere. Hence $u = v$ almost everyhwere, and since both functions are continuous, we obtain $u = v$. Hence, the embedding is closed. By the closed graph theorem, the embedding $W^{1,p} \to C$ is bounded.

**Exercise 4.21** *Let $T : X \supseteq \operatorname{dom} T \to Y$ be a closed, injective operator. Define*

$$\operatorname{dom} T^{-1} := \operatorname{ran} T = \{Tx : x \in \operatorname{dom} T\} \subseteq Y,$$
$$T^{-1}y := x \text{ where } x \in \operatorname{dom} T \text{ is the unique element such that } Tx = y.$$

*Then $T^{-1}$ is a closed operator.*

*If in addition $T$ is surjective, then $T^{-1} : Y \to X$ is bounded.*

## 4.5 * Vector-valued analytic functions

Let $X$ be a complex Banach space and let $\Omega \subseteq \mathbb{C}$ be an open subset. We say that a function $f : \Omega \to X$ is **analytic** (or: **holomorphic**) if

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists for every } z_0 \in \Omega.$$

We say that $f : \Omega \to X$ is **weakly analytic** (or: **weakly holomorphic**) if $x' \circ f : \Omega \to \mathbb{C}$ is analytic for every $x' \in X'$.

**Theorem 4.22.** *A function $f : \Omega \to X$ is analytic if and only if it is weakly analytic.*

*Proof.* Cleary, if $f$ is analytic, then $f$ is weakly analytic. So we only have to prove the other direction.

By considering $X$ as a closed subspace of $X''$ (via the embedding $J$), and by replacing then $X$ by $X''$ (so that the function $f$ becomes $X''$-valued), we can assume that $X$ is a dual space. But doing this, we no longer assume that $f$ is weakly analytic. The assertion which we have to prove is then the following:

Let $X$ be a complex Banach space, and let $X'$ be its dual. Let $f : \Omega \to X'$ be such that $\langle f, x \rangle : \Omega \to \mathbb{C}$ is analytic for every $x \in X$. Then $f$ is analytic.

In fact, it suffices to prove that for fixed $z_0 \in \Omega$ there exists $M \geq 0$ such that for every $y, z \in \Omega \setminus \{z_0\}$ 'close' to $z_0$,

$$\left\| \frac{f(z) - f(z_0)}{z - z_0} - \frac{f(y) - f(z_0)}{y - z_0} \right\| \leq M |z - y|. \tag{4.4}$$

Let $K := \overline{B(z_0, r)} \setminus \{z_0\}$, where $r > 0$ is chosen so small that $K \subseteq \Omega$. Let

$$\tilde{K} = (K \times K) \setminus \{(z, z) : z \in K\}$$

be the cartesian product of $K$ and $K$ from which we take out the 'diagonal'.

By assumption, for every $x \in X$, the function $\langle f, x \rangle$ is analytic. Hence, for every $x \in X$ we have

$$\sup_{(y,z) \in \tilde{K}} \left| \left\langle \frac{\frac{f(z)-f(z_0)}{z-z_0} - \frac{f(y)-f(z_0)}{y-z_0}}{y - z}, x \right\rangle \right| < \infty.$$

By the uniform boundedness principle, this implies

$$\sup_{(y,z) \in \tilde{K}} \left\| \frac{\frac{f(z)-f(z_0)}{z-z_0} - \frac{f(y)-f(z_0)}{y-z_0}}{y - z} \right\| =: M < \infty,$$

which actually implies (4.4) for every $y, z \in K$.

By Theorem 4.22, many important properties of 'classical' analytic functions $\Omega \to \mathbb{C}$ carry over to vector-valued analytic functions $\Omega \to X$. For example:

- Every analytic function $f : \Omega \to X$ is infinitely many times differentiable.
- Every analytic function $f : \Omega \to X$ can be locally developed into a power series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ with $a_n \in X$. In fact: $a_n = \frac{1}{n!} f^{(n)}(z_0)$.
- Cauchy's integral formula $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(y)}{z-y}\, dy$ holds true for appropriate paths $\gamma$. Note, however, that we have not yet defined integrals of vector-valued functions.

An important example of a vector-valued analytic function will be the resolvent of an operator $T \in \mathscr{L}(X)$; see the Chapter 5.

# Chapter 5
# Spectral theory of operators on Banach spaces, compact operators, nuclear operators

## 5.1 The spectrum of closed or bounded operators

Let $X$ and $Y$ be Banach spaces. A **linear operator** from $X$ into $Y$ is a pair $(A, \mathrm{dom}A)$ where $\mathrm{dom}A \subseteq X$ is a linear subspace and $A : \mathrm{dom}A \to Y$ is a linear mapping. We call $\mathrm{dom}A$ the **domain** of $A$. Furthermore, we define the **kernel**, the **range**, and the **graph** of $A$ respectively by

$$\begin{aligned}
\ker A &:= \{x \in X \,:\, Ax = 0\}, \\
\mathrm{ran}A &:= \{y \in Y \,:\, \exists x \in \mathrm{dom}A \text{ s.t. } Ax = y\} \text{ and} \\
\mathrm{graph}A &:= \{(x,y) \in X \times Y \,:\, x \in \mathrm{dom}A \text{ and } Ax = y\}.
\end{aligned}$$

We say that a linear operator from $X$ into $Y$ is **densely defined** if its domain is dense in $X$. If the domain is clear from the context, then we simply speak of a linear operator $A$ from $X$ into $Y$; even if we say "from $X$ into $Y$", the domain need in general not be equal to $X$. However, for a bounded, linear operator $A$ we always assume, unless otherwise stated, that $\mathrm{dom}A = X$. Recall that an operator $A$ from $X$ into $Y$ is **closed** if its graph is closed in $X \times Y$. We recall that an operator $A$ on $X$ is closed if and only if its domain, equipped with the graph norm given by $\|x\|_{\mathrm{dom}A} := \|x\|_X + \|Ax\|_Y$ ($x \in \mathrm{dom}A$), is complete. We also recall the closed graph theorem (Theorem 4.18) which says that every closed operator $A$ with domain $\mathrm{dom}A = X$ is automatically bounded.

Let $A$ be a linear operator on $X$ (from $X$ into $X$). For every $\lambda \in \mathbb{K}$ we write $\lambda - A := \lambda I - A$, where $I$ is the identity operator on $X$ and $\mathrm{dom}(\lambda - A) := \mathrm{dom}A$. We define the **resolvent set** of $A$ by

$$\begin{aligned}
\rho(A) := \{\lambda \in \mathbb{K} :\ & \lambda - A : \mathrm{dom}A \to X \text{ is bijective and} \\
& (\lambda - A)^{-1} \text{ is bounded on } X\}.
\end{aligned}$$

We emphasize that the inverse $(\lambda - A)^{-1}$ is considered as an operator from $X$ into $X$, and not as an operator from $X$ into $\operatorname{dom} A$, although it effectively maps into $\operatorname{dom} A$. For every $\lambda \in \rho(A)$ we write

$$R(\lambda, A) := (\lambda - A)^{-1},$$

and we call $R(\lambda, A)$ the **resolvent** of $A$ at $\lambda$. The mapping $\rho(A) \to \mathscr{L}(X)$, $\lambda \mapsto R(\lambda, A)$ is called the **resolvent** of $A$. The set

$$\sigma(A) := \mathbb{K} \setminus \rho(A)$$

is called the **spectrum** of $A$. Moreover, we define the **point spectrum**, the **approximative point spectrum**, the **continuous spectrum** and the **residual spectrum**, respectively, by

$$\begin{aligned} \sigma_p(A) &:= \{\lambda \in \mathbb{K} : \lambda - A \text{ is not injective}\} \\ &= \{\lambda \in \mathbb{K} : \exists x \in \operatorname{dom} A \setminus \{0\} \text{ s.t. } Ax = \lambda x\} \\ \sigma_{ap}(A) &:= \{\lambda \in \mathbb{K} : \exists (x_n) \subseteq \operatorname{dom} A \text{ s.t.} \|x_n\| = 1 \text{ and } (\lambda - A)x_n \to 0\}, \\ \sigma_c(A) &:= \{\lambda \in \mathbb{K} : \lambda - A \text{ is injective, has dense range, but} \\ &\qquad\qquad (\lambda - A)^{-1} : \operatorname{ran} A \to X \text{ is not bounded}\}, \text{ and} \\ \sigma_r(A) &:= \{\lambda \in \mathbb{K} : \operatorname{ran}(\lambda - A) \text{ is not dense in } X\}. \end{aligned}$$

Elements of $\sigma_p(A)$ are called **eigenvalues**. Any vector $x \in \operatorname{dom} A \setminus \{0\}$ such that $Ax = \lambda x$ is called **eigenvector** for the eigenvalue $\lambda$.

Our first lemma shows that if we look for operators with nonempty resolvent set, then we necessarily have to search in the class of closed operators.

**Lemma 5.1.** *Let $A$ be a linear operator on a Banach space $X$. Then:*

(a) *If the resolvent set of $A$ is nonempty, then $A$ is closed.*

(b) *If $A$ is closed, then*

$$\rho(A) = \{\lambda \in \mathbb{K} : \lambda - A : \operatorname{dom} A \to X \textit{ is bijective}\}.$$

*Proof.* (a) Let $A$ be a linear operator on a Banach space $X$. Assume that the resolvent set is nonempty, and let $\lambda \in \rho(A)$. Then $\lambda - A$ is bijective and $(\lambda - A)^{-1}$ is a bounded, linear operator on $X$. In particular, $(\lambda - A)^{-1}$ is closed. This means that

$$\operatorname{graph}(\lambda - A)^{-1} = \{(y, x) \in X \times X : (\lambda - A)^{-1} y = x\}$$

is closed in $X \times X$. Hence,

$$\operatorname{graph}(\lambda - A) = \{(x, y) \in X \times X : x \in \operatorname{dom} A \text{ and } (\lambda - A)x = y\}$$

is closed in $X \times X$. This easily implies that $A$ has closed graph.

(b) The inclusion "$\subseteq$" is trivial. We only have to prove the converse inclusion. Assume that $A$ is closed, and let $\lambda \in \mathbb{K}$ be such that $\lambda - A$ is bijective. Then $(\lambda - A)^{-1}$ is defined everywhere on $X$ and closed. By the closed graph theorem (Theorem 4.18), $(\lambda - A)^{-1}$ is bounded. Hence, $\lambda \in \rho(A)$.

**Lemma 5.2 (Resolvent identity).** *Let $A$ be a linear operator on a Banach space $X$. For every $\lambda$, $\mu \in \rho(A)$ one has*

$$R(\lambda,A) - R(\mu,A) = (\mu - \lambda)R(\mu,A)R(\lambda,A).$$

*Proof.* For every $\lambda$, $\mu \in \rho(A)$

$$\begin{aligned} R(\lambda,A) - R(\mu,A) &= R(\lambda,A)(\mu - A)R(\mu,A) - R(\lambda,A)(\lambda - A)R(\mu,A) \\ &= R(\lambda,A)(\mu - \lambda)R(\mu,A). \end{aligned}$$

**Lemma 5.3 (The spectrum of a closed operator).** *Let $A$ be a closed, linear operator on a Banach space $X$. Then:*

(a) **Analyticity of the resolvent.** *For every $\lambda \in \rho(A)$ and every $\mu \in \mathbb{K}$ with $|\lambda - \mu| < 1/\|R(\lambda,A)\|$ one has $\mu \in \rho(A)$ and*

$$R(\mu,A) = \sum_{k=0}^{\infty} (\lambda - \mu)^k R(\lambda,A)^{k+1}.$$

*In particular, the resolvent set $\rho(A)$ is open in $\mathbb{K}$ and the spectrum $\sigma(A)$ is closed in $\mathbb{K}$. The resolvent $\rho(A) \to \mathscr{L}(X)$, $\lambda \mapsto R(\lambda,A)$ is analytic, which means that it can locally near every point $\lambda \in \rho(A)$ be developed into a power series.*

(b) **Growth of the resolvent near the spectrum.** *For every $\lambda \in \rho(A)$ one has*

$$\|R(\lambda,A)\| \geq \operatorname{dist}(\lambda,\sigma(A))^{-1}.$$

(c) **The topological boundary of the spectrum belongs to the approximative point spectrum.** *One has*

$$\partial\sigma(A) \subseteq \sigma_{ap}(A).$$

*Proof.* (a) Let $\lambda \in \rho(A)$ and $\mu \in \mathbb{K}$ such that $|\mu - \lambda| < 1/\|R(\lambda,A)\|$. Then

$$\mu - A = \mu - \lambda + \lambda - A = ((\mu - \lambda)R(\lambda,A) + I)(\lambda - A)$$

is boundedly invertible by the Neumann series, that is, $\mu \in \rho(A)$. More precisely, the Neumann series yields

$$R(\mu,A) = \sum_{k=0}^{\infty} (-1)^k (\mu - \lambda)^k R(\lambda,A)^{k+1},$$

so that the resolvent can be locally developed into a power series. As a consequence, the resolvent is analytic.

Assertion (b) follows directly from (a).

(c) If $\lambda \in \partial\sigma(A)$, then there exists $(\lambda_n) \subseteq \rho(A)$ such that $\lim_{n\to\infty} \lambda_n = \lambda$. By assertion (b), $\lim_{n\to\infty} \|R(\lambda_n, A)\| = \infty$. By the definition of the operator norm, there exists a sequence $(y_n) \subseteq X$, $\|y_n\| = 1$, such that

$$\lim_{n\to\infty} \|R(\lambda_n, A)y_n\| = \infty.$$

Put $x_n := \frac{R(\lambda_n,A)y_n}{\|R(\lambda_n,A)y_n\|}$, so that $x_n \in \operatorname{dom} A$ and $\|x_n\| = 1$. Then

$$\lambda x_n - Ax_n = (\lambda - \lambda_n)x_n + \frac{y_n}{\|R(\lambda_n, A)y_n\|} \to 0 \quad (n \to \infty).$$

As a consequence, $\lambda \in \sigma_{ap}(A)$.

**Remark 5.4.** One may also employ the resolvent identity in order to prove that the resolvent is analytic; but in this case one should at least prove its continuity.

Every bounded, linear operator on a Banach space is closed, as one easily shows. The following lemma therefore is additional information compared to Lemma 5.3.

**Lemma 5.5 (The spectrum of a bounded operator).** *Let $T \in \mathscr{L}(X)$ be a bounded, linear operator on a Banach space $X$. Then:*

(a) *For every $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$ one has $\lambda \in \rho(T)$ and*

$$R(\lambda, T) = \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} T^k.$$

*In particular, the spectrum $\sigma(T)$ is contained in the closed disk $\bar{B}(0, \|T\|)$.*

(b) *The spectrum $\sigma(T)$ is compact and, if the Banach space is complex and nontrivial, nonempty.*

*Proof.* (a) Use the identity

$$\lambda - T = \lambda(I - \frac{T}{\lambda})$$

and the Neumann series.

(b) By assertion (a), the spectrum is bounded and, by Lemma 5.3, it is also closed. Hence, the spectrum $\sigma(T)$ is compact.

If $X$ is a complex Banach space and if $\sigma(T)$ was empty, then, by Lemma 5.3, the resolvent is an entire function. On the other hand, by assertion (a),

$$\lim_{|\lambda|\to\infty} \|R(\lambda, T)\| = 0,$$

so that the resolvent is a bounded function which vanishes at infinity. By Liouville's theorem, this implies $R(\lambda, T) \equiv 0$, which is only possible if $X = \{0\}$ is the trivial space.

**Remark 5.6.** In fact, $\lambda \in \rho(T)$ as soon as

$$|\lambda| > \liminf_{n\to\infty} \|T^n\|^{\frac{1}{n}} =: r(T).$$

The number $r(T) \geq 0$ is called the **spectral radius** of $T$. It follows from complex function theory (if $X$ is a complex Banach space) that $r = r(T)$ is actually the smallest radius such that the spectrum of $T$ is contained in the disk $\{\lambda \in \mathbb{C} : |\lambda| \leq r\}$, whence the name.

Let $(A, \operatorname{dom} A)$ be a densely defined, linear operator between two Banach spaces $X$ and $Y$. We defined the **adjoint operator** or **dual operator** $(A', \operatorname{dom} A')$ from $Y'$ into $X'$ by

$$\operatorname{dom} A' := \{y' \in Y' : \exists x' \in X' \forall x \in \operatorname{dom} A : \langle x', x\rangle_{X',X} = \langle y', Ax\rangle_{Y',Y}\} \text{ and}$$
$$A'y' := x'.$$

In other words, $y' \in \operatorname{dom} A'$ if and only if the composition $y' \circ A$, which is initially only defined on $\operatorname{dom} A \subseteq X$, extends to a bounded a linear functional on $X$, and then $A'y'$ is equal to this continuous extension. At this point it is convenient to assume that $A$ is densely defined, because then the composition $y' \circ A$ admits at most one bounded, linear extension and $A'$ is a linear operator. For every $y' \in \operatorname{dom} A$ and every $x \in \operatorname{dom} A$,

$$\langle y', Ax\rangle_{Y',Y} = \langle A'y', x\rangle_{X',X}.$$

If $T : X \to Y$ is a bounded, linear operator between two Banach spaces $X$ and $Y$, then, for *every* $y' \in Y'$, the composition $y' \circ T$ is bounded on $X$. The adjoint operator $T'$ is therefore defined on $Y'$, that is, its domain $\operatorname{dom} T'$ is equal to $Y'$. For every $y' \in Y'$ and every $x \in X$,

$$\langle y', Tx\rangle_{Y',Y} = \langle T'y', x\rangle_{X',X}.$$

**Lemma 5.7.** *For every bounded, linear operator $T \in \mathscr{L}(X,Y)$, the adjoint $T' : Y' \to X'$ is bounded and $\|T\| = \|T'\|$.*

*Proof.* For every $y' \in Y'$,

$$\|T'y'\| = \sup_{\|x\|\leq 1} |\langle T'y', x\rangle| = \sup_{\|x\|\leq 1} |\langle y', Tx\rangle| \leq \|T\| \, \|y'\|,$$

which proves that $T'$ is bounded and that $\|T'\| \leq \|T\|$. On the other hand, by Hahn-Banach (Corollary 3.8 of Chapter 3),

$$\begin{aligned}
\|T'\| &= \sup_{\|y'\|\leq 1} \|T'y'\| \\
&= \sup_{\|y'\|\leq 1} \sup_{\|x\|\leq 1} |\langle T'y', x\rangle| \\
&= \sup_{\|x\|\leq 1} \sup_{\|y'\|\leq 1} |\langle y', Tx\rangle| \\
&= \sup_{\|x\|\leq 1} \|Tx\| \\
&= \|T\|,
\end{aligned}$$

and the claim is proved.

Let $X$ be a Banach space and $X'$ its dual. For every subset $M \subseteq X$ we define the **annihilator**

$$M^{\perp} := \{x' \in X' : \langle x', x\rangle = 0 \,\forall x \in M\}.$$

For every subset $M' \subseteq X'$, we define the **preannihilator**

$$M'_{\perp} := \{x \in X : \langle x', x\rangle = 0 \,\forall x' \in M'\}.$$

It is easy to show that $M^{\perp}$ and $M'_{\perp}$ are closed linear subspaces of $X'$ and $X$, respectively.

**Lemma 5.8.** *Let $X$ be a Banach space and let $(A, \mathrm{dom} A)$ be a closed, linear operator on $X$. Then:*

(a) $(\mathrm{ran} A)^{\perp} = \ker A'$.

(b) $\overline{\mathrm{ran} A} = (\ker A')_{\perp}$.

(c) $(\ker A)^{\perp} \supseteq \overline{\mathrm{ran} A'}$

(d) $\ker A = (\mathrm{ran} A')_{\perp}$.

*Proof.* In order to prove (a), we observe

$$\begin{aligned}
x' \in (\mathrm{ran} A)^{\perp} &\Leftrightarrow \forall x \in X : \langle x', Ax\rangle = 0 \\
&\Leftrightarrow x' \in \mathrm{dom} A' \text{ and } \forall x \in X : \langle A'x', x\rangle = 0 \\
&\Leftrightarrow x' \in \mathrm{dom} A' \text{ and } A'x' = 0 \\
&\Leftrightarrow x' \in \ker A'.
\end{aligned}$$

(b) If $x \in \mathrm{ran} A$, $x = Ay$ for some $y \in \mathrm{dom} A$, and if $x' \in \ker A'$, then

$$\langle x', x\rangle = \langle x', Ay\rangle = \langle A'x', y\rangle = 0.$$

Hence, $\mathrm{ran} A \subseteq (\ker A')_{\perp}$, and since the latter space is closed, we obtain $\overline{\mathrm{ran} A} \subseteq (\ker A')_{\perp}$. Assume that the inclusion is strict. Then there exists $x_0 \in (\ker A')_{\perp}$ which does not belong to $\overline{\mathrm{ran} A}$. By Hahn-Banach (Corollary 3.10 of Chapter 3), there exist $x' \in X'$ such that

$$x'|_{\operatorname{ran}A} = 0 \text{ and } \langle x', x_0 \rangle \neq 0. \tag{5.1}$$

The first equality means that $\langle x', Ax \rangle = 0$ for every $x \in \operatorname{dom} A$. Hence, $x' \in \operatorname{dom} A'$ and $A'x' = 0$, that is, $x' \in \ker A'$. But since $x_0$ is in the preannihilator of $\ker A'$, this implies $\langle x', x_0 \rangle = 0$, and this is a contradiction to (5.1). Hence, we have proved (b).

(c) If $x' \in \operatorname{ran} A'$, $x' = A'y'$ for some $y' \in \operatorname{dom} A'$, and if $x \in \ker A$, then

$$\langle x', x \rangle = \langle A'y', x \rangle = \langle y', Ax \rangle = 0.$$

This implies $\operatorname{ran} A' \subseteq (\ker A)^{\perp}$, and since the latter space is closed, we obtain (c).

(d) Similarly as in (a), we observe

$$\begin{aligned} x \in \ker A &\Leftrightarrow x \in \operatorname{dom} A \text{ and } Ax = 0 \\ &\Leftrightarrow x \in \operatorname{dom} A \text{ and } \forall x' \in X' : \langle x', Ax \rangle = 0 \\ &\Leftrightarrow x \in \operatorname{dom} A \text{ and } \forall x' \in \operatorname{dom} A' : \langle A'x', x \rangle = 0 \\ &\Leftrightarrow x \in (\operatorname{ran} A')_{\perp}. \end{aligned}$$

**Lemma 5.9 (Spectrum of the adjoint operator).** *Let $(A, \operatorname{dom} A)$ be a closed, densely defined, linear operator between two Banach spaces $X$ and $Y$. Then:*

(a) *The adjoint operator $(A', \operatorname{dom} A')$ is closed from $Y'$ into $X'$.*

(b) *If $X = Y$, then for every $\lambda \in \mathbb{K}$ one has $\operatorname{dom}(\lambda - A)' = \operatorname{dom} A'$ and $(\lambda - A)' = \lambda - A'$. If in addition $\lambda - A$ is injective and has dense range, then*

$$((\lambda - A)^{-1})' = (\lambda - A')^{-1}.$$

(c) *If $X = Y$, then $\sigma(A) = \sigma(A')$ and for every $\lambda \in \rho(A)$*

$$R(\lambda, A)' = R(\lambda, A').$$

*Proof.* (a) Let $(y'_n)$ be any sequence in $\operatorname{dom} A'$ such that $y'_n \to y'$ in $Y'$ and $A'y'_n \to x'$ in $X'$. Then, for every $x \in \operatorname{dom} A$,

$$\begin{aligned} \langle x', x \rangle_{X',X} &= \lim_n \langle A'y'_n, x \rangle_{X',X} \\ &= \lim_n \langle y'_n, Ax \rangle_{Y',Y} \\ &= \langle y', Ax \rangle_{Y',Y}. \end{aligned}$$

By definition of the adjoint operator, this equality implies $y' \in \operatorname{dom} A'$ and $A'y' = x'$. As a consequence, $(A', \operatorname{dom} A')$ is closed.

(b) The first part of this assertion is an exercise. For the second part, we consider without loss of generality the operator $A$ instead of $\lambda - A$. So assume that $A$ is injective and has dense range. Then the inverse $A^{-1}$ is densely defined and therefore its adjoint $(A^{-1})'$ is well defined. On the other hand, by Lemma 5.8 (a), the operator $A'$ is injective, and therefore $(A')^{-1}$ is well defined.

Now, for every $x' \in \mathrm{dom}A'$ and for every $x \in \mathrm{ran}A = \mathrm{dom}A^{-1}$ one has $A^{-1}x \in \mathrm{dom}A$ and

$$\langle A'x', A^{-1}x\rangle = \langle x', AA^{-1}x\rangle = \langle x', x\rangle.$$

Hence, for every $x' \in \mathrm{dom}A'$ one has $A'x' \in \mathrm{dom}\,(A^{-1})'$ and $(A^{-1})'A'x' = x'$.

Conversely, for every $x' \in \mathrm{ran}\,(A^{-1})'$ and every $x \in \mathrm{dom}A$ one has $Ax \in \mathrm{dom}A^{-1}$ and

$$\langle (A^{-1})'x', Ax\rangle = \langle x', A^{-1}Ax\rangle = \langle x', x\rangle.$$

Hence, for every $x' \in \mathrm{ran}\,(A^{-1})'$ one has $(A^{-1})'x' \in \mathrm{dom}A'$ and $A'(A^{-1})'x' = x'$.

We have proved that $(A^{-1})'$ is both a left-inverse and a right inverse of $A'$, and hence $(A')^{-1} = (A^{-1})'$.

(c) Let $\lambda \in \rho(A)$. Then, be the preceding step, $R(\lambda, A)'$ is a bounded inverse of $\lambda - A'$, and hence $\lambda \in \rho(A')$.

Conversely, let $\lambda \in \rho(A')$. By Lemma 5.8 (a) and (d), $\lambda - A$ is injective and has dense range. For every $x \in \mathrm{ran}\,(\lambda - A)$ and for every $x' \in X'$,

$$\begin{aligned} |\langle x', (\lambda - A)^{-1}x\rangle| &= |\langle((\lambda - A)^{-1})'x', x\rangle| \\ &= |\langle R(\lambda, A')x', x\rangle| \\ &\leq \|R(\lambda, A')\|\,\|x'\|\,\|x\|, \end{aligned}$$

so that $(\lambda - A)^{-1}$ is bounded. Since this operator is closed and densely defined, it follows that $\lambda \in \rho(A)$.

**Lemma 5.10.** *For every linear operator* $(A, \mathrm{dom}A)$ *on* $X$ *one has*

$$\sigma_r(A) = \sigma_p(A').$$

*Proof.* Let $\lambda \in \sigma_r(A)$. Then, by definition of the residual spectrum, $\mathrm{ran}(\lambda - A)$ is not dense in $X$. By the Hahn-Banach theorem (see in particular Corollary 3.10), there exists a bounded, linear functional $x' \in X' \setminus \{0\}$ which vanishes on $\mathrm{ran}(\lambda - A)$, that is,

$$\langle x', \lambda x - Ax\rangle = 0 \text{ for every } x \in \mathrm{dom}A.$$

In other words, $(\mathrm{ran}(\lambda - A))^{\perp} \neq \{0\}$. By Lemma 5.8 (a), this means $\ker(\lambda - A') \neq \{0\}$, or, by definition of the point spectrum, $\lambda \in \sigma_p(A')$.

Conversely, if $\lambda \in \sigma_p(A')$, then $\ker(\lambda - A') \neq \{0\}$. This implies $(\ker(\lambda - A'))_{\perp} \neq X$. By Lemma 5.8 (b), this means that $\mathrm{ran}(\lambda - A)$ is not dense in $X$. Hence, $\lambda \in \sigma_r(A)$.

## 5.2 Compact operators

A linear operator $T : X \to Y$ between two Banach spaces $X$ and $Y$ is called a **compact operator** if $TB(0,1)$ is relatively compact in $Y$. The set of all compact linear operators from $X$ into $Y$ is denoted by $\mathscr{K}(X,Y)$. We write $\mathscr{K}(X) := \mathscr{K}(X,X)$.

**Remark 5.11.** A linear operator $T : X \to Y$ is compact if and only if for every sequence $(x_n) \subseteq B(0,1)$ there exists a subsequence (again denoted by $(x_n)$) such that $(Tx_n)$ is convergent (or Cauchy).

Since relatively compact subsets of normed spaces are necessarily bounded, every compact operator is bounded.

**Lemma 5.12.** *Let $X$, $Y$, $Z$ be Banach spaces. Then:*

(a) *The set $\mathscr{K}(X,Y)$ is a closed linear subspace of $\mathscr{L}(X,Y)$.*

(b) *If $T \in \mathscr{K}(X,Y)$ and $S \in \mathscr{L}(Y,Z)$, then $ST \in \mathscr{K}(X,Z)$.*

(c) *If $T \in \mathscr{L}(X,Y)$ and $S \in \mathscr{K}(Y,Z)$, then $ST \in \mathscr{K}(X,Z)$.*

(d) *The set $\mathscr{K}(X)$ is a closed, two-sided ideal in $\mathscr{L}(X)$.*

*Proof.* (a) If $T$, $S \in \mathscr{K}(X,Y)$, $\lambda \in \mathbb{K}$, then clearly $\lambda T \in \mathscr{K}(X,Y)$. Moreover, if $(x_n) \subseteq B(0,1)$ is any sequence, then we can choose a subsequence (again denoted by $(x_n)$) such that $(Tx_n)$ converges. From this subsequence, we extract another subsequence (again denoted by $(x_n)$) such that $(Sx_n)$ converges. Then $(Tx_n + Sx_n)$ converges, and therefore $T + S \in \mathscr{K}(X,Y)$. Hence, $\mathscr{K}(X,Y)$ is a linear subspace of $\mathscr{L}(X,Y)$.

In order to see that $\mathscr{K}(X,Y)$ is closed in $\mathscr{L}(X,Y)$, let $(T_n) \subseteq \mathscr{K}(X,Y)$ be convergent to some element in $T \in \mathscr{L}(X,Y)$. Let $(x_j) \subseteq B(0,1)$ be any sequence. A diagonal sequence argument implies that we can choose a subsequence (again denoted by $(x_j)$) such that

$$\lim_{j\to\infty} T_n x_j \text{ exists for every } n \in \mathbb{N}.$$

Let $\varepsilon > 0$ be arbitrary, and choose $n \in \mathbb{N}$ so large such that $\|T - T_n\| < \varepsilon$. Choose $j_0 \in \mathbb{N}$ so large that $\|T_n x_j - T_n x_k\| < \varepsilon$ for every $j, k \geq j_0$. Then, for every $j, k \geq j_0$,

$$\|Tx_j - Tx_k\| \leq \|Tx_j - T_n x_j\| + \|T_n x_j - T_n x_k\| + \|T_n x_k - Tx_k\| < 3\varepsilon.$$

Hence, $(Tx_j)$ is a Cauchy sequence. Since $Y$ is complete, $(Tx_j)$ is convergent. As a consequence, for every sequence $(x_j) \subseteq B(0,1)$ we have extracted a subsequence (again denoted by $(x_j)$) such that $(Tx_j)$ converges. This means that $T \in \mathscr{K}(X,Y)$. Hence, $\mathscr{K}(X,Y)$ is closed in $\mathscr{L}(X,Y)$.

(b), (c) Let $T \in \mathscr{L}(X,Y)$ and $S \in \mathscr{L}(Y,Z)$. If $T$ is compact, then $TB(0,1)$ is relatively compact, and since $S$ is continuous, $STB(0,1)$ is relatively compact in $Z$ by Lemma 0.19 of Chapter 0. Hence, $ST \in \mathscr{K}(X,Z)$. If on the other hand $T$ is only bounded and $S$ is compact, then $TB(0,1)$ is bounded in $Y$, and therefore $STB(0,1)$ is relatively compact in $Z$, that is, $ST \in \mathscr{K}(X,Z)$.

(d) This is an immediate consequence of (a), (b) and (c).

**Lemma 5.13.** *Let X, Y be Banach spaces. Then:*

(a) *If $T \in \mathscr{L}(X,Y)$ has* **finite rank***, that is, if* $\dim \operatorname{ran} T < \infty$*, then* $T \in \mathscr{K}(X,Y)$.

(b) *If $(T_n) \subseteq \mathscr{K}(X,Y)$ is a uniformly convergent sequence of finite rank operators, then $T := \lim_{n\to\infty} T_n \in \mathscr{K}(X,Y)$.*

*Proof.* Assertion (a) follows from the Theorem of Heine-Borel, while (b) is a consequence of Lemma 5.12.

**Example 5.14 (Rank-1-operator).** For every $x' \in X'$ and $y \in Y$ we may define the operator $T : X \to Y$ by

$$Tx := \langle x', x\rangle\, y \quad (x \in X).$$

Then $T$ has rank 1 (unless $x' = 0$ or $y = 0$ in which case $T = 0$), and it is therefore a compact operator. Operators of the form above are also denoted by $x' \otimes y$. Every rank-1-operator is of this form.

**Lemma 5.15.** *A Banach space X is finite dimensional if and only if the identity operator $I \in \mathscr{L}(X)$ is compact.*

*Proof.* This is an immediate consequence of Theorem 1.15 of Chapter 1 which itself was a consequence of the Lemma of Riesz (Lemma 1.14).

A difficult problem is in general to decide which operators are compact. By the very definition of compact operators, it is thus important to know which subsets of (infinite dimensional) Banach spaces are relatively compact. Boundedness of the subset alone does not suffice as the Lemma of Riesz shows (see also the preceeding lemma). In the case when the underlying Banach space is $C(K)$ ($K$ a compact metric space) we have already seen a satisfactory characterization of relatively compact subsets; see the Theorem of Arzela-Ascoli (Theorem 1.36).

**Example 5.16 (Sobolev embedding).** Consider the embedding $J : W^{1,p}(a,b) \to C([a,b])$ from Example 4.20 of Chapter 4. The closed graph theorem showed that $J$ is bounded, i.e. there exists $C \geq 0$ such that

$$\|u\|_\infty \leq C\, \|u\|_{W^{1,p}}, \quad u \in W^{1,p}(a,b).$$

We can show in addition that the embedding is compact if $p > 1$. Let

$$M := \{u \in W^{1,p}(a,b) : \|u\|_{W^{1,p}} < 1\} = JB(0,1) \subseteq C([a,b])$$

be the image of the unit ball under $J$. By boundedness of $J$, $M$ is bounded in $C([a,b])$. Moreover, by Hölder's inequality (we assume $p > 1$), for every $t$, $s \in [a,b]$ ($t \geq s$)and every $u \in M$,

$$|u(t) - u(s)| = |\int_s^t u'(r)\, dr| \leq \int_s^t |u'(r)|\, dr \leq \|u'\|_p\, (t-s)^{\frac{p-1}{p}} \leq (t-s)^{\frac{p-1}{p}}.$$

This implies that $M$ is equicontinuous if $p>1$ (choose for every $\varepsilon>0$ the $\delta$ equal to $\varepsilon^{\frac{p}{p-1}}$ in order to check equicontinuity).

By the Arzela-Ascoli Theorem (Theorem 1.36), $M$ is relatively compact in $C([a,b])$, and therefore the embedding $W^{1,p}(a,b) \hookrightarrow C([a,b])$ is compact if $p>1$.

**Exercise 5.17 (Sobolev embedding)** *Show that the embedding* $W^{1,1}(a,b) \hookrightarrow C([a,b])$ *is not compact.*

**Exercise 5.18 (Multiplication operators in sequence spaces)** *Let* $X = l^p$ *(*$1 \leq p < \infty$*) or let* $X = c_0$*. Let* $m \in l^\infty$ *and define the associated* **multiplication operator** $M \in \mathscr{L}(X)$ *by*

$$Mx = M(x_n) = (m_n x_n), \quad x \in X.$$

*Show that M is compact if and only if* $m \in c_0$*.*

Hint: *Use Lemma 5.13.*

**Exercise 5.19 (Kernel operators)** *Let* $\Omega \subseteq \mathbb{R}^n$ *be a compact (!) set. Let* $k \in C(\Omega \times \Omega)$*, and define the associated* **kernel operator** $K \in \mathscr{L}(C(\Omega))$ *by*

$$Kf(t) = \int_\Omega k(t,s) f(s)\, ds, \quad t \in \Omega,\, f \in C(\Omega).$$

*Then K is compact.*

**Theorem 5.20 (Schauder).** *An operator* $T \in \mathscr{L}(X,Y)$ *is compact if and only if* $T' \in \mathscr{L}(Y',X')$ *is compact.*

*Proof.* Assume that $T \in \mathscr{K}(X,Y)$, and let $K := \overline{TB_X(0,1)} \subseteq Y$. Then $K$ is compact. Let $M := B_{Y'}(0,1)$ be considered as a subset of $C(K)$. Then clearly $M$ is bounded, and it is not difficult to see that $M$ is also equicontinuous. By the theorem of Arzela-Ascoli, $M$ is relatively compact in $C(K)$. This means that for every sequence $(y_n') \in B_{Y'}(0,1)$ there exists a convergent subsequence (convergent in $C(K)$!). If we denote this subsequence again by $(y_n')$, then we obtain

$$0 = \lim_{n,m\to\infty} \|y_n' - y_m'\|_{C(K)} \geq \lim_{n,m\to\infty} \sup_{\|x\|\leq 1} |\langle y_n' - y_m', Tx\rangle| = \lim_{n,m\to\infty} \|T'y_n' - T'y_m'\|_{X'},$$

which just means that $T'$ is compact.

Assume on the other hand that $T' \in \mathscr{K}(Y',X')$. By what we have just proved, this implies $T'' \in \mathscr{K}(X'',Y'')$. Hence, if $(x_n) \in B_X(0,1)$ is any sequence, then there exists a subsequence (again denoted by $(x_n)$) such that $(T''x_n)$ is convergent in $Y''$ (note that we have considered $(x_n)$ also as a sequence in $X''$ via the embedding $J$). However, $T''x_n = Tx_n$, and the claim is proved.

**Theorem 5.21 (Riesz-Schauder).** *Let X be a Banach space, and* $T \in \mathscr{K}(X)$*. Then:*

(a) $\ker(I-T)$ *is finite dimensional.*

(b) $\operatorname{ran}(I-T)$ *is closed and* $\operatorname{ran}(I-T) = \ker(I-T')_\perp$.

(c) $\ker(I-T) = \{0\}$ *if and only if* $\operatorname{ran}(I-T) = X$.

(d) $\dim\ker(I-T) = \dim\ker(I-T') = \dim(X/\operatorname{ran}(I-T))$.

An immediate consequence of the Riesz-Schauder Theorem is Fredholm's alternative.

**Corollary 5.22 (Fredholm alternative).** *Let $X$ be a Banach space, and $T \in \mathscr{K}(X)$. Then, either for every $y \in X$ the equation*

$$x - Tx = y, \tag{5.2}$$

*admits a solution $x \in X$, and in this case the solution $x$ is unique, or the homogeneous equation*

$$x - Tx = 0$$

*has a a finite number of linearly independent solutions $(x_i)_{1 \le i \le n}$ and the equation* (5.2) *has a solution if and only if $y$ satisfies $n$ equations of orthogonality $\langle x_i', y\rangle = 0$, where the $x_i' \in \ker(I-T')$ are linearly independent.*

**Remark 5.23.** If $T \in \mathscr{K}(X)$, then, by property (c) of Theorem 5.21, $I-T$ is injective if and only if $I-T$ is surjective. In finite dimensions, this property of linear mappings is well-known. This property of operators of the form $I-T$ with $T$ compact is however not shared by arbitrary bounded operators on infinite-dimensional Banach spaces. For example, the left-shift $L$ on $l^p(\mathbb{N})$ defined by $Lx = L(x_n) := (x_{n+1})$ is surjective but not injective.

**Remark 5.24.** An operator $S \in \mathscr{L}(X,Y)$ such that $\ker S$ is finite dimensional and such that $\operatorname{ran} S$ is closed and has finite codimension (that is, $\dim(X/\operatorname{ran} S) < \infty$) is called a **Fredholm operator**, and

$$\operatorname{ind} S := \dim\ker S - \dim(X/\operatorname{ran} S)$$

is called the **Fredholm index** of $S$. By Theorem 5.21, $S = I - T \in \mathscr{L}(X)$ is a Fredholm operator of Fredholm index 0 if $T \in \mathscr{K}(X)$.

*Proof (of Theorem 5.21).* (a) On $\ker(I-T)$ we have $T = I$, and since $T$ is compact, $\ker(I-T)$ must be finite dimensional.

(b) Let $(x_n) \subseteq X$ be such that $u_n := x_n - Tx_n \to u \in X$. We have to show that $u \in \operatorname{ran}(I-T)$. Since $\ker(I-T)$ is finite dimensional, for every $n \in \mathbb{N}$ there exists $y_n \in \ker(I-T)$ such that

$$\operatorname{dist}(x_n, \ker(I-T)) = \|x_n - y_n\|.$$

We show that the sequence $(x_n - y_n)$ is bounded. Otherwise, after extracting a subsequence, we may assume that $\lim_{n\to\infty} \|x_n - y_n\| = \infty$. Putting $w_n := \frac{x_n - y_n}{\|x_n - y_n\|}$, we find that $w_n - Tw_n = u_n/\|x_n - y_n\| \to 0$. After extracting a subsequence, we may assume that $Tw_n \to z$ ($T$ is compact). But then $w_n \to z$, too, and therefore $z \in \ker(I-T)$. On the other hand,

$$\operatorname{dist}(w_n, \ker(I-T)) = \frac{\operatorname{dist}(x_n, \ker(I-T))}{\|x_n - y_n\|} = 1,$$

a contradiction. Hence, the sequence $(x_n - y_n)$ is bounded.

But then, by compactness of $T$, we can extract a subsequence (again denoted by $(x_n - y_n)$) such that $T(x_n - y_n) \to v$. Hence,

$$x_n - y_n = u_n + T(x_n - y_n) \to u + v.$$

We deduce that $T(u+v) = v$, or $u = (u+v) - T(u+v)$, so that $u \in \operatorname{ran}(I-T)$. Hence, $\operatorname{ran}(I-T)$ is closed.

Since the equality $\overline{\operatorname{ran}(I-T)} = \ker(I-T')_{\perp}$ always holds true (Lemma 5.8), we have thus proved (b).

(c) Assume first that $I-T$ is injective, i.e. $\ker(I-T) = \{0\}$. Assume that $X_1 := \operatorname{ran}(I-T) \neq X$, that is, $X_1$ is a closed (by (b)) proper subspace of $X$. Then $T|_{X_1} \in \mathscr{K}(X_1)$, so that, by (b) again, $X_2 = (I-T)X_1$ is a closed subspace of $X_1$. Since $I-T$ is injective, $X_2 \neq X_1$. Iterating this argument and putting $X_n = (I-T)^n X$, we obtain a decreasing sequence $(X_n)$ of closed subspaces of $X$ such that $X_{n+1} \neq X_n$. By the Lemma of Riesz, for every $n \geq 1$ there exists $x_n \in X_n$ such that $\|x_n\| = 1$ and $\operatorname{dist}(x_n, X_{n+1}) \geq \frac{1}{2}$. For every $n > m$ we have

$$Tx_n - Tx_m = -(x_n - Tx_n) + (x_m - Tx_m) + x_n - x_m$$

and

$$-(x_n - Tx_n) + (x_m - Tx_m) + x_n \in X_{m+1}.$$

Hence, $\|Tx_n - Tx_m\| \geq \frac{1}{2}$ whenever $n \neq m$, a contradiction to the assumption that $T$ is compact. Hence, $\operatorname{ran}(I-T) = X$.

Assume now on the other hand that $\operatorname{ran}(I-T) = X$. Then, by Lemma 5.8, $\ker(I-T') = \{0\}$. Since $T'$ is compact by Schauder's theorem, this implies $\operatorname{ran}(I-T') = X'$ by the preceeding step. By Lemma 5.8, $\ker(I-T) = \{0\}$.

(d) For every closed subspace $U$ of $X$ the dual $(X/U)'$ is isomorphic to $U^{\perp}$. In particular, for $U = \operatorname{ran}(I-T)$ one obtains (using Lemma 5.8)

$$\ker(I-T') = (\operatorname{ran}(I-T))^{\perp} \cong (X/\operatorname{ran}(I-T))' \cong X/\operatorname{ran}(I-T).$$

The last isomorphy holds since we know by the first isomorphy that $(X/\operatorname{ran}(I-T))'$ is finite dimensional. In particular,

$$\dim \ker(I-T') = \dim X/\operatorname{ran}(I-T),$$

so that we have proved the second inequality.

It remains to prove that

$$\dim X/\operatorname{ran}(I-T) = \dim \ker(I-T).$$

Since $Tx = x$ for every $x \in \ker(I-T)$, we see that $T$ leaves $\ker(I-T)$ invariant. In particular, the operator

$$\begin{aligned}\tilde{T} : X/\ker(I-T) &\to X/\ker(I-T),\\ x+\ker(I-T) &\mapsto Tx+\ker(I-T),\end{aligned}$$

is well-defined and one easily checks that $\tilde{T}$ is compact since $T$ is compact. By construction, $\ker(I-\tilde{T}) = \{0\}$ so that, by (c), $\operatorname{ran}(I-\tilde{T}) = X/\ker(I-T)$. This means that for every $y \in X$ there exists $x \in X$ and $x_0 \in \ker(I-T)$ such that

$$(I-T)x = y - x_0,$$

or

$$y = (I-T)x + x_0 =: x_1 + x_0.$$

In particular, every $y \in X$ can be written as a sum $x_1 + x_0$ of an element $x_1 \in \operatorname{ran}(I-T)$ and an element $x_0 \in \ker(I-T)$. Hence,

$$\dim\ker(I-T') = \dim X/\operatorname{ran}(I-T) \leq \dim\ker(I-T).$$

Replacing $T$ by $T'$ (which is compact by Schauder's theorem), we obtain

$$\dim\ker(I-T'') \leq \dim\ker(I-T') \leq \dim\ker(I-T).$$

On the other hand, since $I-T''$ extends $I-T$, one trivially has

$$\dim\ker(I-T) \leq \dim\ker(I-T'').$$

The claim is proved

**Theorem 5.25 (Spectrum of a compact operator).** *Let $X$ be a Banach space and let $T \in \mathscr{K}(X)$. Then:*

(a) *If $X$ is infinite-dimensional, then $0 \in \sigma(T)$.*

(b) $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.

(c) *Either $\sigma(T)$ is finite or $\sigma(T) \setminus \{0\} = \{\lambda_n : n \in \mathbb{N}\}$ for some sequence $(\lambda_n) \subseteq \mathbb{C}$ such that $\lim_{n\to\infty} \lambda_n = 0$.*

*Proof.* (a) If $0 \in \rho(T)$, then $T^{-1}$ exists and is bounded. Hence, $I = TT^{-1}$ is compact; a contradiction to the assumption that $X$ is infinite dimensional.

(b) Let $\lambda \in \sigma(T) \setminus \{0\}$. If $\lambda \notin \sigma_p(T)$, then $\ker(\lambda - T) = \{0\}$. By the Riesz-Schauder Theorem (Theorem 5.21), this implies $\operatorname{ran}(\lambda - T) = X$ so that $\lambda - T$ is bijective; a contradiction to the assumption $\lambda \in \sigma(T)$.

(c) It suffices to prove that $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq R\}$ is finite for every $R > 0$. If this was not the case, then we find a sequence $(\lambda_n) \subseteq \sigma(T) \setminus \{0\}$ such that $\lambda_n \neq \lambda_m$ for $n \neq m$ and $|\lambda_n| \geq R > 0$. By (b), for every $n \in \mathbb{N}$ there exists $x_n \in X \setminus \{0\}$ such that $\lambda_n x_n - Tx_n = 0$. Note that the family $(x_n)$ are linearly independent. Otherwise, we find a smallest $n \in \mathbb{N}$ such that the family $(x_i)_{1\leq i\leq n}$ is linearly independent, but

$x_{n+1} = \sum_{i=1}^n \alpha_i x_i$ for some scalars $\alpha_i$. Then

$$\sum_{i=1}^n \alpha_i \lambda_{n+1} x_i = \lambda_{n+1} x_{n+1} = T x_{n+1} = \sum_{i=1}^n \alpha_i \lambda_i x_i,$$

and this implies $\alpha_i(\lambda_{n+1} - \lambda_i) = 0$ for every $1 \leq i \leq n$. Since $\lambda_{n+1} \neq \lambda_i$ for $1 \leq i \leq n$, we obtain $\alpha_i = 0$; a contradiction to $x_{n+1} \neq 0$. Let $X_n := \operatorname{span}\{x_i : 1 \leq i \leq n\}$. Then $(X_n)$ is an increasing sequence of closed subspaces of $X$ such that $X_n \neq X_{n+1}$ (the latter by linear independence of the vectors $x_n$). By the Lemma of Riesz, for every $n \geq 2$ there exists $y_n \in X_n$ such that $\|y_n\| = 1$ and $\operatorname{dist}(y_n, X_{n-1}) \geq \frac{1}{2}$. Then, for every $n > m \geq 2$,

$$\begin{aligned} \|Ty_n - Ty_m\| &= \| - (\lambda_n y_n - Ty_n) + (\lambda_m y_m - Ty_m) + \lambda_n y_n - \lambda_m y_m\| \\ &\geq \operatorname{dist}(\lambda_n y_n, X_{n-1}) \\ &\geq \frac{\lambda_n}{2} \quad \geq \quad \frac{R}{2}. \end{aligned}$$

This is a contradiction to the compactness of $T$, and hence (c) is proved.

## 5.3 Nuclear operators

Let $X$ and $Y$ be two Banach spaces. An operator $T : X \to Y$ is called **nuclear operator**, if there exist sequences $(x_k')$ in $X'$ and $(y_k)$ in $Y$ such that

(i) $\sum_k \|x_k'\| \, \|y_k\| < \infty$, and

(ii) $Tx = \sum_k \langle x_k', x\rangle \, y_k$ for every $x \in X$.

Taking up the notation from Example 5.14 (Rank-1-operators), the condition (ii) is equivalent to

$$T = \sum_k x_k' \otimes y_k,$$

the series being absolutely convergent in $\mathscr{L}(X,Y)$, thanks to condition (i). In particular, every nuclear operator is bounded. Note that the representation of $T$ in the above form is not unique in the sense that the sequences $(x_k')$ and $(y_k)$ are not uniquely determined by $T$. We denote by $\mathscr{N}(X,Y)$ the space of all nuclear operators from $X$ into $Y$; $\mathscr{N}(X) := \mathscr{N}(X,X)$. The space $\mathscr{N}(X,Y)$ is equipped with the norm

$$\|T\|_{\mathscr{N}} := \inf\{\sum_k \|x_k'\| \, \|y_k\| \, : \, x_k' \in X', \, y_k \in Y, \, T = \sum_k x_k' \otimes y_k\}.$$

**Theorem 5.26.** *Let $W$, $X$, $Y$ and $Z$ be Banach spaces. Then the following are true:*

(a) *The space of nuclear operators $(\mathscr{N}(X,Y), \|\cdot\|_{\mathscr{N}})$ is a Banach space. Moreover, for every $T \in \mathscr{N}(X,Y)$, $\|T\|_{\mathscr{L}} \leq \|T\|_{\mathscr{N}}$.*

(b) *The space of finite rank operators is a dense subspace of $\mathscr{N}(X,Y)$.*

(c) *Every nuclear operator is compact, that is,* $\mathscr{N}(X,Y) \subseteq \mathscr{K}(X,Y)$*. Moreover, the embedding* $\mathscr{N}(X,Y) \to \mathscr{K}(X,Y)$*,* $T \mapsto T$*, is continuous.*

(d) *For every* $R \in \mathscr{L}(W,X)$*,* $S \in \mathscr{N}(X,Y)$ *and* $T \in \mathscr{L}(Y,Z)$ *one has* $TSR \in \mathscr{N}(W,Z)$ *and* $\|TSR\|_{\mathscr{N}} \leq \|T\|_{\mathscr{L}} \|S\|_{\mathscr{N}} \|R\|_{\mathscr{L}}$*. In particular,* $\mathscr{N}(X)$ *is a two-sided ideal in* $\mathscr{L}(X)$*.*

*Proof.* (a) First, one checks easily that $\|\cdot\|_{\mathscr{N}}$ is indeed a norm. Let $T \in \mathscr{N}(X,Y)$. Then, for every representation of the form $T = \sum_k x'_k \otimes y_k$ and for every $x \in X$,

$$\begin{aligned}\|Tx\| &= \|\sum_k \langle x'_k, x\rangle\, y_k\| \\ &\leq \sum_k \|x'_k\|\, \|x\|\, \|y_k\|,\end{aligned}$$

so that

$$\|T\|_{\mathscr{L}} \leq \sum_k \|x'_k\|\, \|y_k\|.$$

Taking the infimum on the right-hand side over all representations of $T$, it follows that $\|T\|_{\mathscr{L}} \leq \|T\|_{\mathscr{N}}$. In particular, $\mathscr{N}(X,Y)$ embeds continuously into $\mathscr{L}(X,Y)$.

In order to show that $\mathscr{N}(X,Y)$ is a Banach space, we use the criterion from Lemma 1.13. So let the series $\sum_n T_n$ be absolutely convergent in $\mathscr{N}(X,Y)$, that is, $\sum_n \|T_n\|_{\mathscr{N}} < \infty$. By the continuous embedding above, the series $\sum_n T_n$ is then absolutely convergent in the space $\mathscr{L}(X,Y)$, which is a Banach space. By Lemma 1.13, the series $\sum_n T_n$ converges in $\mathscr{L}(X,Y)$ to some element $T \in \mathscr{L}(X,Y)$.

Now, by definition of the nuclear norm, for every $n$ there exist representations $T_n = \sum_k x'_{nk} \otimes y_{nk}$ such that $\sum_k \|x'_{nk}\|\, \|y_{nk}\| \leq \|T_n\|_{\mathscr{N}} + \frac{1}{n^2}$. In particular, the double series $\sum_{n,k} x'_{nk} \otimes y_{nk}$ converges absolutely in the sense that $\sum_{nk} \|x'_{nk}\|\, \|y_{nk}\| \leq \sum_n (\|T_n\|_{\mathscr{N}} + \frac{1}{n^2}) < \infty$, and its limit is $T$. Hence, $T$ is nuclear. Finally, $\|T - \sum_{n=1}^m T_n\|_{\mathscr{N}} \leq \sum_{n=m+1}^\infty \sum_k \|x'_{mk}\|\, \|y_{nk}\| \to 0$ as $n \to \infty$, and we have proved that the series $\sum_n T_n$ converges in $\mathscr{N}(X,Y)$. By Lemma 1.13, $\mathscr{N}(X,Y)$ is a Banach space.

(b) Let $T \in \mathscr{N}(X,Y)$. By definition, there exist sequences $(x'_k)_{k\geq 1}$ in $X'$ and $(y_k)_{k\geq 1}$ in $Y$ such that $\sum_{k=1}^\infty \|x'_k\|\, \|y_k\| < \infty$, and $T = \sum_{k=1}^\infty x'_k \otimes y_k$. Setting $T_K := \sum_{k=1}^K x'_k \otimes y_k$, it follows that $T_K$ has finite rank and

$$\|T - T_K\|_{\mathscr{N}} \leq \sum_{k=K+1}^\infty \|x'_k\|\, \|y_k\| \to 0 \text{ as } K \to \infty,$$

which shows that the finite rank operators are dense in $\mathscr{N}(X,Y)$.

(c) We have already shown in part (a) that $\mathscr{N}(X,Y)$ embeds continuously into $\mathscr{L}(X,Y)$. By part (b), the finite rank operators are dense in $\mathscr{N}(X,Y)$. By Lemma 5.13, every bounded finite rank operator is compact, and by Lemma 5.12, the space $\mathscr{K}(X,Y)$ is closed in $\mathscr{L}(X,Y)$. Hence, $\mathscr{N}(X,Y)$ embeds continuously into $\mathscr{K}(X,Y)$, and in particular, every nuclear operator is compact.

(d) Let $S = \sum_k x'_k \otimes y_k$. Then one easily checks that

$$TSR = \sum_k R'x'_k \otimes Ty_k$$

and

$$\|TSR\|_{\mathscr{N}} \leq \sum_k \|R'x'_k\| \, \|Ty_k\| \leq \|R'\|_{\mathscr{L}} \, \|T\|_{\mathscr{L}} \sum_k \|x'_k\| \, \|y_k\|.$$

Taking the infimum over all representations of $S$ on the right-hand side of this inequality and noting that $\|R\| = \|R'\|$, the claim follows.

**Exercise 5.27 (Multiplication operators in sequence spaces)** *Let $X = l^p$ $(1 \leq p < \infty)$ or let $X = c_0$. Let $m \in l^\infty$ and define the associated multiplication operator $M \in \mathscr{L}(X)$ as in Exercise 5.18:*

$$Mx = M(x_n) = (m_n x_n), \quad x \in X.$$

*Show that $M$ is nuclear if and only if $m \in \ell^1$.*

**Example 5.28 (Kernel operators).** Let $k \in C([0,1] \times [0,1])$ and let $K$ be the associated kernel operator on the Banach space $X = L^\infty(0,1)$, that is, $(Kf)(x) := \int_0^1 k(x,y) \, f(y) \, \mathrm{d}y$. Then $K$ is a nuclear operator.

## 5.4 * The mean ergodic theorem

A bounded, linear operator $T$ on a Banach space $X$ is called **powerbounded** if $\sup_{n \geq 0} \|T^n\| < \infty$. Clearly, the spectral radius of a powerbounded linear operator is less than or equal to 1, which implies that its spectrum is contained in the closed unit disk $\bar{\mathbb{D}} := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Here, we are particularly interested in the asymptotic behaviour of orbits of powers of $T$, or, in other words, in the asymptotic behaviour of the discrete, linear dynamical system $(T^n)$.

**Lemma 5.29.** *Let $T \in \mathscr{L}(X)$ be a powerbounded operator. Then:*

(a) *For every $x \in \ker(I-T)$ and every $n \in \mathbb{N}$ one has $T^n x = x$.*

(b) *For every $x \in \overline{\operatorname{ran}(I-T)}$ one has*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n x = 0,$$

*that is, the orbit $(T^n x)$ converges in the Cesaro mean to 0.*

(c) $\ker(I-T) \cap \overline{\operatorname{ran}(I-T)} = \{0\}$.

*Proof.* (a) If $x \in \ker(I-T)$, then $Tx = x$. An iteration gives $T^n x = x$ for every $n \in \mathbb{N}$.
(b) First let $x \in \operatorname{ran}(I-T)$. Then $x = y - Ty$ for some $y \in X$. Hence,

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1} T^n x &= \frac{1}{N}\sum_{n=0}^{N-1} T^n (y - Ty) \\ &= \frac{1}{N}\sum_{n=0}^{N-1} (T^n y - T^{n+1} y) \\ &= \frac{1}{N}(y - T^N y) \\ &\to 0 \text{ as } N \to \infty,\end{aligned}$$

due to the assumption that $T$ is powerbounded. The assumption that $T$ is powerbounded also implies that the Cesaro means $\frac{1}{N}\sum_{n=0}^{N-1} T^n$ are uniformly bounded. A simple $3\varepsilon$-argument implies that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T^n x = 0$$

for every $x \in \overline{\operatorname{ran}(I-T)}$.
(c) If $x \in \ker(I-T) \cap \overline{\operatorname{ran}(I-T)}$, then, by part (a),

$$x = \frac{1}{N}\sum_{n=0}^{N-1} T^n x \text{ for every } N \in \mathbb{N}.$$

By part (b), the right-hand side of this equality converges to 0 as $N \to \infty$. Hence $x = 0$.

**Theorem 5.30 (Mean ergodic theorem).** *Let $T \in \mathscr{L}(X)$ be a powerbounded operator. Then, for every $x \in X$, the following assertions are equivalent:*

(i) *$x \in \ker(I-T) \oplus \overline{\operatorname{ran}(I-T)}$, that is, $x = x_0 + x_1$ for some $x_0 \in \ker(I-T)$ and some $x_1 \in \overline{\operatorname{ran}(I-T)}$.*

(ii) *The limit $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T^n x$ exists in $X$.*

(iii) *The limit $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T^n x$ exists weakly in $X$.*

(iv) *The sequence $(\frac{1}{N}\sum_{n=0}^{N-1} T^n x)$ of Cesaro means has a weakly convergent subsequence.*

*If one of the equivalent conditions (i)–(iv) holds true, then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T^n x = x_0.$$

We say that a sequence $(x_n)$ in a Banach space $X$ **converges in Cesaro mean** to some element $x \in X$ if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} x_n = x.$$

One can prove (exercise!) that if a sequence $(x_n)$ converges in the usual sense to some element $x \in X$, then it also converges in the Cesaro mean to the same element. However, the converse is not true: the sequence $((-1)^n)$ does obviously not converge in $\mathbb{R}$, but

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (-1)^n = \lim_{N\to\infty} \frac{1}{N} \frac{1}{2} (1 + (-1)^{N+1}) = 0,$$

that is, this sequence converges in the Cesaro mean to 0. We also say that the Cesaro average of this sequence is 0.

If one of the equivalent conditions (i)–(iv) in the Mean Ergodic Theorem above holds true, then the final conclusion is that the sequence $(T^n x)$ of iterates of $T$ applied to $x$ converges in Cesaro mean to $x_0$. Note that the sequence $(T^n x)$ need not converge in the usual sense.

*Proof (of Theorem 5.30).* The implication (i)⇒(ii) follows from Lemma 5.29, while the implications (ii)⇒(iii) and (iii)⇒(iv) are trivial. So let us prove the remaining implication (iv)⇒(i). Assume that the sequence $(\frac{1}{N}\sum_{n=0}^{N-1} T^n x)$ admits a weak accumulation point. Then there exists $x_0 \in X$ and an increasing sequence $(N_k)$ in $\mathbb{N}$ such that

$$w - \lim_{k\to\infty} \frac{1}{N_k} \sum_{n=0}^{N_k-1} T^n x = x_0.$$

Since every bounded, linear operator is also weak-weak continuous, this implies

$$\begin{aligned}(I-T)x_0 &= w - \lim_{k\to\infty} \frac{1}{N_k} \sum_{n=0}^{N_k-1} T^n (I-T)x \\ &= w - \lim_{k\to\infty} \frac{1}{N_k} \sum_{n=0}^{N_k-1} (T^n x - T^{n+1} x) \\ &= w - \lim_{k\to\infty} \frac{1}{N_k} (x - T^{N_k} x) \\ &= 0,\end{aligned}$$

so that $x_0 \in \ker(I-T)$. On the other hand, for every $k$ one has

$$\begin{aligned}x - \frac{1}{N_k} \sum_{n=0}^{N_k-1} T^n x &= \frac{1}{N_k} \sum_{n=0}^{N_k-1} (x - T^n x) \\ &= \frac{1}{N_k} \sum_{0}^{N_k-1} \sum_{j=0}^{n-1} T^j (I-T)x \\ &= (I-T)\,\Big[\frac{1}{N_k} \sum_{0}^{N_k-1} \sum_{j=0}^{n-1} T^j x\Big] \in \operatorname{ran}(I-T).\end{aligned}$$

Hence,

$$\begin{aligned} x - x_0 &= x - \text{weak-}\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=0}^{N_k-1} T^n x \\ &= \text{weak-}\lim_{k\to\infty} \Big[x - \frac{1}{N_k} \sum_{n=0}^{N_k-1} T^n x\Big] \\ &=: x_1 \in \overline{\operatorname{ran}(I-T)}, \end{aligned}$$

and we have proved that (i) holds.

**Corollary 5.31 (Mean ergodic theorem in reflexive spaces).** *Let $T \in \mathscr{L}(X)$ be a powerbounded operator on a reflexive Banach space $X$. Then*

$$X = \ker(I-T) \oplus \overline{\operatorname{ran}(I-T)}$$

*and if $P \in \mathscr{L}(X)$ denotes the projection onto $\ker(I-T)$ along $\overline{\operatorname{ran}(I-T)}$, then, for every $x \in X$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n x = Px,$$

*that is, the iterates of $T$ converge strongly, and in the Cesaro mean, to the projection $P$. If $1$ is not an eigenvalue of $T$, then, for every $x \in X$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n x = 0.$$

*Proof.* If suffices to note that for every $x \in X$ the sequence $(\frac{1}{N}\sum_{n=0}^{N-1} T^n x)$ of Cesaro means is bounded in $X$. Since $X$ is assumed to be reflexive, this sequence thus admits a weakly convergent subsequence by Theorem 3.28. The claims thus follow from the Mean Ergodic Theorem (Theorem 5.30).

Since Hilbert spaces are in particular reflexive spaces, we immediately obtain the following corollary, due to von Neumann.

**Corollary 5.32 (von Neumann mean ergodic theorem).** *Let $T$ be a contraction on a Hilbert space $H$. Then, for every $f \in H$, the Cesaro limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n f =: Pf$$

*exists in $H$, $P$ being the projection onto $\ker(I-T)$ along $\overline{\operatorname{ran}(I-T)}$. If $1$ is not an eigenvalue of $T$, then, for every $f \in H$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n f = 0.$$

## Convergence in the Abel mean of powerbounded operators

Let $T \in \mathscr{L}(X)$ be a powerbounded operator, and let $M \geq 0$ be a constant such that $\|T^n\| \leq M$ for every $n \geq 0$. From the Neumann series (see also the short proof of Lemma 5.5 and the Remark 5.6), we obtain for every $\lambda \in \mathbb{K}$ with $|\lambda| > 1$ the estimate

$$\begin{aligned}\|R(\lambda,T)\| &= \| \sum_{n\geq 0} \frac{T^n}{\lambda^{n+1}} \| \\ &\leq M \sum_{n\geq 0} \frac{1}{|\lambda|^{n+1}} \\ &= M \frac{1}{|\lambda|-1}.\end{aligned}$$

In particular,

$$\|(\lambda-1)R(\lambda,T)\| \leq M \text{ for every real } \lambda > 1. \tag{5.3}$$

**Lemma 5.33.** *Let $T \in \mathscr{L}(X)$ be a powerbounded operator. Then:*

(a) *For every $x \in \ker(I-T)$ and every real $\lambda > 1$ one has $(\lambda-1)R(\lambda,T)x = x$.*

(b) *For every $x \in \overline{\operatorname{ran}(I-T)}$ one has $\lim_{\lambda\to 1+}(\lambda-1)R(\lambda,A)x = 0$.*

(c) $\ker(I-T) \cap \overline{\operatorname{ran}(I-T)} = \{0\}$.

*Proof.* (a) Let $x \in \ker(I-T)$. Then

$$0 = x - Tx = -(\lambda-1)x + (\lambda-T)x \text{ for every real } \lambda > 1.$$

Multiplying this equality with $R(\lambda,T)$ yields the claim.

(b) Assume first that $x \in \operatorname{ran}(I-T)$, that is, $x = y - Ty$ for some $y \in X$. Then

$$\begin{aligned}\lim_{\lambda\to 1+}(\lambda-1)R(\lambda,T)x &= \lim_{\lambda\to 1+}(\lambda-1)R(\lambda,T)((1-\lambda)y+\lambda y - Ty) \\ &= \lim_{\lambda\to 1+}\left[(\lambda-1)^2R(\lambda,T)y + (\lambda-1)y\right] \\ &= 0.\end{aligned}$$

The full claim follows from this equality, from the estimate (5.3), and from a simple density argument (compare with Lemma 2.48).

(c) Let $x \in \ker(I-T) \cap \overline{\operatorname{ran}(I-T)}$. Then the previous two points yield

$$x = (\lambda-1)R(\lambda,T)x \text{ for every real } \lambda > 1,$$

and

$$\lim_{\lambda\to 1+}(\lambda-1)R(\lambda,T)x = 0,$$

which is only possible if $x = 0$.

**Theorem 5.34 (Mean ergodic theorem).** *Let $T \in \mathscr{L}(X)$ be a powerbounded operator. Then, for every $x \in X$, the following assertions are equivalent:*

(i) $x \in \ker(I-T) \oplus \overline{\operatorname{ran}(I-T)}$, *that is,* $x = x_0 + x_1$ *for some* $x_0 \in \ker(I-T)$ *and some* $x_1 \in \overline{\operatorname{ran}(I-T)}$.

(ii) *The limit $\lim_{\lambda\to 1+}(\lambda-1)R(\lambda,A)x$ exists strongly (in $X$).*

(iii) *The limit $\lim_{\lambda\to 1+}(\lambda-1)R(\lambda,A)x$ exists weakly.*

(iv) *The net* $((\lambda-1)R(\lambda,A)x)_{\lambda\searrow 1}$ *admits a weakly convergent subsequence in the sense that there exists a sequence* $(\lambda_n)$ *in* $\mathbb{R}$, $\lambda_n \to 1+$, *such that* $((\lambda_n-1)R(\lambda_n,A)x)_n$ *converges weakly.*

(v) *The limit* $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}T^n x$ *exists strongly.*

*If one of the equivalent conditions (i)–(v) holds true, then*

$$\lim_{\lambda\to 1+}(\lambda-1)R(\lambda,A)x = \lim_{n\to\infty}\frac{1}{n}\sum_{0}^{n-1}T^k x = x_0.$$

We say that a sequence $(x_n)$ in a Banach space $X$ **converges in Abel mean** to some element $x\in X$ if the power series $\sum_{n=0}^{\infty}x_n\lambda^n$ converges (absolutely) for every $\lambda\in\mathbb{D}$, and if

$$\lim_{\lambda\to 1-}(1-\lambda)\sum_{n=0}^{\infty}x_n\lambda^n = x.$$

One can prove that if a sequence $(x_n)$ converges in Cesaro mean to some element $x\in X$, then it also converges in the Abel mean to the same element. The converse, however, is not true. In general, we have thus the implications

$(x_n)$ converges in the usual sense to $x\in X$
$\Downarrow$
$(x_n)$ converges in the Cesaro mean to $x\in X$
$\Downarrow$
$(x_n)$ converges in the Abel mean to $x\in X$.

The second Mean Ergodic Theorem (Theorem 5.34) says that the algebraic condition (i) is equivalent to convergence in the Abel mean of the sequence $(T^n x)$ of iterates of $T$ applied to $x$ (condition (iv)), which in turn is equivalent to convergence in the Cesaro mean (condition (v)). Hence, in this special situation, convergence in the Abel mean and in the Cesaro mean are equivalent.

*Proof (of Theorem 5.34).* The implication (i)⇒(ii) follows from the preceding Lemma 5.29, assertions (a) and (b). The lemma also yields the equality $\lim_{\lambda\to 1+}(\lambda-1)R(\lambda,A)x = x_0$.
The implications (ii)⇒(iii) and (iii)⇒(iv) are trivial.
(iv)⇒(i) We assume that there exists $x_0\in X$ and a sequence $(\lambda_n)$ in $\mathbb{R}$, $\lambda_n\to 1+$, such that weak$-\lim_n(\lambda_n-1)R(\lambda_n,A)x = x_0$.Then, for every $x'\in X'$,

$$\begin{aligned}\langle x',x_0\rangle &= \lim_n\langle x',(\lambda_n-1)R(\lambda_n,T)x\rangle\\ &= \lim_n\langle x',(1-\lambda_n+\lambda_n-T+T)(\lambda_n-1)R(\lambda_n,T)x\rangle\\ &= \lim_n\langle x',-(\lambda_n-1)^2R(\lambda_n,T)x+(\lambda_n-1)x+T(\lambda_n-1)R(\lambda_n,T)x\rangle\\ &= \lim_n\langle x',T(\lambda_n-1)R(\lambda_n,T)x\rangle\\ &= \langle x',Tx_0\rangle.\end{aligned}$$

Hence $x_0 = Tx_0$, or, in other words, $x_0\in\ker(I-T)$. It remains to show that $x_1 := x-x_0\in\overline{\operatorname{ran}(I-T)}$. Note that for every $n$ one has

$$\begin{aligned}x-(\lambda_n-1)R(\lambda_n,T)x &= x-(\lambda_n-T+T-1)R(\lambda_n,T)x\\ &= (I-T)R(\lambda_n,T)x\in\operatorname{ran}(I-T).\end{aligned}$$

Hence,

$$\begin{aligned} x_1 &= x - x_0 \\ &= x - \text{weak-}\lim_{n\to\infty}(\lambda_n - 1)R(\lambda_n, T)x \\ &= \text{weak-}\lim_{n\to\infty}[x - (\lambda_n - 1)R(\lambda_n, T)x] \in \overline{\operatorname{ran}(I-T)}, \end{aligned}$$

which proves that (i) holds.

The equivalence (i)⇔(v) follows from the Mean Ergodic Theorem 5.30.

## The mean ergodic theorem for general resolvents

The preceding situation can still be generalized. We now consider a general closed, linear operator $(A, \operatorname{dom} A)$ on a Banach space $X$, and we study the relation between the behaviour of the resolvent of $A$ near the boundary of the spectrum and some algebraic properties of $A$.

**Lemma 5.35.** *Let $(A, \operatorname{dom} A)$ be a closed, linear operator on a Banach space $X$. Let $\lambda_0 \in \mathbb{K}$ be such that there exists a sequence $(\lambda_n)$ in $\rho(A)$ satisfying $\lim_n \lambda_n = \lambda_0$ and $\|(\lambda_n - \lambda_0)R(\lambda_n, A)\| \leq M$ for every $n$ and some constant $M \geq 0$. Then:*

(a) *For every $x \in \ker(\lambda_0 - A)$ one has $(\lambda_n - \lambda_0)R(\lambda_n, A)x = x$ for every $n$.*

(b) *For every $x \in \overline{\operatorname{ran}(\lambda_0 - A)}$ one has $\lim_n(\lambda_n - \lambda_0)R(\lambda_n, A)x = 0$.*

(c) $\ker(\lambda_0 - A) \cap \overline{\operatorname{ran}(\lambda_0 - A)} = \{0\}$.

*Proof.* (a) Let $x \in \ker(\lambda_0 - A)$. Then $x \in \operatorname{dom} A$ and

$$0 = (\lambda_0 - A)x = (\lambda_0 - \lambda_n)x + (\lambda_n - A)x \text{ for every } n.$$

Multiplying this equality with $R(\lambda_n, A)$ yields the claim.

(b) Assume first that $x \in \operatorname{ran}(\lambda_0 - A)$, that is, $x = (\lambda_0 - A)y$ for some $y \in \operatorname{dom} A$. Then

$$\begin{aligned} \lim_n(\lambda_n - \lambda_0)R(\lambda_n, A)x &= \lim_n(\lambda_n - \lambda_0)R(\lambda_n, A)(\lambda_0 - \lambda_n + \lambda_n - A)y \\ &= \lim_n \left[(\lambda_n - \lambda_0)^2 R(\lambda_n, A)y + (\lambda_n - \lambda_0)y\right] \\ &= 0. \end{aligned}$$

The full claim follows from this equality, from the assumption that the sequence $((\lambda_n - \lambda_0)R(\lambda_n, A))_n$ is bounded in $\mathscr{L}(X)$, and from a simple density argument (compare with Lemma 2.48).

(c) Let $x \in \ker(\lambda_0 - A) \cap \overline{\operatorname{ran}(\lambda_0 - A)}$. Then the previous two points give

$$x = (\lambda_n - \lambda_0)R(\lambda_n, A)x \to 0 \text{ as } n \to \infty,$$

that is, $x = 0$.

**Theorem 5.36 (Mean ergodic theorem for resolvents).** *Let $(A, \operatorname{dom} A)$ be a closed, linear operator on a Banach space $X$. Let $\lambda_0 \in \mathbb{K}$ be such that there exists a sequence $(\lambda_n)$ in $\rho(A)$ satisfying $\lim_n \lambda_n = \lambda_0$ and $\|(\lambda_n - \lambda_0)R(\lambda_n, A)\| \leq M$ for every $n$ and some constant $M \geq 0$. Then, for every $x \in X$, the following assertions are equivalent:*

(i) $x \in \ker(\lambda_n - A) \oplus \overline{\operatorname{ran}(\lambda_0 - A)}$, *that is,* $x = x_0 + x_1$ *for some* $x_0 \in \ker(\lambda_0 - A)$ *and some* $x_1 \in \overline{\operatorname{ran}(\lambda_0 - A)}$.

(ii) *The sequence $((\lambda_n - \lambda_0)R(\lambda_n, A)x)_n$ converges strongly (in $X$).*

(iii) *The sequence $((\lambda_n - \lambda_0)R(\lambda_n, A)x)_n$ converges weakly.*

(iv) *The sequence* $((\lambda_n - \lambda_0)R(\lambda_n,A)x)_n$ *admits a weakly convergent subsequence.*

*If one of the equivalent conditions (i)–(iv) holds true, then*

$$\lim_n (\lambda_n - \lambda_0)R(\lambda_n,A)x = x_0.$$

*Proof.* The implication (i)⇒(ii) follows from the preceding Lemma 5.35, assertions (a) and (b). It also yields the equality $\lim_n (\lambda_n - \lambda_0)R(\lambda_n,A)x = x_0$.
The implications (ii)⇒(iii) and (iii)⇒(iv) are trivial.
So let us prove the implication (iv)⇒(i). We assume that $((\lambda_n - \lambda_0)R(\lambda_n,A)x)_n$ admits a weakly convergent subsequence. After passing to a subsequence, if necessary, we may in fact assume that the sequence $((\lambda_n - \lambda_0)R(\lambda_n,A)x)_n$ itself converges weakly, say, to some element $x_0 \in X$. Then, for every $x' \in X'$,

$$\begin{aligned}
\langle x', \lambda_0 x_0\rangle &= \lim_n \langle x', \lambda_0(\lambda_n - \lambda_0)R(\lambda_n,A)x\rangle \\
&= \lim_n \langle x', (\lambda_0 - \lambda_n + \lambda_n - A + A)(\lambda_n - \lambda_0)R(\lambda_n,A)x\rangle \\
&= \lim_n \langle x', (\lambda_0 - \lambda_n)^2 R(\lambda_n,A)x + (\lambda_0 - \lambda_n)x + A(\lambda_n - \lambda_0)R(\lambda_n,A)x\rangle \\
&= \lim_n \langle x', A(\lambda_n - \lambda_0)R(\lambda_n,A)x\rangle \\
&= \langle x', Ax_0\rangle
\end{aligned}$$

Since $A$ is closed, this equality implies $x_0 \in \operatorname{dom} A$ and $\lambda_0 x = Ax$. In other word, $x_0 \in \ker(\lambda_0 - A)$. It remains to show that $x_1 := x - x_0 \in \overline{\operatorname{ran}(\lambda_0 - A)}$. Note that for every $n$ one has $R(\lambda_n,A)x \in \operatorname{dom} A$ and

$$\begin{aligned}
x - (\lambda_n - 1)R(\lambda_n,A)x &= x - (\lambda_n - A + A - 1)R(\lambda_n,A)x \\
&= (I - A)R(\lambda_n,A)x \in \operatorname{ran}(I - A).
\end{aligned}$$

Hence,

$$\begin{aligned}
x_1 &= x - x_0 \\
&= x - \text{weak-}\lim_{n\to\infty} (\lambda_n - 1)R(\lambda_n,A)x \\
&= \text{weak-}\lim_{n\to\infty} [x - (\lambda_n - 1)R(\lambda_n,A)x] \in \overline{\operatorname{ran}(I - A)},
\end{aligned}$$

which proves that (i) holds.

**Corollary 5.37 (Mean ergodic theorem for resolvents in reflexive spaces).** *In addition to the assumption of the preceding Theorem 5.36, assume that the underlying Banach space X is reflexive. Then* $X = \ker(\lambda_0 - A) \oplus \overline{\operatorname{ran}(\lambda_0 - A)}$ *and the for every* $x \in X$ *the limit*

$$\lim_n (\lambda_n - \lambda_0)R(\lambda_n,A)x =: x_0$$

*exists, and the limit* $x_0$ *coincides with the projection of* $x$ *onto* $\ker(\lambda_0 - A)$ *along* $\overline{\operatorname{ran}(\lambda_0 - A)}$.

*Proof.* By assumption, for every $x \in X$, the sequence $((\lambda_n - \lambda_0)R(\lambda_n,A)x)_n$ is bounded. Since $X$ is reflexive and by Theorem 3.28, for every $x \in X$ the sequence $((\lambda_n - \lambda_0)R(\lambda_n,A)x)_n$ admits a weakly convergent subsequence. The claim follows from Theorem 5.36.

# Chapter 6
# Banach algebras

## 6.1 Banach algebras and the theorem of Gelfand

A normed space $A$ is called a **normed algebra** if it is an algebra, and if

$$\|ab\| \leq \|a\| \, \|b\| \text{ for every } a, b \in A.$$

A complete, normed algebra is also called **Banach algebra**.

**Examples 6.1.** 1. Let $X$ be a normed space. Then the space $A = \mathscr{L}(X)$ of all bounded, linear operators on $X$ is a normed algebra for the usual multiplication which is the composition of operators (Lemma 1.26). It is a Banach algebra as soon as $X$ is a Banach space (Lemma 1.27).

2. Let $X$ be a Banach space. Then the space $A = \mathscr{K}(X)$ of all compact, linear operators on $X$ is a Banach algebra. Actually, $\mathscr{K}(X)$ is a closed, two-sided ideal in $\mathscr{L}(X)$.
3. Let $K$ be a compact space. Then $A = C(K)$ is a Banach algebra for the usual (pointwise) multiplication of functions. Similarly, if $\Omega$ is a locally compact space, then the space of continuous functions $\Omega \to \mathbb{K}$ vanishing at infinity, $C_0(\Omega)$, is a Banach algebra. Finally, if $M$ is an arbitrary topological space, then the space of continuous, bounded functions $M \to \mathbb{K}$, $C_b(M)$, is a Banach algebra. All spaces of continuous functions in this example are equipped with the supremum norm.
4. Let $\Omega$ be a measure space. Then $A = L^\infty(\Omega)$ is a Banach algebra for the usual (pointwise) multiplication.
5. Let $A = L^1(\mathbb{R}^N)$ be equipped with the **convolution product**

$$f * g(x) := \int_{\mathbb{R}^N} f(x-y) g(y) \, dy \quad (f, g \in L^1(\mathbb{R}^N), x \in \mathbb{R}^N).$$

Then $A$ is a Banach algebra.

*Proof.* Let $f, g \in L^1(\mathbb{R}^N)$. By Tonelli's theorem,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(x-y)g(y)| \, dy \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(x-y)g(y)| \, dx \, dy$$
$$= \int_{\mathbb{R}^N} |f(x)| \, dx \int_{\mathbb{R}^N} |g(y)| \, dy$$
$$= \|f\|_{L^1} \|g\|_{L^1} < \infty.$$

This inequality first implies that $f * g(x)$ exists for almost every $x \in \mathbb{R}^N$, and second that

$$\int_{\mathbb{R}^N} |f * g(x)| \, dx \leq \|f\|_{L^1} \|g\|_{L^1} < \infty,$$

that is, $f * g \in L^1(\mathbb{R}^N)$. In particular, the convolution product is well-defined. However, the above inequality also implies a particular case of **Young's inequality**

$$\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1},$$

which implies that $L^1(\mathbb{R}^N)$ equipped with the convolution product is a Banach algebra.

6. Let $A = L^1(\mathbb{R}_+)$ be equipped with the **convolution product**

$$f * g(t) := \int_0^t f(t-s)g(s) \, ds \quad (f, g \in L^1(\mathbb{R}_+), t \in \mathbb{R}_+).$$

Then $A$ is a Banach algebra.

7. Let $A$ be a Banach algebra, and let $I \subseteq A$ be a closed, two-sided ideal. Then the factor space $A/I$ is a Banach algebra for the multiplication

$$(a+I) \cdot (b+I) = ab + I \quad (a, b \in A);$$

note that this product is well-defined since $I$ is a two-sided ideal.

A Banach algebra $A$ is **unital** if it admits a neutral element for the multiplication, usually denoted by 1 or by $e$.

**Remark 6.2 (Adjunction of a unit).** Let $A$ be a Banach algebra without unit. Consider the product space

$$\bar{A} := A \times \mathbb{C},$$

equipped with the sum norm. Then $\bar{A}$ is a unital Banach algebra for the multiplication given by

$$(a, \lambda)(b, \mu) := (ab + \mu a + \lambda b, \lambda \mu) \quad ((a, \lambda), (b, \mu) \in \bar{A}).$$

The unit element is the element $(0, 1)$.

Given a unital Banach algebra $A$, we say that an element $a \in A$ is **invertible** (respectively, **left-invertible**, **right-invertible**), if there exists an element $b \in A$ such that

$$ab = ba = 1 \quad (\text{respectively, } ba = 1 \text{ or } ab = 1).$$

If $a$ is invertible, then the element $b \in A$ satisfying $ab = ba = 1$ is uniquely determined. We write $b =: a^{-1}$, and we call $a^{-1}$ the **inverse** of $a$. We define the **resolvent set** of an element $a \in A$ by

$$\rho(a) := \{\lambda \in \mathbb{K} : \lambda - a \text{ is invertible}\},$$

and the **spectrum** by

$$\sigma(a) := \mathbb{K} \setminus \rho(a).$$

For every $\lambda \in \rho(a)$ we write $R(\lambda, a) := (\lambda - a)^{-1}$, and we call $R(\lambda, a)$ the **resolvent** of $a$ at $\lambda$. The function $R(\cdot, a)$ is simply called the resolvent of $a$.

Several of the lemmas on the structure of the resolvent set and the spectrum of a bounded, linear operator on a Banach space, which are stated in the preceding chapter, remain true in the general context of Banach algebras and elements in Banach algebras. We start with the resolvent identity.

**Lemma 6.3 (Resolvent identity).** *Let $A$ be a unital Banach algebra, and $a \in A$. Then, for every $\lambda$, $\mu \in \rho(a)$ one has*

$$R(\lambda, a) - R(\mu, a) = (\mu - \lambda) R(\mu, a) R(\lambda, a).$$

*Proof.* For every $\lambda$, $\mu \in \rho(a)$

$$\mu - \lambda = (\mu - a) - (\lambda - a).$$

Multiplying both sides by $R(\mu, a)$ and $R(\lambda, a)$, one obtains the claim.

**Lemma 6.4 (Neumann series).** *Let $A$ be a unital Banach algebra, and let $a \in A$ be such that $\|a\| < 1$. Then $1 - a$ is invertible, and*

$$(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n,$$

*the series being absolutely convergent in $A$.*

**Lemma 6.5 (The resolvent is analytic).** *Let $A$ be a unital Banach algebra. For every $a \in A$ the resolvent set $\rho(a)$ is open in $\mathbb{K}$ and the resolvent $\rho(a) \to A$, $\lambda \mapsto R(\lambda, a)$ is analytic.*

*Proof.* Let $\lambda \in \rho(a)$ and $\mu \in \mathbb{K}$. Then

$$\mu - a = \mu - \lambda + \lambda - a = ((\mu - \lambda) R(\lambda, a) + I)(\lambda - a),$$

and the right-hand side is invertible if $|\mu - \lambda| < 1/\|R(\lambda, a)\|$ by the Neumann series. Hence, $\rho(a)$ is open in $\mathbb{K}$. The Neumann series (Lemma 6.4) precisely yields

$$R(\mu, a) = \sum_{n=0}^{\infty} (-1)^n R(\lambda, a)^{n+1} (\mu - \lambda)^n,$$

that is, the resolvent $\lambda \mapsto R(\lambda, a)$ can be locally developped into a power series. In other words, the resolvent is analytic.

**Lemma 6.6 (Growth of the resolvent near the spectrum).** *For every $\lambda \in \rho(a)$ one has*

$$\|R(\lambda, a)\| \geq \operatorname{dist}(\lambda, \sigma(a))^{-1}.$$

*Proof.* As we have seen in the proof of the preceding Lemma 6.5, for $\lambda \in \rho(a)$ the condition

$$|\mu - \lambda| \, \|R(\lambda, a)\| < 1$$

implies $\mu \in \rho(a)$. The claim follows.

**Lemma 6.7.** *For every $a \in A$ one has*

$$\{\lambda \in \mathbb{K} : |\lambda| > \|a\|\} \subseteq \rho(a),$$

*and*

$$R(\lambda, a) = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}} \quad (|\lambda| > \|a\|).$$

*Proof.* Use the identity

$$\lambda - a = \lambda (I - \frac{a}{\lambda})$$

and the Neumann series.

**Remark 6.8.** Similarly as in Remark 5.6, we can remark here that $\lambda \in \rho(a)$ as soon as

$$|\lambda| > \liminf_{n \to \infty} \|a^n\|^{\frac{1}{n}} =: r(a).$$

As in the case of bounded, linear operators, the number $r(a) \geq 0$ is called the **spectral radius** of $a$.

**Lemma 6.9.** *Let $A \neq \{0\}$ be a complex, unital Banach algebra. Then for every $a \in A$ the spectrum $\sigma(a)$ is nonempty and compact, and*

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

*Proof.* The compactness of $\sigma(a)$ follows from Lemma 6.5 and 6.7. If $\sigma(a)$ was empty, then, by Lemma 6.5, the resolvent $\lambda \mapsto R(\lambda, a)$ is an entire function. On the other hand, by Lemma 6.7,

$$\lim_{|\lambda| \to \infty} \|R(\lambda, a)\| = 0.$$

By Liouville's theorem, this implies $R(\lambda, a) \equiv 0$, which is only possible if $A = \{0\}$ is the trivial algebra.

**Theorem 6.10 (Gelfand-Mazur).** *Let $A \neq \{0\}$ be a complex, unital Banach algebra such that every element $a \neq 0$ is invertible (that is, $A$ is a* **division algebra***). Then $A = \mathbb{C}$.*

*Proof.* Let $a \in A$. Then, by Lemma 6.9, there exists $\lambda \in \mathbb{C}$ such that $\lambda - a$ is not invertible. By assumption, this implies $\lambda - a = 0$, or, in other words, $a = \lambda$ is a scalar multiple of the unit element.

A (two-sided) ideal $I$ in a Banach algebra is called **maximal ideal** if $I \neq A$ and if there does not exist an other (two-sided) ideal $J$ in $A$ such that $I \subsetneq J \subsetneq A$.

**Lemma 6.11.** *Every ideal in a unital Banach algebra is contained in a maximal ideal.*

*Proof.* Let $I$ be an ideal in a unital Banach algebra $A$ with unit denoted by 1. Define the set $\mathscr{M} := \{J : J \text{ is an ideal in } A \text{ and } I \subseteq J \subsetneq A\}$, and equip it with the order relation $\leq$ given by inclusion: $J_1 \leq J_2 \Leftrightarrow J_1 \subseteq J_2$. Let $\mathscr{J} \subseteq \mathscr{M}$ be a totally ordered subset and define $\bar{J} := \bigcup_{J \in \mathscr{J}} J$. Then clearly $\bar{J}$ is an ideal in $A$ which contains $I$. On the other hand, $\bar{J} \neq A$, since all the ideals $J$ are strictly contained in $A$, and since therefore $1 \notin J$ for every $J \in \mathscr{J}$. Hence, $\bar{J} \in \mathscr{M}$. Clearly, $\bar{J}$ is a supremum for $\mathscr{J}$, and we have proved that every totally ordered set admits a supremum. By the Lemma of Zorn, $\mathscr{M}$ admits a maximal element which, by definition, must be a maximal ideal of $A$.

**Lemma 6.12.** *Every maximal ideal in a Banach algebra is closed.*

*Proof.* Let $A$ be a Banach algebra, and let $I$ be a maximal ideal. Assume first that $A$ is unital. By the Neumann series, the set $G(A)$ of all invertible elements in $A$ is open, and since $1 \in G(A)$, this set is also nonempty. Clearly, $I \cap G(A) = \emptyset$, for if $I$ contained an invertible element, then $1 \in I$, which is only possible if $I = A$. By the preceding two arguments, $I \subseteq \bar{I} \subseteq A \setminus G(A) \neq A$, and clearly, the closure of $I$ is also an ideal. Since $I$ is a maximal ideal, we obtain $I = \bar{I}$, that is, $I$ is closed.

Now if $A$ is not unital, then we consider the unital algebra $\bar{A}$ from Remark 6.2, which results from $A$ by adjunction of a unit element. Then $I$ is also an ideal in $\bar{A}$, which is contained, by Lemma 6.11, in a maximal ideal $J$. By the first part of this proof, $J$ is closed. As a consequence, $I = J \cap A$ is closed.

Let $A$ be a Banach algebra. A **character** is a nonzero algebra homomorphism $A \to \mathbb{K}$.

**Lemma 6.13.** *Every character on a Banach algebra is automatically continuous.*

*Proof.* Let $A$ be a Banach algebra, and let $\chi : A \to \mathbb{K}$ be a character. Assume first that $A$ is unital. Since $\chi$ is an algebra homomorphism, then $\ker \chi$ is an ideal. Consider the associated, commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \chi\ } & \mathbb{K} \\ \Big\downarrow{\scriptstyle q_\chi} & & \Big\uparrow{\scriptstyle i_\chi} \\ A/\ker\chi & \xrightarrow{\ b_\chi\ } & \mathbb{K} \end{array}$$

where $q_\chi$, $b_\chi$ and $i_\chi$ are the canonical surjection (quotient map) onto $A/\ker\chi$, the canonical bijection onto $\operatorname{ran}\chi$ (here, $\mathbb{K}$), and the canonical injection from $\operatorname{ran}\chi$ into $\mathbb{K}$ (here, the identity map). The kernel $\ker\chi$ having codimension 1 ($b_\chi$ being bijective), it must be a maximal ideal. By Lemma 6.12, $\ker\chi$ is closed, and hence the canonical surjection $q_\chi$ is continuous onto the normed quotient space $A/\ker\chi$. Since the other two homomorphisms $b_\chi$ and $i_\chi$ are homomorphisms between finite-dimensional (in fact: one-dimensional) normed spaces, they are continuous, too. Hence, $\chi$ is continuous.

If $A$ is not a unital Banach algebra, then we consider the unital algebra $\bar{A}$ from Remark 6.2, which results from $A$ by adjunction of a unit element. We then define the linear functional

$$\begin{aligned} \bar{\chi} : \bar{A} &\to \mathbb{K}, \\ (a,\lambda) &\mapsto \chi(a) + \lambda. \end{aligned}$$

For every $(a,\lambda)$, $(b,\mu) \in \bar{A}$ we have, since $\chi$ is an algebra homomorphism,

$$\begin{aligned} \bar{\chi}((a,\lambda)(b,\mu)) &= \bar{\chi}(ab+\lambda b+\mu a, \lambda\mu) \\ &= \chi(ab+\lambda b+\mu a) + \lambda\mu \\ &= \chi(a)\chi(b) + \lambda\chi(b) + \mu\chi(a) + \lambda\mu \\ &= (\chi(a)+\lambda)(\chi(b)+\mu) \\ &= \bar{\chi}(a,\lambda)\,\bar{\chi}(b,\mu), \end{aligned}$$

so that $\bar{\chi}$ is a character (= algebra homomorphism). By the first part of the proof, $\bar{\chi}$ is continuous, which implies that $\chi$ is continuous, too.

Let $A$ be a Banach algebra, and let $A'$ be its dual space. The set of all characters is denoted by $\sigma(A)$, and it is called the **spectrum** of the algebra $A$, or the **Gelfand space** of the algebra $A$. By the preceding lemma, the Gelfand space is a subset of $A'$. The following lemma says that the Gelfand space is in fact a subset of the unit ball of $A'$.

**Lemma 6.14.** *Let $A$ be a Banach algebra. Then, for every character $\chi \in \sigma(A)$ one has $\|\chi\|_{A'} \le 1$, with equality if $A$ is a unital Banach algebra and if $\|1\| = 1$.*

*Proof.* Let $\chi \in \sigma(A)$, and let $a \in A$ be such that $\|a\| \le 1$. Then, for every $n \in \mathbb{N}$,

$$\begin{aligned} |\langle\chi,a\rangle|^n &= |\langle\chi,a\rangle^n| \\ &= |\langle\chi,a^n\rangle| \\ &\le \|\chi\|\,\|a^n\| \\ &\le \|\chi\|\,\|a\|^n \\ &\le \|\chi\|. \end{aligned}$$

Since the right-hand side is finite, we necessarily obtain $|\langle\chi,a\rangle| \le 1$, and hence $\|\chi\| \le 1$.

If $A$ is unital, and if $\|1\| = 1$, then $\|\chi\| \geq |\langle \chi, 1\rangle| = 1$, which together with the preceding inequality implies $\|\chi\| = 1$.

**Remark 6.15.** If $A$ is a unital Banach algebra, then one does not necessarily have $\|1\| = 1$. However, there always exists an equivalent Banach algebra norm $\|\cdot\|_1$ for which one has $\|1\|_1 = 1$. Such a norm is for example given by

$$\|a\|_1 := \sup_{\|b\| \leq 1} \|ab\|.$$

By the preceding lemma, the Gelfand space of any Banach algebra $A$ is a subset of the closed unit ball in $A'$. The closed unit ball in $A'$, however, when being equipped with the topology which is induced by the weak$^*$ topology on $A'$, is, by the Theorem of Banach-Alaoglu, a compact space. In the following, we shall always consider the Gelfand space as a topological space, equipped with the topology which is induced by the weak$^*$ topology on $A'$, too. The following theorem, in combination with Lemma 6.11, shows in particular that the Gelfand space of a nontrivial, complex, commutative, unital Banach algebra is nonempty.

**Theorem 6.16 (Gelfand-Mazur).** *In a complex, commutative Banach algebra, there is a one-to-one correspondence between the set of maximal ideals and the Gelfand space. In fact, every maximal ideal is the kernel of some unique character, and, conversely, the kernel of every character is a closed, maximal ideal.*

*In particular, the Gelfand space of a nontrivial, complex, commutative, unital Banach algebra is nonempty.*

*Proof.* Let $I$ be a maximal ideal of a complex, commutative Banach algebra $A$. Assume first that $A$ is unital. Since $A$ is commutative, the left, right and two-sided ideals all coincide, and hence there is no other left or right ideal strictly included between $I$ and $A$. As a consequence, the quotient algebra $A/I$ is a complex, commutative, unital Banach algebra without any left or right ideals, except the trivial ones. However, if $B$ is a complex, commutative, unital Banach algebra with any left or right ideal, except the trivial ones, then for every $a \in B$, the ideal $aB$ generated by $a$ must be either equal to $\{0\}$ or $B$. Since $B$ is unital, $aB$ contains $a$, and thus $aB = B$ for every nonzero $B$. But then every nonzero element is invertible which implies, by the first theorem of Gelfand-Mazur (Theorem 6.10), that $B$ is isomorphic to $\mathbb{C}$. As a consequence, $A/I$ is isomorphic to $\mathbb{C}$. Now, the quotient map $\chi : A \to A/I = \mathbb{C}$ is a character, and $I = \ker \chi$.

If $A$ is not unital, then we consider the unital Banach algebra $\bar{A}$ from Remark 6.2, which we obtain from $A$ by adjunction of a unit. By Lemma 6.11, there exists a maximal ideal $J \subseteq \bar{A}$ such that $J \supseteq I \times \{0\}$. By the preceding argument, there exists a character $\bar{\chi}$ on $\bar{A}$ such that $\ker \bar{\chi} = J$. The restriction of $\bar{\chi}$ to $A \times \{0\} = A$ is a character on $A$ such that $\ker \chi = I$.

Conversely, if $\chi \in \sigma(A)$ is a character, then $\ker \chi$ is an ideal of codimension 1, hence a maximal ideal. Moreover, since $\chi$ is continuous by Lemma 6.13, $\ker \chi$ is closed.

The existence of a character in a complex, commutative, *unital* Banach algebra now follows from this first part of the proof and the fact that there exists a maximal ideal (Lemma 6.11) and that every maximal ideal is closed (Lemma 6.12). Hence, $\sigma(A)$ is nonempty.

**Lemma 6.17.** *Let $A$ be a Banach algebra. The set $\sigma(A)\cup\{0\}$ is a closed subset of the closed unit ball $\bar B_{A'}(0,1)$. If $A$ is a unital Banach algebra, then the Gelfand space $\sigma(A)$ itself is a closed subset of $\bar B_{A'}(0,1)$. In particular, if $A$ is a unital Banach algebra, then the Gelfand space $\sigma(A)$ is a compact, nonempty space. In general, the Gelfand space is a locally compact space (which may, however, be empty).*

*Proof.* Let $(\chi_\alpha)_\alpha$ be a net in $\sigma(A)\cup\{0\}$, which converges in $\bar B_{A'}(0,1)$ to some element $a'$. Then, for every $a$, $b\in A$

$$\begin{aligned}\langle a',ab\rangle &= \lim_\alpha \langle \chi_\alpha, ab\rangle\\ &= \lim_\alpha \langle \chi_\alpha, a\rangle\,\langle \chi_\alpha, b\rangle\\ &= \lim_\alpha \langle a', a\rangle\,\langle a', b\rangle.\end{aligned}$$

In other words, $a'$ is a multiplicative functional, which means either $a'\in\sigma(A)$, or $a'=0$. As a consequence, $\sigma(A)\cup\{0\}$ is closed in $\bar B_{A'}(0,1)$.

If, in addition, $A$ is a unital Banach algebra, and if $(\chi_\alpha)_\alpha$ is a net in $\sigma(A)$ which converges to some $a'\in\bar B_{A'}(0,1)$, then, by the preceding argument, $a'\in\sigma(A)$, or $a'=0$. However,

$$\langle a',1\rangle = \lim_\alpha \langle \chi_\alpha, 1\rangle = 1,$$

which actually excludes the possibility $a'=0$. Hence, $\sigma(A)$ is closed in $\bar B_{A'}(0,1)$.

**Example 6.18 (Gelfand space of $C(K)$).** We consider the Banach algebra $C(K)$, where $K$ is a compact Hausdorff space. We claim that

$$\sigma(C(K)) \text{ is homeomorphic to } K,$$

or, with a slight abuse of language, the Gelfand space of $C(K)$ is equal to $K$. In fact, for every $x\in K$ the Dirac functional $\delta_x : C(K)\to\mathbb{K}$, $f\mapsto f(x)$ is a character, so that $K$ can be naturally identified with a subset of $\sigma(C(K))$. On the other hand, every character in $\sigma(C(K))$ must be a Dirac functional. In fact, let us argue from the point of view of maximal ideals. If $I$ is a maximal ideal, then there must be some $x\in K$ such that $f(x)=0$ for every $f\in I$. In fact, if this was not true, then there exists $f\in I$ which never vanishes on $K$ (sic!). By continuity of $f$ and compactness of $K$, $|f|$ is uniformly bounded away from 0, and $f^{-1}$ exists in $C(K)$. Since $I$ is an ideal, we obtain $1=ff^{-1}\in I$, and therefore $I=C(K)$, a contradiction to the assumption that $I$ is a maximal ideal. On the other hand, again since $I$ is a maximal ideal, there exists exactly one $x\in K$ such that $f(x)=0$ for every $f\in I$. Hence, $I=\ker\delta_x$ for the corresponding Dirac functional. Thus, the mapping $K\mapsto\sigma(C(K))$, $x\mapsto\delta_x$ is a bijection, which is clearly also continuous thanks to continuity of elements in $C(K)$

and the definition of the weak$*$ topology. By compactness of $K$ and $\sigma(C(K))$, this bijection is a homeomorphism.

**Examples 6.19 (Gelfand space of $L^1(\mathbb{R}^N)$ or $L^1(\mathbb{R}_+)$).**

1. We consider the Banach algebra $L^1(\mathbb{R}^N)$, equipped with the convolution product $*$, as in Example 6.1.5. The dual space of $L^1(\mathbb{R}^N)$ can be identified with $L^\infty(\mathbb{R}^N)$, the duality being given by

$$\langle f, g\rangle_{L^\infty,L^1} := \int_{\mathbb{R}^N} fg.$$

Let $\chi \in L^\infty(\mathbb{R}^N)$ be a character. Then, by Lemma 6.14, $\|\chi\|_\infty \leq 1$, and by definition of character, for every $f$, $g \in L^1(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} \chi(x)f(x)\,dx \int_{\mathbb{R}^N} \chi(y)g(y)\,dy &= \langle \chi, f\rangle_{L^\infty,L^1} \langle \chi, g\rangle_{L^\infty,L^1} \\ &= \langle \chi, f * g\rangle_{L^\infty,L^1} \\ &= \int_{\mathbb{R}^N} \chi(x) \int_{\mathbb{R}^N} f(x-y)g(y)\,dy\,dx \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \chi(x+y)f(x)g(y)\,dy\,dx.\end{aligned}$$

It is not difficult to deduce from this equality, that every character $\chi$ satisfies the functional equation

$$\chi(x+y) = \chi(x)\,\chi(y) \text{ for almost every } x,\, y \in \mathbb{R}^N.$$

Since $\chi$ is measurable, bounded and nonzero, this functional equation implies that there exists $\xi \in \mathbb{R}^N$ such that

$$\chi(x) = e^{i\xi x} \text{ for every } x \in \mathbb{R}^N.$$

Thus, the Gelfand space of $L^1(\mathbb{R}^N)$ is given by

$$\sigma(L^1(\mathbb{R}^N)) = \{e^{i\xi\cdot} : \xi \in \mathbb{R}^N\}.$$

One can show that this space, equipped with the weak$^*$ topology, is homeomorphic to the space $\mathbb{R}^N$, equipped with the usual Euclidean topology.

2. Now we consider the Banach algebra $L^1(\mathbb{R}_+)$, equipped with the convolution product, as in Example 6.1.6. As in the previous example, one shows that every character $\chi \in L^\infty(\mathbb{R}_+)$ satisfies the functional equation

$$\chi(t+s) = \chi(t)\,\chi(s) \text{ for almost every } t,\, s \in \mathbb{R}_+.$$

This implies that there exists $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$ such that

$$\chi(t) = e^{-\lambda t} \text{ for every } t \in \mathbb{R}_+.$$

Hence,

$$\sigma(L^1(\mathbb{R}_+)) = \{e^{-\lambda\cdot} : \lambda \in \mathbb{C}, \operatorname{Re}\lambda \geq 0\}.$$

One can show that this space is homeomorphic to the closed right half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\}$.

Let $A$ be a Banach algebra with Gelfand space $\sigma(A)$, and let $a \in A$. Then we define the function

$$\begin{aligned} \hat{a} : \sigma(A) \cup \{0\} &\to \mathbb{K}, \\ \chi &\mapsto \hat{a}(\chi) := \langle \chi, a \rangle, \end{aligned}$$

and we note that this function is continuous and vanishing at infinity. In fact, if $(\chi_\alpha)_\alpha$ is a convergent net in $\sigma(A) \cup \{0\}$, $\lim_\alpha \chi_\alpha =: \chi$, then, by definition of the weak$^*$ topology,

$$\lim_\alpha \hat{a}(\chi_\alpha) = \lim_\alpha \langle \chi_\alpha, a \rangle = \langle \chi, a \rangle = \hat{a}(\chi).$$

As a consequence, $\hat{a} \in C(\sigma(A) \cup \{0\})$. In the following, we consider the function $\hat{a}$ only to be defined on the Gelfand space itself. If $A$ is a unital Banach algebra, then $\sigma(A)$ is already compact by the preceding lemma, and $\hat{a} \in C(\sigma(A))$. If $A$ is a non-unital Banach algebra, then the Gelfand space $\sigma(A)$ is only locally compact, and $\hat{a} \in C_0(\sigma(A))$, the space of continuous functions vanishing at infinity. Since $C(K) = C_0(K)$ for every compact space $K$, we may always write $\hat{a} \in C_0(\sigma(A))$.

**Theorem 6.20 (Gelfand).** *Let $A$ be a complex, commutative Banach algebra, and let $\sigma(A)$ be its Gelfand space (considered as a locally compact space for the weak$^*$ topology). Then the* **Gelfand transform**

$$\begin{aligned} \hat{} : A &\to C_0(\sigma(A)), \\ a &\mapsto \hat{a}, \end{aligned}$$

*where $\hat{a}(\chi) := \langle \chi, a \rangle$ ($\chi \in \sigma(A)$), is a bounded Banach algebra homomorphism.*

*Proof.* We have already shown above that the Gelfand transform is well-defined. By Lemma 6.14,

$$\begin{aligned} \|\hat{a}\|_{C_0(\sigma(A))} &= \sup_{\chi \in \sigma(A)} |\hat{a}(\chi)| \\ &= \sup_{\chi \in \sigma(A)} |\langle \chi, a \rangle| \\ &\leq \sup_{\chi \in \sigma(A)} \|\chi\| \, \|a\| \\ &\leq \|a\|, \end{aligned}$$

so that $\hat{}$ is actually a contraction. It is clear that $\hat{}$ is linear. Moreover, for every $a$, $b \in A$ and every $\chi \in \sigma(A)$ one has

$$\begin{aligned}\widehat{ab}(\chi) &= \langle \chi, ab\rangle \\ &= \langle \chi, a\rangle \langle \chi, b\rangle \\ &= \hat{a}(\chi)\,\hat{b}(\chi),\end{aligned}$$

that is, $\widehat{ab} = \hat{a}\,\hat{b}$. We have proved that $\hat{}$ is an algebra homomorphism.

**Theorem 6.21.** *Let A be a complex, commutative, unital Banach algebra. Then, for every $a \in A$,*

$$\sigma(a) = \{\langle \chi, a\rangle \, : \, \chi \in \sigma(A)\}.$$

*Proof.* "$\subseteq$" Let $\lambda \in \sigma(a)$. Then $\lambda - a$ is not invertible, which means that $\lambda - a$ is contained in some maximal ideal. Hence, there exists a character $\chi \in \sigma(A)$ such that $\langle \chi, \lambda - a\rangle = 0$. However, $\langle \chi, \lambda\rangle = \lambda \langle \chi, 1\rangle = \lambda$, and hence $\lambda \in \{\langle \chi, a\rangle \, : \, \chi \in \sigma(A)\}$.
"$\supseteq$" Now assume that $\lambda \in \{\langle \chi, a\rangle \, : \, \chi \in \sigma(A)\}$. Then there exists $\chi \in \sigma(A)$ such that $0 = \lambda - \langle \chi, a\rangle = \langle \chi, \lambda - a\rangle$. In other words, $\lambda - a$ is contained in the kernel of some character $\chi$, or, equivalently, in some maximal ideal. As a consequence, $\lambda - a$ is not invertible, that is, $\lambda \in \sigma(a)$.

## 6.2 $C^*$-algebras and the theorem of Gelfand-Naimark

An **involution** on a (complex) Banach algebra $A$ is a mapping $* : A \to A$, $a \mapsto a^*$ such that

(a) $(a+b)^* = a^* + b^*$ and $(\lambda a)^* = \bar{\lambda}\, a^*$ for every $a, b \in A$, $\lambda \in \mathbb{C}$,
(b) $(ab)^* = b^* a^*$ for every $a, b \in A$,
(c) $a^{**} = a$ for every $a \in A$.

If $A$ is unital, and if $e$ is the unit element, then automatically

$$e^* = e.$$

Indeed, for every $a \in A$, by properties ((b)) and ((c)),

$$\begin{aligned}e^* a &= e^* a^{**} \\ &= (a^* e)^* \\ &= a^{**} \\ &= a,\end{aligned}$$

and similarly, $a e^* = a$. Hence, $e^*$ is a unit element, too. By uniqueness of the unit element, $e = e^*$.

A Banach algebra with involution is called a **Banach $*$-algebra**. A Banach $*$-algebra such that

$$\|a a^*\| = \|a\|^2 \text{ for every } a \in A \tag{6.1}$$

is called a $C^*$**-algebra**. An example of a (commutative) $C^*$-algebra is the algebra $C(K)$ ($K$ a compact space) with the natural involution $f \mapsto \bar{f}$ (pointwise complex conjugation). The following theorem says that – up to isomorphism – this is the only example of a commutative, unital $C^*$-algebra.

**Theorem 6.22 (Gelfand-Naimark).** *Let $A$ be a unital, commutative $C^*$-algebra. Then the Gelfand transform* $\hat{}\colon A \to C(\sigma(A))$ *is an isometric $*$-isomorphism, that is, it is an isometric isomorphism and*

$$\bar{\hat{a}} = \hat{a^*} \text{ for every } a \in A.$$

*Proof.* Let $\chi \in \sigma(A)$ be a character. Let $a \in A$ be a selfadjoint element, that is, $a = a^*$, and write $\chi(a) = \alpha + i\beta$ with $\alpha$, $\beta \in \mathbb{R}$. Let $e \in A$ be the unit element and define, for every $t \in \mathbb{R}$

$$b := a + it\,e.$$

Then

$$\chi(b) = \chi(a) + it\chi(e) = \alpha + i(\beta + t)$$

and

$$b\,b^* = (a + it\,e)(a + it\,e)^* = (a + it\,e)(a - ite) = a^2 + t^2\,e.$$

As a consequence,

$$\begin{aligned} \alpha^2 + (\beta + t)^2 &= |\chi(b)|^2 \\ &\le \|b\|^2 \\ &= \|b\,b^*\| \\ &\le \|a\|^2 + t^2, \end{aligned}$$

and therefore

$$\alpha^2 + \beta^2 + 2\beta t \le \|a\|^2 .$$

This is only possible if $\beta = 0$. Hence, we have proved $\chi(a) \in \mathbb{R}$.

If $a \in A$ is an arbitrary element, then $a = u + iv$ with $u$, $v \in A$ such that $u = u^*$ and $v = v^*$ (in fact, take $u = \frac{a+a^*}{2}$ and $v = \frac{a-a^*}{2i}$). Then $a^* = u - iv$ and therefore, using the first step,

$$\begin{aligned} \chi(a^*) &= \chi(u) - i\chi(v) \\ &= \overline{\chi(u) + i\chi(v)} \\ &= \overline{\chi(a)}. \end{aligned}$$

By the definition of the Gelfand transform, this is equivalent to saying that $\bar{\hat{a}} = \hat{a^*}$, that is, $\hat{}$ is a $*$-homomorphism.

Next, we prove that the Gelfand transform is isometric. For every $a \in A$ we have

$$\begin{aligned}\|\hat{a}\|_\infty &= \sup_{\chi\in\sigma(A)} |\chi(a)| \\ &= \sup_{\lambda\in\sigma(a)} |\lambda| && \text{(by Theorem 6.21)} \\ &= r(a) && (r(a) \text{ being the spectral radius}) \\ &= \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} .\end{aligned}$$

Note that, by properties ((b)) and ((c)), $(a\,a^*)^* = a^{**}\,a^* = a\,a^*$, and therefore, by property (6.1),

$$\|a\,a^*\|^2 = \|(a\,a^*)\,(a\,a^*)^*\| = \left\|(a\,a^*)^2\right\|.$$

By induction, this inequality implies

$$\|a\,a^*\|^{2^m} = \left\|(a\,a^*)^{2^m}\right\| \text{ for every } m \in \mathbb{N}.$$

This equality, again property (6.1) and the commutativity imply

$$\left\|a^{2^m}\right\|^2 = \left\|a^{2^m}\,(a^*)^{2^m}\right\| = \left\|(a\,a^*)^{2^m}\right\| = \|a\,a^*\|^{2^m} = \|a\|^{2\cdot 2^m},$$

and thus

$$\|\hat{a}\|_\infty = \|a\|,$$

which yields that the Gelfand transform is isometric.

Like any isometric, linear operator, the Gelfand transform is injective and the range $\hat{A}$ is a closed subalgebra of $C(\sigma(A))$. Since, by the first step, the Gelfand transform is a $*$-homomorphism, the algebra $\hat{A}$ is closed under taking complex conjugation. Clearly, $1 = \hat{e} \in \hat{A}$. Also, $\hat{A}$ separates the points of $\sigma(A)$. Thus, by the Stone-Weierstraß theorem, $\hat{A} = C(\sigma(A))$, that is, the Gelfand transform is surjective.

**Corollary 6.23.** *Let $A$ be a unital, commutative $C^*$-algebra generated by a single element $a \in A$, that is, the linear span of elements of the form $a^n\,(a^*)^m$ ($n$, $m \in \mathbb{N}_0$) is dense in $A$. Then*

$$\begin{aligned}\hat{a} : \sigma(A) &\to \sigma(a), \\ \chi &\mapsto \chi(a) = \hat{a}(\chi),\end{aligned}$$

*is a homeomorphism. Moreover, if we denote by $\check{}$ the inverse of the Gelfand transform (see Theorem 6.22), then*

$$\begin{aligned}\Phi : C(\sigma(a)) &\to A, \\ f &\mapsto \check{}(f\circ\hat{a})\end{aligned}$$

*is an isometric $*$-isomorphism such that*

$$\Phi(1) = e \text{ and } \Phi(id) = a.$$

*Proof.* Note that by Theorem 6.21, $\hat{a}$ maps the Gelfand space continuously onto the spectrum $\sigma(a) \subseteq \mathbb{C}$. We show that the mapping $\hat{a}$ is injective. In fact, if $\chi_1$, $\chi_2 \in \sigma(A)$ are such that $\chi_1(a) = \chi_2(a)$, then, by the fact that the Gelfand transform is a $*$-homomorphism (Theorem 6.22), $\chi_1(a^*) = \chi_2(a^*)$. Hence, by the multiplicativity of $\chi_1$ and $\chi_2$, $\chi_1(a^n(a^*)^m) = \chi_2(a^n(a^*)^m)$ for every $n$, $m \in \mathbb{N}_0$. By assumption, $\chi_1$ and $\chi_2$ therefore coincide on a dense subspace of $A$. By continuity of the characters, $\chi_1 = \chi_2$, and injectivity of $\hat{a}$ is proven. The function $\hat{a}$ being a continuous bijection between the compact spaces $\sigma(A)$ and $\sigma(a)$, it is necessarily a homeomorphism.

The mapping $C(K) \to C(\sigma(A))$, $f \mapsto f \circ \hat{a}$ is then an isometric $*$-isomorphism, and the same is true for the inverse $\check{}: C(\sigma(A)) \to A$ of the Gelfand transform. Thus, $\Phi$ is an isometric $*$-isomorphism. The properties $\Phi(1) = e$ and $\Phi(\mathrm{id}) = a$ follow easily from the definition of $\Phi$.

# Chapter 7
# Operators on Hilbert spaces

## 7.1 The numerical range of an operator on a Hilbert space

Let $A$ be a linear operator on a Hilbert space $H$. The **numerical range** of $A$ is the set

$$w(A) := \{\langle Au, u\rangle_H \in \mathbb{C} : u \in \operatorname{dom} A \text{ and } \|u\|_H = 1\}.$$

The following possibly surprising result gives some structure of the numerical range.

**Theorem 7.1 (Toeplitz-Hausdorff).** *The numerical range of an operator on a Hilbert space is convex.*

*Proof.* For every $\lambda$, $\mu \in \mathbb{C}$ one has $w(\lambda A + \mu) = \lambda\, w(A) + \mu$. By dilating, rotating and shifting the numerical range (multiplicating by $\lambda$ and adding $\mu$) it suffices to prove that if $x_0$, $x_1 \in \operatorname{dom} A$ are such that $\|x_0\|_H = \|x_1\|_H = 1$, $\langle Ax_0, x_0\rangle = 0$ and $\langle Ax_1, x_1\rangle = 1$, then $[0,1] \subseteq w(A)$. So let $x_0$ and $x_1$ as in the preceding sentence and let $\lambda \in ]0,1[$. We now would like to find $x = \alpha x_0 + \beta x_1 \in \operatorname{dom} A$ ($\alpha$, $\beta \in \mathbb{R}$) such that $\|x\|_H = 1$ and $\langle Ax, x\rangle = \lambda$, that is, we are seeking for $\alpha$, $\beta \in \mathbb{R}$ such that

I. $\alpha^2 + \beta^2 + 2\alpha\beta \operatorname{Re}\langle x_0, x_1\rangle = 1$, and

II. $\beta^2 + \alpha\beta\,(\langle Ax_0, x_1\rangle + \langle Ax_1, x_0\rangle) = \lambda$.

Assume that $B := \langle Ax_0, x_1\rangle + \langle Ax_1, x_0\rangle$ is real. Then the set of all solutions of the first equality I is an ellipse through the points $(0,1)$, $(0,-1)$, $(1,0)$ and $(-1,0)$, while the set of all solutions of the second equality II is a hyperbola through the points $(0,\sqrt{\lambda})$ and $(0,-\sqrt{\lambda})$. Since $\lambda \in ]0,1[$, the two solution sets intersect in four points, that is, the system I and II has four solutions. Hence, $\lambda \in w(A)$.

If $B$ is not real, then we can replace $x_0$ by $e^{i\theta}x_0$. Then we still have $\langle Ax_0, x_0\rangle = 0$, and for an appropriate choice of $\theta$, the number $B$ is real.

By the Hausdorff-Toeplitz theorem, the complement of the numerical range is always connected unless the numerical range is a line or a strip.

The following lemma is a motivation to consider the numerical range.

**Lemma 7.2.** *Let $A$ be a linear operator on a Hilbert space $H$. Then:*

(a) *For every $\lambda \in \mathbb{C}$ and for every $x \in \operatorname{dom} A$,*

$$\|(\lambda - A)x\| \geq \operatorname{dist}(\lambda, w(A))\,\|x\|.$$

*In particular, if $\lambda \notin \overline{w(A)}$, then $\lambda - A$ is injective and, if $A$ is in addition closed, has closed range.*

(b) *For every connected $B \subseteq \mathbb{C} \setminus \overline{w(A)}$, if $B \cap \rho(A)$ is nonempty, then $B \subseteq \rho(A)$ and for every $\lambda \in B$,*

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{dist}(\lambda, w(A))}.$$

*Proof.* (a) For every $x \in \operatorname{dom} A$ with $\|x\| = 1$,

$$\begin{aligned} \|(\lambda - A)x\| &= \|(\lambda - A)x\|\,\|x\| \\ &\geq |\langle (\lambda - A)x, x\rangle| \\ &= |\lambda - \langle Ax, x\rangle| \\ &\geq \operatorname{dist}(\lambda, w(A)), \end{aligned}$$

which implies the first claim.

(b) Let $B \subseteq \mathbb{C} \setminus \overline{w(A)}$ be connected. Since the resolvent set $\rho(A)$ is open in $\mathbb{C}$, the intersection $B \cap \rho(A)$ is open in $B$. We show that the intersection is also closed in $B$. For this, let $(\lambda_n)$ be a sequence in $B \cap \rho(A)$ which converges to some $\lambda \in B$. Then the $\lambda_n - A$ are boundedly invertible, and $\lambda$ has positive distance to $w(A)$. By (a), the sequence $(R(\lambda_n, A))$ is bounded in $\mathscr{L}(H)$. Since the resolvent blows up near the boundary of the spectrum, it follows that $\lambda \in \rho(A)$, or more precisely, $\lambda \in B \cap \rho(A)$. The rest of the argument follows from connectedness of $B$.

**Lemma 7.3.** *Let $T \in \mathscr{L}(H)$ be a bounded operator on a Hilbert space $H$. Then*

$$\sigma(T) \subseteq \overline{w(T)}. \tag{7.1}$$

*Proof.* For a bounded operator $T \in \mathscr{L}(H)$, the Cauchy-Schwarz inequality implies that its numerical range is included in the closed disk $\overline{B(0, \|T\|)}$. In fact, by the Toeplitz-Hausdorff theorem (Theorem **??**), the numerical range $w(T)$ is a convex subset of this closed disk, and therefore the complement of its closure is connected. By Lemma **??**, $\mathbb{C} \setminus \overline{B(0, \|T\|)}$ is contained in the resolvent set $\rho(A)$, and therefore, by Lemma 7.2 (b), $\mathbb{C} \setminus \overline{w(A)}$ is contained in this resolvent set.

## 7.2 Spectral theorem for compact selfadjoint operators

Let $H$, $K$ be two Hilbert spaces, $T \in \mathscr{L}(H, K)$. For every $y \in K$ the mapping $H \to \mathbb{K}$, $x \mapsto \langle Tx, y\rangle_K$ is a bounded linear functional on $H$ which admits a unique representation by $T^*y \in H$ such that

$$\langle Tx, y\rangle_K = \langle x, T^*y\rangle_H \quad (x \in H).$$

The resulting linear operator $T^* : K \to H$ is called the **(Hilbert space) adjoint** of $T$.

**Lemma 7.4.** *Let $H_1$, $H_2$, and $H_3$ be three Hilbert spaces, $T$, $S \in \mathscr{L}(H_1, H_2)$, $R \in \mathscr{L}(H_2, H_3)$ and $\lambda \in \mathbb{K}$. Then:*

(a) $(T+S)^* = T^* + S^*$.

(b) $(\lambda T)^* = \bar{\lambda} T^*$.

(c) $(RT)^* = T^* R^*$.

(d) $T^* \in \mathscr{L}(H_2, H_1)$ *and* $\|T^*\| = \|T\|$.

(e) $T^{**} = T$.

(f) $\|T^*T\| = \|TT^*\| = \|T\|^2$.

(g) $\ker T = (\operatorname{ran} T^*)^\perp$ *and* $\ker T^* = (\operatorname{ran} T)^\perp$ *(orthogonal spaces).*

*Proof.* The properties (a)–(c) are simple exercises. Concerning (d), note that

$$\begin{aligned} \|T^*\| &:= \sup_{\|y\|_{H^2} \le 1} \|T^*y\|_{H_1} \\ &= \sup_{\|y\|_{H^2} \le 1} \sup_{\|x\|_{H^1} \le 1} |\langle T^*y, x\rangle_{H_1}| \\ &= \sup_{\|x\|_{H^1} \le 1} \sup_{\|y\|_{H^2} \le 1} |\langle y, Tx\rangle_{H_2}| \\ &= \sup_{\|x\|_{H^1} \le 1} \|Tx\|_{H_2} \\ &= \|T\|. \end{aligned}$$

Next, for every $x \in H_1$, $y \in H_2$,

$$\begin{aligned} \langle T^{**}x, y\rangle_{H_2} &= \langle x, T^*y\rangle_{H_1} \\ &= \overline{\langle T^*y, x\rangle_{H_1}} \\ &= \overline{\langle y, Tx\rangle_{H_2}} \\ &= \langle Tx, y\rangle_{H_2}, \end{aligned}$$

which implies (e). Finally, note that

$$\begin{aligned}
\|T^*T\| &= \sup_{\|x\|\le 1} \|T^*Tx\| \\
&= \sup_{\|x\|\le 1} \sup_{\|y\|\le 1} \big|\langle T^*Tx, y\rangle\big| \\
&= \sup_{\|x\|\le 1} \sup_{\|y\|\le 1} \big|\langle Tx, Ty\rangle\big| \\
&\ge \sup_{\|x\|\le 1} \big|\langle Tx, Tx\rangle\big| \\
&= \sup_{\|x\|\le 1} \|Tx\|^2 \\
&= \|T\|^2,
\end{aligned}$$

while the inequality $\|T^*T\| \le \|T^*\|\,\|T\| = \|T\|^2$ (using also (d)) is trivial. Hence, we have proved (f). The property (g) is also left as an exercise.

**Remark 7.5.** Let $\mathscr{A}$ be a complex Banach algebra. A mapping $* : \mathscr{A} \to \mathscr{A}$ is called an **involution** if for every $a, b \in \mathscr{A}$, $\lambda \in \mathbb{C}$,

$$(a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*,\ (\lambda a)^* = \bar{\lambda} a^*,\ (a^*)^* = a.$$

If a complex Banach algebra $\mathscr{A}$ admits an involution $*$ such that for every $a \in \mathscr{A}$,

$$\|a^*a\| = \|a\|^2,$$

then $\mathscr{A}$ is called a $C^*$**-algebra**.

If $H$ is a Hilbert space, then $\mathscr{L}(H)$ is a $C^*$-algebra for the involution $T \mapsto T^*$, where $T^*$ is the (Hilbert space) adjoint of $T$. This follows from Lemma 7.4.

The simplest $C^*$-algebra is $\mathbb{C}$ (the involution being the complex conjugation). In the space of matrices $\mathbb{C}^{N\times N} = \mathscr{L}(\mathbb{C}^N)$, the involution as defined above, that is, the Hilbert space adjoint with respect to the Euclidean inner product, is given by $A^* = \bar{A}^t$ (complex conjugation and transposition).

Given a compact space $K$, the space $C(K)$ is also a $C^*$-algebra for the usual algebra structure and the involution $f \mapsto f^*$ given by $f^*(x) := \bar{f}(x)$ $(x \in K)$.

Let $H$ be a complex Hilbert space. An operator $T \in \mathscr{L}(H)$ is called **selfadjoint** if $T = T^*$, or, equivalently, if for every $x, y \in H$,

$$\langle Tx, y\rangle = \langle x, Ty\rangle.$$

We say that the operator $T$ is **positive semidefinite** and we write $T \ge 0$, if it is selfadjoint, and

$$\langle Tx, x\rangle \ge 0 \quad \text{for every } x \in H.$$

An operator $T \in \mathscr{L}(H)$ is called **normal** if $TT^* = T^*T$. An operator $U \in \mathscr{L}(H,K)$ between two Hilbert spaces is called **unitary** if $U$ is an isomorphism and $U^*U = I_H$ and $UU^* = I_K$. Clearly, every selfadjoint operator and every unitary operator (on a single Hilbert space) is normal.

**Remark 7.6.** In every $C^*$-algebra $\mathscr{A}$ one can define that an element $a$ is selfadjoint if $a = a^*$. The selfadjoint elements of $\mathscr{A} = \mathbb{C}$ are the real numbers. The selfadjoint elements of $\mathbb{C}^{N\times N}$ are the hermitian matrices, that is, the matrices $A$ for which $A = \bar{A}^t$.

**Theorem 7.7 (Hellinger-Toeplitz).** *Let $T : H \to H$ be linear and* **symmetric***, that is,*

$$\langle Tx, y\rangle = \langle x, Ty\rangle \text{ for every } x,\ y \in H.$$

*Then $T$ is bounded.*

*Proof.* Let $(x_n) \subseteq H$ be convergent to $x \in H$ and such that $(Tx_n)$ converges to $y \in H$. Then, for every $z \in H$,

$$\langle Tx, z\rangle = \langle x, Tz\rangle = \lim_{n\to\infty} \langle x_n, Tz\rangle = \lim_{n\to\infty} \langle Tx_n, z\rangle = \langle y, z\rangle.$$

Hence, $Tx = y$. This means that $T$ is closed, and by the closed graph theorem, $T$ is bounded.

**Lemma 7.8.** *Let $T \in \mathscr{L}(H)$ be a selfadjoint operator on a Hilbert space $H$. Then:*

(a)

$$\sigma(T) \subseteq \overline{w(T)} \subseteq \mathbb{R}. \tag{7.2}$$

(b)

$$\sup w(T) \in \sigma(T) \text{ and } \inf w(T) \in \sigma(T),$$

*Proof.* (a) The first inclusion holds for general bounded operators (Lemma 7.3), and it remains only to show that the numerical range is a subset of $\mathbb{R}$. This, however, follows from $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$ by selfadjointness and by symmetry of the inner product.

(b) Let $\lambda := \sup w(T)$. By definition of $w(T)$, the form $a(x,y) := \lambda \langle x, y\rangle - \langle Tx, y\rangle$ is sesquilinear in the case of a complex Hilbert space, or bilinear and symmetric in the case of a real Hilbert space. Moreover, this form is positive semidefinite, that is, $a(x,x) \geq 0$ for every $x \in H$.

By the Cauchy-Schwarz inequality applied to the form $a(x,y)$, for every $x$, $y \in H$,

$$|\langle \lambda x - Tx, y\rangle| \leq \langle \lambda x - Tx, x\rangle^{\frac{1}{2}} \langle \lambda y - Ty, y\rangle^{\frac{1}{2}}.$$

This inequality implies that there exists a constant $C \geq 0$ such that for every $x \in H$,

$$\|\lambda x - Tx\| \leq C \langle \lambda x - Tx, x\rangle^{\frac{1}{2}}.$$

Let $(x_n) \subseteq H$, $\|x_n\| = 1$ be such that $\langle Tx_n, x_n\rangle \to \lambda$. Then the preceeding inequality implies that $\lim_{n\to\infty} \|\lambda x_n - Tx_n\| = 0$. Hence, $\lambda \in \sigma_{ap}(T) \subseteq \sigma(T)$.

The proof that $\inf w(T) \in \sigma(T)$ is similar.

**Lemma 7.9.** *Let $T \in \mathscr{L}(H)$ be a selfadjoint operator on a Hilbert space $H$. Then*

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle| = \sup_{\lambda\in\sigma(T)} |\lambda|.$$

*Proof.* The second equality follows from Lemma **??** combined with Lemma 7.3. Moreover, the inequality

$$\sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \|T\|$$

is obvious, by the definition of $\|T\|$ and the Cauchy-Schwarz inequality. Using the fact that $T = T^*$, one easily calculates for every $x$, $y \in H$,

$$4\,\mathrm{Re}\langle Tx, y\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle.$$

Hence,

$$\begin{aligned}
\|T\| &= \sup_{\|x\|=1} \|Tx\| \\
&= \sup_{\|x\|=1} \sup_{\|y\|=1} \mathrm{Re}\langle Tx, y\rangle \\
&= \frac{1}{4} \sup_{\|x\|=1} \sup_{\|y\|=1} [\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle] \\
&\leq \frac{1}{4} \sup_{\|x\|=1} \sup_{\|y\|=1} [|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle|] \\
&\leq \sup_{\|z\|=1} |\langle Tz, z\rangle| \frac{1}{4} \sup_{\|x\|=1} \sup_{\|y\|=1} [\|x+y\|^2 + \|x-y\|^2] \\
&\leq \sup_{\|z\|=1} |\langle Tz, z\rangle| \frac{1}{2} \sup_{\|x\|=1} \sup_{\|y\|=1} [\|x\|^2 + \|y\|^2] \\
&\leq \sup_{\|z\|=1} |\langle Tz, z\rangle|,
\end{aligned}$$

which is just the remaining inequality.

**Lemma 7.10.** *Let $T \in \mathscr{L}(H)$ be a selfadjoint operator on a complex Hilbert space, and let $x$, $y \in H$ be two eigenvectors corresponding to two distinct eigenvalues $\lambda$, $\mu \in \sigma_p(T)$. Then $\langle x, y\rangle = 0$.*

*Proof.* Since $T$ is selfadjoint and $\lambda$, $\mu \in \mathbb{R}$ (Lemma 7.3),

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle = \langle x, \mu y\rangle = \mu\langle x, y\rangle.$$

Since $\lambda \neq \mu$, this equality can only hold if $\langle x, y\rangle = 0$.

**Theorem 7.11 (Spectral theorem for compact, selfadjoint operators).** *Let $H$ be a separable Hilbert space, and let $T \in \mathscr{K}(H)$ be a compact, selfadjoint operator. Then there exists an orthonormal basis $(e_n)_n$ of $H$, and a family $(\lambda_n)_n$ of real numbers such that* $\lim_{n\to\infty} \lambda_n = 0$ *and*

$$Te_n = \lambda_n e_n \quad \textit{for every } n,$$

*that is, there is an orthonormal basis $(e_n)_n$ consisting only of eigenvectors of $T$. In other words, $T$ is* **unitarily equivalent** *to the multiplication operator $M : \ell^2 \to \ell^2$, $M(x_n)_n := (\lambda_n x_n)_n$, that is, there exists a unitary operator $U : H \to \ell^2$ such that the diagram*

$$\begin{array}{ccc} H & \xrightarrow{T} & H \\ \downarrow U & & \uparrow U^* = U^{-1} \\ \ell^2 & \xrightarrow{M} & \ell^2 \end{array}$$

*commutes*

*Proof.* By the spectral theory of compact operators, $\sigma(T)$ is at most countable, every $\mu \in \sigma(T) \setminus \{0\}$ is an eigenvalue, and its eigenspace $H_\mu := \ker(\mu - T)$ is finite-dimensional.

Let $(\mu_n)$ be the (finite or countable) family of *all* nonzero eigenvalues of $T$ ($\mu_n \neq \mu_m$ if $n \neq m$), and let $d_n := \dim \ker(\mu_n - T)$ be their multiplicities. Let $(f_k^n)_{1 \leq k \leq d_n}$ be an orthonormal basis of $H_{\mu_n} = \ker(\mu_n - T)$. If the kernel $H_0 := \ker T$ is nontrivial, then choose also an orthonormal basis $(f_k)_{0 \leq k < \dim H_0}$ of $H_0$. Next, let $(e_n)$ be the family which is obtained by taking successively the union over all eigenvectors $f_k^n$ and $f_k$, and let $(\lambda_n)$ be the family which is obtained by taking the eigenvalues corresponding to $f_k^n$ or $f_k$. For simplicity, assume that the kernel $H_0 = \ker T$ is trivial. Then $e_1 = f_1^1, \dots, e_{d_1} = f_{d_1}^1, e_{d_1+1} = f_1^2, \dots, e_{d_1+d_2} = f_{d_2}^2$, etc., and $\lambda_1 = \mu_1$, $\dots, \lambda_{d_1} = \mu_1, \lambda_{d_1+1} = \mu_2, \dots, \lambda_{d_1+d_2} = \mu_2$, etc.

The family $(e_n)$ thus obtained is an orthonormal system by construction and by Lemma 7.10. Moreover, by construction, $Te_n = \lambda_n e_n$ for every $n$. It remains only to show that $\operatorname{span}\{e_n : n\} =: H^0$ is dense in $H$.

Let $H^1 := (H^0)^\perp$ be the orthogonal complement. For every $x \in H^1$ and every $n$, since $T$ is selfadjoint,

$$\langle Tx, e_n \rangle = \langle x, Te_n \rangle = \langle x, \lambda_n e_n \rangle = \bar{\lambda}_n \langle x, e_n \rangle = 0.$$

Hence, $TH^1 \subset H^1$, that is, $T$ leaves the space $H^1$ invariant. We may thus consider the restriction $T^1 := T|_{H_1} \in \mathscr{L}(H^1)$ which inherits the property from $T$ to be compact and selfadjoint. Since *all* eigenvectors of $T$ are contained in $H^0$, $T^1$ does not have any eigenvalue. In other words, $\sigma(T^1) \subseteq \{0\}$. By Lemma 7.9, this implies $T^1 = 0$. However, as we just remarked, $T^1$ does also not admit any eigenvector for the only possible eigenvalue 0. Hence, $H^1 = \ker T^1 = \{0\}$, which implies that $H^0$ is dense in $H$.

To complete the proof, consider the operator $U : H \to \ell^2$ given by $Ux := (\langle x, e_n \rangle)_n$. This operator does the work, that is, $U$ is unitary and $T = U^* M U$, as one easily shows.

**Remark 7.12.** Let $T \in \mathscr{K}(H)$ be a compact, selfadjoint operator on a general (not necessarily separable) Hilbert space. Then $H = \ker T \oplus (\ker T)^\perp$, where $(\ker T)^\perp = \overline{\operatorname{ran} T}$ is separable (any compact, metric space is separable, and $\operatorname{ran} T$ is spanned by the relatively compact set $TB_H(0,1)$. Applying the above spectral theorem (which holds only on separable Hilbert spaces) to the restriction of $T$ to $\overline{\operatorname{ran} T}$, we obtain an orthonormal basis of $\operatorname{ran} T$ which consists only of eigenvectors of $T$. This (at most countable) orthonormal basis can be completed by an orthonormal basis of $\ker T$, which consists necessarily of eigenvectors to the eigenvalue 0. As a conclusion, we obtain an orthonormal eigenbasis of $H$ which consists only of eigenvectors of $T$. We thus see that the assumption of separability of $H$ can be dropped in the spectral theorem.

We may immediately generalize the spectral theorem to the larger class of normal operators. For this, we also need the following variant of Schauder's theorem.

**Lemma 7.13.** *Let $H$, $K$ be two Hilbert spaces and $T \in \mathscr{L}(H,K)$. Then $T$ is compact if and only if $T^*$ is compact.*

*Proof.* It is intructive to represent the Hilbert space adjoint $T^*$ by using the Banach space adjoint $T' \in \mathscr{L}(K',H')$ and the (antilinear) isomorphisms $\Phi_H : H' \to H$ and $\Phi_K : K' \to K$ which one obtains from the Theorem of Riesz-Fréchet (Theorem 2.44). In fact,

$$T^* = \Phi_H T' \Phi_K^{-1}.$$

If $T$ is compact, then $T'$ is compact by Schauder's theorem (Theorem 5.20), and hence $T^*$ is compact due to the above representation. Conversely, if $T^*$ is compact, then, by what has just been said, $T^{**}$ is compact. However, $T^{**} = T$ (Lemma 7.4 (e)), and the claim is proved.

**Theorem 7.14 (Spectral theorem for compact, normal operators).** *Let $H$ be a complex, separable Hilbert space, and let $T \in \mathscr{K}(H)$ be a compact, normal operator. Then there exists an orthonormal basis $(e_n)_{n\in I} \subseteq H$ ($I \subseteq \mathbb{N}$) of $H$, and a sequence $(\lambda_n)_{n\in I} \subseteq \mathbb{C}$ such that $\lim_{n\to\infty} \lambda_n = 0$ and*

$$Te_n = \lambda_n e_n \quad \text{for every } n \in I,$$

*that is, $(e_n)$ is an orthonormal basis consisting only of eigenvectors of $T$.*

*Proof.* We define

$$\operatorname{Re} T := \frac{T+T^*}{2} \text{ and } \operatorname{Im} T := \frac{T-T^*}{2i}.$$

Since $T$ is normal, the operators $\operatorname{Re} T$ and $\operatorname{Im} T$ commute. Moreover, they are easily seen to be selfadjoint and compact (for compactness, we use Lemma 7.13). We show that $\operatorname{Re} T$ and $\operatorname{Im} T$ can be diagonalized simultaneously.

By the spectral theory of compact operators, $\sigma(\operatorname{Re} T)$ is at most countable, every $\alpha \in \sigma(\operatorname{Re} T) \setminus \{0\}$ is an eigenvalue, and its eigenspace $H_\alpha := \ker(\alpha - T)$ is finite-dimensional.

Let $(\alpha_n)$ be the (finite or countable) family of *all* nonzero eigenvalues of $\operatorname{Re} T$ ($\alpha_n \neq \alpha_m$ if $n \neq m$), and let $d_n := \dim \ker(\alpha_n - T)$ be their multiplicities. For every $e \in H_{\alpha_n}$ one has

$$\operatorname{Re} Te = \alpha_n e.$$

We apply $\operatorname{Im} T$ on both sides of this equality, and use the fact that $\operatorname{Re} T$ and $\operatorname{Im} T$ commute, and we find that the vector $\operatorname{Im} Te$ is also an eigenvector of $\operatorname{Re} T$ for the eigenvalue $\alpha_n$. In other words, the eigenspaces $H_{\alpha_n}$ are left invariant under $\operatorname{Im} T$. By applying the spectral theorem for compact, selfadjoint operators to the restrictions of $\operatorname{Im} T$ to $H_{\alpha_n}$, we find for every $n$ an orthonormal basis $(f_k^n)_{1\leq k\leq d_n}$ of $H_{\alpha_n}$, and we find a family $(\beta_k^n)_{1\leq k\leq d_n}$ of real numbers such that

$$\operatorname{Im} T f_k^n = \beta_k^n f_k^n \text{ for every } 1 \leq k \leq d_n.$$

Of course, we still have

$$\operatorname{Re} T f_k^n = \alpha_n f_k^n \text{ for every } 1 \leq k \leq d_n.$$

If $H_0 := \ker \operatorname{Re} T$ is nontrivial, then we may repeat the arguments from above in order to see that $\operatorname{Im} T$ leaves $H_0$ invariant. We may also apply the spectral theorem for compact, selfadjoint operators to the restriction of $\operatorname{Im} T$ to $H_0$, and we find an orthonormal basis $(f_k)_{0\leq k<\dim H_0}$ and a sequence $(\beta_k)_{0\leq k<\dim H_0}$ of real numbers such that

$$\operatorname{Im} T f_k = \beta_k f_k \text{ for every } 0 \leq k < \dim H_0.$$

Of course, we have

$$\operatorname{Re} T f_k = 0 \text{ for every } 0 \leq k < \dim H_0.$$

From the above relations and from the equality $T = \operatorname{Re} T + i \operatorname{Im} T$ we obtain

$$T f_k^n = (\alpha_n + i\beta_k^n) f_k^n =: \mu_k^n f_k^n \text{ for every } 1 \leq k \leq d_n$$

and

$$T f_k = i\beta_k f_k =: \mu_k f_k \text{ for every } 0 \leq k < \dim H_0,$$

that is, the $f_k^n$ and $f_k$ are eigenvectors of $T$ for the complex eigenvalues $\mu_k^n$ and $\mu_k$, respectively.

Next, let $(e_n)$ be the family which is obtained by taking successively the union over all eigenvectors $f_k^n$ and $f_k$, and let $(\lambda_n)$ be the family which is obtained by taking the eigenvalues corresponding to $f_k^n$ or $f_k$. For simplicity, assume that the kernel $H_0 = \ker \operatorname{Re} T$ is trivial. Then $e_1 = f_1^1, \ldots, e_{d_1} = f_{d_1}^1, e_{d_1+1} = f_1^2, \ldots, e_{d_1+d_2} = f_{d_2}^2$, etc., and $\lambda_1 = \mu_1^1, \ldots, \lambda_{d_1} = \mu_{d_1}^1, \lambda_{d_1+1} = \mu_1^2, \ldots, \lambda_{d_1+d_2} = \mu_{d_2}^2$, etc.

The family $(e_n)$ thus obtained is orthonormal by construction and by Lemma 7.10 (applied to $\operatorname{Re} T$). Moreover, by construction, $Te_n = \lambda_n e_n$ for every $n$. It remains only to show that $\operatorname{span}\{e_n : n \in \mathbb{N}\} =: H^0$ is dense in $H$. For this, one proceeds similarly as in the proof of the spectral theorem for compact, selfadjoint operators. One shows that $\operatorname{Re} T$ and $\operatorname{Im} T$ leave $H^1 = (H^0)^\perp$ invariant but admit no eigenvectors in $H^1$. This implies for example $\operatorname{Re} T = 0$ in $H^1$, and thus $H^1 = \{0\}$. As a consequence, $H^0$ is dense, and $(e_n)$ an orthonormal basis.

## 7.3 Spectral theorem for bounded, normal operators

### The continuous functional calculus

**Theorem 7.15 (Spectral theorem for bounded, normal operators - the continuous functional calculus).** *Let $T \in \mathscr{L}(H)$ be a normal operator, and let $K := \sigma(T)$ be its spectrum. Then there exists an $C^*$-algebra homomorphism*

$$\Phi : C(K) \to \mathscr{L}(H)$$

*with the following properties:*

(i) *$\Phi(\mathrm{id}) = T$ and in particular $\Phi(p) = p(T)$ for every polynomial $p$.*

(ii) *$\Phi$ is isometric, that is, $\|\Phi(f)\|_{\mathscr{L}(H)} = \|f\|_\infty$ for every $f \in C(K)$.*

(iii) *$\Phi$ is positive in the sense that if $f \geq 0$, then $\Phi(f) \geq 0$.*

(iv) *(Spectral mapping theorem) For every $f \in C(K)$ one has $\sigma(\Phi(f)) = f(\sigma(T)) = f(K)$.*

(v) *(Spectral mapping theorem for the point spectrum) For every $f \in C(K)$ and every $\lambda \in \sigma_p(T)$ and every corresponding eigenvector $x \in H$ (that is, $Tx = \lambda x$) one has $\Phi(f)x = f(\lambda)x$.*

*Proof.* This theorem is a direct consequence of Corollary 6.23 to the Gelfand-Naimark theorem (Theorem 6.22, applied to the commutative $C^*$-subalgebra of $\mathscr{L}(H)$ generated by $T$ (commutativity of this subalgebra follows from the assumption that $T$ is normal).

### The Riesz-Markov representation theorem

Let $K$ be a compact space. We denote by $\mathscr{B}(K)$ the Borel-$\sigma$-algebra on $K$, that is, the smallest $\sigma$-algebra on $K$ which contains the open sets. A **Borel measure** on $K$ is a measure on the Borel-$\sigma$-algebra $\mathscr{B}(K)$, that is, a $\sigma$-additive function $\mu : \mathscr{B}(K) \to [0, +\infty]$ (we consider here only nonnegative measures). A Borel measure $\mu$ on $K$ is **regular** if for every Borel measurable set $A \subseteq K$

(i) $\mu(A) = \inf\{\mu(O) : O \supseteq A,\, O \text{ open}\}$, and

(ii) $\mu(A) = \sup\{\mu(C) : C \subseteq A,\, C \text{ compact}\}$.

We say that $\mu$ is **finite** if $\mu(K) < \infty$. The following Riest-Markov representation theorem characterizes positive, linear functionals on $C(K)$. We say that a functional $\varphi \in C(K)'$ is **positive** if $\varphi(f) \geq 0$ for every function $f \in C(K)$ taking its values in $\mathbb{R}_+$ (the notion of positivity makes also sense on the complex space $C(K)$). Finally, we define $B(K)$ to be the space of all bounded, Borel measurable functions $K \to \mathbb{C}$.

Equipped with the sup-norm, $B(K)$ is a Banach space which contains $C(K)$ as a closed, linear subspace.

**Theorem 7.16 (Riesz-Markov representation theorem).** *Let $K$ be a compact space. Then, for every positive functional $\varphi \in C(K)'$ there exists a finite, regular Borel measure $\mu$ on $K$ such that*

$$\varphi(f) = \int_K f\, d\mu \text{ for every } f \in C(K).$$

*Proof.* Let $\varphi \in C(K)'$ be a positive functional. If necessary, we restrict $\varphi$ to the (real) subspace of real-valued continuous functions. By Hahn-Banach, we may extend the functional $\varphi$ to a functional $\tilde{\varphi}$ on the space $B(K)$ of bounded Borel measurable functions, such that $\|\tilde{\varphi}\|_{B(K)'} = \|\varphi\|_{C(K)'}$.

Since $\varphi$ is positive, then $\varphi(1) = \|\varphi\|_{C(K)'}$. Hence, for every Borel function $f \in B(K)$ satisfying $\|f\|_\infty \leq 1$ one has

$$|\tilde{\varphi}(f)| \leq \|\tilde{\varphi}\|_{B(K)'} = \|\varphi\|_{C(K)'} = \varphi(1).$$

In particular, if $f \in B(K)$ is a *positive* Borel function such that $\|f\|_\infty \leq 1$, then $1-f$ is also a positive Borel function, $\|1-f\|_\infty \leq 1$, and thus $|\tilde{\varphi}(f)| \leq \varphi(1)$ and $|\tilde{\varphi}(1-f)| \leq \varphi(1)$. On the other hand,

$$0 \leq \varphi(1) = \tilde{\varphi}(1) = \tilde{\varphi}(f) + \tilde{\varphi}(1-f),$$

which, together with the preceding estimates, is only possible if $\tilde{\varphi}(f) \geq 0$ (and $\tilde{\varphi}(1-f) \geq 0$). We have thus proved that the extension $\tilde{\varphi}$ is a positive linear functional on $B(K)$.

For every Borel measurable set $A \subseteq K$, we now define

$$\mu(A) := \tilde{\varphi}(\chi_A) \geq 0,$$

where $\chi_A \in B(K)$ is the characteristic function of the set $A$. We claim that $\mu$ is a bounded, regular, Borel measure which represents $\varphi$ as stated in the theorem.

First, $\mu$ is finitely additive by additivity of $\tilde{\varphi}$, and $\mu$ is monotone ($\mu(A) \leq \mu(B)$ whenever $A \subseteq B$) by positivity of $\tilde{\varphi}$.

## The spectral theorem for bounded, normal operators

**Lemma 7.17.** *Let $T \in \mathscr{L}(H)$ be a normal operator on a separable Hilbert space, for which there exists $x \in X$ such that* $\operatorname{span}\{T^n(T^*)^m x : n, m \in \mathbb{N}_0\}$ *is dense in $H$. Let $K = \sigma(T)$ be the spectrum of $T$. Then there exists a regular, finite Borel measure $\mu$ on $K$ and a unitary operator $U : H \to L^2(K; d\mu)$ such that the diagram*

$$\begin{array}{ccc} H & \xrightarrow{\quad T \quad} & H \\ \Big\downarrow{\scriptstyle U} & & \Big\uparrow{\scriptstyle U^*=U^{-1}} \\ L^2(K;d\mu) & \xrightarrow{\quad M \quad} & L^2(K;d\mu) \end{array}$$

*commutes. Here, $M : L^2(K;d\mu) \to L^2(K;d\mu)$ is the multiplication operator given by*

$$Mf(\omega) = \omega f(\omega) \quad (f \in L^2(K;d\mu),\, \omega \in K).$$

*In other words, $T$ is unitarily equivalent to a multiplication operator.*

*Proof.* Let $x \in H$ be any vector, and let $\Phi : C(K) \to \mathscr{L}(H)$ be the continuous functional calculus associated with $T$ (Theorem 7.15). Then the linear mapping

$$\begin{aligned} \varphi_x : C(K) &\to \mathbb{C}, \\ f &\mapsto \langle \Phi(f)x, x\rangle, \end{aligned}$$

is bounded and positive (Theorem 7.15 (ii), (iii)). By the Riesz-Markov representation theorem (Theorem 7.16), there exists a finite, regular Borel measure $\mu_x$ on $K$ such that

$$\varphi_x(f) = \langle \Phi(f)x, x\rangle = \int_K f \, d\mu \text{ for every } f \in C(K).$$

As a consequence of this equality and by using the properties of $\Phi$, for every $f \in C(K)$,

$$\begin{aligned} \|\Phi(f)x\|_H^2 &= \langle \Phi(f)^* \Phi(f)x, x\rangle_H \\ &= \langle \Phi(\bar{f} f)x, x\rangle_H \\ &= \int_K |f|^2 \, d\mu. \end{aligned}$$

This equality shows first that if $f_1, f_2 \in C(K)$ coincide $\mu$-almost everywhere, then $\Phi(f_1)x = \Phi(f_2)x$. Hence, the operator

$$\begin{aligned} U^* : L^2(K;d\mu) &\to H, \\ f &\mapsto U^* f = \Phi(f)x \end{aligned}$$

is well defined first for equivalence classes of continuous functions, but then, by the above equality and by continuous extension, everywhere on $L^2(K;d\mu)$. Moreover, the operator thus defined is isometric.

Now we suppose that the vector $x \in H$, which was arbitrary in the beginning, is as in the statement. Then the operator $U^*$ is isometric and invertible, and thus a unitary operator. In fact, $U^*$ being isometric, it is injective and has closed range. Moreover, the range of $U^*$ contains the set $\{T^n (T^*)^m x : n, m \in \mathbb{N}_0\} = \{\Phi(\mathrm{Id}^n \, \bar{id}^m)x : n, m \in \mathbb{N}_0\}$ which is dense in $H$ by the assumption and the choice of $x$. Hence, $U^*$ is surjective.

Finally, for every $f \in C(K)$,

$$\begin{aligned} TU^*f &= T\Phi(f)x \\ &= \Phi(id)\Phi(f)x \\ &= \Phi(id\cdot f)x \\ &= U^*(id\cdot f), \end{aligned}$$

and thus $UTU^*$ ($U = (U^*)^{-1}$) is the multiplication operator given in the statement.

**Lemma 7.18.** *Let $T \in \mathscr{L}(H)$ be a normal operator on a Hilbert space $H$. Then there exists a family $(H_i)_{i\in I}$ of closed subspaces such that*

(a) *the $H_i$ are mutually orthogonal,*
(b) $H = \bigoplus_{i\in I} H_i$,
(c) *each $H_i$ is invariant under $T$ and $T^*$, and*
(d) *for every $i \in I$ there exists $x \in H_i$ such that $\{T^n(T^*)^m x : n,m \in \mathbb{N}_0\}$ is dense in $H_i$.*

*Proof.*

**Theorem 7.19 (Spectral theorem for bounded, normal operators).** *Let $T \in \mathscr{L}(H)$ be a normal operator on a separable Hilbert space $H$. Then there exists a measure space $(\Omega, \mathscr{A}, \mu)$, a function $m \in L^\infty(\Omega; d\mu)$, and a unitary operator $U : H \to L^2(\Omega; d\mu)$ such that the diagram*

$$\begin{array}{ccc} H & \xrightarrow{T} & H \\ \big\downarrow{\scriptstyle U} & & \big\uparrow{\scriptstyle U^*=U^{-1}} \\ L^2(\Omega;d\mu) & \xrightarrow{M} & L^2(\Omega;d\mu) \end{array}$$

*commutes. Here, $M : L^2(\Omega; d\mu) \to L^2(\Omega; d\mu)$ is the multiplication operator given by*

$$Mf(\omega) = m(\omega)f(\omega) \quad (f \in L^2(\Omega; d\mu),\ \omega \in \Omega).$$

*In other words, $T$ is unitarily equivalent to a multiplication operator.*

*Proof.* Choose a family $(H_i)_{i\in I}$ (with $I \subseteq \mathbb{N}$) as in Lemma 7.18. By Lemma 7.17, for every $i \in I$ there exists a finite, regular Borel measure $\mu_i$ on $\sigma(T|_{H_i}) \subseteq \sigma(T) =: K$ and a unitary operator $U_i^* : L^2(\sigma(T); d\mu_i) \to H_i$ such that $U_i T|_{H_i} U_i^* = M_i$, where $M_i : L^2(K; d\mu_i) \to L^2(K; d\mu_i)$ is the multiplication operator given by $M_i f(\omega) = \omega f(\omega)$.

Set $\Omega := K \times I = \bigcup_{i\in I} \sigma(T) \times \{i\}$, and let $\mu$ be the Borel measure on $\Omega$ whose restriction to $\sigma(T) \times \{i\} \cong \sigma(T)$ coincides with $\mu_i$, that is,

$$\mu\Big(\bigcup_{i\in I} B_i \times \{i\}\Big) := \sum_{i\in I} \mu_i(B_i) \quad (B_i \in \mathscr{B}(K)).$$

Note that

$$L^2(\Omega;d\mu) \cong \bigoplus_{i\in I} L^2(K;d\mu_i)$$

in a canonical way, and that, via this identification, $U^* = \bigoplus_{i\in I} U_i^*$ defines a unitary operator from $L^2(\Omega;d\mu)$ onto $H = \bigoplus_{i\in I} H_i$. It is now a short exercise to show that $UTU^* = M$, where $ML^2(\Omega;d\mu) \to L^2(\Omega;d\mu)$ is the multiplication operator given by

$$Mf(\omega,i) = \omega f(\omega,i).$$

## The measurable functional calculus

In the following, given a Borel measurable $K \subseteq \mathbb{R}$, we define the space

$$B(K) := \{f : K \to \mathbb{C} : f \text{ is bounded and Borel measurable}\}.$$

Equipped with the supremum norm $\|\cdot\|_\infty$, this space is a $C^*$-algebra for the natural (pointwise) scalar multiplication, addition and multiplication. Clearly, if $K$ is compact, $B(K)$ contains $C(K)$ as a closed subspace.

**Theorem 7.20 (Spectral theorem - the measurable functional calculus).** *Let $T \in \mathscr{L}(H)$ be a normal operator on a separable Hilbert space $H$. Let the measure space $(\Omega,\mathscr{A},\mu)$, the unitary operator $U : H \to L^2(\Omega;d\mu)$, the function $m \in L^\infty(\Omega;d\mu)$ and the multiplication operator $M \in \mathscr{L}(L^2(\Omega;d\mu))$ be as in the Spectral Theorem (Theorem 7.19). Let $K := \sigma(T)$. Then the operator*

$$\begin{aligned} \tilde{\Phi} : B(K) &\to \mathscr{L}(H), \\ f &\mapsto \tilde{\Phi}(f) := U^* f(M) U, \end{aligned}$$

*where $f(M) \in \mathscr{L}(L^2(\Omega;d\mu))$ is the multiplication operator given by*

$$f(M)g(\omega) := f(m(\omega))\,g(\omega) \quad (g \in L^2(\Omega;d\mu),\, \omega \in \Omega),$$

*is a $C^*$-algebra homomorphism which extends the continuous functional calculus $\Phi$ from Theorem 7.15, and which has the properties:*

(i) $\|\tilde{\Phi}\| = 1$,

(ii) $\tilde{\Phi}(f) \geq 0$ *whenever* $f \geq 0$, *and*

(iii) *if $(f_n)$ is a bounded sequence in $B(K)$ which converges $\mu$-almost everywhere to a function $f \in B(K)$, then, for every $x \in H$,*

$$\lim_n \tilde{\Phi}(f_n)x = \tilde{\Phi}(f)x.$$

**Remark 7.21.** Note that we can choose the multiplication operator $M$ such that the range of $m$ is a subset of $K$, so that the expression $f(m(\omega))$ is well defined.

*Proof.* In the special case $T = M$, that is, when $T$ already is a multiplication operator (and $U = U^* = I$), the properties of $\tilde{\Phi}$ are easy to verify, even property (iii), which relies only on Lebesgue's dominated convergence theorem. The case of general $T$ follows then from this special case and the Spectral Theorem (Theorem 7.19).

## Spectral measures and spectral decomposition

## 7.4 Spectral theorem for unbounded selfadjoint operators

In the preceding two sections, we have actually proved more than just solvability of an elliptic and a hyperbolic partial differential equation. We have proved that the Dirichlet-Laplace operator is selfadjoint, that it has a compact resolvent, and that therefore it is diagonalisable. In this last section, we discuss the spectral theorem for unbounded selfadjoint operators with compact resolvent.

Let $H$ be a complex Hilbert space, and let $A : H \supseteq \mathrm{dom} A \to H$ be a densely defined (that is, the domain $\mathrm{dom} A$ is dense in $H$) and linear operator. We define

$$\begin{aligned} \mathrm{dom} A^* &:= \{x \in H : \exists y \in H\, \forall z \in \mathrm{dom} A : \langle Az, x\rangle_H = \langle z, y\rangle_H\}, \\ A^* x &:= y. \end{aligned}$$

The operator $(A^*, \mathrm{dom} A^*)$ is called the **(Hilbert space) adjoint** of $A$. For every $x \in \mathrm{dom} A$, $y \in \mathrm{dom} A^*$ one has

$$\langle Ax, y\rangle = \langle x, A^* y\rangle.$$

**Remark 7.22.** The adjoint $A^*$ is well-defined in the sense that the element $y \in H$ is uniquely determined (use that $\mathrm{dom} A$ is dense in $H$).

**Lemma 7.23.** *Let $A : \mathrm{dom} A \to H$ be a densely defined, linear operator. Then $A^* : \mathrm{dom} A^* \to H$ is closed.*

*Proof.* Let $(x_n) \subseteq \mathrm{dom} A^*$ be convergent to some $x \in H$ and such that $(A^* x_n)$ converges to $y \in H$. Then, for every $z \in \mathrm{dom} A$,

$$\begin{aligned} \langle z, y\rangle &= \lim_{n\to\infty} \langle z, A^* x_n\rangle \\ &= \lim_{n\to\infty} \langle Az, x_n\rangle \\ &= \langle Az, x\rangle. \end{aligned}$$

By definition of $A^*$ this implies $x \in \mathrm{dom} A^*$ and $A^* x = y$. Hence, $A^*$ is closed.

Let $H$ be a complex Hilbert space, and let $A : H \supseteq \mathrm{dom} A \to H$ be a densely defined, linear operator. We say that $A$ is **symmetric** if for every $x, y \in \mathrm{dom} A$,

$$\langle Ax, y\rangle = \langle x, Ay\rangle.$$

We say that $A$ is **selfadjoint** if $A = A^*$.

**Remark 7.24.** Saying that $A$ is selfadjoint, that is, that $A = A^*$, means that $\operatorname{dom} A = \operatorname{dom} A^*$ *and* $A = A^*$. By Lemma 7.23, every selfadjoint operator is necessarily closed. Note, however, that a symmetric closed linear operator $A$ need in general not be selfadjoint! However, if $\operatorname{dom} A = H$, then symmetric implies selfadjoint by the Theorem of Hellinger-Toeplitz (Theorem 7.7).

**Remark 7.25.** If a bounded operator $A : H \to H$ ($\operatorname{dom} A = H$!) is selfadjoint in the sense of the definition for unbounded operators (see page 127), then $A$ is selfadjoint in the sense of the definition for bounded operators (see page 116), and vice versa.

**Lemma 7.26.** *Let* $A : \operatorname{dom} A \to H$ *be densely defined and symmetric. Then the following are equivalent:*

(i) *$A$ is selfadjoint.*

(ii) *$A$ is closed and* $\ker(A^* \pm i) = \{0\}$.

(iii) $\operatorname{ran}(A \pm i) = H$.

*Proof.* We first remark that if $(A, \operatorname{dom} A)$ is symmetric, then $\ker(A \pm i) = \{0\}$. In fact, let $x \in H$ be such that $(A - i)x = 0$. Since $A$ is symmetric,

$$i\|x\|^2 = \langle ix, x\rangle = \langle Ax, x\rangle = \langle x, Ax\rangle = -i\|x\|^2.$$

Hence, $x = 0$. Similarly, one proves $\ker(A + i) = \{0\}$.

(i)⇒(ii). Now assume that $A$ is selfadjoint. By Lemma 7.23, $A^*$ is closed, and therefore $A$ $(= A^*)$ is closed. Since $A$ is symmetric, and since $A^* = A$, we find $\ker(A^* \pm i) = \{0\}$ by the above argument.

(ii)⇒(iii). Similarly as in Lemma 5.8 one proves that

$$\ker(A^* - i) = (\operatorname{ran}(A + i))^\perp,$$

where $\perp$ now means the Hilbert space orthogonal. Hence, if $\ker(A^* - i) = \{0\}$, then $\operatorname{ran}(A + i)$ is dense in $H$. We prove that $\operatorname{ran}(A + i)$ is also closed. Since $A$ is symmetric, we have $\langle Ax, x\rangle \in \mathbb{R}$ for every $x \in \operatorname{dom} A$. Hence, for every $x \in \operatorname{dom} A$,

$$\begin{aligned}\|(A + i)x\| &= \|Ax\|^2 + \|x\|^2 + 2\operatorname{Re}\langle Ax, ix\rangle \\ &= \|Ax\|^2 + \|x\|^2 \quad \geq \quad \|x\|^2.\end{aligned}$$

Let $(x_n) \subseteq \operatorname{dom} A$ be such that $\lim_{n\to\infty}(A + i)x_n = y \in H$ exists. By the preceding inequality, this implies that $(x_n)$ is a Cauchy sequence in $H$. Hence, $x := \lim_{n\to\infty} x_n \in H$ exists. Since $A + i$ is closed, we obtain $x \in \operatorname{dom} A$ and $(A + i)x = y$. We have shown that $\operatorname{ran}(A + i)$ is closed. Similarly, one shows that $\operatorname{ran}(A - i)$ is closed.

(iii)⇒(i). Since $A$ is symmetric, $\operatorname{dom} A \subseteq \operatorname{dom} A^*$ and $Ax = A^*x$ for every $x \in \operatorname{dom} A$. It remains to show that $\operatorname{dom} A^* \subseteq \operatorname{dom} A$. Let $y \in \operatorname{dom} A^*$. Since

$\operatorname{ran}(A+i)=H$, there exists $x\in\operatorname{dom}A$ such that $(A^*+i)y=(A+i)x$. By the inclusion $(A,\operatorname{dom}A)\subseteq(A^*,\operatorname{dom}A^*)$, $(A^*+i)y=(A^*+i)x$. Since $\operatorname{ran}(A-i)=H$ implies $\ker(A^*+i)=\{0\}$ (compare again with Lemma 5.8), this implies $x=y\in\operatorname{dom}A$.

**Exercise 7.27** *The Dirichlet-Laplace operator A defined in* (7.6) *is selfadjoint.*

**Lemma 7.28.** *Let $A:\operatorname{dom}A\to H$ be densely defined and closed. Then, for every $\lambda\in\rho(A)$ one has $\bar\lambda\in\rho(A^*)$ and*

$$R(\lambda,A)^*=R(\bar\lambda,A^*).$$

*Proof.* For every $x\in\operatorname{dom}A$ and every $y\in\operatorname{dom}A^*$ one has

$$\begin{aligned}\langle x,R(\lambda,A)^*(\bar\lambda-A^*)y\rangle &= \langle R(\lambda,A)x,(\bar\lambda-A^*)y\rangle\\ &= \langle(\lambda-A)R(\lambda,A)x,y\rangle\\ &= \langle x,y\rangle\end{aligned}$$

and

$$\begin{aligned}\langle x,(\bar\lambda-A^*)R(\lambda,A)^*y\rangle &= \langle(\lambda-A)x,R(\lambda,A)^*y\rangle\\ &= \langle R(\lambda,A)(\lambda-A)x,y\rangle\\ &= \langle x,y\rangle,\end{aligned}$$

so that $\bar\lambda-A^*$ is invertible and $R(\bar\lambda,A^*)=R(\lambda,A)^*$.

**Theorem 7.29 (Spectral mapping theorem).** *Let $A:\operatorname{dom}A\to H$ be densely defined, closed. Assume that $\rho(A)$ is not empty. Then, for every $\lambda\in\rho(A)$,*

$$(\lambda-\sigma(A))^{-1}=\sigma((\lambda-A)^{-1})\setminus\{0\}.$$

*Proof.* The proof is an exercise.

We say that a closed, linear operator $(A,\operatorname{dom}A)$ on a Banach space $X$ **has compact resolvent** if $\rho(A)$ is nonempty, and if there exists $\lambda\in\rho(A)$ such that $R(\lambda,A)$ is compact.

**Lemma 7.30.** *Let $(A,\operatorname{dom}A)$ be a closed, linear operator on a Banach space $X$ such that $\rho(A)\neq\emptyset$. Then the following are equivalent:*

(i) *A has compact resolvent.*

(ii) *For every $\lambda\in\rho(A)$, the resolvent $R(\lambda,A)$ is compact.*

(iii) *The embedding $j:(\operatorname{dom}A,\|\cdot\|_{\operatorname{dom}A})\to(X,\|\cdot\|_X)$, $x\mapsto x$ is compact.*

*Proof.* The implication (ii)⇒(i) is trivial, while the converse (i)⇒(ii) is a consequence of the resolvent identity

$$R(\mu,A)=R(\lambda,A)+(\lambda-\mu)R(\mu,A)R(\lambda,A).$$

(i)⇒(iii) Assume that $\lambda \in \rho(A)$ is such that $R(\lambda,A)$ is compact. Let $(x_n)$ be a bounded sequence in $(\operatorname{dom} A, \|\cdot\|_{\operatorname{dom} A})$, that is, there exists $C \geq 0$ such that

$$\|x_n\|_X + \|Ax_n\|_X \leq C \text{ for every } n.$$

Since $R(\lambda,A)$ is invertible from $X$ onto $\operatorname{dom} A$, there exists a sequence $(y_n)$ in $X$ such that $R(\lambda,A)y_n = x_n$. Using the equality $AR(\lambda,A) = \lambda R(\lambda,A) - I$, the above estimate for the $x_n$ yields

$$\|R(\lambda,A)y_n\|_X + \|\lambda R(\lambda,A)y_n - y_n\|_X \leq C \text{ for every } n.$$

This estimate yields that $(y_n)$ is necessarily bounded in $X$. Since $R(\lambda,A)$ is compact, there exists a subsequence of $(y_n)$ (which we denote for simplicity again by $(y_n)$) such that $(R(\lambda,A)y_n) = (x_n)$ converges in $X$. In other words, for every bounded sequence $(x_n)$ in $(\operatorname{dom} A, \|\cdot\|_{\operatorname{dom} A})$ we can extract a subsequence which converges in $X$. Hence, the embedding $j : (\operatorname{dom} A, \|\cdot\|_{\operatorname{dom} A}) \to (X, \|\cdot\|_X)$, $x \mapsto x$ is compact.
(iii)⇒(i) Choose any $\lambda \in \rho(A)$. Then the operator $j : (\operatorname{dom} A, \|\cdot\|_{\operatorname{dom} A}) \to (X, \|\cdot\|_X)$, $x \mapsto \lambda x - Ax$ is continuous (by definition of the graph norm) and invertible (by the choice of $\lambda$). By the bounded inverse theorem, $R(\lambda,A)$ is a bounded linear operator from $(X, \|\cdot\|_X)$ onto $(\operatorname{dom} A, \|\cdot\|_{\operatorname{dom} A})$. Composing this operator with $j$, we obtain that $R(\lambda,A)$ is a compact operator on $X$.

**Lemma 7.31.** *Consider the meromorphic functions $f$ and $g$ on $\mathbb{C}$ given by*

$$f(z) := \frac{i-z}{i+z} \quad \text{and}$$
$$g(z) := i\,\frac{1-z}{1+z} \quad (z \in \mathbb{C}).$$

*Then, for every $z \in \mathbb{C}$:*

(a) *If $z \in \mathbb{R}$, then $|f(z)| = 1$. If $|z| = 1$, then $f(z) \in i\mathbb{R}$.*

(b) *If $z \in i\mathbb{R}$, then $|g(z)| = 1$. If $|z| = 1$, then $g(z) \in \mathbb{R}$.*

(c) *$f(g(z)) = g(f(z)) = z$.*

The functions $f$ and $g$ in the preceding lemma are two special Möbius transforms. A general Möbius transform has the form $f(z) = \frac{az+b}{cz+d}$ and it always has the property that it maps straight lines to straight lines or circles, and circles to straight lines or circles. Möbius transforms are the affine mappings on the Riemann sphere. We use their properties in the following lemma in order to transform selfadjoint operators (which have spectrum in the real line) to unitary operators (which have spectrum in the unit circle) and back.

**Lemma 7.32 (Cayley transform).** *Let $H$ be a Hilbert space.*

(a) *Let $A : \operatorname{dom} A \to H$ be a densely defined, selfadjoint operator. Then its* **Cayley transform**

$$f(A) := U := (i-A)(i+A)^{-1}$$

*is a unitary operator such that* $\operatorname{rg}(I+U) = \operatorname{dom} A$*. In particular,* $I+U$ *has dense range.*

(b) *If* $U \in \mathscr{L}(H)$ *is a unitary operator such that* $I+U$ *has dense range, then*

$$g(U) := A := i(I-U)(I+U)^{-1}$$

*with maximal domain* $\operatorname{dom} A := \operatorname{rg}(I+U) \to H$ *is selfadjoint.*

(c) *Let* $A : \operatorname{dom} A \to H$ *be a densely defined, selfadjoint operator. Then* $g(f(A)) = A$*.*

**Theorem 7.33 (Spectral theorem for unbounded selfadjoint operators with compact resolvent).** *Let* $A : \operatorname{dom} A \to H$ *be densely defined, selfadjoint, having compact resolvent. Then there exists an orthonormal basis* $(e_n) \subseteq H$ *and a sequence* $(\lambda_n) \subseteq \mathbb{R}$ *such that* $\lim_{n\to\infty} |\lambda_n| = \infty$*,*

$$e_n \in \operatorname{dom} A \text{ and } Ae_n = \lambda_n e_n \text{ for every } n.$$

*Moreover,* $\sigma(A) = \sigma_p(A) = \{\lambda_n : n\}$*.*

*Proof.* Let $\lambda \in \rho(A)$ be such that $R(\lambda,A) \in \mathscr{K}(H)$. By Theorem 5.25, $\sigma(R(\lambda,A))$ is countable. Hence, by Theorem 7.29, $\sigma(A)$ is countable. In particular, there exists $\mu \in \rho(A) \cap \mathbb{R}$. By Lemma 7.30 (that is, by the resolvent identity), $R(\mu,A)$ is compact, too. Moreover, since $\mu \in \mathbb{R}$, for every $x, y \in H$,

$$\begin{aligned}\langle R(\mu,A)x,y\rangle &= \langle R(\mu,A)x,(\mu-A)R(\mu,A)y\rangle \\ &= \langle(\mu-A)R(\mu,A)x,R(\mu,A)y\rangle \\ &= \langle x,R(\mu,A)y\rangle,\end{aligned}$$

so that $R(\mu,A)$ is selfadjoint. By the spectral theorem for selfadjoint compact operators, there exists an orthonormal basis $(e_n)$ of $H$ and a sequence $(\mu_n) \subseteq \mathbb{R}\setminus\{0\}$ such that $\lim_{n\to\infty} \mu_n = 0$ and such that

$$\mu_n e_n = R(\mu,A)e_n \text{ for every } n.$$

This equation implies on the one hand that $e_n \in \operatorname{dom} A$ and on the other hand, when we multiply by $\mu - A$,

$$\lambda_n e_n = Ae_n \text{ for every } n,$$

with $\lambda_n = \mu - \frac{1}{\mu_n}$. Clearly, $\lim_{n\to\infty} |\lambda_n| = \infty$, and by the spectral mapping theorem (Theorem 7.29), $\sigma(A) = \sigma_p(A) = \{\lambda_n : n\}$. The claim is proved.

## 7.5 Hilbert-Schmidt operators and trace class operators

In this section, let $H$ be a separable Hilbert space. An operator $T : H \to H$ is a **Hilbert-Schmidt operator** if there exists an orthonormal basis $(e_n)_n$ in $H$ such that

$\sum_n \|Te_n\|_H^2 < \infty$. We denote by $\mathscr{HS}(H)$ the set of all Hilbert-Schmidt operators on $H$.

Whenever $T$ is a bounded, linear operator on a separable Hilbert space $H$, then $T^*T$ is selfadjoint and positive semidefinite. If $T$ is in addition compact, then $T^*T$ is compact, too. By the spectral theorem for selfadjoint, compact operators, there exists an orthonormal basis $(e_n)$ of eigenvectors and a null sequence $(\lambda_n)$ of nonnegative, real numbers such that $T^*Te_n = \lambda_n e_n$. We call the numbers

$$s_n(T) := \sqrt{\lambda_n}$$

the **singular numbers** of $T$. Like $(\lambda_n)$, the sequence of singular numbers is a null sequence.

**Theorem 7.34.** *Let H be a separable Hilbert space. Then:*

(a) *For every pair of orthonormal bases $(e_n)_n$ and $(\hat{e}_n)_n$ of H, and for every $T \in \mathscr{HS}(H)$,*
$$\sum_n \|Te_n\|^2 = \sum_n \|T\hat{e}_n\|^2 =: \|T\|_{\mathscr{HS}}^2,$$
*and $\|T\|_{\mathscr{L}} \leq \|T\|_{\mathscr{HS}}$.*

(b) *For every pair of orthonormal bases $(e_n)_n$ and $(\hat{e}_n)_n$ of H, and for every T, $S \in \mathscr{HS}(H)$,*
$$\sum_n \langle Te_n, Se_n\rangle = \sum_n \langle T\hat{e}_n, S\hat{e}_n\rangle =: \langle T, S\rangle_{\mathscr{HS}}.$$

(c) *The space $(\mathscr{HS}(H), \langle\cdot,\cdot\rangle_{\mathscr{HS}})$ is a Hilbert space.*

(d) *The space of finite rank operators is dense in $\mathscr{HS}(H)$, and $\mathscr{HS}(H)$ embeds continuously into $\mathscr{K}(H)$. In particular, every Hilbert-Schmidt operator is compact.*

(e) *For every $T \in \mathscr{HS}(H)$, the adjoint $T^*$ is a Hilbert-Schmidt operator and $\|T^*\|_{\mathscr{HS}} = \|T\|_{\mathscr{HS}}$.*

(f) *For every $S \in \mathscr{HS}(H)$ and every T, $R \in \mathscr{L}(H)$, $TSR \in \mathscr{HS}(H)$ and $\|TSR\|_{\mathscr{HS}} \leq \|T\|_{\mathscr{L}} \|S\|_{\mathscr{HS}} \|R\|_{\mathscr{L}}$. In particular, $\mathscr{HS}(H)$ is a two-sided ideal in $\mathscr{L}(H)$.*

(g) *A compact operator $T \in \mathscr{K}(H)$ is a Hilbert-Schmidt operator if and only if the sequence of its singular numbers belongs to $\ell^2$, and then $\|T\|_{\mathscr{HS}}^2 = \|(s_n(T))\|_{\ell^2}$.*

*Proof.* (a) Let $T$ be a Hilbert-Schmidt operator, and let $(e_n)$ be an orthonormal basis of $H$ such that $\sum_n \|Te_n\|^2$ is finite. Let $(\hat{e}_n)_n$ be another orthonormal basis of $H$. Then

$$\begin{aligned}\sum_n \|T\hat{e}_n\|^2 &= \sum_n \langle T\hat{e}_n, T\hat{e}_n\rangle \\ &= \sum_n \langle T(\sum_k \langle \hat{e}_n, e_k\rangle e_k), T(\sum_l \langle \hat{e}_n, e_l\rangle e_l)\rangle \\ &= \sum_{n,k,l} \langle \hat{e}_n, e_k\rangle \, \langle Te_k, Te_l\rangle \, \overline{\langle \hat{e}_n, e_l\rangle} \\ &= \sum_{k,l} \langle Te_k, Te_l\rangle \, \langle e_k, e_l\rangle \\ &= \sum_k \|Te_k\|^2.\end{aligned}$$

In particular, the quantity $\sum_n \|T\hat{e}_n\|^2$ is finite, too, and the quantity $\|T\|_{\mathscr{HS}}$ does not depend on the choice of the orthonormal basis. Moreover, for every $x \in H$ and for every orthonormal basis $(e_n)$ of $H$,

$$\begin{aligned}\|Tx\| &= \|\sum_n \langle x, e_n\rangle Te_n\| \\ &\leq \sum_n |\langle x, e_n\rangle| \, \|Te_n\| \\ &\leq (\sum_n |\langle x, e_n\rangle|^2)^{\frac{1}{2}} \, (\sum_n \|Te_n\|^2)^{\frac{1}{2}} \\ &= \|T\|_{\mathscr{HS}} \, \|x\|,\end{aligned}$$

which proves that $T$ is bounded and $\|T\|_{\mathscr{L}} \leq \|T\|_{\mathscr{HS}}$.

The proof of (b) is analogous to the proof of the first part of (a).

(c) It follows easily from (a) that $\mathscr{HS}(H)$ is a linear space. It is also straightforward to show that $\langle \cdot, \cdot \rangle_{\mathscr{HS}}$ is an inner product, and that $\| \cdot \|_{\mathscr{HS}}$ is the induced norm. One only has to prove that $\mathscr{HS}(H)$ is complete for this norm. So let $(T_n)$ be a Cauchy sequence in $\mathscr{HS}(H)$. Then, by part (a), $(T_n)$ is a Cauchy sequence in $\mathscr{L}(H)$, and therefore it converges in $\mathscr{L}(H)$ to some operator $T$. Let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that, for every $n$, $m \in \mathbb{N}_{\geq n_0}$, $\|T_n - T_m\|_{\mathscr{HS}} \leq \varepsilon$. This is equivlent to saying that for any fixed orthonormal basis $(e_k)$ and for every $n$, $m \in \mathbb{N}_{\geq n_0}$, $\sum_k \|(T_n - T_m)e_k\|^2 \leq \varepsilon^2$. Sending $m \to \infty$ in this inequality implies that for every $n \in \mathbb{N}_{\geq n_0}$, $\sum_k \|(T_n - T)e_k\|^2 \leq \varepsilon^2$. From here it follows in particular that $T \in \mathscr{HS}(H)$ and $\|T_n - T\|_{\mathscr{HS}} \leq \varepsilon$ for every $n \in \mathbb{N}_{\geq n_0}$. We have proved convergence of $(T_n)$ in $\mathscr{HS}(H)$ and thus the completeness of $\mathscr{HS}(H)$.

(d) Let $T \in \mathscr{HS}(H)$, let $(e_n)$ be any orthonormal basis, and let $P_N$ be the orthogonal projection onto the finite dimensional space spanned by $\{Te_n : n \leq N\}$. Then $T_N := P_N T$ is a finite rank operator and $\|T - T_N\|^2_{\mathscr{HS}} = \sum_{n \geq N+1} \|Te_n\|^2 \to 0$ as $N \to \infty$. Hence, the finite rank operators are dense in the Hilbert-Schmidt operators. Since every finite rank operator is compact, and since the space of Hilbert-Schmidt operators embeds continuously into $\mathscr{L}(H)$, it follows actually that $\mathscr{HS}(H)$ embeds continuously into $\mathscr{K}(H)$.

(e) Let $T \in \mathscr{HS}(H)$ and let $(e_n)$ be any orthonormal basis of $H$. Then, by Parseval's identity,

$$\begin{aligned}\sum_n \|T^*e_n\|^2 &= \sum_{n,m} |\langle T^*e_n, e_m\rangle|^2 \\ &= \sum_{n,m} |\langle e_n, Te_m\rangle|^2 \\ &= \sum_m \|Te_m\|^2 \\ &= \|T\|^2_{\mathscr{HS}}.\end{aligned}$$

(f) Let $T$, $R \in \mathscr{L}(H)$ and $S \in \mathscr{HS}(H)$. Then, for every orthonormal basis $(e_n)$ of $H$,

$$\sum_n \|TSe_n\|^2 \leq \|T\|^2_{\mathscr{L}} \sum_n \|Se_n\|^2 = \|T\|^2_{\mathscr{L}} \|S\|^2_{\mathscr{HS}},$$

and from here and from part (e) follows

$$\|SR\|_{\mathscr{HS}} = \|R^*S^*\|_{\mathscr{HS}} \leq \|R^*\|_{\mathscr{L}} \|S^*\|_{\mathscr{HS}} = \|S\|_{\mathscr{HS}} \|R\|_{\mathscr{L}}.$$

(g) Let $T$ be a compact operator on a Hilbert space, and let $(e_n)$ be the orthonormal basis of eigenvectors of $T^*T$. Then

$$\sum_n \|Te_n\|^2 = \sum_n \langle Te_n, Te_n\rangle = \sum_n \langle T^*Te_n, e_n\rangle = \sum_n \lambda_n = \sum_n s_n(T)^2.$$

The proof of Theorem 7.34 is complete.

Examples of Hilbert-Schmidt operators are described in the following theorem. Actually, this theorem fully characterizes the Hilbert-Schmidt operators on separable $L^2$-spaces.

**Theorem 7.35.** *Let $(\Omega, \mu)$ be a measure space such that $L^2(\Omega)$ is separable. Let $k \in L^2(\Omega \times \Omega)$ be a kernel. Then the associated kernel operator $K : L^2(\Omega) \to L^2(\Omega)$ given by $(Kf)(x) = \int_\Omega k(x,y) f(y)\,\mathrm{d}y$ ($f \in L^2(\Omega)$, $x \in \Omega$) is a Hilbert-Schmidt operator and $\|K\|_{\mathscr{HS}} = \|k\|_{L^2}$. The operator $J : L^2(\Omega \times \Omega) \to \mathscr{HS}(L^2(\Omega))$ which assigns to every kernel $k$ the associated kernel operator $K$ is an isometric isomorphism.*

*Proof.* For $k \in L^2(\Omega \times \Omega)$, the associated kernel operator $K$ and an orthonormal basis $(e_n)$ of $L^2(\Omega)$ we compute using Parselval's identity,

$$\begin{aligned}\sum_n \|Ke_n\|_{L^2}^2 &= \sum_n \int_\Omega \left| \int_\Omega k(x,y) e_n(y) \, \mathrm{d}y \right|^2 \mathrm{d}x \\ &= \int_\Omega \sum_n |\langle k(x,\cdot), e_n \rangle_{L^2}|^2 \, \mathrm{d}x \\ &= \int_\Omega \|k(x,\cdot)\|_{L^2}^2 \, \mathrm{d}x \\ &= \int_\Omega \int_\Omega |k(x,y)|^2 \, \mathrm{d}y \, \mathrm{d}x \\ &= \|k\|_{L^2}^2 .\end{aligned}$$

This proves that $K$ is a Hilbert-Schmidt operator and $\|K\|_{\mathscr{HS}} = \|k\|_{L^2}$. This also proves that the operator $J$ from the statement is well-defined and isometric. In particular, $J$ is injective has closed range. However, $J$ has also dense range, since every finite rank operator $T = \sum_{k=1}^n \langle \cdot, f_k \rangle g_k$ is a kernel operator associated with the kernel $k(x,y) = \sum_{k=1}^n g_k(x) \overline{f_k(y)} \in L^2(\Omega \times \Omega)$. It follows that $J$ is surjective, and it is thus an isometric isomorphism.

An analogue of Theorem 7.34 (g), the characterisation of Hilbert-Schmidt operators by their singular numbers, is the assertion following theorem.

**Theorem 7.36.** *Let $H$ be a separable Hilbert space. Then:*

(a) *A compact operator $T$ on $H$ is a nuclear operator if and only if the sequence of its singular numbers belongs to $\ell^1$, and then $\|T\|_{\mathscr{N}} = \|(s_n(T))\|_{\ell^1}$.*

(b) *For every $T$, $S \in \mathscr{HS}(H)$, the product $TS$ is a nuclear operator and*

$$\|TS\|_{\mathscr{N}} \leq \|T\|_{\mathscr{HS}}^{\frac{1}{2}} \, \|S\|_{\mathscr{HS}}^{\frac{1}{2}} .$$

*Proof.*

**Remark 7.37.** (a) In fact, for a compact operator $T$, we define the **trace**

$$\operatorname{tr} T := \sum_n s_n(T).$$

(not quite) The trace plays the role of an integral; not an integral of functions but an integral of operators. The **trace class operators** are those compact operators for which the trace $\operatorname{tr} T$ is finite. By the Theorem 7.36 **??**, the trace class operators are precisely the nuclear operators on Hilbert spaces.

(b) The norm inequality in Theorem 7.36 **??** very much ressembles a Cauchy-Schwarz inequality if one translates it into the corresponding inequality for singular numbers

$$\sum_n s_n(TS) \leq \left( \sum_n s_n(T)^2 \right)^{\frac{1}{2}} \left( \sum_n s_n(S)^2 \right)^{\frac{1}{2}} ,$$

or, in terms of the trace,

$$\operatorname{tr}(ST) \le (\operatorname{tr} T)^{\frac{1}{2}} (\operatorname{tr} S)^{\frac{1}{2}}.$$

(c) Recall the identities $(c_0)' = \ell^1$ and $(\ell^1)' = \ell^\infty$. These identities have the following analogues for operator spaces on Hilbert spaces: $\mathscr{K}(H)' = \mathscr{N}(H)$ and $\mathscr{N}(H)' = \mathscr{L}(H)$ (without proof).

(d) Schatten classes.

## 7.6 Operators associated with sesquilinear forms

Let $V$ and $H$ be two complex Hilbert spaces such that $V \subseteq H$ with continuous and dense embedding. A sesquilinear form $a : V \times V \to \mathbb{C}$ is called

- **bounded**, if there exists $C \ge 0$ such that

$$|a(u,v)| \le C \|u\|_V \|v\|_V \text{ for every } u, v \in V,$$

- **coercive**, if there exists $\eta > 0$ such that

$$\operatorname{Re} a(u,u) \ge \eta \|u\|_V^2 \text{ for every } u \in V,$$

- **accretive**, if

$$\operatorname{Re} a(u,u) \ge 0 \text{ for every } u \in V,$$

- $H$**-elliptic** or simply **elliptic**, if there exist $\omega \in \mathbb{R}$, $\eta > 0$ such that

$$\operatorname{Re} a(u,u) + \omega \|u\|_H^2 \ge \eta \|u\|_V^2 \text{ for every } u, v \in V,$$

- **symmetric**, if

$$a(u,v) = \overline{a(v,u)} \text{ for every } u, v \in V,$$

- **sectorial**, if there exists $\theta \in ]0, \frac{\pi}{2}[$ such that

$$a(u,u) \in \overline{\Sigma_\theta} \text{ for every } u \in V,$$

where

$$\Sigma_\theta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\},$$

is the open sector of opening (half-) angle $\theta$ around the positive real axis. Let us actually for the following also formally define the closed sector of angle 0:

$$\overline{\Sigma_0} := [0, \infty[.$$

Here are some easy observations.

**Lemma 7.38.** *Let $a : V \times V \to \mathbb{C}$ be a sesquilinear form. Then:*

(a) *The form a is bounded if and only if it is continuous.*

(b) *If a is bounded / continuous and coercive, then it is sectorial.*

*Proof.* The proof of assertion (a) is an easy exercise in the spirit of the corresponding result for linear forms / operators.

(b) If $a$ is bounded / continuous and coercive, then

$$|\operatorname{Im} a(u,u)| \leq |a(u,u)| \leq C\,\|u\|_V^2 \leq \frac{C}{\eta}\operatorname{Re} a(u,u),$$

which implies $a(u,u) \in \overline{\Sigma_\theta}$ with $\tan\theta = \frac{C}{\eta}$.

Given a bounded, sesquilinear form $a : V \times V \to \mathbb{C}$, we define the associated operator $A$ on $H$ by setting

$$\begin{aligned} \operatorname{dom} A &:= \{u \in V \,:\, \exists f \in H \forall v \in V : a(u,v) = \langle f, v\rangle\}, \\ Au &:= f. \end{aligned}$$

[1] The element $f$ in the definition of $\operatorname{dom} A$ is uniquely determined since $V$ is *dense* in $H$! In fact, if $u \in V$ and $f, \hat{f} \in H$ are such that $a(u,v) = \langle f, v\rangle$ and $a(u,v) = \langle \hat{f}, v\rangle$ for all $v \in V$, then $\langle f - \hat{f}, v\rangle = 0$ for all $v \in V$, that is, $f - \hat{f} \in V^\perp = \{0\}$.

The main result of this section is the following.

**Theorem 7.39.** *Let $a : V \times V \to \mathbb{C}$ be a bounded, coercive, sesquilinear form, and let $A$ be the associated operator on $H$. Let $\theta \in [0, \frac{\pi}{2}[$ be such that $a(u,u) \in \overline{\Sigma_\theta}$ for every $u \in V$ (such a $\theta$ exists by Lemma 7.38). Then $\sigma(A) \subseteq \overline{\Sigma_\theta}$ and, for every $\lambda \in \mathbb{C} \setminus \overline{\Sigma_\theta}$,*

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{dist}(\lambda, \overline{\Sigma_\theta})}.$$

**Lemma 7.40.** *Let $A$ be a linear operator on a Hilbert space $H$, and let $D$ be a connected subset of $\mathbb{C} \setminus \overline{w(A)}$. If there exists $\lambda_0 \in D$ such that $\lambda_0 - A$ is surjective, then $D \subseteq \rho(A)$ and, for every $\lambda \in D$,*

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{dist}(\lambda, \overline{w(A)})}. \tag{7.3}$$

*Proof.* Let $u \in \operatorname{dom} A$ be such that $\|u\| = 1$. Then, by the Cauchy-Schwarz inequality and by definition of the numerical range, for every $\lambda \in \mathbb{C}$,

[1] Viewing this operator as a relation, one has $A = \{(u,f) \in H \times H \,:\, u \in V \forall v \in V : a(u,v) = \langle f, v\rangle\}$

$$\begin{aligned}\|(\lambda - A)u\| &= \|(\lambda - A)u\| \, \|u\| \\ &\geq |\langle \lambda u - Au, u\rangle| \\ &= |\lambda \|u\|^2 - \langle Au, u\rangle| \\ &= |\lambda - \langle Au, u\rangle| \\ &\geq \inf\{|\lambda - z| \, : \, z \in w(A)\} \\ &= \operatorname{dist}(\lambda, w(A)) \\ &= \operatorname{dist}(\lambda, \overline{w(A)}) \, \|u\|.\end{aligned}$$

By homogeneity of the norm, the inequality

$$\|(\lambda - A)u\| \geq \operatorname{dist}(\lambda, \overline{w(A)}) \, \|u\|$$

then actually holds for every $u \in \operatorname{dom} A$. Hence, for every $\lambda \in \mathbb{C} \setminus \overline{w(A)}$, the operator $\lambda - A$ is injective and has closed range. Moreover, if $\lambda - A$ is in addition surjective, then $\lambda \in \rho(A)$ and the resolvent estimate (7.3) holds.

Now the rest of the proof is a usual argument using connectedness. Let $D' = \{\lambda \in D \, : \, \lambda - A \text{ is surjective}\}$. By assumption, $D'$ is a nonempty subset of $D$ which in turn is a subset of $\mathbb{C} \setminus \overline{w(A)}$. If $\lambda \in D'$, then by the preceding argument, $\lambda \in \rho(A)$ which is an open subset of $\mathbb{C}$ (Lemma 5.3) and it follows that $D'$ is an open subset of $D$. Finally, if $(\lambda_n)$ is a sequence in $D'$ which converges to some element $\lambda \in D$, then $(\lambda_n)$ is a sequence in $\rho(A)$ and $(R(\lambda_n, A))$ is bounded in $\mathscr{L}(X)$ by the resolvent estimate (7.3). By Lemma 5.3, $\lambda \in \rho(A)$. Since $\lambda$ also belongs to $D$, it follows that $\lambda \in D'$, and we have proved that $D'$ is closed in $D$. Since $D$ is connected, it follows that $D' = D$.

The following theorem generalises the Riesz-Fréchet theorem, which applies to inner products or *symmetric* sesquilinear forms, to the case of nonsymmetric sesquilinear forms.

**Theorem 7.41 (Lax-Milgram).** *Let $a : V \times V \to \mathbb{C}$ be a continuous, coercive, sesquilinear form on a complex Hilbert space $V$. Then, for every continuous, antilinear $f : V \to \mathbb{C}$ there exists a unique element $u \in V$ such that*

$$a(u, v) = \langle f, v\rangle_{V', V} \text{ for all } v \in V.$$

*Proof.* For every $u \in V$ we denote by $Au$ the continuous, antilinear form $V \to \mathbb{C}$, $v \mapsto a(u, v)$. We thus obtain a continuous, linear operator $A : V \to V'$ (where $V'$ is in this proof exceptionally the space of all continuous, *antilinear* forms on $V$). We have to show that $A$ is bijective.

*The operator $A$ is injective and has closed range.* By coercivity of $a$, for every $v \in V$,

$$\begin{aligned}\eta\,\|v\|_V^2 &\leq \operatorname{Re} a(v,v)\\ &\leq |a(v,v)|\\ &= |\langle Av, v\rangle_{V',V}|\\ &\leq \|Av\|_{V'}\,\|v\|_V,\end{aligned}$$

which implies

$$\eta\,\|v\|_V \leq \|Av\|_{V'} \text{ for every } v\in V.$$

As a consequence of this inequality, $A$ is injective and has closed range.

The operator $A$ is surjective. Since $A$ has closed range, it suffices to show that $A$ has dense range in $V'$. If the range of $A$ is not dense in $V'$, then, by the Hahn-Banach theorem and by reflexivity of $V$, there exists $u \in V\setminus\{0\}$ such that

$$\langle Av, u\rangle_{V',V} = a(v,u) = 0 \text{ for every } v\in V.$$

Choosing $v = u$ in this equality, we obtain a contradiction to coercivity of $a$. Hence, the range of $A$ is dense in $V'$, and the proof is complete.

**Remark 7.42.** The operator $A : V \to V'$ appearing in the proof of the Theorem of Lax-Milgram (sometimes called *Lemma* of Lax-Milgram) and the operator $A$ : $\operatorname{dom} A \to H$ associated with the sesquilinear form $a$ are of course not independent of each other.

*Proof (of Theorem 7.39).* By Lemma 7.38, the form $a$ is sectorial, that is, there exists $\theta \in [0, \frac{\pi}{2}[$ such that $a(u,u) \in \overline{\Sigma_\theta}$ for every $u \in H$. By definition of the operator $A$ associated with the form $a$, $\langle Au, u\rangle = a(u,u)$ for every $u \in \operatorname{dom} A$. Hence, the numerical range of $A$ is contained in $\overline{\Sigma_\theta}$.

Let $D = \mathbb{C}\setminus\overline{\Sigma_\theta}$. Then $D$ is a connected subset of $\mathbb{C}$. Let $f \in H$. Then the mapping $V \to \mathbb{C}$, $v \mapsto \langle f, v\rangle_H$ is continuous and antilinear. By the Lax-Milgram theorem, there exists a unique $u \in V$ such that $a(u,v) = \langle f, v\rangle_H$ for every $v \in V$. By definition of $A$ this is equivalent to saying that for every $f \in H$ there exists a unique $u \in \operatorname{dom} A$ such that $Au = f$. Hence, the operator $A$ is invertible, that is, $0 \in \rho(A)$. Since the resolvent set is open, the intersection $\rho(A)\cap D$ is therefore nonempty. By Lemma 7.40, this implies $D \subseteq \rho(A)$ and for every $\lambda \in \mathbb{C}\setminus\overline{\Sigma_\theta}$,

$$\begin{aligned}\|R(\lambda, A)\| &\leq \frac{1}{\operatorname{dist}(\lambda, \overline{w(A)})}\\ &\leq \frac{1}{\operatorname{dist}(\lambda, \overline{\Sigma_\theta})}.\end{aligned}$$

The proof is complete

**Lemma 7.43.** *Let $a : V\times V \to \mathbb{C}$ be a bounded, elliptic, sesquilinear form and let $A$ be the operator on $H$ associated with $a$. If the embedding $V \to H$ is compact, then $A$ has compact resolvent.*

*Proof.* The embedding $\operatorname{dom} A \to V$, $u \mapsto u$ ($\operatorname{dom} A$ is equipped with the graph norm) is bounded by the Closed Graph Theorem, and the embedding $V \to H$, $u \mapsto u$ is compact by assumption. Hence, the embedding $\operatorname{dom} A \to H$ is compact, and the claim now follows from Lemma 7.30.

## 7.7 * Elliptic partial differential operators

### 7.7.1 The Dirichlet-Laplace operator

Let $\Omega \subseteq \mathbb{R}^N$ be open, $\lambda \in \mathbb{C}$, and consider the elliptic partial differential equation

$$\begin{cases} \lambda u - \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{7.4}$$

where $\Delta = \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2}$ stands for the Laplace operator and $f \in L^2(\Omega)$. While the first line in (7.4) is a partial differential equation in which the unknown function $u$ and its partial derivatives (here, the second, not mixed partial derivatives) appear, the second line in (7.4) is a boundary condition. It is called (homogeneous) Dirichlet boundary condition.

Note that if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a **classical solution** of (7.4), that is, $u$ satisfies (7.4) in the usual sense (using classical partial derivatives), then we may multiply the first line in (7.4) by the complex conjugate of a test function $\varphi \in C_c^\infty(\Omega)$ and integrate over $\Omega$. An integration by parts (simple form of the theorem of Gauss, due to the Dirichlet boundary conditions) then yields

$$\begin{aligned} \int_\Omega f\bar{\varphi} &= \lambda \int_\Omega u\bar{\varphi} - \int_\Omega \Delta u\bar{\varphi} \\ &= \lambda \int_\Omega u\bar{\varphi} - \int_\Omega \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}\bar{\varphi} \\ &= \lambda \int_\Omega u\bar{\varphi} + \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i}\overline{\frac{\partial \varphi}{\partial x_i}} \\ &= \lambda \int_\Omega u\bar{\varphi} + \int_\Omega \nabla u\overline{\nabla\varphi}. \end{aligned}$$

Given $f \in L^2(\Omega)$, we now call a function $u \in H_0^1(\Omega)$ a **weak solution** of (7.4) if

$$\lambda \int_\Omega u\bar{\varphi} + \int_\Omega \nabla u\overline{\nabla\varphi} = \int_\Omega f\bar{\varphi} \text{ for every } \varphi \in H_0^1(\Omega). \tag{7.5}$$

Let $H := L^2(\Omega)$ and define

$$\begin{aligned} \operatorname{dom} A := \{u \in H_0^1(\Omega) : \exists f \in L^2(\Omega) \forall \varphi \in H_0^1(\Omega) : \\ \int_\Omega \nabla u \overline{\nabla \varphi} = \int_\Omega f \bar{\varphi}\} \\ Au := f, \end{aligned} \tag{7.6}$$

so that $A : \operatorname{dom} A \to L^2(\Omega)$ is a linear operator on $L^2(\Omega)$. Actually, $A$ is the operator associated with the sesquilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ given by

$$a(u,v) = \int_\Omega \nabla u \overline{\nabla v}.$$

By definition, $u \in \operatorname{dom} A$ and $Au = f$ if and only if $u$ is a weak solution of (7.4) for $\lambda = 0$. Moreover, a function $u \in H_0^1(\Omega)$ is a weak solution of (7.4) if and only if

$$u \in \operatorname{dom} A \text{ and } \lambda u + Au = f. \tag{7.7}$$

In this sense, we may say that $-A$ is the *realization* of the Laplace operator with Dirichlet boundary conditions. We therefore call $-A$ the **Dirichlet-Laplace operator**. The problem (7.7) is a *functional analytic* reformulation of (7.4). Instead of solving a partial differential equation we now have to solve an algebraic equation. Clearly, the operator $A$ is linear.

**Theorem 7.44.** *The form a is sesquilinear, bounded, accretive, $L^2$-elliptic and symmetric. As a consequence, the negative Dirichlet-Laplace operator A is selfadjoint, positive semidefinite.*

*Proof.* For every $u$, $v \in H_0^1(\Omega)$,

$$|a(u,v)| \leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \leq \|u\|_{H^1} \|v\|_{H^1},$$

so that $a$ is bounded. For every $u \in H_0^1(\Omega)$,

$$\operatorname{Re} a(u,u) = \int_\Omega |\nabla u|^2 \geq 0,$$

so that $a$ is accretive. For every $u \in H_0^1(\Omega)$,

$$\operatorname{Re} a(u,u) + \|u\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 = \|u\|_{H^1}^2,$$

so that $a$ is elliptic. For every $u$, $v \in H_0^1(\Omega)$,

$$a(u,v) = \int_\Omega \nabla u \overline{\nabla v} = \overline{\int_\Omega \nabla v \overline{\nabla u}} = \overline{a(v,u)},$$

so that $a$ is symmetric (and we could have left out the real parts in front of $a(u,u)$ above).

By the preceding theorem, the spectrum of the negative Dirichlet-Laplace operator $A$ is contained in the intervall $[0,\infty[$. One can show that if $\Omega = \mathbb{R}^N$, then $\sigma(A) = [0,\infty[$. For bounded domains, the situation is however much different.

**Theorem 7.45.** *If the domain $\Omega \subseteq \mathbb{R}^N$ is bounded, then there exists an orthonormal basis $(e_n)$ of $L^2(\Omega)$ and a sequence $(\lambda_n) \subset \mathbb{R}_-$ such that $\lim_{n\to\infty} \lambda_n = -\infty$ and for every $n \in \mathbb{N}$*

$$e_n \in \operatorname{dom} A \text{ and } \lambda_n e_n - Ae_n = 0.$$

*Moreover,* $\sigma(A) = \sigma_p(A) = \{\lambda_n : n \in \mathbb{N}\}$.

**Remark 7.46.** Theorem 7.45 gives also a description of the *spectrum* of the Dirichlet-Laplace operator $A$. Every spectral value is an eigenvalue. Every eigenspace is finite dimensional and there exists an orthonormal basis consisting only of eigenvectors. For every $\lambda \notin \sigma(A)$ and every $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ of (7.4).

Theorem 7.45 also implies that the Dirichlet-Laplace operator is unitarily equivalent to a multiplication operator on an $l^2$ space, that is, the Dirichlet-Laplace operator is *diagonalizable*.

In order to prove Theorem 7.45, we need the following theorem which will not be proved here. We only remark that in the case when $\Omega \subset \mathbb{R}$ is a bounded interval we have given a proof in Example 5.16. For a proof for general $\Omega$, see [**?**].

**Theorem 7.47 (Rellich-Kondrachov).** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Then the embedding*

$$H_0^1(\Omega) \to L^2(\Omega), \quad u \mapsto u,$$

*is compact.*

*Proof (of Theorem 7.45).* Let $u$, $v \in \operatorname{dom} A$. Then,

$$\begin{aligned} \langle Au, v\rangle_{L^2} = \int_\Omega Au\bar{v} \quad &= \quad -\int_\Omega \nabla u \overline{\nabla v} \\ = -\overline{\int_\Omega \nabla v \overline{\nabla u}} \quad &= \quad \overline{\int_\Omega Av\bar{u}} \\ = \overline{\langle Av, u\rangle_{L^2}} \quad &= \quad \langle u, Av\rangle. \end{aligned}$$

This equality means that $A$ is *symmetric*.

By Theorem 10.23 of Chapter 2, for every $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ of (7.4) with $\lambda = 1$. This means that $I - A : \operatorname{dom} A \to H$ is bijective. Let $J := (I-A)^{-1} : H \to \operatorname{dom} A \subseteq H$ be the inverse. For every $u$, $v \in H$, $u = u_1 - Au_1$, $v = v_1 - Av_1$, by the symmetry of $A$,

$$\langle Ju, v\rangle = \langle u_1, v_1 - Av_1\rangle = \langle u_1 - Au_1, v_1\rangle = \langle u, v_1\rangle = \langle u, Jv\rangle.$$

Hence, $J$ is symmetric. By the Theorem of Hellinger-Toeplitz (Theorem 7.7), $J : H \to H$ is bounded, and thus also selfadjoint. Since $J$ is also a linear operator from

$H$ into $H^1_0(\Omega)$, and since $J$ is closed when considered as such an operator, we obtain in fact that $J : H \to H^1_0(\Omega)$ is bounded by the closed graph theorem. Since the embedding $H^1_0(\Omega) \to L^2(\Omega)$ is compact by the Rellich-Kondrachov theorem, we obtain that $J \in \mathscr{K}(H)$.

By the spectral theorem for selfadjoint compact operators, there exists an orthonormal basis $(e_n)$ of $H = L^2(\Omega)$ and a sequence $(\mu_n) \subset \mathbb{R}$ such that $\lim_{n\to\infty} \mu_n = 0$ and

$$\mu_n e_n = Je_n \text{ for every } n \in \mathbb{N}.$$

Since $\operatorname{ran} J = \operatorname{dom} A$, we obtain also that $e_n \in \operatorname{dom} A$ for every $n \in \mathbb{N}$. Multiplying the above equation by $I - A$, we obtain

$$\lambda_n e_n - Ae_n = 0 \text{ for every } n \in \mathbb{N},$$

with $\lambda_n := \frac{\mu_n - 1}{\mu_n} \in \mathbb{R}$. Since, by Theorem 10.23 of Chapter 2, $\lambda - A$ is invertible for every $\lambda > 0$, we obtain $\lambda_n \in \mathbb{R}_-$. Clearly, the sequence $(\lambda_n)$ is unbounded since $\mu_n \to 0$.

Now let $\lambda \notin \{\lambda_n : n \in \mathbb{N}\}$, and let $f \in L^2(\Omega)$. If $\lambda = 1$ (or even $\lambda > 0$), then we have seen above that the operator $\lambda - A : \operatorname{dom} A \to H$ is bijective. So we can assume that $\lambda \neq 1$. Then $\frac{1}{1-\lambda} \in \rho(J)$ and we can define $u := R(1,A)R(\frac{1}{1-\lambda}, J)\frac{f}{\lambda - 1}$. Clearly, $u \in \operatorname{dom} A$, and an easy calculation shows that $\lambda u - Au = f$. Moreover, every solution of $\lambda u - Au = f$ is of the form above, and thus $\lambda - A$ is bijective.

The claim is proved.

**Corollary 7.48.** *The operator A is closed and*

$$\operatorname{dom} A = \{u \in L^2(\Omega) : (\lambda_n \langle u, e_n \rangle) \in \ell^2\}.$$

*Proof.* If an operator $A : X \supseteq \operatorname{dom} A \to X$ on a Banach space $X$ has nonempty resolvent set, then $A$ is necessarily closed. In fact, $(\lambda - A)^{-1}$ is bounded for some $\lambda \in \rho(A) \neq \emptyset$; in particular, $(\lambda - A)^{-1}$ is closed, and thus $\lambda - A$ is closed.

Note that the Dirichlet-Laplace operator $A$ defined above has nonempty resolvent set by Theorem 7.45, and thus $A$ is closed.

The remaining claim follows easily from the fact that, by Theorem 7.45, $A$ is unitarily equivalent to the (unbounded) multiplication operator

$$\begin{aligned} \operatorname{dom} M &:= \{(x_n) \in l^2 : (\lambda_n x_n) \in \ell^2\}, \\ M(x_n) &:= (\lambda_n x_n), \end{aligned}$$

where the unitary operator is given by

$$\begin{aligned} U : L^2(\Omega) &\to \ell^2, \\ u &\mapsto (\langle u, e_n \rangle), \end{aligned}$$

that is, $A = U^{-1}MU$.

### 7.7.2 General elliptic operators in divergence form and inhomogeneous Dirichlet boundary conditions

Consider the elliptic operator $L$ which is formally given by

$$Lu = -\sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j u) + \sum_{i=1}^{N} [\partial_i(b_i(x)u) + c_i(x)\partial_i u] + d(x)u,$$

where

$$a_{ij}, b_i, c_i, d \in L^\infty(\Omega) \quad (1 \le i, j \le N),$$

and $\Omega \subseteq \mathbb{R}^N$ is an open set. We assume that the coefficients $a_{ij}$ are **uniformly elliptic** in the sense that there exists $\eta > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_j\overline{\xi_i} \ge \eta\,|\xi|^2 \text{ for all } \xi \in \mathbb{C}^N, x \in \Omega. \tag{7.8}$$

There are no further conditions on the lower order conditions $b_i$, $c_i$ and $d$, and in fact, the boundedness condition on these coefficients may be relaxed a little bit. All coefficients may be complex valued. We then consider the problem

$$\begin{aligned} \lambda u + Lu &= f + \operatorname{div} g \text{ in } \Omega, \\ u &= h \text{ on } \partial\Omega, \end{aligned} \tag{7.9}$$

where

$$\begin{aligned} &f, g_1, \dots, g_N \in L^2(\Omega) \text{ and} \\ &h \in H^1(\Omega). \end{aligned}$$

We say that a function $u \in H^1(\Omega)$ is a **weak solution** of the problem

$$\lambda u - Lu = f + \operatorname{div} g \text{ in } \Omega, \tag{7.10}$$

if, for all $\varphi \in \mathrm{C}_c^\infty(\Omega)$,

$$\begin{aligned} &\lambda \int_\Omega u\bar{\varphi} + \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x)\partial_j u\overline{\partial_i\varphi} + \\ &+ \sum_{i=1}^{N} \int_\Omega [-b_i(x)u\overline{\partial_i\varphi} + c_i(x)\partial_i u\bar{\varphi} + \int_\Omega d(x)u\bar{\varphi} \\ &= \int_\Omega f\bar{\varphi} + \sum_{i=1}^{N} \int_\Omega g_i\overline{\partial_i\varphi}. \end{aligned} \tag{7.11}$$

Note that by an approximation argument, if the above equality holds for all test functions $\varphi \in \mathrm{C}_c^\infty(\Omega)$, then it holds for all $\varphi \in H_0^1(\Omega)$, and vice versa. Next, we say that the inhomogeneous Dirichlet boundary condition

$$u = h \text{ on } \partial\Omega \tag{7.12}$$

is satisfied, if

$$u - h \in H_0^1(\Omega). \tag{7.13}$$

Accordingly, $u \in H^1(\Omega)$ is a weak solution of (7.9) if it satisfies both (7.11) and (7.13).

**Theorem 7.49.** *Assume that $\Omega$, $a_{ij}$, $b_i$, $c_i$, $d$, $f$, $g_i$ and $h$ are as above. Then there exists a real number $\hat{\lambda}$ such that for all $\lambda \in \mathbb{C}$ with* $\operatorname{Re}\lambda > \hat{\lambda}$ *the problem* (7.9) *admits a unique weak solution $u \in H^1(\Omega)$.*

*Proof.* Assume first that $h = 0$. Then we define the sesquilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ by

$$\begin{aligned} a(u,v) = \lambda \int_\Omega u\bar{v} + \sum_{i,j=1}^N \int_\Omega a_{ij}(x)\partial_j u\overline{\partial_i v} + \\ + \sum_{i=1}^N \int_\Omega [-b_i(x)u\overline{\partial_i v} + c_i(x)\partial_i u\bar{v} + \int_\Omega d(x)u\bar{v}. \end{aligned}$$

Then $a$ is continuous since for every $u$, $v \in H_0^1(\Omega)$, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |a(u,v)| \le |\lambda| \, \|u\|_{L^2} \|v\|_{L^2} + \sum_{i,j=1}^N \|a_{ij}\|_{L^\infty} \|\partial_j u\|_{L^2} \|\partial_i v\|_{L^2} + \\ + \sum_{i=1}^N [\|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_i v\|_{L^2} + \|c_i\|_{L^\infty} \|\partial_i u\|_{L^2} \|v\|_{L^2}] + \|d\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2} \\ \le C \|u\|_{H^1} \|v\|_{H^1}, \end{aligned}$$

where $C \ge 0$ is for example the sum of $|\lambda|$ and the $L^\infty$-norms of the coefficients $a_{ij}$, $b_i$, $c_i$, $d$.

We show that $a$ is also coercive whenever

$$\operatorname{Re}\lambda > \hat{\lambda} := \|d\|_{L^\infty} + \frac{1}{2\eta} \sum_{i=1}^N (\|b_i\|_{L^\infty}^2 + \|c_i\|_{L^\infty}^2).$$

In fact, for every such $\lambda \in \mathbb{C}$ and every $u \in H_0^1(\Omega)$, by the uniform ellipticity condition, by the Cauchy-Schwarz inequality and by Young's inequality,

$$\begin{aligned}
\operatorname{Re} a(u,u) &= \operatorname{Re}\lambda \int_\Omega |u|^2 + \operatorname{Re}\sum_{i,j=1}^N \int_\Omega a_{ij}\partial_j u \overline{\partial_i u} + \\
&\quad + \operatorname{Re}\sum_{i=1}^N \int_\Omega [-b_i(x) u \overline{\partial_i u} + c_i(x)\partial_i u \bar{u} + \int_\Omega \operatorname{Re} d(x)|u|^2 \\
&\geq (\operatorname{Re}\lambda - \|d\|_{L^\infty})\,\|u\|_{L^2}^2 + \eta\,\|\nabla u\|_{L^2}^2 - \\
&\quad - \left(\sum_{i=1}^N (\|b_i\|_{L^\infty}^2 + \|c_i\|_{L^\infty}^2)\right)^{\frac{1}{2}} \|u\|_{L^2}\|\nabla u\|_{L^2} \\
&\geq (\operatorname{Re}\lambda - \hat{\lambda})\|u\|_{L^2}^2 + \frac{\eta}{2}\|\nabla u\|_{L^2}^2 \\
&\geq \tilde{\eta}\,\|u\|_{H_0^1}^2,
\end{aligned}$$

where $\tilde{\eta} = \min\{\operatorname{Re}\lambda - \hat{\lambda}, \frac{\eta}{2}\} > 0$. Hence, $a$ is coercive.

Consider next the mapping $\ell : H^1(\Omega) \to \mathbb{C}$ given by

$$\ell(v) = \int_\Omega f\bar{v} - \sum_{i=1}^N \int_\Omega g_i \overline{\partial_i v},$$

which is well defined and continuous by the Cauchy-Schwarz inequality, and anti-linear. Existence and uniqueness of a weak solution of (7.9) thus follows from the Lax-Milgram lemma (Lemma 7.41) applied to $a$ and $\ell$.

Let now $h \in H^1(\Omega)$ be arbitrary, and define

$$\hat{f} := f + \lambda h + \sum_{i=1}^N c_i \partial_i h + dh \in L^2(\Omega) \text{ and}$$

$$\hat{g}_i := g_i + \sum_{j=1}^N a_{ij}\partial_j h + b_i h \in L^2(\Omega) \quad (1 \leq i \leq N).$$

Then one easily verifies that $u \in H^1(\Omega)$ is a weak solution of (7.9) if and only if $w = u - h \in H_0^1(\Omega)$ is a weak solution of

$$\begin{aligned}
\lambda w - Lw &= \hat{f} + \operatorname{div}\hat{g} \text{ in } \Omega, \\
w &= 0 \text{ on } \partial\Omega,
\end{aligned}$$

and from this equivalence and the first step one obtains existence and uniqueness of a weak solution of (7.9).

## 7.8 * The heat equation

Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and consider the **heat equation**

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{in } \mathbb{R}_+ \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{7.14}$$

where $\Delta$ denotes the Laplace operator, and $u_0 \in L^2(\Omega)$.

We call a function $u \in C(\mathbb{R}_+; L^2(\Omega))$ a **mild solution** of (7.14) if $u(0) = u_0$ and if for every $\varphi \in \operatorname{dom} A$ the function $t \mapsto \langle u(t), \varphi \rangle_{L^2}$ is continuously differentiable and if

$$\frac{d}{dt}\langle u, \varphi \rangle_{L^2} = \langle u, A\varphi \rangle_{L^2}.$$

Here, $A$ is the realization of the Dirichlet-Laplace operator on $L^2(\Omega)$ defined in (7.6).

**Theorem 7.50.** *For every $u_0 \in L^2(\Omega)$ there exists a unique mild solution $u$ of* (7.14).

*Proof.* Let $A$ be the realization of the Dirichlet-Laplace operator as defined in the previous section. By Theorem 7.45, there exists an orthonormal basis $(e_n)$ and an unbounded sequence $(\lambda_n) \subset \mathbb{R}_-$ such that for every $n \in \mathbb{N}$ one has $\lambda_n e_n = Ae_n$.

Assume that $u$ is a mild solution of the heat equation (7.14). Then, for every $n \in \mathbb{N}$,

$$\frac{d}{dt}\langle u(t), e_n \rangle_{L^2} = \langle u(t), Ae_n \rangle_{L^2} = \lambda_n \langle u(t), e_n \rangle_{L^2}.$$

This implies

$$\langle u(t), e_n \rangle_{L^2} = e^{\lambda_n t} \langle u_0, e_n \rangle_{L^2}, \quad t \geq 0.$$

Hence, since $(e_n)$ is an orthonormal basis,

$$u(t) = \sum_{n\in\mathbb{N}} e^{\lambda_n t} \langle u_0, e_n \rangle_{L^2} \, e_n, \quad t \geq 0. \tag{7.15}$$

This proves uniqueness of mild solutions.

On the other hand, let $u_0 \in L^2(\Omega)$ and define $u(t)$ as in (7.15). Since $|e^{\lambda_n t}| \leq 1$ for every $t \geq 0$ and since $t \mapsto e^{\lambda_n t}$ is continuous, $u(t) \in L^2(\Omega)$ for every $t \geq 0$, and the function $t \mapsto u(t)$, $\mathbb{R}_+ \to L^2(\Omega)$ is continuous. Moreover, $u(0) = u_0$.

Let $\varphi \in \operatorname{dom} A$. By Corollary 7.48, $(\lambda_n \langle \varphi, e_n \rangle) \in \ell^2$. As a consequence, $t \mapsto \langle u(t), \varphi \rangle_{L^2}$ is continuously differentiable and, by the symmetry of $A$,

$$\begin{aligned} \frac{d}{dt}\langle u, \varphi \rangle_{L^2} &= \sum_{n\in\mathbb{N}} \lambda_n e^{\lambda_n t} \langle u_0, e_n \rangle_{L^2} \, \langle e_n, \varphi \rangle_{L^2} \\ &= \sum_{n\in\mathbb{N}} e^{\lambda_n t} \langle u_0, e_n \rangle_{L^2} \, \langle Ae_n, \varphi \rangle_{L^2} \\ &= \sum_{n\in\mathbb{N}} e^{\lambda_n t} \langle u_0, e_n \rangle_{L^2} \, \langle e_n, A\varphi \rangle_{L^2} \\ &= \langle u, A\varphi \rangle_{L^2}, \quad t \geq 0. \end{aligned}$$

This proves existence of mild solutions.

**Remark 7.51.** The concrete form (7.15) of the solution $u$ of the heat equation (7.14) allows us to prove that in fact

$$u \in C^\infty((0,\infty); L^2(\Omega)),$$

or even

$$u \in C^\infty((0,\infty); \operatorname{dom} A^k) \text{ for every } k \geq 1,$$

where $\operatorname{dom} A^k$ is the domain of $A^k$ equipped with the graph norm. The heat equation thus has a regularizing effect in space and time; even if $u_0$ belongs 'only' to $L^2(\Omega)$, then $u(t)$ belongs already to $\operatorname{dom} A^k$ for every $k \geq 1$. Moreover, the solution is $C^\infty$ with values in $\operatorname{dom} A^k$ for every $k \geq 1$.

## 7.9 * The wave equation

Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and consider the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{in } \mathbb{R}_+ \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u_t(0,x) = u_1(x) & \text{in } \Omega, \end{cases} \tag{7.16}$$

where $\Delta$ denotes the Laplace operator, $u_0 \in H_0^1(\Omega)$, and $u_1 \in L^2(\Omega)$.

We call a function $u \in C(\mathbb{R}_+; H_0^1(\Omega)) \cap C^1(\mathbb{R}_+; L^2(\Omega))$ a *mild solution* of (7.16) if $u(0) = u_0$, $u_t(0) = u_1$, if for every $\varphi \in H_0^1(\Omega)$ the function $t \mapsto \langle u, \varphi \rangle_{L^2}$ is twice continuously differentiable and if

$$\frac{d^2}{dt^2} \langle u(t), \varphi \rangle_{L^2} + \int_\Omega \nabla u(t) \overline{\nabla \varphi} = 0.$$

**Theorem 7.52.** *For every $u_0 \in H_0^1(\Omega)$ and every $u_1 \in L^2(\Omega)$ there exists a unique mild solution of* (7.16).

For the proof of Theorem 7.52, we need the following result which we shall not prove here; compare with Corollary 7.48.

**Lemma 7.53.** *Let $A$ be the Dirichlet-Laplace operator as defined in* (7.6)*, and let $(e_n)$ and $(\lambda_n)$ be as in Theorem 7.45. Then*

$$H_0^1(\Omega) = \{u \in L^2(\Omega) : (\sqrt{-\lambda_n} \langle u, e_n \rangle) \in \ell^2\}.$$

*Proof (of Theorem 7.52).* Let $A$ be the realization of the Dirichlet-Laplace operator as defined in Section 7.7. By Theorem 7.45, there exists an orthonormal basis $(e_n)$

and an unbounded sequence $(\lambda_n) \subset \mathbb{R}_-$ such that for every $n \in \mathbb{N}$ one has $\lambda_n e_n = Ae_n$.

Assume that $u$ is a mild solution of the wave equation (7.16). Then, for every $n \in \mathbb{N}$,

$$\frac{d^2}{dt^2} \langle u(t), e_n \rangle_{L^2} = \langle u(t), Ae_n \rangle_{L^2} = \lambda_n \langle u(t), e_n \rangle_{L^2}.$$

Setting $\alpha_n := \sqrt{-\lambda_n}$, this implies

$$\langle u(t), e_n \rangle_{L^2} = \cos(\alpha_n t) \langle u_0, e_n \rangle_{L^2} + \frac{1}{\alpha_n} \sin(\alpha_n t) \langle u_1, e_n \rangle_{L^2}, \quad t \geq 0.$$

Hence, since $(e_n)$ is an orthonormal basis,

$$u(t) = \sum_{n \in \mathbb{N}} \cos(\alpha_n t) \langle u_0, e_n \rangle_{L^2} e_n + \sum_{n \in \mathbb{N}} \frac{1}{\alpha_n} \sin(\alpha_n t) \langle u_1, e_n \rangle_{L^2} e_n, \quad t \geq 0. \tag{7.17}$$

This proves uniqueness of mild solutions.

On the other hand, let $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, and define $u(t)$ as in (7.17). Since $|\cos(\alpha_n t)| \leq 1$ and $|\sin(\alpha_n t)| \leq 1$ for every $t \geq 0$ and since cos and sin are continuous, by Lemma 7.53, $u(t) \in H_0^1(\Omega)$ for every $t \geq 0$, and the function $t \mapsto u(t)$, $\mathbb{R}_+ \to H_0^1(\Omega)$ is continuous. Moreover, $u(0) = u_0$. By the same reasons, $t \mapsto u(t)$, $\mathbb{R}_+ \to L^2(\Omega)$ is continuously differentiable and $u_t(0) = u_1$.

Let $\varphi \in H_0^1(\Omega)$. By Lemma 7.53, $(\alpha_n \langle \varphi, e_n \rangle) \in \ell^2$. As a consequence, $t \mapsto \langle u(t), \varphi \rangle$ is twice continuously differentiable and, by the symmetry of $A$,

$$\begin{aligned}
\frac{d^2}{dt^2} \langle u, \varphi \rangle &= -\sum_{n \in \mathbb{N}} \lambda_n \cos(\alpha_n t) \langle u_0, e_n \rangle_{L^2} \langle e_n, \varphi \rangle_{L^2} - \\
&\quad - \sum_{n \in \mathbb{N}} \alpha_n \sin(\alpha_n t) \langle u_1, e_n \rangle_{L^2} \langle e_n, \varphi \rangle_{L^2} \\
&= -\sum_{n \in \mathbb{N}} \cos(\alpha_n t) \langle u_0, e_n \rangle_{L^2} \langle Ae_n, \varphi \rangle_{L^2} - \\
&\quad - \sum_{n \in \mathbb{N}} \frac{1}{\alpha_n} \sin(\alpha_n t) \langle u_1, e_n \rangle_{L^2} \langle Ae_n, \varphi \rangle_{L^2} \\
&= -\sum_{n \in \mathbb{N}} \cos(\alpha_n t) \langle u_0, e_n \rangle_{L^2} \int_\Omega \nabla e_n \nabla \varphi - \\
&\quad - \sum_{n \in \mathbb{N}} \frac{1}{\alpha_n} \sin(\alpha_n t) \langle u_1, e_n \rangle_{L^2} \int_\Omega \nabla e_n \nabla \varphi \\
&= -\int_\Omega \nabla u \nabla \varphi, \quad t \geq 0.
\end{aligned}$$

This proves existence of mild solutions.

**Remark 7.54.** The concrete form (7.17) of the solution $u$ of the wave equation (7.16) shows that it can be uniquely extended to a solution $u$ defined on $\mathbb{R}$. However, for the wave equation (7.16) there is no regularizing effect as for the heat equation (7.14).

## 7.10 * The Schrödinger equation

Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and consider the **Schrödinger equation**

$$\begin{cases} u_t - i\Delta u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{in } \mathbb{R}_+ \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{7.18}$$

where $\Delta$ denotes the Laplace operator, $i = \sqrt{-1}$ is the complex unity, and $u_0 \in L^2(\Omega)$.

We call a function $u \in C(\mathbb{R}_+; L^2(\Omega))$ a **mild solution** of (7.18) if $u(0) = u_0$ and if for every $\varphi \in \operatorname{dom} A$ the function $t \mapsto \langle u, \varphi \rangle_{L^2}$ is continuously differentiable and if

$$\frac{d}{dt} \langle u, \varphi \rangle_{L^2} = i \langle u, A\varphi \rangle_{L^2}, \quad t \geq 0.$$

Here, $A$ is the realization of the Dirichlet-Laplace operator on $L^2(\Omega)$ defined in (7.6).

**Theorem 7.55.** *For every $u_0 \in L^2(\Omega)$ there exists a unique mild solution $u$ of* (7.18).

*Proof.* Let $A$ be the realization of the Dirichlet-Laplace operator as defined in (7.6). By Theorem 7.45, there exists an orthonormal basis $(e_n)$ and an unbounded sequence $(\lambda_n) \subset \mathbb{R}_-$ such that for every $n \in \mathbb{N}$ one has $\lambda_n e_n = A e_n$.

Assume that $u$ is a mild solution of the Schrödinger equation (7.18). Then, for every $n \in \mathbb{N}$,

$$\frac{d}{dt} \langle u(t), e_n \rangle_{L^2} = i \langle u(t), A e_n \rangle_{L^2} = i\lambda_n \langle u(t), e_n \rangle_{L^2}.$$

This implies

$$\langle u(t), e_n \rangle_{L^2} = e^{i\lambda_n t} \langle u_0, e_n \rangle_{L^2}, \quad t \geq 0.$$

Hence, since $(e_n)$ is an orthonormal basis,

$$u(t) = \sum_{n \in \mathbb{N}} e^{i\lambda_n t} \langle u_0, e_n \rangle_{L^2} e_n, \quad t \geq 0. \tag{7.19}$$

This proves uniqueness of mild solutions.

On the other hand, let $u_0 \in L^2(\Omega)$ and define $u(t)$ as in (7.19). Since $|e^{i\lambda_n t}| \leq 1$ for every $t \geq 0$ and since $t \mapsto e^{i\lambda_n t}$ is continuous, $u(t) \in L^2(\Omega)$ for every $t \geq 0$, and the function $t \mapsto u(t)$, $\mathbb{R}_+ \to L^2(\Omega)$ is continuous. Moreover, $u(0) = u_0$.

Let $\varphi \in \operatorname{dom} A$. By Corollary 7.48, $(\lambda_n \langle \varphi, e_n \rangle) \in \ell^2$. As a consequence, $t \mapsto \langle u(t), \varphi \rangle_{L^2}$ is continuously differentiable and, by the symmetry of $A$,

$$\begin{aligned}\frac{d}{dt}\langle u, \varphi \rangle_{L^2} &= \sum_{n\in\mathbb{N}} i\lambda_n e^{i\lambda_n t} \langle u_0, e_n \rangle_{L^2} \langle e_n, \varphi \rangle_{L^2} \\ &= i \sum_{n\in\mathbb{N}} e^{i\lambda_n t} \langle u_0, e_n \rangle_{L^2} \langle A e_n, \varphi \rangle_{L^2} \\ &= i \sum_{n\in\mathbb{N}} e^{i\lambda_n t} \langle u_0, e_n \rangle_{L^2} \langle e_n, A\varphi \rangle_{L^2} \\ &= i \langle u, A\varphi \rangle_{L^2}, \quad t \geq 0.\end{aligned}$$

This proves existence of mild solutions.

**Remark 7.56.** The concrete form (7.19) of the solution $u$ of the Schrödinger equation (7.18) shows that it can be uniquely extended to a solution $u$ defined on $\mathbb{R}$. However, similarly as for the wave equation (7.16), there is no regularizing effect for the Schrödinger equation (7.18).

# Chapter 8
# $C_0$-semigroups

## 8.1 $C_0$-semigroups and their generators

Let $X$ be a Banach space. A $C_0$**-semigroup** or **strongly continuous semigroup** is a function $T : \mathbb{R}_+ \to \mathscr{L}(X)$ satisfying the following three properties:

(a) $T(0) = I$,

(b) $T(t+s) = T(t)T(s)$ for every $t$, $s \in \mathbb{R}_+$,

(c) $\lim_{t\to 0+} T(t)x = x$ for every $x \in X$.

Sometimes we also write $(T(t))_{t\geq 0}$ instead of $T$. The first two properties are actually the semigroup properties; they say that $T$ is a semigroup homomorphism from the additive semigroup $\mathbb{R}_+$ into the space of bounded, linear operators $\mathscr{L}(X)$. A semigroup satisfies therefore a functional equation like the exponential function, and we will see that it actually shares many properties of exponential functions, the most important being the property that the exponential function solves a linear ordinardy differential equation. The third property is the strong continuity of the semigroup at time $t = 0$. We will see below that this already implies the strong continuity on $\mathbb{R}_+$. First, we show the following fundamental estimate.

**Lemma 8.1.** *Let $T$ be a $C_0$-semigroup on a Banach space $X$. Then there exist constants $M \geq 1$, $\omega \in \mathbb{R}$ such that, for every $t \in \mathbb{R}_+$,*

$$\|T(t)\| \leq M e^{\omega t}.$$

*Proof.* For every null sequence $(t_n)$ in $\mathbb{R}_+$ and for every $x \in X$, the sequence $(T(t_n)x)$ is convergent by strong continuity of the semigroup (property (c)). In particular, this sequence is bounded for every $x$. By the uniform boundedness principle, the sequence $(T(t_n))$ is bounded in $\mathscr{L}(X)$. Since the sequence $(t_n)$ is arbitrary, it follows that there exists a time $t_0 > 0$ such that $M := \sup_{0\leq t\leq t_0} \|T(t)\| \geq 1$ is finite.

Now let $t \in \mathbb{R}_+$. Then $t = nt_0 + s$ for some $n \in \mathbb{N}_0$, $s \in [0,t_0)$. By the semigroup property, it follows that

$$\begin{aligned}\|T(t)\| &= \|T(nt_0+s)\| \\ &= \|T(t_0)^n T(s)\| \\ &\le \|T(t_0)\|^n \, \|T(s)\| \\ &\le M^{n+1} \\ &= M\, e^{nt_0 \frac{\log M}{t_0}} \\ &\le M\, e^{\omega t},\end{aligned}$$

where $\omega := \frac{\log M}{t_0}$.

**Lemma 8.2.** *Let $T$ be a $C_0$-semigroup on a Banach space $X$. Then, for every $x \in X$, the function $t \mapsto T(t)x$ is continuous on $\mathbb{R}_+$.*

*Proof.* Let $M \ge 1$ and $\omega \in \mathbb{R}$ be as in Lemma 8.1. Fix $x \in X$. Then, for every $t \in \mathbb{R}_+$ and $h > 0$, $\|T(t+h)x - T(t)x\| = \|T(t)(T(h)x - x)\| \le Me^{\omega t}\|T(h)x - x\| \to 0$ as $h \to 0+$. Hence, $T(\cdot)x$ is continuous from the right at $t \in \mathbb{R}_+$. On the other hand, as long as $t - h \ge 0$, $\|T(t-h)x - T(t)x\| = \|T(t-h)(x - T(h)x)\| \le Me^{\omega(t-h)}\|x - T(h)x\| \to 0$ as $h \to 0+$. This proves that $T(\cdot)x$ is also continuous from the left at $t \in \mathbb{R}_+$. Altogether, $T(\cdot)x$ is continuous on $\mathbb{R}_+$.

Let $T$ be a $C_0$-semigroup on a Banach space $X$. Then we define the **(infinitesimal) generator** $A$ of $T$ by setting

$$\begin{aligned}\operatorname{dom} A &:= \{x \in X : \lim_{t \to 0+} \frac{T(t)x - x}{t} \text{ exists}\}, \\ Ax &:= \lim_{t \to 0+} \frac{T(t)x - x}{t} \quad (x \in \operatorname{dom} A).\end{aligned}$$

**Theorem 8.3.** *Let $T$ be a $C_0$-semigroup on a Banach space $X$, with generator $A$. Then:*

(a) *For every $x \in X$, $\lim_{h \to 0+} \frac{1}{h} \int_t^{t+h} T(s)x \, \mathrm{d}s = T(t)x$.*

(b) *For every $x \in X$, $\int_0^t T(s)x \, \mathrm{d}s \in \operatorname{dom} A$ and*

$$A\left(\int_0^t T(s)x \, \mathrm{d}s\right) = T(t)x - x.$$

(c) *For every $x \in \operatorname{dom} A$ and every $t \in \mathbb{R}_+$, $T(t)x \in \operatorname{dom} A$ and*

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax.$$

(d) *For every $x \in \operatorname{dom} A$ and for every $0 \le s \le t$,*

$$T(t)x - T(s)x = \int_s^t T(r)Ax \, \mathrm{d}r = \int_s^t AT(r) \, \mathrm{d}r.$$

*Proof.* Assertion (a) follows from the continuity of $T(\cdot)x$ (Lemma 8.2).

By assertion (a), for every $t \in \mathbb{R}_+$ and $h > 0$,

$$\begin{aligned}\frac{T(h)-I}{h}\int_0^t T(s)x\,\mathrm{d}s &= \frac{1}{h}\left[\int_0^t T(h)T(s)x\,\mathrm{d}s - \int_0^t T(s)x\,\mathrm{d}s\right]\\ &= \frac{1}{h}\left[\int_0^t T(s+h)x\,\mathrm{d}s - \int_0^t T(s)x\,\mathrm{d}s\right]\\ &= \frac{1}{h}\left[\int_h^{t+h} T(s)x\,\mathrm{d}s - \int_0^t T(s)x\,\mathrm{d}s\right]\\ &= \frac{1}{h}\left[\int_t^{t+h} T(s)x\,\mathrm{d}s - \int_0^h T(s)x\,\mathrm{d}s\right]\\ &\to T(t)x - x \text{ as } h \to 0+,\end{aligned}$$

and assertion (b) follows.

In order to prove (c), let $x \in \operatorname{dom}A$, $t \in \mathbb{R}_+$ and $h > 0$. Then

$$\begin{aligned}\frac{T(h)T(t)x - T(t)x}{h} &= \frac{T(t+h)x - T(t)x}{h}\\ &= \frac{T(t)T(h)x - T(t)x}{h}\\ &= T(t)\frac{T(h)x - x}{h}\\ &\to T(t)Ax \text{ as } h \to 0+.\end{aligned}$$

Thus, $T(t)x \in \operatorname{dom}A$ and $AT(t)x = T(t)Ax$. The first line, however, alo shows that $T(\cdot)x$ is differentiable at $t$ and that $\frac{d}{dt}T(t)x = AT(t)x$.

Part (d) follows by integrating the identity from (c) from $s$ to $t$.

**Theorem 8.4.** *If $A$ is the infinitesimal generator of a $C_0$-semigroup $T$, then $A$ is closed. Moreover, if* $\operatorname{dom}A^n$ *denotes the domain of $A^n$, then* $\bigcap_{n\geq 1}\operatorname{dom}A^n$ *is dense in $X$. In particular,* $\operatorname{dom}A$ *is dense in $X$.*

*Proof.* Let $x_n \in \operatorname{dom}A$ be such that $x_n \to x$ and $Ax_n \to y$. For every $n$ and every $t \in \mathbb{R}_+$, by Theorem 8.3 (d),

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n\,\mathrm{d}s$$

Letting $n \to \infty$ yields, by the continuity of the operators $T(s)$,

$$T(t)x - x = \int_0^t T(s)y\,\mathrm{d}s.$$

Dividing this equality by $t > 0$, and by Theorem 8.3 (a),

$$\lim_{t\to 0+}\frac{T(t)x - x}{t} = y,$$

that is, $x \in \operatorname{dom} A$ and $Ax = y$ Hence, $A$ is closed.

By Theorem 8.3 (b) and (a), for every $x \in X$, $\frac{1}{t}\int_0^t T(s)x \, \mathrm{d}s \in \operatorname{dom} A$ and $\lim_{t\to 0+} \frac{1}{t}\int_0^t T(s)x \, \mathrm{d}s = x$. Hence, $\operatorname{dom} A$ is dense in $X$.

This argument can be slightly changed to obtain the stronger statement about the density of $\bigcap_{n\geq 1} \operatorname{dom} A^n$. In fact, let $x \in X$ and let $\varphi \in C_c^\infty((0,\infty))$ be a test function. Set $x_\varphi = \int_0^\infty \varphi(s)T(s)x \, \mathrm{d}s$. Then

$$\begin{aligned} \frac{T(h)x_\varphi - x_\varphi}{h} &= \int_0^\infty \varphi(s)\frac{T(h)T(s)x - T(s)x}{h} \, \mathrm{d}s \\ &= \int_0^\infty \varphi(s)\frac{T(s+h)x - T(s)x}{h} \, \mathrm{d}s \\ &= \int_0^\infty \frac{\varphi(s-h) - \varphi(s)}{h} T(s)x \, \mathrm{d}s \\ &\to -\int_0^\infty \varphi'(s)T(s)x \, \mathrm{d}s, \end{aligned}$$

that is, $x_\varphi \in \operatorname{dom} A$ and $Ax_\varphi = -x_{\varphi'}$. However, this argument can be iterated and it follows that, for every $n \geq 1$, $x_\varphi \in \operatorname{dom} A^n$ and $A^n x_\varphi = (-1)^n x_{\varphi^{(n)}}$. Hence, $x_\varphi \in \bigcap_{n\geq 1} \operatorname{dom} A^n$. Now choose a (positive) test function $\varphi \in C_c^\infty((0,\infty))$ such that $\int_0^\infty \varphi(s) \, \mathrm{d}s = 1$ and put $\varphi_k(s) := k\varphi(ks)$ $(k \in \mathbb{N})$. Then $\int_0^\infty \varphi_k(s) \, \mathrm{d}s = 1$ and one shows that $x_{\varphi_k} \to x$ as $k \to \infty$. This proves the density of $\bigcap_{n\geq 1} \operatorname{dom} A^n$ in $X$.

By definition, every $C_0$-semigroup has a unique infinitesimal generator. Conversely, a closed, densely defined linear operator can be infinitesimal generator of at most one $C_0$-semigroup.

**Theorem 8.5.** *Let $T$ and $S$ be two $C_0$-semigroups on a Banach space $X$, with generators $A$ and $B$, respectively. If $A = B$, then $T = S$.*

*Proof.* Let $x \in \operatorname{dom} A = \operatorname{dom} B$. From Theorem 8.3 (c) it follows easily that for fixed $t \in \mathbb{R}_+$ the function $s \mapsto T(t-s)S(s)x$ is differentiable. Moreover,

$$\begin{aligned} \frac{d}{ds}(T(t-s)S(s)x) &= -AT(t-s)S(s)x + T(t-s)BS(s)x \\ &= -T(t-s)AS(s)x + T(t-s)BS(s)x = 0, \end{aligned}$$

so that this function is constant. In particular, its values at $s = 0$ and $s = t$ are equal, that is, $S(t)x = T(t)x$. This equality holds for every $x \in \operatorname{dom} A = \operatorname{dom} B$. By Theorem 8.4, the domain $\operatorname{dom} A$ is dense in $X$. By continuity of $S(t)$ and $T(t)$, this implies $S(t) = T(t)$.

We end this section by considering the class of uniformly continuous semigroups and some examples of $C_0$-semigroups. A $C_0$-semigroup is **uniformly continuous**, if $\lim_{t\to 0+} \|T(t) - I\|_{\mathscr{L}(X)} = 0$. Obviously, this type of continuity at $t = 0$ is stronger than just strong continuity.

**Example 8.6 (Operator exponential function).** Let $A \in \mathscr{L}(X)$ be a bounded, linear operator. Set

$$T(t) := e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

The series on the right-hand side converges absolutely for every $t \in \mathbb{R}_+$ (actually, it converges absolutely for every $t \in \mathbb{K}$) and defines a bounded, linear operator. Similarly as in the scalar case, one shows that this operator exponential function satisfies the semigroup property $T(t+s) = T(t)T(s)$ and clearly $T(0) = I$. It is also an exercise to show that $T$ is uniformly continuous and even differentiable (even analytic due to the power series representation) and $\frac{d}{dt}T(t) = AT(t)$. In particular, $A$ is the infinitesimal generator of this semigroup.

**Theorem 8.7.** *A $C_0$-semigroup on a Banach space is uniformly continuous if and only if its infinitesimal generator is a bounded, linear operator.*

*Proof.* One implication follows from Example 8.6 above. If the infinitesimal generator $A$ of a $C_0$-semigroup $T$ is bounded, then it generates also the uniformly continuous semigroup $S(t) = e^{tA}$. By Theorem 8.5, necessarily $T = S$, that is, $T$ is uniformly continuous.

Conversely, assume that $T$ is a uniformly continuous semigroup and let $A$ be its infinitesimal generator. Then there exists $\rho > 0$ such that

$$\|I - \frac{1}{\rho}\int_0^{\rho} T(s)\,\mathrm{d}s\| < 1. \tag{8.1}$$

Then by the Neumann series it follows that $\frac{1}{\rho}\int_0^{\rho} T(s)\,\mathrm{d}s$ is invertible, and therefore $\int_0^{\rho} T(s)\,\mathrm{d}s$ is invertible. Now,

$$\begin{aligned}\frac{T(h)-I}{h}\int_0^{\rho} T(s)\,\mathrm{d}s &= \frac{1}{h}\left(\int_0^{\rho} T(s+h)\,\mathrm{d}s - \int_0^{\rho} T(s)\,\mathrm{d}s\right)\\ &= \frac{1}{h}\left(\int_{\rho}^{\rho+h} T(s)\,\mathrm{d}s - \int_0^{h} T(s)\,ds\right),\end{aligned}$$

or

$$\frac{T(h)-I}{h} = \left(\frac{1}{h}\int_{\rho}^{\rho+h} T(s)\,\mathrm{d}s - \frac{1}{h}\int_0^{h} T(s)\,\mathrm{d}s\right)\left(\int_0^{\rho} T(s)\,\mathrm{d}s\right)^{-1}.$$

From here one sees that $\lim_{h\to 0+}\frac{T(h)-I}{h}$ exists in the space of bounded, linear operators. But this limit is the infinitesimal generator $A$ of $T$, and therefore the infinitesimal generator is bounded.

**Example 8.8 (Shift semigroups I).** Let $X$ be one of the spaces $L^p(\mathbb{R})$ (with $1 \leq p < \infty$), $C_0(\mathbb{R})$ (the space of continuous functions vanishing at $\infty$) or $BUC(\mathbb{R})$ (the space of bounded and uniformly continuous functions). Define $T : \mathbb{R}_+ \to \mathscr{L}(X)$ by

$$(T(t)f)(x) := f(x+t) \qquad (f \in X, t \in \mathbb{R}_+, x \in \mathbb{R}).$$

Then $T$ is a $C_0$-semigroup on $X$. We call it **left-shift semigroup**. Its infinitesimal generator is given by $Af = f'$ with $\mathrm{dom}A = W^{1,p}(\mathbb{R})$, $\mathrm{dom}A = \{f \in C_0(\mathbb{R}) \cap C^1(\mathbb{R}) :$

$f' \in C_0(\mathbb{R})\}$ and $\operatorname{dom} A = \{f \in BUC(\mathbb{R}) \cap C^1(\mathbb{R}) : f' \in BUC(\mathbb{R})\}$. Similarly, one defines the **right-shift semigroup** $S$ by setting

$$(S(t)f)(x) := f(x-t) \qquad (f \in X, t \in \mathbb{R}_+, x \in \mathbb{R}),$$

which has infinitesimal generator $B := -A$, that is, $Bf = -f'$.

**Example 8.9 (Shift semigroups II).** More generally, let $-\infty \le a < b \le +\infty$, and let $X$ be one of the spaces $L^p(a,b)$ (with $1 \le p < \infty$) or $C_0([a,b))$ (the space of continuous functions on the closure of the interval $(a,b)$, vanishing at $b$). Define the **left-shift semigroup** $T : \mathbb{R}_+ \to \mathscr{L}(X)$ by

$$(T(t)f)(x) := \begin{cases} f(x+t) & \text{if } x+t \in (a,b) \\ 0 & \text{if } x+t > b. \end{cases} \qquad (f \in X, t \in \mathbb{R}_+, x \in (a,b)).$$

Then $T$ is indeed a $C_0$-semigroups on $X$. The infinitesimal generator of $T$ is given by $Af = f'$ with $\operatorname{dom} A = \{f \in W^{1,p}(a,b) : f(b) = 0\}$ or $\operatorname{dom} A = \{f \in C_0([a,b)) \cap C^1([a,b]) : f' \in C_0([a,b))\}$.

Similarly, one defines the **right-shift semigroup** $S : \mathbb{R}_+ \to \mathscr{L}(X)$ by

$$(S(t)f)(x) := \begin{cases} f(x-t) & \text{if } x-t \in (a,b) \\ 0 & \text{if } x+t < a. \end{cases} \qquad (f \in X, t \in \mathbb{R}_+, x \in (a,b)),$$

with the only change, in the case when $X$ is a space of continuous functions, that $X = C_0((a,b])$ is the appropriate choice; otherwise, $X = L^p(a,b)$ when $1 \le p < \infty$.

**Example 8.10 (Rescaled semigroup).** Let $A$ be the generator of a $C_0$-semigroup $T$ on a Banach space $X$, and let $\lambda \in \mathbb{K}$. Then $A + \lambda$ is the generator of the semigroup $T_\lambda$, where $T_\lambda(t) = e^{\lambda t} T(t)$.

## 8.2 The theorems of Hille-Yosida and Lumer-Phillips

In the preceding section we defined $C_0$-semigroups and their infinitesimal generators. From the point of view of applications, for example to the ordinary differential equation $\dot{u} = Au$ very often an operator $A$ is given, and one would like to know whether it generates a $C_0$-semigroup. If it does, then the orbits of the semigroup are the solutions of the ordinary differentiable equations by Theorem 8.3 (c) (we will come back later to the question what we mean by a "solution" of this ordinary differential equation). In other words, when we have shown that an operator generates a $C_0$-semigroup, then we have proved existence (and uniqueness) of solutions of an associated ordinary differential equation which can be an abstract reformulation of a partial differential equation. We saw in the previous section, that every infinitesimal generator of a $C_0$-semigroup is closed and densely defined (Theorem 8.4). Here is a further, fundamental property.

**Theorem 8.11.** *Let $A$ be the generator of a $C_0$-semigroup $T$ on a (complex) Banach space $X$. Let $M \geq 1$ and $\omega \in \mathbb{R}$ be such that $\|T(t)\| \leq Me^{\omega t}$ ($t \geq 0$). Then $\mathbb{C}_{\mathrm{Re}>\omega} = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > \omega\} \subseteq \rho(A)$ and, for every $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega}$ and $x \in X$,*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t.$$

*Proof.* Let $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega}$. Then

$$|e^{-\lambda t}| \, \|T(t)x\| \leq Me^{-(\mathrm{Re}\,\lambda - \omega)t} \|x\|, \tag{8.2}$$

so that the function $t \mapsto e^{-\lambda t} T(t)x$ is integrable on $(0, \infty)$. Let $R_\lambda x := \int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t$. Then $R_\lambda$ is a bounded linear operator. Recall from Example 8.10, that $t \mapsto e^{-\lambda t} T(t)$ is a $C_0$-semigroup with generator $A - \lambda$. By Theorem 8.3 (b), for every $x \in X$ and for every $t \geq 0$,

$$(A - \lambda) \int_0^t e^{-\lambda s} T(s)x \, \mathrm{d}s = e^{-\lambda t} T(t)x - x.$$

Letting $t \to \infty$ (and using again the above estimate), it follows that

$$(\lambda - A) R_\lambda x = x,$$

that is, $R_\lambda$ is a right-inverse of $\lambda - A$. Similarly, it follows from Theorem 8.3 (c) that $R_\lambda$ is a left-inverse of $\lambda - A$. Hence, $\lambda \in \rho(A)$ and $R(\lambda, A) = R_\lambda$.

**Remarks 8.12.** (a) Theorem 8.11 is of course also true in real Banach spaces, when one replaces the right-half plane $\mathbb{C}_{\mathrm{Re}>\omega}$ by the interval $]\omega, \infty[$.

(b) Let $f : \mathbb{R}_+ \to X$ be a measurable function such that $\|f(t)\| \leq Me^{\omega t}$ for some constants $M \geq 0$, $\omega \in \mathbb{R}$ and every $t \geq 0$. Then, for every $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega}$, the integral

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) \, \mathrm{d}t$$

converges absolutely. The function $\hat{f} : \mathbb{C}_{\mathrm{Re}>\omega} \to X$ is called **Laplace transform** of $f$. It is an analytic function in the right-half plane $\mathbb{C}_{\mathrm{Re}>\omega}$. Theorem 8.11 says that the resolvent of the generator of a $C_0$-semigroup exists in a right-half plane and, moreover, that the resolvent is, in an appropriate right-half plane, the Laplace transform of the semigroup.

**Theorem 8.13 (Hille-Yosida).** *For an operator $A$ on a Banach space $X$, and for constants $M \geq 1$, $\omega \in \mathbb{R}$, the following assertions are equivalent:*

(i) *The operator $A$ generates a $C_0$-semigroup $T$ satisfying $\|T(t)\| \leq Me^{\omega t}$ ($t \geq 0$).*

(ii) *The operator $A$ is closed, densely defined, $]\omega, \infty[ \subseteq \rho(A)$ and*

$$\sup_{k \in \mathbb{N}, \lambda > \omega} \|(\lambda - \omega)^k R(\lambda, A)^k\| \leq M.$$

*Proof.* (i)⇒(ii) Assume that $A$ generates a $C_0$-semigroup $T$ satisfying $\|T(t)\| \le Me^{\omega t}$ $(t \ge 0)$. Then $A$ is necessarily closed and densely defined (Theorem 8.4) and $]\omega,\infty[\subseteq \rho(A)$ (Theorem 8.11). Moreover, by Theorem 8.11, the resolvent of $A$ in the right-half plane $\mathbb{C}_{\mathrm{Re}>\omega}$ is the Laplace transform of the semigroup:

$$R(\lambda,A)x = \int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t \quad (\lambda \in \mathbb{C}_{\mathrm{Re}>\omega}, x \in X).$$

On the other hand, recall from the power series representation of the resolvent (Lemma 5.3), which was an application of the Neumann series, that for every $k \in \mathbb{N}_0$

$$\left(\frac{d}{d\lambda}\right)^k R(\lambda,A) = (-1)^k k! R(\lambda,A)^{k+1}.$$

In other words, for every $k \in \mathbb{N}_{\ge 1}$ and every $x \in X$,

$$\begin{aligned} R(\lambda,A)^k x &= \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{d}{d\lambda}\right)^{k-1} \int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t \\ &= \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-\lambda t} T(t)x \, \mathrm{d}t. \end{aligned}$$

Hence, by an iterated integration by parts, for every $k \in \mathbb{N}_{\ge 1}$ and every $\lambda > \omega$,

$$\|(\lambda-\omega)^k R(\lambda,A)^k x\| \le \frac{(\lambda-\omega)^k}{(k-1)!} \int_0^\infty M t^{k-1} e^{-(\lambda-\omega)t} \|x\| \, \mathrm{d}t = M\|x\|,$$

which gives the missing estimate in assertion (ii).

(ii)⇒(i) Replacing the operator $A$ by $A-\omega$, we may without loss of generality assume that $\omega = 0$ in our assumption, that is, $A$ is closed, densely defined, $]0,\infty[\subseteq \rho(A)$ and $\sup_{k\in\mathbb{N},\lambda>0} \|\lambda^k R(\lambda,A)^k\| \le M$. This simplifies a little bit our computations. Let us then define, for $\lambda > 0$, the **Yosida approximation** $A_\lambda := \lambda A R(\lambda,A) = \lambda^2 R(\lambda,A) - \lambda$. Then $A_\lambda$ is a bounded linear operator. Let us show that $A_\lambda$ approximates the operator $A$ in the sense that

$$\lim_{\lambda\to\infty} A_\lambda x = Ax \text{ for every } x \in \operatorname{dom} A. \tag{8.3}$$

Actually, for every $x \in \operatorname{dom} A$,

$$\begin{aligned} \lambda R(\lambda,A)x &= (\lambda - A + A)R(\lambda,A)x \\ &= x - R(\lambda,A)Ax \\ &\to x \quad \text{as } \lambda \to \infty, \end{aligned}$$

where we have used the assumption $\|\lambda R(\lambda,A)\| \le M$. By the same assumption and since $\operatorname{dom} A$ is dense in $X$, it follows that

$$\lim_{\lambda\to\infty} \lambda R(\lambda,A)x = x \text{ for every } x \in X,$$

and from here directly follows (8.3).

The Yosida approximation $A_\lambda$ being a bounded linear operator, it generates a $C_0$-semigroup via the exponential function, that is, it generates the even uniformly continuous semigroup $t \mapsto e^{tA_\lambda}$. We first note that for every $\lambda > 0$ and $t \geq 0$,

$$\begin{aligned} \|e^{tA_\lambda}\| &= \|e^{t(\lambda^2 R(\lambda,A)-\lambda)}\| \\ &= e^{-\lambda t}\|\sum_{k=0}^{\infty} \frac{(\lambda t)^k (\lambda R(\lambda,A))^k}{k!}\| \\ &\leq e^{-\lambda t}\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} M \\ &= e^{-\lambda t} e^{\lambda t} M = M. \end{aligned}$$

Hence, the semigroups $e^{\cdot A_\lambda}$ are all uniformly bounded by the constant $M$. Note that the Yosida approximations $A_\lambda$ and $A_\mu$ commute. Now, for every $\lambda$, $\mu > 0$, $t \geq 0$ and $x \in X$,

$$\begin{aligned} \|e^{tA_\lambda}x - e^{tA_\mu}x\| &= \|\int_0^1 \frac{d}{ds}(e^{tsA_\lambda} e^{t(1-s)A_\mu} x)\, \mathrm{d}s\| \\ &= \|t \int_0^1 e^{tsA_\lambda} e^{t(1-s)A_\mu}(A_\lambda - A_\mu)x\, \mathrm{d}s\| \\ &\leq M^2 t\, \|A_\lambda x - A_\mu x\|. \end{aligned}$$

From this estimate and (8.3) follows that, for every $x \in \operatorname{dom} A$,

$$\lim_{\lambda,\mu\to\infty} \|e^{tA_\lambda}x - e^{tA_\mu}x\| = 0,$$

and the limit is uniform in $t$ from compact subsets of $\mathbb{R}_+$. Again, since the semigroups $e^{\cdot A_\lambda}$ are uniformly bounded and since $A$ is densely defined, this estimate actually holds for every $x \in X$. It follows that

$$\lim_{\lambda\to\infty} e^{tA_\lambda}x =: T(t)x \text{ exists for every } x \in X,$$

and the limit is uniform in $t$ from compact subsets of $\mathbb{R}_+$. The operators $T(t)$ thus defined are bounded, linear and $\|T(t)\| \leq M$. Moreover, one can easily check that $T$ is a $C_0$-semigroup. The semigroup property and the strong continuity easily carry over from the semigroups $e^{\cdot A_\lambda}$ and the fact that the limit in the definition of $T$ is uniform in $t$ from compact subsets of $\mathbb{R}_+$. To conclude the proof we show that $A$ is the infinitesimal generator of $T$. Let $x \in \operatorname{dom} A$. Then, by (8.3) and Theorem 8.3 (b),

$$\begin{aligned} T(t)x - x &= \lim_{\lambda\to\infty} e^{tA_\lambda}x - x \\ &= \lim_{\lambda\to\infty} \int_0^t e^{sA_\lambda}A_\lambda x \,\mathrm{d}s \\ &= \int_0^t T(s)Ax \,\mathrm{d}s \end{aligned}$$

and hence, by Theorem 8.3 (a), for every $x \in \operatorname{dom} A$,

$$\lim_{t\to 0+} \frac{T(t)x - x}{t} = Ax.$$

So if $B$ is the generator of $T$, it follows that $x \in \operatorname{dom} B$ and $Bx = Ax$. In other words, $A \subseteq B$. Since the semigroup $T$ is uniformly bounded, it follows that $1 \in \rho(B)$ (Theorem 8.11). On the other hand, by our assumption, $1 \in \rho(A)$. Since $A \subseteq B$, $(I-B)\operatorname{dom} A = (I-A)\operatorname{dom} A = X$, which implies $\operatorname{dom} B = (I-B)^{-1}X = \operatorname{dom} A$, and therefore $A = B$. The proof is complete.

An operator $A$ on a Banach space $X$ is **dissipative** if for every $x \in \operatorname{dom} A$ there exists $x' \in J(x)$ such that

$$\operatorname{Re}\langle x', Ax\rangle \le 0.$$

Here, $J$ is the **duality map** which assigns to every $x \in X$ the set

$$J(x) := \{x' \in X' : \|x'\|^2 = \|x\|^2 = \langle x', x\rangle\}.$$

The elements of $J(x)$ coincide, up to the factor $\|x\|$, with the norm attaining functionals, that is, functionals $z' \in X'$ satisfying $\|z'\| = 1$ and $\langle z', x\rangle = \|x\|$. It follows from Hahn-Banach that $J(x)$ is non-empty.

**Lemma 8.14.** *An operator $A$ on a Banach space $X$ is dissipative if and only if, for every $\lambda > 0$ and every $x \in \operatorname{dom} A$,*

$$\|\lambda x - Ax\| \ge \lambda \|x\|.$$

*Proof.* Assume that $A$ is dissipative. Fix $\lambda > 0$ and $x \in \operatorname{dom} A$. Let $x' \in J(x)$ be such that $\operatorname{Re}\langle x', Ax\rangle \le 0$. Then,

$$\begin{aligned} \lambda\|x\|^2 &= \lambda \operatorname{Re}\langle x', x\rangle \\ &\le \operatorname{Re}\langle x', \lambda x - Ax\rangle \\ &\le \|x'\| \, \|\lambda x - Ax\| \\ &= \|x\| \, \|\lambda x - Ax\|. \end{aligned}$$

Conversely, assume that for every $\lambda > 0$ an every $x \in \operatorname{dom} A$, $\|\lambda x - Ax\| \ge \lambda\|x\|$. Let $\lambda > 0$, $x \in \operatorname{dom} A$, and choose $x'_\lambda \in J(\lambda x - Ax)$. Set $z'_\lambda := x'_\lambda / \|x'_\lambda\|$, so that $\|z'_\lambda\| = 1$. Then

$$\begin{aligned}\lambda\|x\| &\le \|\lambda x - Ax\| \\ &= \langle z'_\lambda, \lambda x - Ax\rangle \\ &= \lambda\,\mathrm{Re}\langle z'_\lambda, x\rangle - \mathrm{Re}\langle z'_\lambda, Ax\rangle \\ &\le \lambda\|x\| - \mathrm{Re}\langle z'_\lambda, Ax\rangle.\end{aligned}$$

From here follows

$$\mathrm{Re}\langle z'_\lambda, Ax\rangle \le 0 \text{ and } \mathrm{Re}\langle z'_\lambda, x\rangle \ge \|x\| - \frac{1}{\lambda}\|Ax\|.$$

By Banach-Alaoglu, the unit ball of $X'$ is weak$^*$ compact. Hence, the net $(z'_\lambda)_{\lambda\nearrow\infty}$ admits a weak$^*$ accumulation point $z' \in X'$. Then $\|z'\| \le 1$,

$$\mathrm{Re}\langle z', Ax\rangle \le 0 \text{ and } \mathrm{Re}\langle z', x\rangle \ge \|x\|.$$

The latter inequality and $\|z'\| \le 1$ imply $\|z'\| = 1$ and $\langle z', x\rangle = \|x\|$, that is, $z'$ is a norm attaining functional for $x$. Setting $x' = \|x\|\, z'$, it follows $x' \in J(x)$ and $\mathrm{Re}\langle x', Ax\rangle \le 0$. Hence, $A$ is dissipative.

**Lemma 8.15.** *For a dissipative operator $A$ on a Banach space $X$, the following assertions are equivalent:*

(i) *For some $\lambda > 0$, the operator $\lambda - A$ is surjective.*

(ii) *For every $\lambda > 0$, the operator $\lambda - A$ is surjective.*

*Moreover, if $A$ is closed, then these assertions are equivalent to*

(iii) *For some $\lambda > 0$, the operator $\lambda - A$ has dense range.*

*Proof.* (i)⇒(ii) Set $\Lambda := \{\lambda > 0 : \lambda - A \text{ is surjective}\}$. Since $A$ is dissipative and by Lemma 8.14, $\lambda - A$ is injective for every $\lambda > 0$. Hence, $\lambda - A$ is invertible for every $\lambda \in \Lambda$. Still by Lemma 8.14, for every $\lambda \in \Lambda$, the inverse $(\lambda - A)^{-1} = R(\lambda, A)$ is bounded (so that $\Lambda \subseteq \rho(A)$!), and $\|\lambda R(\lambda_n, A)\| \le 1$

Now, by assumption, the set $\Lambda$ is a non-empty subset of $]0,\infty[$. We show that $\Lambda$ is open and closed in $]0,\infty[$. Since the interval $]0,\infty[$ is connected, this yields $\Lambda = ]0,\infty[$.

Let $(\lambda_n)$ be a sequence in $\Lambda$ which converges to some element $\lambda_0 \in ]0,\infty[$. Then, by the above remarks, $\lambda_n \in \rho(A)$ and $\|R(\lambda_n, A)\| \le 1/\lambda_n$. It follows from this norm estimate that the sequence $(R(\lambda_n, A))$ is bounded in $\mathscr{L}(X)$, and by Lemma 5.3, $\lambda_0 \in \rho(A)$. In particular, $\lambda_0 - A$ is surjective, that is, $\lambda_0 \in \Lambda$. We have proved that $\Lambda$ is closed in $]0,\infty[$.

Let $\lambda_0 \in \Lambda \subseteq \rho(A)$. The resolvent set being open in $\mathbb{K}$ by Lemma 5.3, it follows that $\Lambda$ is open in $]0,\infty[$.

The implications (ii)⇒(i) and (ii)⇒(iii) are trivial.

(iii)⇒(i) Now assume that $A$ is closed and that $\lambda - A$ has dense range. By Lemma 8.14, for every $x \in \mathrm{dom} A$, $\|\lambda x - Ax\| \ge \lambda\|x\|$. An inequality of the type $\|Bx\| \ge \eta\|x\|$ for every $x \in \mathrm{dom} B$ and for some $\eta > 0$, however, implies for a *closed* linear operator $B$ that $B$ is injective and has closed range. Hence, $\lambda - A$ has closed range.

Together with the assumption that $\lambda - A$ has dense range, it follows that $\lambda - A$ is surjective.

An operator $A$ on a Banach space $X$ is $m$-**dissipative** if it is dissipative and if it satisfies in addition one of the equivalent conditions of Lemma 8.15, for example, the condition that $I - A$ is surjective.

**Lemma 8.16.** (a) *An operator A on a Banach space is m-dissipative if and only if* $]0,\infty[\subseteq \rho(A)$ *and* $\sup_{\lambda>0} \|\lambda R(\lambda,A)\| \leq 1$.

(b) *A dissipative, continuously invertible operator on a Banach space is m-dissipative.*

(c) *An m-dissipative operator on a reflexive Banach space is densely defined.*

*Proof.* (a) The sufficiency part is basically proved in the proof of Lemma 8.15; see the proof of the implication (i)⇒(ii), properties of the set $\Lambda$. Conversely, if $]0,\infty[\subseteq \rho(A)$ and $\sup_{\lambda>0} \|\lambda R(\lambda,A)\| \leq 1$, then for every $\lambda > 0$ and every $y \in X$, $\|\lambda R(\lambda,A)y\| \leq \|y\|$. Setting $y = \lambda x - Ax$ ($x \in \operatorname{dom} A$) in this inequality and using Lemma 8.14 shows that $A$ is dissipative. Moreover, the assumption $]0,\infty[\subseteq \rho(A)$ implies that $\lambda - A$ is surjective for every $\lambda > 0$, and hence $A$ is $m$-dissipative.

(b) Let $A$ be dissipative and continuously invertible. Then $0 \in \rho(A)$. The resolvent set being open, this implies that there exists $\lambda > 0$ such that $\lambda \in \rho(A)$. In particular, $\lambda - A$ is surjective for some $\lambda > 0$, and hence $A$ is $m$-dissipative.

(c) Let $A$ be an $m$-dissipative operator on a reflexive Banach space $X$. Then, by assertion (a), $]0,\infty[\subseteq \rho(A)$ and $\sup_{\lambda>0} \|\lambda R(\lambda,A)\| \leq 1$.

Let $x \in X$. Then the net $(\lambda R(\lambda,A)x)_{\lambda\nearrow\infty}$ is bounded in $X$. The space $X$ being reflexive, there exists $y \in X$ and a sequence $(\lambda_n) \nearrow \infty$ such that weak $-\lim_{n\to\infty} \lambda_n R(\lambda_n,A)x = y$. Then

$$\lim_{n\to\infty} R(\lambda_n,A)x = 0 \quad \text{and}$$
$$\text{weak} - \lim_{n\to\infty} AR(\lambda_n,A)x = \text{weak} - \lim_{n\to\infty} (\lambda_n R(\lambda_n,A)x - x) = y - x.$$

Since $A$ is closed (that is, its graph is closed in $X \times X$), and by Hahn-Banach, $A$ is weakly closed (that is, its graph weakly closed in $X \times X$; note that the graph is a linear subspace and therefore convex). This and the preceding convergences imply that $y - x = A0 = 0$, that is, $y = x$. In other words, weak $-\lim_{n\to\infty} \lambda_n R(\lambda_n,A)x = x$. Since $x \in X$ was arbitrary and since $\lambda_n R(\lambda_n,A)x \in \operatorname{dom} A$, this shows that $\operatorname{dom} A$ is weakly dense in $X$. By Hahn-Banach, this implies that $\operatorname{dom} A$ is norm dense in $X$ (in fact, the weak closure and the norm closure of the convex set $\operatorname{dom} A$ coincide).

Recall that an operator $T \in \mathscr{L}(X)$ is a contraction if $\|T\| \leq 1$.

**Theorem 8.17 (Lumer-Phillips).** *An operator on a Banach space is the generator of a $C_0$-semigroup of contractions if and only if it is m-dissipative and densely defined.*

*Proof.* This theorem is a direct consequence of the characterization in Lemma 8.16 (a) and the Theorem of Hille-Yosida (Theorem 8.13 with $M = 1$ and $\omega = 0$). Note that by the sub-multiplicativity of the operator norm, the condition $\sup_{\lambda>0} \|\lambda R(\lambda,A)\| \leq 1$ is equivalent to the condition $\sup_{k\in\mathbb{N},\lambda>0} \|\lambda^k R(\lambda,A)^k\| \leq 1$.

**Example 8.18.** On the *real* Banach space $X = C([0,1])$ (equipped with the supremum norm) we consider the operator $A$ given by

$$\begin{aligned} \operatorname{dom} A &:= \{f \in C^2([0,1]) \, : \, f(0) = f(1) = 0\}, \\ Af &:= f''. \end{aligned}$$

Let us show that this operator is dissipative. In fact, for every function $f \in \operatorname{dom} A$, the absolute value $|f|$ attains its supremum in some point $x_0 \in [0,1]$, that is, $|f(x_0)| = \sup_{x\in[0,1]} |f(x)| = \|f\|_\infty$. The function $f$ has in $x_0$ either a global maximum (and then $f(x_0) \geq 0$) or a global minimum (and then $f(x_0) \leq 0$). Define the Dirac functional $\delta \in C([0,1])'$ by

$$\langle \delta, g\rangle = \delta(g) := \begin{cases} g(x_0) & \text{if } f(x_0) \geq 0, \\ -g(x_0) & \text{if } f(x_0) < 0. \end{cases}$$

Then $\|\delta\| = 1$ and $\delta(f) = \|f\|_\infty$, that is, $\delta$ is a norm attaining functional for $f$. Hence, $\|f\|_\infty \, \delta \in J(f)$. Now,

$$\langle \delta, Af\rangle = \delta(Af) = \begin{cases} f''(x_0) & \text{if } f(x_0) \geq 0, \\ -f''(x_0) & \text{if } f(x_0) < 0. \end{cases}$$

Recall from the necessary conditions for a local maximum or minimum of a real-valued function of one real variable, that $f''(x_0) \leq 0$ if $x_0$ is a local maximum, and $f''(x_0) \geq 0$ if $x_0$ is a local minimum. Hence, $\langle \delta, Af\rangle \leq 0$. We have proved that $A$ is dissipative.

Moreover, the operator $A$ is invertible. The inverse $A^{-1} =: K$ is the kernel operator given by $(Kf)(x) = -\int_0^1 k(x,y)f(y)\,\mathrm{d}y$ where

$$k(x,y) = \begin{cases} (1-x)y & \text{if } y \geq x, \\ x(1-y) & \text{if } y < x; \end{cases}$$

see the Exercises! By Lemma 8.16 (b), the operator $A$ is thus $m$-dissipative. Unfortunately, its domain is, due to the Dirichlet boundary conditions $f(0) = f(1) = 0$ not dense in $C([0,1])$, so that we can not conclude that $A$ generates a $C_0$-semigroup on this space. In fact, we can conclude that it does not generate a $C_0$-semigroup. However, when we define the **part** $A_0$ of $A$ in the space $X_0 := \{f \in C([0,1]) \, : \, f(0) = f(1) = 0\}$, that is,

$$\begin{aligned} \operatorname{dom} A_0 &:= \{f \in C^2([0,1]) : f(0) = f(1) = 0 \text{ and } f'' \in X_0\}, \\ A_0 f &:= f'', \end{aligned}$$

then $A_0$ is $m$-dissipative and densely defined. By the Theorem of Lumer-Phillips (Theorem 8.17), $A_0$ therefore generates a $C_0$-semigroup of contractions on the space $X_0$. We will see that this means that the heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} && \text{in } (0,\infty) \times (0,1), \\ &u(t,0) = u(t,1) = 0 && \text{for } t \in (0,\infty), \\ &u(0,x) = u_0(x) && \text{for } x \in (0,1), \end{aligned}$$

is well-posed in the sense that for every $u_0 \in X_0$ it admits a unique solution $u$, and this solution is given by $u(t,x) = (T(t)u_0)(x)$, where $T$ is the semigroup generated by $A_0$. Of course, there is the question in which sense this function $u$ is a solution of the above heat equation, but we do not discuss this question for the moment.

An operator $A$ on a Banach space is **closable** if the closure of its graph in is again the graph of an operator, and then the latter operator is denoted by $\bar{A}$ and is called the **closure** of $A$. Note that the closure of the graph in $X \times X$ always exists, and since the graph is a linear subspace of $X \times X$, its closure is a linear subspace, too. The problem which might occur is that this closure is not the graph of a single-valued operator. This happens exactly if the closure of the graph contains a subspace of the form $\{0\} \times U$ for some nonzero subspace $U \subseteq X$. If we allowed to work with multi-valued operators, then every operator would be closable.

**Lemma 8.19.** *Let $A$ be a dissipative operator on a Banach space $X$. Then:*

(a) *If $A$ is closable, then its closure is dissipative.*

(b) *If $A$ is densely defined, then it is closable.*

*Proof.* (a) Let $x \in \operatorname{dom}\bar{A}$, where $\bar{A}$ denotes the closure of $A$, and let $\lambda > 0$. By definition of the closure, there exists a sequence $(x_n)$ in $\operatorname{dom}A$ such that $x_n \to x$ and $Ax_n \to \bar{A}x$. Since $A$ is dissipative and by Lemma 8.14, for every $n$, $\|\lambda x_n - Ax_n\| \geq \lambda \|x_n\|$. Passing to the limit yields $\|\lambda x - \bar{A}x\| \geq \lambda \|x\|$. By Lemma 8.14, $\bar{A}$ is dissipative.

(b) Assume that $A$ was not closable. Then there exists a sequence $(x_n)$ in $\operatorname{dom}A$ such that $x_n \to 0$ and $Ax_n \to y$ with $\|y\| = 1$. By Lemma 8.14, for every $x \in \operatorname{dom}A$ and for every $\lambda > 0$,

$$\|(x + \lambda^{-1}x_n) + \lambda A(x + \lambda^{-1}x_n)\| \geq \|x + \lambda^{-1}x_n\|.$$

Letting first $n \to \infty$ and second $\lambda \to 0+$ yields $\|x - y\| \geq \|x\|$ for every $x \in \operatorname{dom}A$. But this is impossible since $\operatorname{dom}A$ is dense in $X$. Hence, $A$ is closable.

## 8.3 The abstract Cauchy problem

Let $A$ be a linear operator on a Banach space $X$. We consider the homogeneous Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) \quad (t \in \mathbb{R}_+), \\ u(0) &= x. \end{aligned} \tag{CP}$$

Here, $x \in X$ is a given initial value. The function $u : \mathbb{R}_+ \to X$ is the unknown, and $\dot{u}$ stands for its time derivative (usually, the variable $t$ plays the role of a time variable). A function $u \in C^1(\mathbb{R}_+;X)$ is called a **classical solution** of the Cauchy problem (CP) if $u(0) = x$, $u(t) \in \operatorname{dom} A$, and $\dot{u}(t) = Au(t)$ for every $t \in \mathbb{R}_+$. A function $u \in C(\mathbb{R}_+;X)$ is called a **mild solution** of the Cauchy problem (CP) if $\int_0^t u(s)\, \mathrm{d}s \in \operatorname{dom} A$ and $u(t) = x + A\int_0^t u(s)\, \mathrm{d}s$ for every $t \geq 0$. As a motivation for this notion, formally integrate the differential equation in (CP) over the interval $[0,t]$ and interchange the operator $A$ with the integral, which is at least possible when $A$ is a closed operator. In this way one also sees that every classical solution is a mild solution, if $A$ is closed. Note that by definition, if $u$ is a classical solution, then necessarily $x \in \operatorname{dom} A$.

**Lemma 8.20.** *Let $A$ be a closed, linear operator on a Banach space $X$. Then a mild solution of the Cauchy problem* (CP) *is a classical solution if and only if it is continuously differentiable*

*Proof.* By definition, every classical solution is continuously differentiable. Conversely, let $u$ be a mild solution of of the Cauchy problem (CP) which is in addition continuously differentiable. Then, for every $t \geq 0$, $h > 0$, $\int_t^{t+h} u(s)\, \mathrm{d}s \in \operatorname{dom} A$ and $u(t+h) - u(t) = A\int_t^{t+h} u(s)\, \mathrm{d}s$. Dividing this equality by $h$, letting $h \to 0+$ and using the closedness of $A$, it follows that $u(t) \in \operatorname{dom} A$ and $\dot{u}(t) = Au(t)$. Hence, $u$ is a classical solution.

**Theorem 8.21.** *Let $A$ be a linear operator on a Banach spae $X$. Then the following assertions are equivalent:*

(i) *The operator $A$ is closed, and for every initial value $x \in X$ the Cauchy problem* (CP) *admits a unique mild solution $u_x$.*

(ii) *The resolvent set $\rho(A)$ is nonemtpy, and for every initial value $x \in \operatorname{dom} A$ the Cauchy problem* (CP) *admits a unique classical solution.*

(iii) *The operator $A$ generates a $C_0$-semigroup.*

*Proof.* If $A$ generates the $C_0$-semigroup $T$, then by Theorem 8.3 the functions $u_x(t) := T(t)x$ are mild and classical solutions, respectively, depending on whether $x \in X$ or $x \in \operatorname{dom} A$. They are actually the only mild / classical solutions. In fact, by linearity, it suffices to show that mild solutions for the initial value $x = 0$ are unique. So let $v$ be a mild solution for $x = 0$, that is, $\int_0^t v(s)\, \mathrm{d}s \in \operatorname{dom} A$ and $v(t) = A\int_0^t v(s)\, \mathrm{d}s$ for every $t \in \mathbb{R}_+$. Then the primitive $V(t) = \int_0^t v(s)\, \mathrm{d}s$ is a classical solution for the initial value 0. Define now, for fixed $t \in \mathbb{R}_+$, the function $W(s) := T(t-s)V(s)$ $(s \in [0,t])$. Then $W$ is continuously differentiable and

$\dot{W}(s) = T(t-s)\dot{V}(s) - AT(t-s)V(s) = 0$, that is, $W$ is constant. In particular, $V(t) = W(t) = W(0) = 0$, so that $V = 0$ and $v = 0$. We have proved uniqueness of mild /classical solutions. The continuous dependence of mild /classical solutions follows from the continuity of the operators $T(t)$ and the exponential bound from Lemma 8.1. We have proved the implications (iii)⇒(i) and (iii)⇒(ii).

(i)⇒(iii) Assume that for every $x \in X$ the Cauchy problem (CP) admits a unique mild solution $u_x$. We first show that the mapping $S : x \mapsto u_x|_{[0,\tau]}$, $X \to C([0,\tau];X)$ is continuous for every $\tau \geq 0$. For this, assume that $x_n \to x$ in $X$ and $Sx_n = u_{x_n}|_{[0,\tau]} \to v$ in $C([0,\tau];X)$. By definition of mild solution, for every $n$ and every $t \in [0,\tau]$, $\int_0^t u_{x_n}(s)\, \mathrm{d}s \in \operatorname{dom} A$ and

$$u_{x_n}(t) - x_n = A \int_0^t u_{x_n}(s)\, \mathrm{d}s.$$

It follows from this identity and the assumptions on $x_n$ and $u_{x_n}$, that

$$\int_0^t u_{x_n}(s)\, \mathrm{d}s \to \int_0^t v(s)\, \mathrm{d}s \text{ and}$$
$$A \int_0^t u_{x_n}(s)\, \mathrm{d}s \to v(t) - x.$$

Since $A$ is closed, this implies $\int_0^t v(s)\, \mathrm{d}s \in \operatorname{dom} A$ and $A \int_0^t v(s)\, \mathrm{d}s = v(t) - x$. In other words, $v$ is a mild solution for the initial value $x$. By uniqueness, $v = u_x = Sx$. We have proved that the operator $S$ is closed. By the closed graph theorem, $S$ is continuous. Since $\tau \geq 0$ was arbitrary, we have actually proved that the mapping $S : X \to C(\mathbb{R}_+;X)$, $x \mapsto u_x$ is continuous, where $C(\mathbb{R}_+;X)$ is equipped with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$.

Now set, for every $x \in X$ and every $t \in \mathbb{R}_+$, $T(t)x := u_x(t)$. Then, by linearity of $A$ and by the continuity of the operator $S$, the $T(t)$ are linear and continuous. Clearly, $T(0)x = x$, and the function $t \mapsto T(t)$ is strongly continuous. Moreover, one easily shows that for every $x \in X$ and every $t \geq 0$, the function $u_x(t + \cdot)$ is a mild solution for the initial value $u_x(t) =: y$. By uniqueness of mild solutions, $u_x(t+s) = u_y(s)$ for every $s \in \mathbb{R}_+$. In other words, $T(t+s)x = u_x(t+s) = u_y(s) = T(s)u_y(0) = T(s)u_x(t) = T(s)T(t)x$. We have proved that $T$ is a $C_0$-semigroup.

Let $B$ be its infinitesimal generator. Then, for every $x \in \operatorname{dom} B$, the function $u_x = T(\cdot)x$ is a classical solution of the Cauchy problem $\dot{u} = Bu$, $u(0) = x$. In particular $u_x$ is continuously differentiable. By Lemma 8.20, $u_x$ is then also a classical solution of the Cauchy problem (CP). In particular, $u_x(0) = x \in \operatorname{dom} A$ and $Ax = Au_x(0) = \dot{u}_x(0) = Bu_x(0) = Bx$. We have proved that $B \subseteq A$, that is, $A$ is an extension of $B$. If $A$ is a proper extension (that is, $B \subsetneq A$), then there exists $x \in \operatorname{dom} A \setminus \operatorname{dom} B$. Let $M \geq 1$ and $\omega \in \mathbb{R}$ be such that $\|T(t)\| \leq Me^{\omega t}$ $(t \in \mathbb{R}_+)$. Then $\mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(B)$. Hence, for fixed $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega}$, there exists $y \in \operatorname{dom} B$ such that $\lambda y - By = \lambda x - Ax$. If we set $z = x - y$, then $z \neq 0$ and $\lambda z - Az = 0$ (since $A$ is an extension of $B$), that is, $\lambda$ is an eigenvalue of $A$, and $z$ is an associated eigenvector. But then $u(t) = e^{\lambda t} z$ is classical solution of the Cauchy problem (CP) for the initial value $u(0) = z$. By assumption, this classical solution is unique, and by definition, $T(t)z = e^{\lambda t} z$. This, however, contradicts the choice of $\lambda$ and the exponential bound on the semigroup $T$.

(ii)⇒(i) Assume that $\rho(A) \neq \emptyset$ and that for every $x \in \mathrm{dom} A$ the Cauchy problem (CP) admits a unique classical solution $u_x$. Recall from Lemma 5.1 that $A$ is closed. Fix $\lambda \in \rho(A)$. Let $x \in X$. Set $y := R(\lambda, A)x \in \mathrm{dom} A$, and let $v = u_y$ be the unique classical solution of the Cauchy problem (CP) with initial value $v(0) = y$. Then $v$ is continuously differentiable, $v(t) \in \mathrm{dom} A$ and $Av(t) = \dot{v}(t)$ for every $t \in \mathbb{R}_+$. Set $u(t) := \lambda v(t) - \dot{v}(t) = (\lambda - A)v(t)$. Then $u \in C(\mathbb{R}_+; X)$,

$$\begin{aligned} \int_0^t u(s)\, ds &= \int_0^\lambda (\lambda v(s) - \dot{v}(s))\, \mathrm{d}s \\ &= \lambda \int_0^t v(s)\, \mathrm{d}s - v(t) + y \in \mathrm{dom} A \end{aligned}$$

and

$$\begin{aligned} A\int_0^t u(s)\, ds &= \lambda A \int_0^t v(s)\, \mathrm{d}s - Av(t) + Ay \\ &= \lambda(v(t) - y) - \dot{v}(t) + (\lambda y - x) \\ &= (\lambda v(t) - \dot{v}(t)) - x \\ &= u(t) - x, \end{aligned}$$

that is, $u$ is a mild solution of the Cauchy problem (CP) with initial value $u(0) = x$. It only remains to show uniqueness of mild solutions. By linearity, it suffices to show that there is only one mild solution $w$ for the initial value $w(0) = 0$, namely the function $w = 0$. However, if $w$ is a mild solution for the initial value $w(0) = 0$, then its primitive $W(t) = \int_0^t w(s)\, \mathrm{d}s$ is a classical solution for the same initial value. By uniqueness of classical solutions, it follows $W = 0$, and hence $w = 0$.

We say that the Cauchy problem (CP) is **wellposed in the sense of Hadamard** if it admits for every $x \in X$ a unique mild solution $u_x$ and if the mapping $X \to C(\mathbb{R}_+; X)$, $x \mapsto u_x$ is continuous, that is, one has existence and uniqueness of solutions and continuous dependence on the initial data. Theorem 8.21 then says, given a closed, linear operator $A$, that the Cauchy problem (CP) is wellposed in the sense of Hadamard if and only if $A$ generates a $C_0$-semigroup. The continuous dependence of solutions from initial values is either hidden in the proof of the implication (i)⇒(iii) (the application of the closed graph theorem) or follows from assertion (iii).

We may also consider the inhomogeneous Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) + f(t) \quad (t \in \mathbb{R}_+), \\ u(0) &= x. \end{aligned} \tag{iCP}$$

Here $x \in X$ is a given initial value and $f \in C(\mathbb{R}_+; X)$ a given function. Similarly as in the case of the homogeneous Cauchy problem, we call a function $u \in C^1(\mathbb{R}_+; X)$ a **classical solution** of (iCP) if $u(0) = x$, $u(t) \in \mathrm{dom} A$ and $\dot{u}(t) = Au(t) + f(t)$

for every $t \in \mathbb{R}_+$. We call a function $u \in C(\mathbb{R}_+;X)$ a **mild solution** of (iCP) if $\int_0^t u(s)\,\mathrm{d}s \in \operatorname{dom} A$ and $u(t) = x + A\int_0^t u(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s$ for every $t \in \mathbb{R}_+$.

**Theorem 8.22.** *Let A be the generator of a $C_0$-semigroup T on a Banach space X. Then, for every $x \in X$ and $f \in C(\mathbb{R}_+;X)$, the inhomogeneous Cauchy problem* (iCP) *admits a unique mild solution u, and this mild solution is given by*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds \quad (t \in \mathbb{R}_+). \tag{8.4}$$

The formula (8.4) is often called the **variation-of-constants formula** or **Duhamel's formula**.

*Proof.* Uniqueness of mild solutions follows by linearity and by uniqueness of mild solutions for the homogeneous Cauchy problem (Theorem 8.21.

For existence it suffices to check that the function $u$ given by Duhamel's formula is indeed a mild solution. This is left as an exercise.

**Remark 8.23.** One can also prove a converse of Theorem 8.21 in the spirit of Theorem 8.21 (i)⇒(iii). Indeed, if $A$ is a closed, linear operator and if for every $f \in C(\mathbb{R}_+;X)$ the inhomogeneous Cauchy problem (iCP) with $x = 0$ admits a unique mild solution $u_f$, then $A$ generates a $C_0$-semigroup. We omit the proof of this result.

## 8.4 * The exponential formula

The Theorems of Hille-Yosida and Lumer-Phillips characterize infinitesimal generators of (contraction) $C_0$-semigroups. Under certain assumptions on $A$ and its resolvent, $A$ generates a $C_0$-semigroup. Our proof of the Hille-Yosida theorem is in some way also constructive in the sense that it shows that the $C_0$-semigroup generated by $A$ is the limit of exponential functions generated by the Yosida approximations of $A$. In this section, we present another approximation for the semigroup in terms of the resolvent of $A$. This approximation formula follows directly from an inversion formula for Laplace transforms. Recall from Theorem 8.11 that the resolvent of the generator of a $C_0$-semigroup is the Laplace transform of the semigroup. Therefore, any general result for Laplace transforms directly applies to resolvents and semigroups, and in the case of the exponential formula, the proof for the general result does not add any difficulty.

Let $X$ be a complex Banach space. A function $f : \mathbb{R}_+ \to X$ is **exponentially bounded** if there exist constants $M \geq 0$, $\omega \in \mathbb{R}$ such that $\|f(t)\| \leq Me^{\omega t}$. For a measurable, exponentially bounded function we define the **Laplace transform** $\hat{f} : \mathbb{C}_{\mathrm{Re}>\omega} \to X$ by

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,\mathrm{d}t \quad (\mathrm{Re}\,\lambda > \omega).$$

The Laplace transform $\hat{f}$ is an analytic in the right-half plane $\mathbb{C}_{\mathrm{Re}>\omega}$ as one can see by differentiating the Laplace integral with respect to $\lambda$. It follows from each of the following inversion formulas that the Laplace transform is injective, that is, if $\hat{f} = 0$, then $f = 0$.

**Theorem 8.24 (Post-Widder inversion formula).** *Let $f \in C(\mathbb{R}_+;X)$ be exponentially bounded. Then, for every $t > 0$,*

$$f(t) = \lim_{k\to\infty} (-1)^k \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} \hat{f}^{(k)}(\frac{k}{t}),$$

*where $\hat{f}^{(k)}$ denotes the k-th derivative of the Laplace transform $\hat{f}$.*

*Proof.* Assume that $\|f(t)\| \leq Me^{\omega t}$ $(t \in \mathbb{R}_+)$. Without loss of generality, we may assume that $\omega \geq 0$. Then, for every $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega}$ and every $k \geq 0$,

$$\hat{f}^{(k)}(\lambda) = \int_0^\infty (-s)^k e^{-\lambda s} f(s)\,\mathrm{d}s.$$

Hence, for every $t > 0$ and every $k \geq 0$,

$$\begin{aligned}(-1)^k \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} \hat{f}^{(k)}(\frac{k}{t}) &= \frac{k^{k+1}}{k!} \int_0^\infty \frac{s^k}{t^{k+1}} e^{-\frac{ks}{t}} f(s)\,\mathrm{d}s \\ &= \frac{k^{k+1}}{k!} \int_0^\infty s^k e^{-ks} f(st)\,\mathrm{d}s.\end{aligned}$$

Note that, for every $k \geq 1$,

$$\frac{k^{k+1}}{k!} \int_0^\infty s^k e^{-ks}\,\mathrm{d}s = 1,$$

as one can see from integration by parts and substitution. Hence, for every $0 < a < 1 < b$,

$$\begin{aligned}&\limsup_{k\to\infty} \left\| (-1)^k \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} \hat{f}^{(k)}(\frac{k}{t}) - f(t) \right\| \\ &= \limsup_{k\to\infty} \left\| \frac{k^{k+1}}{k!} \int_0^\infty s^k e^{-ks} (f(st) - f(t))\,\mathrm{d}s \right\| \\ &\leq \limsup_{k\to\infty} \frac{k^{k+1}}{k!} \left( \int_0^a + \int_b^\infty \right) s^k e^{-ks} M(e^{\omega st} + e^{\omega t})\,\mathrm{d}s \\ &\quad + \limsup_{k\to\infty} \frac{k^{k+1}}{k!} \int_a^b s^k e^{-ks}\,\mathrm{d}s \sup_{a\leq s\leq b} \|f(st) - f(t)\| \\ &\leq \sup_{a\leq s\leq b} \|f(st) - f(t)\|,\end{aligned}$$

because

$$\frac{k^{k+1}}{k!}\int_0^a s^k e^{-ks} M(e^{\omega st}+e^{\omega t})\,\mathrm{d}s \le 2Me^{\omega t}\frac{k^{k+1}}{k!}(ae^{-a})^k$$
$$\to 0 \text{ as } k\to\infty$$

and

$$\ldots$$

However, by continuity of $f$, for every $t>0$,

$$\lim_{\substack{a\to 1-\\ b\to 1+}} \sup_{a\le s\le b} \|f(st)-f(t)\| = 0.$$

We finally deduce that

$$\limsup_{k\to\infty} \left\| (-1)^k \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \hat{f}^{(k)}(\frac{k}{t}) - f(t) \right\| = 0,$$

which yields the claim.

From the Post-Widder inversion formula we immediately deduce the following uniqueness result for the Laplace transform.

**Corollary 8.25 (Injectivity of the Laplace transform).** *Let $f\in C(\mathbb{R}_+;X)$ be exponentially bounded such that $\hat{f}=0$. Then $f=0$.*

The Post-Widder inversion formula also implies the following exponential formula.

**Theorem 8.26 (Exponential formula).** *Let $A$ be the infinitesimal generator of a $C_0$-semigroup $T$ on a Banach space $X$. Then, for every $x\in X$ and every $t>0$,*

$$T(t)x = \lim_{k\to\infty}(I-\frac{t}{k}A)^{-k}x = \lim_{k\to\infty}\left(\frac{k}{t}R(\frac{k}{t},A)\right)^k x.$$

**Remarks 8.27.** (a) Recall, for every $a\in\mathbb{C}$ and every $t>0$, the exponential formula $e^{ta}=\lim_{k\to\infty}(1-\frac{ta}{k})^{-k}$. Theorem 8.26 says that this particular exponential formula remains true when $a$ is replaced by the generator $A$ of a $C_0$-semigroup. The Hille-Yosida resolvent condition yields the bound

$$\begin{aligned}\|T(t)x\| &\le \limsup_{k\to\infty}\|\frac{(\frac{k}{t}-\omega)^k}{(\frac{k}{t}-\omega)^k}\left(\frac{k}{t}R(\frac{k}{t},A)\right)^k\| \\ &\le M\limsup_{k\to\infty}\left(\frac{\frac{k}{t}}{\frac{k}{t}-\omega}\right)^k \\ &= Me^{\omega t}.\end{aligned}$$

(b) The formulas

$$e^{ta} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \text{ and } e^{ta} = \lim_{k\to\infty} (1 + \frac{ta}{k})^k$$

also represent the exponential function, but they are inappropriate when one replaces $a$ by an unbounded operator $A$. However, besides the exponential formula above, there are other representations of the exponential function which can be generalised to the case of semigroups.

(c) Consider the implicit Euler scheme

$$\begin{aligned} \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} &= Au(t_{i+1}) \quad (i = 0, \dots, k-1), \\ u(0) &= x, \end{aligned} \tag{E}$$

as an approximation of the Cauchy problem CP. Here, $0 = t_0 < t_1 < \cdots < t_k = t$ is a partition of the interval $[0,t]$ $(k \geq 1)$. If one takes an equidistant partition, then the time step $t_{i+1} - t_i = t/k$ is independent of $i$. The first line in this Euler scheme can be rewritten in the form $u(t_{i+1}) - \frac{t}{k} Au(t_{i+1}) = u(t_i)$ or, if $I + \frac{t}{k} A$ is invertible, as $u(t_{i+1}) = (I - \frac{t}{k} A)^{-1} u(t_i)$. Iterating this identity yields $u(t) = (I - \frac{t}{k} A)^{-k} x$. The exponential formula thus says that, if $A$ is the infinitesimal generator of a $C_0$-semigroup, then the solutions of the implicit Euler scheme (E) converge (as $k \to \infty$) to the unique mild solution of the Cauchy problem (CP) One can show that this convergence is uniform for $t$ in compact subsets of $\mathbb{R}_+$. Using the implicit Euler scheme one can actually give an alternative proof of the Hille-Yosida theorem.

*Proof (of Theorem 8.26).* Assume that $\|T(t)\| \leq Me^{\omega t}$ $(t \in \mathbb{R}_+)$. Let $f(t) := T(t)x$, so that $f$ is continuous and exponentially bounded. By Theorem 8.11, $\hat{f}(\lambda) = R(\lambda, A)x$. Differentiating the resolvent yields for every $k \in \mathbb{N}$ and every $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega}$,

$$\hat{f}^{(k)}(\lambda) = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^k R(\lambda, A)x = (-1)^k k! R(\lambda, A)^{k+1} x,$$

and therefore, for every $k \in \mathbb{N}$ and every $t > 0$

$$(-1)^k \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} \hat{f}^{(k)}(\frac{k}{t}) = \left(\frac{k}{t}\right)^{k+1} R(\frac{k}{t}, A)^{k+1} x.$$

The claim now follows from the Post-Widder inversion formula (Theorem 8.24).

## 8.5 Holomorphic $C_0$-semigroups

In the first section of this chapter, we have introduced $C_0$-semigroups, that is, semigroups which are strongly continuous, but also uniformly continuous semigroups, which are continuous on $\mathbb{R}_+$ with respect to the operator norm and which are precisely the $C_0$-semigroups with bounded infinitesimal generators. In this section, we consider a class of semigroups in between these two extremal cases and still representing high regularity.

Given an angle $\theta \in ]0,\pi[$, we define the open sector

$$\Sigma_\theta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\}.$$

A holomorphic (= complex differentiable) function $T\colon \Sigma_\theta \to \mathscr{L}(X)$ ($X$ being a complex Banach space) is called a **holomorphic $C_0$-semigroup** if

(a) $\lim_{\substack{z\to 0\\ z\in\Sigma_{\theta'}}} T(z)x = x$ for every $x \in X$ and every $\theta' \in ]0,\theta[$, and

(b) $T(z_1+z_2) = T(z_1)\cdot T(z_2)$ for every $z_1, z_2 \in \Sigma_\theta$,

The restriction to $\mathbb{R}_+$ of every holomorphic $C_0$-semigroup *is* a $C_0$-semigroup if we set $T(0) = I$, and therefore it admits an infinitesimal generator, as usual. We also call a $C_0$-semigroup defined on $\mathbb{R}_+$ holomorphic if it extends to a holomorphic $C_0$-semigroup on some sector $\Sigma_\theta$. Note that it is sufficient just to have a holomorphic extension on some sector $\Sigma_\theta$. The semigroup property on the sector $\Sigma_\theta$ and the strong continuity at 0 are then automatic.

**Lemma 8.28.** *Let $T$ be a $C_0$-semigroup on a complex Banach space, and assume that $T$ admits a holomorphic extension to $\Sigma_\theta$. Then this extension is a holomorphic $C_0$-semigroup.*

*Proof.* For every $z' \in \Sigma_\theta$ the functions $z \mapsto T(z+z')$, $z \mapsto T(z)T(z')$ and $z \mapsto T(z')T(z)$ are holomorphic on the sector $\Sigma_\theta$ and they coincide on $]0,\infty[$, at least if $z'$ is real. By the identity theorem for holomorphic functions, they coincide everywhere on $\Sigma_\theta$. Once this semigroup property is proved for real $z'$, one can apply this argument again and one gets the semigroup property for all $z', z \in \Sigma_\theta$.

For the strong continuity at 0, we define for every $n \in \mathbb{N}$ the function $T_n(z) := T(z/n)$ ($z \in \Sigma_\theta$). Fix $x \in X$. Then, by assumption, $\lim_{n\to\infty} T_n(z)x = x$ for every real $z \in \Sigma_\theta$. By Vitali's theorem, $\lim_{n\to\infty} T_n(z)x = x$ uniformly for $z$ in compact subsets of $\Sigma_\theta$. Given $\theta' \in ]0,\theta[$, we may for example choose the compact subset $K = \{z \in \mathbb{C} : 1 \le |z| \le 2, |\arg z| \le \theta'\}$. Then, for $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}_{\ge n_0}$ and every $z \in K$,

$$\|T(z/n)x - x\| = \|T_n(z)x - x\| \le \varepsilon,$$

and looking at the set $\{z/n \in \mathbb{C} : z \in K \text{ and } n \in \mathbb{N}_{\ge n_0}\}$ it follows that we have proved

$$\lim_{\substack{z\to 0\\ z\in\Sigma_{\theta'}}} T(z)x = x.$$

**Theorem 8.29.** *Let $A$ be a closed, densely defined operator on a complex Banach space $X$, and let $\theta \in ]0, \frac{\pi}{2}[$, $\omega \in \mathbb{R}$. The following assertions are equivalent:*

(i) *The operator $A$ generates a holomorphic $C_0$-semigroup $T$ on $\Sigma_\theta$ and for every $\theta' \in ]0,\theta[$*

$$\sup_{z\in\Sigma_{\theta'}} \|e^{-\omega \operatorname{Re} z}T(z)\| < \infty.$$

(ii) *For every $\theta' \in ]-\theta,\theta[$, the operator $e^{i\theta'}A$ generates a $C_0$-semigroup $T_{\theta'}$ and*

$$\sup_{t\in\mathbb{R}_+} \|e^{-\omega t}T_{\theta'}(t)\| < \infty.$$

(iii) *The shifted sector $\omega+\Sigma_{\theta+\frac{\pi}{2}}$ is contained in $\rho(A)$ and for every $\theta' \in ]0,\theta[$*

$$\sup_{\lambda\in\omega+\Sigma_{\theta'+\frac{\pi}{2}}} \|(\lambda-\omega)R(\lambda,A)\| < \infty.$$

*Proof.* By replacing $A$ with $A-\omega$ we may without loss of generality assume that $\omega = 0$.

(i) $\Longrightarrow$ (ii) One easily verifies that for every $\theta' \in ]-\theta,\theta[$ the function $T_{\theta'}(t) := T(e^{i\theta'}t)$ *is* a $C_0$-semigroup, which is bounded by assumption ($\omega = 0$), and that $e^{i\theta'}A$ is its infinitesimal generator.

(ii) $\Longrightarrow$ (iii) If $e^{i\theta'}A$ is the infinitesimal generator of a bounded $C_0$-semigroup for every $\theta' \in ]-\theta,\theta[$, then, by Theorem 8.11, $\mathbb{C}_{\mathrm{Re}>0} \subseteq \rho(e^{i\theta'}A)$ and

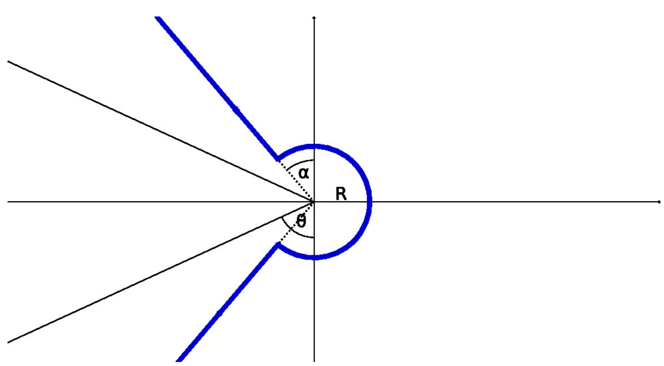
$$\sup_{\operatorname{Re}\lambda>0} \|(\operatorname{Re}\lambda)R(\lambda,e^{i\theta'}A)\| < \infty.$$

for every $\theta' \in ]-\theta,\theta[$. From here follows $\Sigma_{\theta+\frac{\pi}{2}} \subseteq \rho(A)$ and the resolvent estimate from (iii).

(iii) $\Longrightarrow$ (ii) Fix $\theta' \in ]0,\theta[$. Choose then $\alpha \in ]\theta',\theta[$ and define, for every $z \in \Sigma_{\theta'}$,

$$T(z) := \frac{1}{2\pi i}\int_{\gamma_{1/|z|}} e^{\lambda z}R(\lambda,A)\,\mathrm{d}\lambda,$$

where for $R>0$ the contour $\gamma_R$ is the union of the two rays $e^{\mp i(\alpha+\frac{\pi}{2})}\cdot[R,\infty[$ and the arc $\{\lambda\in\mathbb{C} : |\lambda| = R, |\arg\lambda| \le \alpha+\frac{\pi}{2}\}$, and the orientation runs from $e^{-i\alpha}\infty$ to $e^{i\alpha}\infty$.



The contour integral converges absolutely in $\mathscr{L}(X)$. In fact, by assumption,

$$C_\alpha := \sup_{\lambda \in \Sigma_{\alpha+\frac{\pi}{2}}} \|\lambda R(\lambda, A)\| < \infty.$$

Thus, on the rays and for $z = |z|e^{i\beta}$ with $\beta \in ]-\theta', \theta'[$ we have the estimates

$$\begin{aligned}
&\frac{1}{2\pi}\int_{e^{\mp i(\alpha+\frac{\pi}{2})}\cdot[\frac{1}{|z|},\infty[} \left\| e^{\lambda z} R(\lambda, A)\right\| \, |\mathrm{d}\lambda| \\
&= \frac{1}{2\pi}\int_{\frac{1}{|z|}}^{\infty} \left\| e^{re^{\mp i(\alpha+\frac{\pi}{2})}|z|e^{i\beta}} R(re^{\mp i(\alpha+\frac{\pi}{2})}, A)\right\| \, \mathrm{d}r \\
&\le \frac{1}{2\pi}\int_{\frac{1}{|z|}}^{\infty} e^{-r|z|\sin(\alpha+\beta)} \frac{C_\alpha}{r} \, \mathrm{d}r \\
&\le \frac{C_\alpha}{2\pi}\int_{1}^{\infty} e^{-r\sin(\alpha-\theta')} \frac{1}{r} \, \mathrm{d}r \\
&\le \frac{C_\alpha}{2\pi \sin(\alpha-\theta')}.
\end{aligned}$$

On the arc, we have the estimate

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\substack{|\lambda|=\frac{1}{|z|} \\ |\arg\lambda| \le \alpha+\frac{\pi}{2}}} \left\| e^{\lambda z} R(\lambda, A)\right\| \, |\mathrm{d}\lambda| \\
&= \frac{1}{2\pi}\int_{-(\alpha+\frac{\pi}{2})}^{\alpha+\frac{\pi}{2}} \left\| e^{\frac{1}{|z|}e^{i\theta'}|z|e^{i\beta}} R(\frac{1}{|z|}e^{i\theta'}, A)\frac{i}{|z|}e^{i\theta'}\right\| \, \mathrm{d}\theta' \\
&\le \frac{C_\alpha}{2\pi}\int_{-(\alpha+\frac{\pi}{2})}^{\alpha+\frac{\pi}{2}} \left| e^{e^{i\theta'}e^{i\beta}}\right| \, \mathrm{d}\theta' \\
&\le C_\alpha\, e.
\end{aligned}$$

Altogether, these estimates also show that $T$ is uniformly bounded on $\Sigma_{\theta'}$.

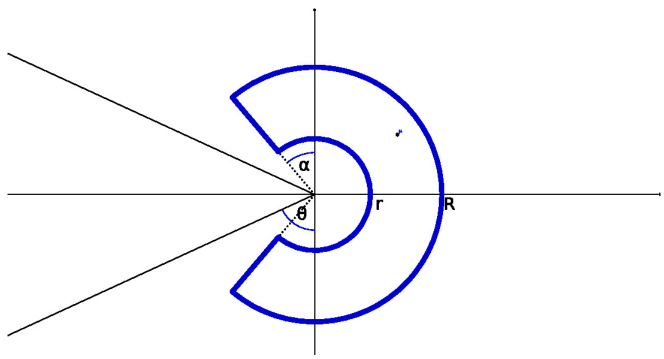
By Cauchy's formula, for every $r > 0$,

$$T(z) = \frac{1}{2\pi i}\int_{\gamma_r} e^{\lambda z} R(\lambda, A) \, \mathrm{d}\lambda.$$

This representation over the contour $\gamma_r$ has the advantage that the contour of integration no longer depends on $z$ or the length of $z$, and one sees that $T$ is holomorphic. We show that $T$ is the holomorphic $C_0$-semigroup generated by $A$. For this, we compute the Laplace transform of $T$. Let $\mu \in \mathbb{C}_{\mathrm{Re}>0}$ and choose $r > 0$ so small (actually, $r < \mathrm{Re}\,\mu$) that the contour $\gamma_r$ lies on the left of $\mu$. Then

$$\begin{aligned}\int_0^\infty e^{-\mu t} T(t)\, \mathrm{d}t &= \int_0^\infty e^{-\mu t} \left( \frac{1}{2\pi i} \int_{\gamma_r} e^{\lambda t} R(\lambda, A)\, \mathrm{d}\lambda \right) \mathrm{d}t \\ &= \frac{1}{2\pi i} \int_{\gamma_r} \int_0^\infty e^{-(\mu-\lambda)t}\, \mathrm{d}t R(\lambda, A)\, \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \int_{\gamma_r} \frac{1}{\mu - \lambda} R(\lambda, A)\, \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \left( \int_{\gamma_r} - \int_{\gamma_R} + \int_{\gamma_R} \right) \frac{1}{\mu - \lambda} R(\lambda, A)\, \mathrm{d}\lambda,\end{aligned}$$

where $R > r$. The difference of the contour integrals $\int_{\gamma_r} - \int_{\gamma_R}$ is an integral over a closed contour and, for $R > r$ large enough, $\mu$ lies inside this contour.




By Cauchy's formula,

$$\frac{1}{2\pi i} \left( \int_{\gamma_r} - \int_{\gamma_R} \right) \frac{1}{\mu - \lambda} R(\lambda, A)\, \mathrm{d}\lambda = R(\mu, A).$$

On the other hand, elementary estimates show that

$$\lim_{R \to \infty} \frac{1}{2\pi i} \int_{\gamma_R} \frac{1}{\mu - \lambda} R(\lambda, A)\, \mathrm{d}\lambda = 0.$$

But the integrals on the left-hand side do not depend on $R$, for $R$ large enough, so that, for $R$ large enough,

$$\frac{1}{2\pi i} \int_{\gamma_R} \frac{1}{\mu - \lambda} R(\lambda, A)\, \mathrm{d}\lambda = 0.$$

We have shown that

$$\int_0^\infty e^{-\mu t} T(t)\, \mathrm{d}t = R(\mu, A).$$

From here one easily deduces that for every $k \in \mathbb{N}$ and every $\mu > 0$

$$\begin{aligned}
\left\|\mu^k R(\mu,A)^k\right\| &= \left\|\frac{\mu^k}{(k-1)!}\left(\frac{d}{d\mu}\right)^{k-1} R(\mu,A)\right\| \\
&= \left\|\frac{\mu^k}{(k-1)!}\int_0^\infty e^{-\mu t}(-t)^{k-1}T(t)\,\mathrm{d}t\right\| \\
&\le \frac{\mu^k}{(k-1)!}\int_0^\infty e^{-\mu t}t^{k-1}M\mathrm{d}t \\
&= M,
\end{aligned}$$

where $M = \sup_{t\ge 0}\|T(t)\| < \infty$. Now it follows from the Theorem of Hille-Yosida that $A$ is the generator of a (bounded) $C_0$-semigroup $S$. By Theorem 8.11, the Laplace transform of $S$ is the resolvent of $A$ and thus coincides with the Laplace transform of $T$. By uniqueness of the Laplace transform (Corollary 8.25), $T = S$, and we have proved that $A$ generates a bounded holomorphic $C_0$-semigroup on $\Sigma_{\theta'}$. Since $\theta' \in ]0,\theta[$ was arbitrary, we have proved assertion (iii), and the proof is complete.

A $C_0$-semigroup $T$ on a Banach space $X$ is **differentiable**, if it is (real) differentiable from the open interval $]0,\infty[$ with values in $\mathscr{L}(X)$. Obviously, every holomorphic semigroup is differentiable.

The following theorem also characterises generators of holomorphic semigroups, but the quantitative estimates on the sector of holomorphy are not as explicit as in the previous theorem.

**Theorem 8.30.** *Let $A$ be a closed, densely defined operator on a complex Banach space $X$. The following assertions are equivalent:*

(i) *The operator $A$ generates a holomorphic $C_0$-semigroup.*

(ii) *For some $\omega \in \mathbb{R}$, the right-half plane $\mathbb{C}_{\mathrm{Re}>\omega}$ is contained in $\rho(A)$ and*

$$\sup_{\mathrm{Re}\,\lambda>\omega} \|(\lambda-\omega)R(\lambda,A)\| < \infty.$$

(iii) *The operator $A$ generates a differentiable $C_0$-semigroup $T$, and*

$$\sup_{t\in(0,1)} \|tAT(t)\|_{\mathscr{L}(X)} < \infty.$$

*Proof.* The implication (i) $\Longrightarrow$ (ii) follows from the implication (i) $\Longrightarrow$ (iii) in Theorem 8.29.

(ii) $\Longrightarrow$ (i) and (ii) $\Longrightarrow$ (iii). For simplicity, assume that $\omega = 0$. The estimate $\|R(\lambda,A)\|_{\mathscr{L}(X)} \le C/|\lambda|$ in the open right half plane $\mathbb{C}_{\mathrm{Re}>0}$ implies that the resolvent remains bounded near every point $is \in i\mathbb{R}$, $s \ne 0$. By Lemma 5.3, $(i\mathbb{R})\setminus\{0\} \subseteq \rho(A)$ and $\|R(is,A)\|_{\mathscr{L}(X)} \le C/|s|$ for all $s \in \mathbb{R}\setminus\{0\}$. By Lemma 5.3,

$$R(\lambda,A) = \sum_{k=0}^\infty R(is,A)^{k+1}(-1)^k(\lambda-is)^k$$

and the series converges absolutely for every $\lambda \in \mathbb{C}$ with $|\lambda - is| < |s|/C$. In particular, the radius of converges grows linearly in $s$. This implies $\Sigma_{\theta+\frac{\pi}{2}} \subseteq \rho(A)$ with $\theta = \arctan(1/C)$, and a simple estimate of the power series shows that $\sup_{\lambda\in\Sigma_{\theta'+\frac{\pi}{2}}} \|\lambda R(\lambda,A)\|_{\mathscr{L}(X)} < \infty$ for every $\theta' \in ]0,\theta[$. By Theorem 8.29 (iii) $\Longrightarrow$ (i), $A$ generates a holomorphic $C_0$-semigroup $T$. We have seen in the proof of that implication in Theorem 8.29 that

$$T(t) = \frac{1}{2\pi i}\int_{\gamma_{1/t}} e^{\lambda t} R(\lambda,A)\,\mathrm{d}\lambda,$$

and estimating the integral like in that proof yields (iii).

(iii) $\Longrightarrow$ (i) Assume that $A$ generates a differentiable $C_0$-semigroup and that $C := \sup_{t\in(0,1)} \|tAT(t)\|_{\mathscr{L}(X)} < \infty$. Then $T$ is infinitely differentiable on $]0,\infty[$ and, for every $t \in (0,1)$ and every $k$,

$$\|T^{(k)}(t)\| = \|A^k T(t)\| = \|(AT(t/k))^k\| \le \|T'(t/k)\|^k \le \left(\frac{Ck}{t}\right)^k.$$

Dividing this inequality by $k! \ge \left(\frac{k}{e}\right)^k$ yields

$$\frac{1}{k!}\|T^{(k)}(t)\| \le \left(\frac{Ce}{t}\right)^k,$$

which implies that the power series

$$T(z) := \sum_{k=0}^{\infty} \frac{T^{(k)}(t)}{k!}(z-t)^k$$

converges absolutely for $z \in \mathbb{C}$ with $|z-t| < \frac{t}{Ce}$. In other words, $T$ extends analytically to the set $\Sigma_\theta \cap \{\mathrm{Re}\, z < 1\}$ with $\theta = \arcsin(1/Ce)$. Now the semigroup property yields that $T$ extends analytically to the whole sector $\Sigma_\theta$ and we have proved the theorem.

**Remarks 8.31.** (a) We state without proof that if $A$ is the generator of a $C_0$-semigroup $T$ and if

$$\limsup_{t\to 0+} \|tAT(t)\| < \frac{1}{e},$$

then $A$ is bounded. Compare with the proof of the implication (iii) $\Longrightarrow$ (i) in Theorem 8.30.

(b) We state without proof that if $A$ is the generator of a $C_0$-semigroup $T$ and if

$$\limsup_{t\to 0+} \|T(t) - I\| < 2,$$

then the semigroup is holomorphic. The bound 2 is sharp as one can see by considering a contraction semigroup which is not holomorphic, like for example the shift semigroups on $L^p(\mathbb{R})$.

We conclude this section with an important class of generators of holomorphic $C_0$-semigroups. Let $V$ and $H$ be two Hilbert spaces such that $V \subseteq H$ with continuous and dense embedding. Let $a : V \times V \to \mathbb{C}$ be a bounded, elliptic, sesquilinear form and let $A$ be the associated operator; see Section 7.6. We recall that $a$ is bounded if there exists a constant $C \geq 0$ such that

$$|a(u,v)| \leq C\,\|u\|_V\,\|v\|_V \text{ for every } u, v \in V,$$

and it is elliptic if there exist constants $\omega \in \mathbb{R}$, $\eta > 0$ such that

$$\operatorname{Re} a(u,u) + \omega\,\|u\|_H^2 \geq \eta\,\|u\|_V^2 \text{ for every } u \in V.$$

The associated operator is given by

$$\begin{aligned} \operatorname{dom} A &:= \{u \in V \,:\, \exists f \in H \forall v \in V : a(u,v) = \langle f, v\rangle\}, \\ Au &:= f. \end{aligned}$$

If the form $a$ is accretive, then $\operatorname{Re}\langle Au, u\rangle = \operatorname{Re} a(u,u) \geq 0$ for every $u \in \operatorname{dom} A$, that is, $-A$ is dissipative. In fact, we call an operator $A$ on a Banach space **accretive** if $-A$ is dissipative. It is a matter of taste whether one prefers to work with dissipative or accretive operators; in any case, the difference is only a sign.

**Theorem 8.32.** *Let the sesquilinear form a, the associated operator A, and the constants C, ω, η be as above. Then the operator $-A$ generates a holomorphic $C_0$-semigroup T satisfying*

$$\|T(t)\| \leq e^{\omega t} \textit{ for all } t \in \mathbb{R}_+.$$

*Proof.* Besides the statement that $-A$ is the generator of a $C_0$-semigroup, the theorem contains two more statements, namely that the semigroup is holomorphic and that it satisfies the norm estimate for real $t$.

Note that the operator $B := A + \omega I$ is the operator associated with the "shifted" sesquilinear form $b : (u,v) \mapsto a(u,v) + \omega\langle u, v\rangle_H$, as one can easily check. The form $b$ is bounded, coercive, and also accretive. It follows that the operator $B$ is accretive or, equivalently, that $-B$ is dissipative. However, by Theorem 7.39, $-B$ is even $m$-dissipative because $\sigma(-B) \subseteq -\overline{\Sigma_\theta}$ for some $\theta \in [0, \frac{\pi}{2}[$ and, for every $\lambda \in \mathbb{C} \setminus (-\overline{\Sigma_\theta})$,

$$\|R(\lambda, -B)\| \leq \frac{1}{\operatorname{dist}(\lambda, -\overline{\Sigma_\theta})}.$$

Since $H$ is reflexive, and by Lemma 8.16 (c), $B$ is densely defined. By the Lumer-Phillips theorem (Theorem 8.17), $-B$ generates a $C_0$-semigroup $S$ of contractions. By a simple rescaling, $-A = -B + \omega I$ generates the $C_0$-semigroup $T$ given by

$T(t) = e^{\omega t} S(t)$, and it therefore satisfies the norm estimate $\|T(t)\| \leq e^{\omega t}$ ($t \in \mathbb{R}_+$). Finally, the above resolvent estimate and Theorem 8.30 further imply that the semigroups $S$ and $T$ are holomorphic.

## 8.6 Spectral mapping theorems

Let $A$ be a closed operator on a Banach space $X$. Then, for every power function $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^k$ ($k \in \mathbb{N}$),

$$f(\sigma(A)) = \sigma(f(A)), \tag{8.5}$$

and for the inverse function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$, $z \mapsto 1/z$,

$$f(\sigma(A) \setminus \{0\}) = \sigma(f(A)) \setminus \{0\},$$

where for any function $f : U \to \mathbb{C}$ on an open set $U \subseteq \mathbb{C}$ and for every subset $D \subseteq U$ we denote by $f(D) = \{f(z) : z \in D\}$ the image of $f$ under $D$. The operators $f(A)$ have here a natural meaning – for the inverse function at least when $A$ is injective. The above equalities are instances of so-called spectral mapping theorems.

In the special context when $A$ is the infinitesimal generator of a $C_0$-semigroup $T$ on a Banach space $X$, one may ask whether the equality (8.5) holds for $f(z) = e^{tz}$ when we formally interpret $f(A) = T(t)$. Since in this case $f(\sigma(A))$ never contains the origin 0, we in general do not expect the equality to be true as it stands. We say that $T$ satisfies the **spectral mapping theorem** if

$$e^{t\sigma(A)} = \sigma(T(t)) \setminus \{0\} \text{ for every } t \geq 0,$$

and it satisfies the **weak spectral mapping theorem** if

$$\overline{e^{t\sigma(A)}} = \sigma(T(t)) \text{ for every } t \geq 0.$$

The following first result shows that one inclusion is always true, and it shows equality at least for parts of the spectrum.

**Theorem 8.33.** *Let $T = (T(t))_{t \geq 0}$ be a $C_0$-semigroup on a Banach space $X$, with generator $A$. Then, for every $t > 0$,*

(a) $e^{t\sigma(A)} \subseteq \sigma(T(t)) \setminus \{0\}$,

(b) $e^{t\sigma_p(A)} = \sigma_p(T(t)) \setminus \{0\}$,

(c) $e^{t\sigma_r(A)} = \sigma_r(T(t)) \setminus \{0\}$,

(d) $e^{t\sigma_{ap}(A)} \subseteq \sigma_{ap}(T(t)) \setminus \{0\}$.

*Proof.* All assertions follows from the midnight formula (Lemma 8.3 (b)), which says that for every $x \in X$, $t > 0$ and $\lambda \in \mathbb{C}$,

$$(A-\lambda)\int_0^t e^{\lambda(t-s)}T(s)x\,\mathrm{d}s = T(t)x - e^{\lambda t}x. \tag{8.6}$$

If $e^{\lambda t} \in \rho(T(t))$, then

$$\begin{aligned}&(A-\lambda)\int_0^t e^{\lambda(t-s)}T(s)(T(t)-e^{-\lambda t})^{-1}x\,\mathrm{d}s\\ &= \int_0^t e^{\lambda(t-s)}T(s)(T(t)-e^{-\lambda t})^{-1}(A-\lambda)x\,\mathrm{d}s = x,\end{aligned}$$

so that $\lambda \in \rho(A)$. This yields the inclusion (a).

If $\lambda \in \sigma_p(A)$, then there exists an eigenvector $x \in \operatorname{dom} A$, $x \neq 0$, such that $(A-\lambda)x = 0$. Inserting this into (8.6) yields $T(t)x = e^{\lambda t}x$, and hence $e^{\lambda t} \in \sigma_p(T(t))$. In order to prove the converse inclusion, assume that $e^{\lambda t} \in \sigma_p(T(t))$. Let $x \neq 0$ be an eigenvector, that is, $T(t)x = e^{\lambda t}x$. Then the function $s \mapsto e^{-\lambda s}T(s)x$ is $t$-periodic and nonzero. This means, that for some $k \in \mathbb{Z}$ the Fourier coefficient

$$y := \int_0^t e^{-2\pi iks/t}e^{-\lambda s}T(s)x\,\mathrm{d}s$$

is nonzero. By Theorem 8.3 (b), $y \in \operatorname{dom} A$ and, by (8.6), $(A-\lambda-2\pi ik/t)y = (e^{-\lambda t}T(t)-I)x = 0$. Hence, $\lambda + 2\pi ik/t$ is an eigenvalue of $A$, and we have proved (b).

For the residual spectrum (assertion (c)) one can proceed by a duality argument. Note that $\sigma_r(A) = \sigma_p(A')$ (which actually holds true for every closed operator $A$!) and prove the equality $e^{t\sigma_p(A')} = \sigma_p(T(t)') \setminus \{0\}$ along the lines of the proof of assertion (b). The easy inclusion $e^{t\sigma_r(A)} \subseteq \sigma_r(T(t)) \setminus \{0\}$ can also be proved as follows: if $\lambda \in \sigma_r(A)$, then $A-\lambda$ does not have dense range. By (8.6), $T(t)-e^{\lambda t}$ does not have dense range, too, that is, $e^{\lambda t} \in \sigma_r(T(t))$.

The proof of (d) is very similar to the proof of the corresponding inclusion for the point spectrum. If $\lambda \in \sigma_{ap}(A)$, then there exists a sequence $(x_n)$ in $\operatorname{dom} A$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty} Ax_n - \lambda x_n = 0$ in $X$. Inserting this into (8.6) yields $\lim_{n\to\infty} T(t)x_n - e^{\lambda t}x_n = 0$, and hence $e^{\lambda t} \in \sigma_{ap}(T(t))$. We have proved the inclusion (d).

We say that a $C_0$-semigroup $T$ on a Banach space $X$ is **immediately norm continuous** if it is continuous from the *open* interval $]0,\infty[$ with values in $\mathscr{L}(X)$, that is, with respect to the operator norm, and we say that the semigroup is **eventually norm continuous** if it is continuous from $]t_0,\infty[$ with values in $\mathscr{L}(X)$ for some $t_0 \geq 0$. Clearly, holomorphic semigroups and differentiable semigroups are immediately norm continuous.

**Theorem 8.34.** *Let $T$ be an immediately norm continuous $C_0$-semigroup on a Banach space $X$. Then*

$$e^{t\sigma_{ap}(A)} = \sigma_{ap}(T(t)) \setminus \{0\}.$$

*Proof.* The inclusion $\subseteq$ is already proved in Theorem 8.33 (d). For the converse inclusion we imitate the proof of the corresponding inclusion for the point spectrum (see the proof of Theorem 8.33 (b)), however, by working on a different Banach space.

Let $\ell^\infty(X)$ be the Banach space of all $X$-valued, bounded sequences, equipped with the supremum norm, and define the "diagonal" semigroup $\tilde{T}$ on $\ell^\infty(X)$ by setting

$$\tilde{T}(t)(x_n) := (T(t)x_n) \quad ((x_n) \in \ell^\infty(X), t \geq 0).$$

The function $\tilde{T}$ indeed satisfies the semigroup property, but it is in general not strongly continuous. Our assumption that $T$ is immediately norm continuous however implies that $\tilde{T}$ is immediately norm continuous, that is, continuous from the open interval $]0,\infty[$ into $\mathscr{L}(\ell^\infty(X))$. We say nothing about any type of continuity of $\tilde{T}$ at $t = 0$. Let $c_0(X)$ be the closed subspace of $\ell^\infty(X)$ consisting of all $X$-valued sequences converging to 0. Clearly, this space $c_0(X)$ is left invariant under $\tilde{T}$, that is $\tilde{T}(t)c_0(X) \subseteq c_0(X)$ for every $t \geq 0$.

This allows us to consider on the quotient space $\hat{X} = \ell^\infty(X)/c_0(X)$ (with quotient map denoted by $\hat{}$) the quotient semigroup $\hat{T}$ given by

$$\hat{T}(t)\hat{x} := \widehat{\tilde{T}(t)x} \quad (\hat{x} = x + c_0(X), x \in \ell^\infty(X)).$$

This semigroup is also immediately norm continuous, but like the semigroup $\tilde{T}$ it is possibly not strongly continuous at $t = 0$.

After these preparations, assume now that $e^{\lambda t} \in \sigma_{ap}(T(t))$. Then there exists a sequence $x = (x_n)$ in $X$ such that

$$\|x_n\| = 1 \text{ and } \lim_{n\to\infty}(T(t) - e^{\lambda t})x_n = 0.$$

In other words, $x \in \ell^\infty(X)$, $x \notin c_0(X)$ and $(\tilde{T}(t) - e^{\lambda t})x \in c_0(X)$. After taking quotients, we have found $\hat{x} \in \hat{X}$ such that $\hat{x} \neq 0$ and $(\hat{T}(t) - e^{\lambda t})\hat{x} = 0$. Thus, the (continuous!) function $s \mapsto e^{-\lambda s}\hat{T}(s)\hat{x}$ is $t$-periodic and nonzero. This means, that for some $k \in \mathbb{Z}$ the Fourier coefficient

$$\hat{y} := \int_0^t e^{-2\pi iks/t} e^{-\lambda s}\hat{T}(s)\hat{x}\,\mathrm{d}s$$

is nonzero. The sequence $y = (y_n) \in \ell^\infty(X)$, where $y_n$ is given by

$$y_n := \int_0^t e^{-2\pi iks/t} e^{-\lambda s} T(s)x_n\,\mathrm{d}s \quad (n \in \mathbb{N}),$$

is a representative of $\hat{y}$. By Theorem 8.3 (b), $y_n \in \operatorname{dom} A$ for every $n$. The element $\hat{y}$ being nonzero means that the sequence $(y_n)$ does not converge to 0. Hence, after passing to a subsequence of $(y_n)$ (which we denote for simplicity again by $(y_n)$), we may assume that $\inf_n \|y_n\| > 0$. Setting $z_n := y_n/\|y_n\|$ we have $z_n \in \operatorname{dom} A$, $\|z_n\| = 1$ and

$$(A - \lambda - 2\pi ik/t)z_n = \frac{1}{\|y_n\|}(e^{-\lambda t}T(t) - I)x_n \to 0 \text{ as } n \to \infty.$$

It follows that $\lambda + 2\pi ik/t \in \sigma_{ap}(A)$, and the proof is complete.

The spectral mapping theorem is related to the asymptotic behaviour (the behaviour for large times) of semigroups.

Given a $C_0$-semigroup $T$ on a Banach space $X$, with generator $A$, we define the **exponential growth bound**

$$\omega(T) := \inf\{\omega \in \mathbb{R} : \exists M \geq 0 \forall t \in \mathbb{R}_+ : \|T(t)\| \leq M e^{\omega t}\}.$$

By Lemma 8.1, $\omega(T) < +\infty$, that is, the exponential growth bound is either finite or equal to $-\infty$.

**Remark 8.35.** The infimum in the definition of the exponential growth bound is in general not a minimum as one can see by choosing $X = \mathbb{C}^2$ and $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$T(t) = e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

and $\|T(t)\| \simeq 1 + t$. In particular $\|T(t)\| \leq M_\omega e^{\omega t}$ for all $\omega > 0$ (and actually $\omega(T) = 0$), but one does not have $\|T(t)\| \leq M_0 e^{0t} = M_0$ for some $M_0 \geq$ and for all $t \geq 0$.

We further define the *spectral bounds*

$$s(A) := \inf\{\omega \in \mathbb{R} : \mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(A)\}$$

and

$$s_0(A) := \inf\{\omega \in \mathbb{R} : \mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(A) \text{ and } R(\cdot, A) \text{ is bounded on } \mathbb{C}_{\mathrm{Re}>\omega}\}.$$

It follows from Theorem 8.11 and a simple estimate of the Laplace integral that $s_0(A) \leq \omega(T)$, and trivially one has $s(A) \leq s_0(A)$, so that

$$s(A) \leq s_0(A) \leq \omega(T). \tag{8.7}$$

We say that the semigroup $T$ is **exponentially stable** if there exist $a > 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{-at}$. Exponential stability can equivalently be expressed by the property that $\omega(T) < 0$. Here are some characterisations of exponential stability of which only the characterisation (iii) is used in the sequel.

**Theorem 8.36 (Datko).** *Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup on a Banach space $X$. Then the following assertions are equivalent:*

(i) *The semigroup $T$ is exponentially stable, that is, $\omega(T) < 0$.*

(ii) *There exists $t \in \mathbb{R}_+$ such that $\|T(t)\| < 1$.*

(iii) *There exists $t \in \mathbb{R}_+$ such that $r(T(t)) < 1$, where $r(T(t))$ is the spectral radius of $T(t)$ (see Remark 5.6).*

(iv) *For every/some* $1 \leq p < \infty$ *and every* $x \in X$ *one has* $T(\cdot)x \in L^p(\mathbb{R}_+;X)$.

(v) *For every/some* $1 \leq p \leq \infty$ *and every* $f \in L^p(\mathbb{R}_+;X)$ *one has* $T * f \in L^p(\mathbb{R}_+;X)$.

*Proof.* The implications (i)⇒(ii) and (i)⇒(iv) are obvious. The implication (i)⇒(v) follows from Young's inequality.

The implication (ii)⇒(iii) follows from the inequality $r(T(t)) \leq \|T(t)\|$ which is valid for every bounded operator.

(iii)⇒(ii) If $r(T(t)) < 1$, then radius of convergence of the series

$$\sum_{k=0}^{\infty} T(t)^k \lambda^{k+1} = R(\frac{1}{\lambda}, T(t)),$$

which obviously converges for $|\lambda| \, \|T(t))\| < 1$, is strictly larger than 1. In particular, the series converges also for $\lambda = 1$, and then necessarily

$$\lim_{k\to\infty} \|T(t)^k\| = \lim_{k\to\infty} \|T(kt)\| = 0,$$

from where (ii) follows.

(ii)⇒(i) Assume that $\|T(t_0)\| < 1$, and put

$$M := \sup_{0\leq\tau\leq t_0} \|T(\tau)\| < +\infty \text{ and } \omega := \frac{\log \|T(t_0)\|}{t_0} < 0.$$

(If $\|T(t_0)\| = 0$, then put $\omega$ any negative real number). Let $t \in \mathbb{R}_+$. Then $t = kt_0 + \tau$ for some $k \in \mathbb{N}$ and $\tau \in [0,t_0]$. Then

$$\begin{aligned} \|T(t)\| &= \|T(kt_0 + \tau)\| \\ &\leq \|T(t_0)\|^k \, \|T(\tau)\| \\ &\leq M\, e^{\omega k t_0} \\ &= (M\, e^{-\omega t_0})\, e^{\omega t}, \end{aligned}$$

and hence $\omega(T) \leq \omega < 0$.

In order to prove the implication (iv)⇒(ii), assume that for some $p \in [1,\infty)$ and every $x \in X$ one has $T(\cdot)x \in L^p(\mathbb{R}_+;X)$. We first note that by the Closed Graph Theorem there exists a constant $C \geq 0$ such that for every $x \in X$

$$\|T(\cdot)x\|_{L^p} \leq C\, \|x\|.$$

Let $M := \sup_{0\leq s\leq 1} \|T(s)\| < \infty$. For every $x \in X$ and every $t \geq 1$ one has, by Hölder's inequality,

$$\|T(t)x\| = \left\| \int_{t-1}^{t} T(t-s)T(s)x\, ds \right\| \leq MC\, \|x\|,$$

so that the semigroup $(T(t))_{t\geq 0}$ is bounded. Applying Hölder's inequality again, we obtain for $x \in X$ and every $t \in \mathbb{R}_+$

$$\begin{aligned}\|t\,T(t)x\| &= \left\|\int_0^t T(t-s)T(s)x\,ds\right\| \\ &\le M\,(C+1)\,C\,\|x\|\,t^{\frac{1}{p'}},\end{aligned}$$

so that

$$\|T(t)\| \le MC(C+1)\,t^{-\frac{1}{p}} \text{ for every } t \in \mathbb{R}_+.$$

From here follows $\omega(T) < 0$.

(v)⇒(iv)/(ii) Assume that for some $p \in [1,\infty]$ and every $f \in L^p(\mathbb{R}_+;X)$ one has $T * f \in L^p(\mathbb{R}_+;X)$. Again by the Closed Graph Theorem, there exists a constant $C \ge 0$ such that for every $f \in L^p(\mathbb{R}_+;X)$

$$\|T * f\|_{L^p} \le C\,\|f\|_{L^p}.$$

For $\tau \in \mathbb{R}_+$ and $x \in X$ we put

$$f_\tau = T(\cdot)x\,\chi_{[0,\tau]}.$$

Then, for every $0 \le t \le \tau$

$$T * f_\tau(t) = \int_0^t T(t-s)T(s)x\,ds = t\,T(t)x.$$

The above estimate thus implies ($p < \infty$) for every $\tau \ge 0$ and every $x \in X$,

$$\int_0^\tau \|(1+t)\,T(t)x\|^p \,\mathrm{d}t \le (C+1)^p \int_0^\tau \|T(t)x\|^p \,\mathrm{d}t.$$

From this estimate one easily deduces (iv). For $p = \infty$ one deduces directly (ii).

In general, we are interested in situations in which *"the spectrum determines the exponential growth"*, that is, when

$$s(A) = \omega(T).$$

If this equality holds, then the location of the spectrum of the infinitesimal generator determines the exponential growth bound.

**Theorem 8.37.** *One has $s(A) = \omega(T)$ for*

(a) *semigroups satisfying the weak spectral mapping theorem,*

(b) *holomorphic semigroups,*

(c) *immediately differentiable semigroups,*

(d) *eventually differentiable semigroups,*

(e) *immediately norm continuous semigroups,*

(f) *eventually norm continuous semigroups,*

(g) *asymptotically norm continuous semigroups,*

(h) *immediately compact semigroups,*

(i) *eventually compact semigroups,*

(j) *compact perturbations of one of the above.*

*Proof.* We give here only the proof for immediately norm continuous semigroups (which includes the case of immediately differentiable semigroups and holomorphic semigroups); the proof for semigroups satisfying the weak spectral mapping theorem is very similar.

Let $t > 0$ be arbitrary. Recall the definition of the spectral radius from Remark 5.6 which says that

$$r(T(t)) = \liminf_{n\to\infty} \|T(t)^n\|^{\frac{1}{n}} = \liminf_{n\to\infty} \|T(nt)\|^{\frac{1}{n}}.$$

It is an exercise to show that

$$\omega(T) = \liminf_{t\to\infty} \frac{\log \|T(t)\|}{t}.$$

As a consequence,

$$r(T(t)) = \liminf_{n\to\infty} e^{\frac{1}{n}\log \|T(nt)\|} \geq e^{\omega(T)t}.$$

If $r(T(t)) = 0$, then this implies $\omega(T) = -\infty$. Since $s(A) \leq \omega(T)$, this implies the equality $s(A) = \omega(T)$.

So let us assume that $r(T(t)) > 0$. By the characterisation of the spectral radius (see Remark 5.6), there exists $\lambda \in \mathbb{C}$ such that $e^{\lambda t} \in \sigma(T(t))$ and $|e^{\lambda t}| = e^{\mathrm{Re}\,\lambda t} = r(T(t))$. Necessarily, $e^{\lambda t}$ belongs to the topological boundary of $\sigma(T(t))$ and therefore, by Lemma 5.3, $e^{\lambda t} \in \sigma_{ap}(T(t))$. Since $T$ is immediately norm continuous, and by Theorem 8.34, there exists $k \in \mathbb{Z}$ such that $\lambda + 2\pi ik/t \in \sigma_{ap}(A)$. By definition of the spectral bound, this implies $s(A) \geq \mathrm{Re}\,\lambda$, and hence

$$e^{s(A)t} \geq e^{\mathrm{Re}\,\lambda t} = r(T(t)) \geq e^{\omega(T)t}.$$

Taking the logarithm yields $s(A) \geq \omega(T)$. Together with the inequality $s(A) \leq \omega(T)$ which is always true, the proof is complete.

The following corollary in the case when the Banach space $X$ is finite dimensional is perhaps known from the theory of linear ordinary differential equations.

**Corollary 8.38 (Lyapunov).** *Let $T$ be an immediately norm continuous $C_0$-semigroup on a Banach space $X$ with generator $A$. Then $T$ is exponentially stable if and only if $s(A) < 0$.*

We mention without proof two other results in which the spectrum determines the exponential growth.

**Theorem 8.39 (Weis, 1996).** *Let $T$ be a positive $C_0$-semigroup with generator $A$ on $L^p(\Omega)$ (arbitrary measure space, $p \in [1,\infty]$), that is, $T(t)f \geq 0$ for all $t \geq 0$ and all $f \geq 0$. Then $s(A) = \omega(T)$.*

**Theorem 8.40 (Herbst, Gearhart, Prüß, ..., 1978-84).** *Let $T$ be a $C_0$-semigroup on a Hilbert space, with generator $A$. Then $s_0(A) = \omega(T)$.*

We finally give here an example showing that $\omega(T)$ need not be determined by the spectral bounds $s(A)$ or $s_0(A)$.

**Example 8.41 (Greiner-Voigt-Wolff).** Let $1 \leq p < q < \infty$. On the space $X := L^p(1,\infty) \cap L^q(1,\infty)$, we consider the positive semigroup $T = (T(t))_{t\geq 0}$ given by

$$(T(t)f)(x) := f(e^t x) \qquad (f \in X, t \in \mathbb{R}_+, x \in (1,\infty)).$$

Then

$$\omega(T) = -\frac{1}{q} \quad \text{and} \tag{8.8}$$

$$s(A) = -\frac{1}{p}. \tag{8.9}$$

Indeed, for every $f \in X$ and every $t \in \mathbb{R}_+$ one has

$$\begin{aligned} \|T(t)f\|_X &= \|T(t)f\|_{L^p} + \|T(t)f\|_{L^q} \\ &= e^{-\frac{1}{p}t}\,\|f\|_{L^p} + e^{-\frac{1}{q}t}\,\|f\|_{L^q}, \end{aligned}$$

so that

$$\|T(t)\| \leq e^{-\frac{1}{q}t}.$$

On the other hand, by choosing $f = \chi_{(e^t,e^t+1)}$, we obtain

$$\|T(t)\| = e^{-\frac{1}{q}t},$$

from which (8.8) follows.

For every $\operatorname{Re}\lambda < -\frac{1}{p}$, let $f_\lambda(x) := x^\lambda$. Then $f_\lambda \in X$ and $T(t)f_\lambda = e^{\lambda t} f_\lambda$ $(t \in \mathbb{R}_+)$. This implies $f_\lambda \in D(A)$ and $Af_\lambda = \lambda f_\lambda$. Hence,

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -\frac{1}{p}\} \subseteq \sigma(A)$$

and

$$s(A) \geq -\frac{1}{p}.$$

Together with (8.8), this implies $s(A) = \omega(T)$ if $p = q$. Hence,

$$R(\lambda,A)f \in L^p(1,\infty) \text{ for every } f \in X \text{ and every } \operatorname{Re}\lambda > -\frac{1}{p}.$$

Note that

$$(Af)(x) = x f'(x) \qquad (f \in X,\, x \in (1,\infty)),$$

so that

$$(R(\lambda,A)f)(x) = x^{\lambda} \int_x^{\infty} \frac{f(s)}{s^{\lambda+1}}\, ds \qquad (f \in X,\, \operatorname{Re}\lambda > \omega(T),\, x \in (1,\infty)).$$

Hence, for every $f \in X$ and every $\operatorname{Re}\lambda > -\frac{1}{p}$ we may estimate, using Hölder's inequality,

$$\begin{aligned} \|R(\lambda,A)f\|_{L^q}^q &= \int_1^{\infty} \left| x^{\lambda} \int_x^{\infty} \frac{f(s)}{s^{\lambda+1}}\, ds \right|^q dx \\ &\le C(\operatorname{Re}\lambda, p, q)\, \|f\|_{L^p}^q \int_1^{\infty} x^{-\frac{q}{p}}\, dx \\ &\le C(\operatorname{Re}\lambda, p, q)\, \|f\|_{L^p}^q . \end{aligned}$$

As a consequence, for every $\operatorname{Re}\lambda > -\frac{1}{p}$ and every $f \in X$ we have $R(\lambda,A)f \in X$. From this we finally obtain (8.9).

## 8.7 A relation between $C_0$-semigroups and stochastic processes

Let $(\Omega, \mathscr{A}, P)$ be a probability space with a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. Let $(S,\mathscr{B})$ be a measurable space which we call the **state space**; it could consist of a topological or metric space equipped $S$ equipped with the Borel-$\sigma$-algebra $\mathscr{B}$. We then consider a **stochastic process** $(X_t)_{t\in\mathbb{R}_+}$ in $S$, that is, each $X_t$ is a measurable function from $\Omega$ into $S$.

The **finite dimensional marginal distributions** are the probability measures $P_{t_1,\dots,t_N}$ on $S^N$ given by

$$P_{t_1,\dots,t_N}(A) := P((X_{t_1},\dots X_{t_N})^{-1}(A)) \quad (A \subseteq S^N \text{ measurable});$$

here, $0 \le t_1 < \cdots < t_N$. The process is **time homogeneous** if

$$P_{t_1,\dots,t_N} = P_{t_1+\tau,\dots,t_N+\tau}$$

for all choices of $0 \le t_1 < \cdots < t_N$ and $\tau \ge 0$. The process is a **Markov process**, if

$$P(X_t \in A | \mathscr{F}_s) = P(X_t \in A | X_s) \quad (s < t, A \in \mathscr{B});$$

in other words, a Markov process has no memory. If $(X_t)$ is a Markov process, then we call

$$\mu_{s,t}(A) := P(X_t \in A | X_s) \quad (s < t, A \in \mathscr{B})$$

**transition probability**, and we set $\nu_0 := P_0$, the one-dimensional marginal distribution for $t_1 = 0$. Then we define

$$(\mu \otimes \nu)(x,A) := \int_S \int_S 1_A(y,z)\nu(y,\mathrm{d}z)\mu(x,\mathrm{d}y) \quad (x \in S, A \subseteq S^2 \text{ measurable}),$$

and

$$\begin{aligned}(\mu * \nu)(x,A) &:= (\mu \otimes \nu)(x, S \times A) \\ &= \int_S \int_S 1_{S\times A}(y,z)\nu(y,\mathrm{d}z)\mu(x,\mathrm{d}y) \\ &= \int_S \nu(y,A)\, \mu(x,\mathrm{d}y) \quad (x \in S, A \in \mathscr{B}).\end{aligned}$$

# Chapter 9
# Calculus on Banach spaces

## 9.1 Differentiable functions between Banach spaces

Let $X$, $Y$ be two Banach spaces, and let $U \subseteq X$ be open. A function $f : U \to Y$ is **differentiable** at $x \in U$ if there exists a bounded linear operator $T \in \mathscr{L}(X,Y)$ such that

$$\lim_{\|h\| \to 0} \frac{f(x+h) - f(x) - Th}{\|h\|} = 0. \tag{9.1}$$

We say that $f$ is **differentiable** if it is differentiable at every point $x \in U$. If $f$ is differentiable at a point $x \in U$, then $T \in \mathscr{L}(X,Y)$ is uniquely determined. We write $Df(x) := f'(x) := T$ and call $Df(x) = f'(x)$ the **derivative** of $f$ at $x$.

**Lemma 9.1.** *If a function $f : U \to Y$ is differentiable at $x \in U$, then it is continuous at $x$. In particular, every differentiable function is continuous.*

*Proof.* Let $(x_n) \subseteq U$ be convergent to $x$. By definition (equation (9.1)) and continuity of $f'(x)$,

$$\begin{aligned} \|f(x_n) - f(x)\| &\leq \|f(x_n) - f(x) - f'(x)(x - x_n)\| + \|f'(x)(x - x_n)\| \\ &\to \quad 0, \end{aligned}$$

as $n \to \infty$.

Let $X$, $Y$ be two Banach spaces, and let $U \subseteq X$ be open. A function $f : U \to Y$ is called **continuously differentiable** if it is differentiable and if $f' : U \to \mathscr{L}(X,Y)$ is continuous. We denote by

$$C^1(U;Y) := \{f : U \to Y : f \text{ differentiable and } f' \in C(U;\mathscr{L}(X,Y))\}$$

the space of all continuously differentiable functions. Moreover, for $k \geq 2$, we denote by

$$C^k(U;Y) := \{f : U \to Y : f \text{ differentiable and } f' \in C^{k-1}(U;\mathscr{L}(X,Y))\}$$

the space of all $k$ times continuously differentiable functions.

Let $X_i$ $(1 \leq i \leq n)$ and $Y$ be Banach spaces. Let $U \subseteq \bigotimes_{i=1}^n X_i$ be open. We say that a function $f : U \to Y$ is at $a = (a_i)_{1 \leq i \leq n} \in U$ **partially differentiable** with respect to the $i$-th coordinate if the function

$$f_i : U_i \subseteq X_i \to Y, \quad x_i \mapsto f(a_1, \dots, x_i, \dots, a_n)$$

is differentiable in $a_i$. We write $\frac{\partial f}{\partial x_i}(a) := f_i'(a_i) \in \mathscr{L}(X_i, Y)$.

## 9.2 Local inverse function theorem and implicit function theorem

Let $X$ and $Y$ be two Banach spaces and let $U \subseteq X$ be an open subset. The following are two classical theorems in differential calculus.

**Theorem 9.2 (Local inverse function theorem).** *Let $f : U \to Y$ be continuously differentiable and $\bar{x} \in U$ such that $f'(\bar{x}) : X \to Y$ is an isomorphism, that is, bounded, bijective and the inverse is also bounded. Then there exist neighbourhoods $V \subseteq U$ of $\bar{x}$ and $W \subseteq Y$ of $f(\bar{x})$ such that $f : V \to W$ is a $C^1$ diffeomorphism, that is $f$ is continuously differentiable, bijective and the inverse $f^{-1} : W \to V$ is continuously differentiable, too.*

**Theorem 9.3 (Implicit function theorem).** *Assume that $X = X_1 \times X_2$ for two Banach spaces $X_1$, $X_2$, and let $f : X \supset U \to Y$ be continuously differentiable. Let $\bar{x} = (\bar{x}_1, \bar{x}_2) \in U$ be such that $\frac{\partial f}{\partial x_2}(\bar{x}) : X_2 \to Y$ is an isomorphism. Then there exist neighbourhoods $U_1 \subseteq X_1$ of $\bar{x}_1$ and $U_2 \subseteq X_2$ of $\bar{x}_2$, $U_1 \times U_2 \subseteq U$, and a continuously differentiable function $g : U_1 \to U_2$ such that*

$$\{x \in U_1 \times U_2 : f(x) = f(\bar{x})\} = \{(x_1, g(x_1)) : x_1 \in U_1\}.$$

For the proof of the local inverse theorem, we need the following lemma.

**Lemma 9.4.** *Let $f : U \to Y$ be continuously differentiable such that $f : U \to f(U)$ is a homeomorphism, that is, continuous, bijective and with continuous inverse. Then $f$ is a $C^1$ diffeomorphism if and only if for every $x \in U$ the derivative $f'(x) : X \to Y$ is an isomorphism.*

*Proof.* Assume first that $f$ is a $C^1$ diffeomorphism. When we differentiate the identities $x = f^{-1}(f(x))$ and $y = f(f^{-1}(y))$, which are true for every $x \in U$ and every $y \in f(U)$, then we find

$$\begin{aligned} I_X &= (f^{-1})'(f(x)) f'(x) \quad \text{for every } x \in U \text{ and} \\ I_Y &= f'(f^{-1}(y))(f^{-1})'(y) \\ &= f'(x)(f^{-1})'(f(x)) \quad \text{for every } x = f^{-1}(y) \in U. \end{aligned}$$

As a consequence, $f'(x)$ is an isomorphism for every $x \in U$.

For the converse, assume that $f'(x)$ is an isomorphism for every $x \in U$. For every $x_1, x_2 \in U$ one has, by differentiability,

$$f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1) + o(x_2 - x_1),$$

where $o$ depends on $x_1$ and $\lim_{x_2 \to x_1} \frac{o(x_2 - x_1)}{\|x_2 - x_1\|} = 0$. We have $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$ if we put $y_i := f(x_i)$. Hence, the above identity becomes

$$y_2 = y_1 + f'(f^{-1}(y_1))(f^{-1}(y_2) - f^{-1}(y_1)) + o(f^{-1}(y_2) - f^{-1}(y_1)).$$

To this identity, we apply the inverse operator $(f'(f^{-1}(y_1)))^{-1}$ and we obtain

$$f^{-1}(y_2) = f^{-1}(y_1) + (f'(f^{-1}(y_1)))^{-1}(y_2 - y_1) - (f'(f^{-1}(y_1)))^{-1} o(f^{-1}(y_2) - f^{-1}(y_1)).$$

Since $f^{-1}$ is continuous, the last term on the right-hand side of the last equality is sublinear. Hence, $f^{-1}$ is differentiable and

$$(f^{-1})'(y_1) = (f'(f^{-1}(y_1)))^{-1}.$$

From this identity (using that $f^{-1}$ and $f'$ are continuous) we obtain that $f^{-1}$ is continuously differentiable. The claim is proved.

*Proof (Proof of the local inverse function theorem).* Consider the function

$$\begin{aligned} g : U &\to X, \\ x &\mapsto f'(\bar{x})^{-1} f(x). \end{aligned}$$

It suffices to show that $g : V \to W$ is a $C^1$ diffeomorphism for appropriate neighbourhoods $V$ of $\bar{x}$ and $W$ of $g(\bar{x})$.

Consider also the function

$$\begin{aligned} \varphi : U &\to X, \\ x &\mapsto x - g(x). \end{aligned}$$

This function $\varphi$ is continuously differentiable and $\varphi'(x) = I - f'(\bar{x})^{-1} f'(x)$ for every $x \in U$. In particular, $\varphi'(\bar{x}) = 0$. By continuity of $\varphi'$, there exists $r > 0$ and $L < 1$ such that $\|\varphi'(x)\| \le L$ for every $x \in \bar{B}(\bar{x}, r) \subseteq U$. Hence,

$$\|\varphi(x_1) - \varphi(x_2)\| \le L \|x_1 - x_2\| \quad \text{for every } x_1, x_2 \in \bar{B}(\bar{x}, r).$$

By the definition of $\varphi$, this implies

$$\begin{aligned} \|g(x_1) - g(x_2)\| &= \|x_1 - x_2 - (\varphi(x_1) - \varphi(x_2))\| \\ &\ge \|x_1 - x_2\| - L\|x_1 - x_2\| \\ &= (1 - L)\|x_1 - x_2\|. \end{aligned} \tag{9.2}$$

We claim that for every $y \in \bar{B}(g(\bar{x}),(1-L)r)$ there exists a unique $x \in \bar{B}(\bar{x},r)$ such that $g(x) = y$.

The uniqueness follows from (9.2).

In order to prove existence, let $x_0 = \bar{x}$, and then define recursively $x_{n+1} = y + \varphi(x_n) = y + x_n - f'(\bar{x})^{-1} f(x_n)$ for every $n \geq 0$. Then

$$\begin{aligned}
\|x_n - \bar{x}\| &= \|\sum_{k=0}^{n-1} x_{k+1} - x_k\| \\
&\leq \|x_1 - x_0\| + \sum_{k=1}^{n-1} \|\varphi(x_k) - \varphi(x_{k-1})\| \\
&\leq \sum_{k=0}^{n-1} L^k \|x_1 - x_0\| \\
&= \frac{1-L^n}{1-L} \|y - g(\bar{x})\| \\
&\leq (1-L^n)\, r \quad \leq \quad r,
\end{aligned}$$

which implies $x_n \in \bar{B}(\bar{x},r)$ for every $n \geq 0$. Similarly, for every $n \geq m \geq 0$,

$$\|x_n - x_m\| \leq \sum_{k=m}^{n-1} L^k \|y - g(\bar{x})\|,$$

so that the sequence $(x_n)$ is a Cauchy sequence in $\bar{B}(\bar{x},r)$. Since $\bar{B}(\bar{x},r)$ is complete, there exists $\lim_{n\to\infty} x_n =: x \in \bar{B}(\bar{x},r)$. By continuity,

$$x = y + \varphi(x) = y + x - g(x),$$

or

$$g(x) = y.$$

This proves the above claim, that is, $g$ is locally invertible. It remains to show that $g^{-1}$ is continuous (then $g$ is a homeomorphism, and therefore a $C^1$ diffeomorphism by Lemma 9.4). Contiunity of the inverse function, however, is a direct consequence of (9.2) (which even implies Lipschitz continuity).

**Remark 9.5.** The iteration formula

$$x_{n+1} = y + x_n - f'(\bar{x})^{-1} f(x_n)$$

used in the proof of the local inverse theorem in order to find a solution of $g(x) = f'(\bar{x})^{-1} f(x) = y$ should be compared to the discrete Newton iteration

$$x_{n+1} = y + x_n - f'(x_n)^{-1} f(x_n);$$

see Theorem 9.8 below.

*Proof (Proof of the implicit function theorem).* Consider the function

$$\begin{aligned} F : U &\to X_1 \times Y, \\ (x_1,x_2) &\mapsto (x_1, f(x_1,x_2)). \end{aligned}$$

Then $F$ is continuously differentiable and

$$F'(\bar{x})(h_1,h_2) = (h_1, \frac{\partial f}{\partial x_1}(\bar{x})h_1 + \frac{\partial f}{\partial x_2}(\bar{x})h_2).$$

In particular, by the assumption, $F'(\bar{x})$ is locally invertible with inverse

$$F'(\bar{x})^{-1}(y_1,y_2) = (y_1, (\frac{\partial f}{\partial x_2}(\bar{x}))^{-1}(y_2 - \frac{\partial f}{\partial x_1}(\bar{x})y_1)).$$

By the local inverse theorem (Theorem 9.2), there exists a neighbourhood $U_1$ of $\bar{x}_1$, a neighbourhood $U_2$ of $\bar{x}_2$ and a neighbourhood $V$ of $(\bar{x}_1, f(\bar{x})) = F(\bar{x})$ such that $F : U_1 \times U_2 \to V$ is a $C^1$ diffeomorphism. The inverse is of the form

$$F^{-1}(y_1,y_2) = (y_1, h_2(y_1,y_2)),$$

where $h_2$ is a function such that $f(y_1, h_2(y_1,y_2)) = y_2$. Let

$$\tilde{U}_1 := \{x_1 \in U_1 : (x_1, f(\bar{x})) \in V\}.$$

Then $\tilde{U}_1$ is open by continuity of the function $x_1 \mapsto (x_1, f(\bar{x}))$, and $\bar{x}_1 \in \tilde{U}_1$. We restrict $F$ to $\tilde{U}_1 \times U_2$, and we define

$$\begin{aligned} g : \tilde{U}_1 &\to X_2, \\ x_1 &\mapsto g(x_1) = F^{-1}(x_1, f(\bar{x}))_2, \end{aligned} \tag{9.3}$$

where $F^{-1}(\cdot)_2$ denotes the second component of $F^{-1}(\cdot)$. Then $g$ is continuously differentiable, $g(\tilde{U}_1) \subseteq U_2$ and $g$ satisfies the required property of the implicit function.

**Lemma 9.6 (Higher regularity of the local inverse).** *Let $f \in C^k(U;Y)$ for some $k \geq 1$ and assum that $f : U \to f(U)$ is a $C^1$ diffeomorphism. Then $f$ is a $C^k$ diffeomorphism, that is, $f^{-1}$ is $k$ times continuously differentiable.*

*Proof.* For every $y \in f(U)$ we have

$$(f^{-1})'(y) = f'(f^{-1}(y))^{-1}.$$

The proof therefore follows by induction on $k$.

**Lemma 9.7 (Higher regularity of the implicit function).** *If, in the implicit function theorem (Theorem 9.3), the function $f$ is $k$ times continuously differentiable, then the implicit function $g$ is also $k$ times continuously differentiable.*

*Proof.* This follows from the previous lemma (Lemma 9.6) and the definition of the implicit function in the proof of the implicit function theorem.

## 9.3 * Newton's method

**Theorem 9.8 (Newton's method).** *Let $X$ and $Y$ be two Banach spaces, $U \subseteq X$ an open set. Let $f \in C^1(U;Y)$ and assume that there exists $\bar{x} \in U$ such that (i) $f(\bar{x}) = 0$ and (ii) $f'(\bar{x}) \in \mathscr{L}(X,Y)$ is an isomorphism. Then there exists a neighbourhood $V \subseteq U$ of $\bar{x}$ such that for every $x_0 \in V$ the operator $f'(x_0)$ is an isomorphism, the sequence $(x_n)$ defined iteratively by*

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n), \quad n \geq 0, \tag{9.4}$$

*remains in $V$ and $\lim_{n\to\infty} x_n = \bar{x}$.*

*Proof.* By Corollary 1.35 and continuity, there exists a neighbourhood $\tilde{V} \subseteq U$ of $\bar{x}$ such that $f'(x)$ is isomorphic for all $x \in \tilde{V}$. Next, it will be useful to define the auxiliary function $\varphi : \tilde{V} \to X$ by

$$\varphi(x) := x - f'(x)^{-1} f(x), \quad x \in \tilde{V}.$$

Since $f(\bar{x}) = 0$, we find that for every $x \in \tilde{V}$

$$\begin{aligned}\varphi(x) - \varphi(\bar{x}) &= x - f'(x)^{-1}(f(x) - f(\bar{x})) - \bar{x} \\ &= x - \bar{x} - f'(x)^{-1}(f'(\bar{x})(x-\bar{x}) + r(x-\bar{x})),\end{aligned}$$

so that by the continuity of $f'(\cdot)^{-1}$

$$\lim_{x\to\bar{x}} \frac{\|\varphi(x) - \varphi(\bar{x})\|}{\|x-\bar{x}\|} = 0.$$

Hence, there exists $r > 0$ such that $V := B(\bar{x}, r) \subseteq \tilde{V} \subseteq U$ and such that for every $x \in V$

$$\|\varphi(x) - \bar{x}\| = \|\varphi(x) - \varphi(\bar{x})\| \leq \frac{1}{2}\,\|x - \bar{x}\|.$$

This implies that for every $x_0 \in V$ one has $\varphi(x_0) \in V$ and if we define iteratively $x_{n+1} = \varphi(x_n) = \varphi^{n+1}(x_0)$, then

$$\|x_n - \bar{x}\| \leq \left(\frac{1}{2}\right)^n \|x_0 - \bar{x}\| \to 0 \text{ as } n \to \infty.$$

# Chapter 10
# Sobolev spaces

## 10.1 Test functions, convolution and regularization

Let $\Omega \subseteq \mathbb{R}^d$ be an *open* set. For every continuous function $\varphi \in C(\Omega)$ we define the **support**

$$\operatorname{supp} \varphi := \overline{\{x \in \Omega : \varphi(x) \neq 0\}},$$

where the closure is to be understood in $\mathbb{R}^d$. Thus, the support is by definition always closed in $\mathbb{R}^d$, but it is not necessarily a subset of $\Omega$. Next we let

$$\mathscr{D}(\Omega) := C_c^\infty(\Omega) := \{\varphi \in C^\infty(\Omega) : \operatorname{supp} \varphi \subseteq \Omega \text{ is compact}\}$$

be the space of **test functions** on $\Omega$, and

$$L^1_{loc}(\Omega) := \{f : \Omega \to \mathbb{K} \text{ measurable} : \int_K |f| < \infty \forall K \subseteq \Omega \text{ compact}\}$$

the space of **locally integrable functions** on $\Omega$. For every $f \in L^1_{loc}(\mathbb{R}^d)$ and every $\varphi \in \mathscr{D}(\mathbb{R}^d)$ we define the **convolution** $f * \varphi$ by

$$\begin{aligned} f * \varphi(x) &:= \int_{\mathbb{R}^d} f(x-y)\varphi(y) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(y)\varphi(x-y) \, \mathrm{d}y. \end{aligned}$$

**Lemma 10.1.** *For every $f \in L^1_{loc}(\mathbb{R}^d)$ and every $\varphi \in \mathscr{D}(\mathbb{R}^d)$ one has $f * \varphi \in C^\infty(\mathbb{R}^d)$ and for every $1 \leq i \leq d$,*

$$\frac{\partial}{\partial x_i}(f * \varphi) = f * \frac{\partial \varphi}{\partial x_i}.$$

*Proof.* Let $e_i \in \mathbb{R}^d$ be the $i$-th unit vector. Then

$$\lim_{h \to 0} \frac{1}{h}(\varphi(x + he_i) - \varphi(x)) = \frac{\partial \varphi}{\partial x_i}(x)$$

uniformly in $x \in \mathbb{R}^d$ (note that $\varphi$ has compact support). Hence, for every $x \in \mathbb{R}^d$

$$\begin{aligned}
&\frac{1}{h}(f * \varphi(x+he_i) - f * \varphi(x)) \\
&= \frac{1}{h}\int_{\mathbb{R}^d} f(y)(\varphi(x+he_i-y) - \varphi(x-y))\,\mathrm{d}y \\
&\to \int_{\mathbb{R}^d} f(y)\frac{\partial\varphi}{\partial x_i}(x-y)\,\mathrm{d}y.
\end{aligned}$$

The following theorem is proved in courses on measure theory. We omit the proof.

**Theorem 10.2 (Young's inequality).** *Let $f \in L^p(\mathbb{R}^d)$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$. Then $f * \varphi \in L^p(\mathbb{R}^d)$ and*

$$\|f * \varphi\|_p \leq \|f\|_p \, \|\varphi\|_1.$$

**Theorem 10.3.** *For every $1 \leq p < \infty$ and every open $\Omega \subseteq \mathbb{R}^d$ the space $\mathscr{D}(\Omega)$ is dense in $L^p(\Omega)$.*

*Proof.* The technique of this proof (*regularization* and *truncation*) is important in the theory of partial differential equations, distributions and Sobolev spaces. The first step (regularization) is based on Lemma 10.1. The truncation step is in this case relatively easy.

*Regularization.* Let $\varphi \in \mathscr{D}(\mathbb{R}^d)$ be a positive function such that $\|\varphi\|_1 = \int_{\mathbb{R}^d} \varphi = 1$. One may take for example the function

$$\varphi(x) := \begin{cases} c\,e^{1/(1-|x|^2)} & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases} \tag{10.1}$$

with an appropriate constant $c > 0$. Then let $\varphi_n(x) := n^d \varphi(nx)$, so that $\|\varphi_n\|_1 = \int_{\mathbb{R}^d} \varphi_n = 1$ for every $n \in \mathbb{N}$.

Let $f \in L^p(\mathbb{R}^d)$. By Lemma 10.1 and Young's inequality (Theorem 10.2), for every $n \in \mathbb{N}$, $f_n := f * \varphi_n \in C^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and $\|f_n\|_p \leq \|f\|_p$. Hence, for every $n \in \mathbb{N}$ the operator $T_n : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, $f \mapsto f * \varphi_n$ is linear and bounded and $\|T_n\| \leq 1$. Moreover, if $f = 1_I$ for some bounded interval $I = (a_1, b_1) \times \cdots \times (a_d, b_d) \subseteq \Omega$, then

$$\begin{aligned}
\|f_n - f\|_p^p &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)\varphi(ny)n^d \,\mathrm{d}y - f(x) \right|^p \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (f(x-\frac{y}{n}) - f(x))\varphi(y)\,\mathrm{d}y \right|^p \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(x-\frac{y}{n}) - f(x)|\varphi(y)\,\mathrm{d}y \right)^p \mathrm{d}x \to 0
\end{aligned}$$

as $n \to \infty$ by Lebesgue's dominated convergence theorem. In other words, $\lim_{n\to\infty} \|T_n f - f\|_p = 0$ for every $f = 1_I$ with $I$ as above. Since $\operatorname{span}\{1_I : I \subseteq \mathbb{R}^d$

bounded interval} is dense in $L^p(\mathbb{R}^d)$, we find that $\lim_{n\to\infty}\|T_n f - f\|_p = 0$ for every $f$ from a dense subset $M$ of $L^p(\mathbb{R}^d)$. Since the $T_n$ are bounded, we conclude from Lemma 2.48 that $T_n f \to f$ in $L^p(\mathbb{R}^d)$ for every $f \in L^p(\mathbb{R}^d)$. This proves that $L^p \cap C^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.

*Truncation.* Now we consider a general open set $\Omega \subseteq \mathbb{R}^d$ and prove the claim. Let $\varphi \in \mathscr{D}(\mathbb{R}^d)$ be a positive test function such that $\operatorname{supp}\varphi \subseteq \overline{B(0,1)}$ and $\int_{\mathbb{R}^d}\varphi = 1$ (one may take for example the function from (10.1)). Then let $\varphi_n(x) := n^d\varphi(nx)$.

For every $n \in \mathbb{N}$ we let

$$K_n := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \frac{1}{n}\} \cap \overline{B(0,n)},$$

so that $K_n \subseteq \Omega$ is compact for every $n \in \mathbb{N}$.

Now let $f \in L^p(\Omega) \subseteq L^p(\mathbb{R}^d)$ and $\varepsilon > 0$. Let

$$f1_{K_n}(x) = \begin{cases} f(x) & \text{if } x \in K_n, \\ 0 & \text{if } x \in \Omega \setminus K_n. \end{cases}$$

By Lebesgue's dominated convergence theorem (since $\bigcup_n K_n = \Omega$),

$$\|f - f1_{K_n}\|_p^p = \int_\Omega |f|^p (1 - 1_{K_n})^p \to 0 \text{ as } n \to \infty.$$

In particular, there exists $n \in \mathbb{N}$ such that $\|f - f1_{K_n}\|_p \leq \varepsilon$.

For every $m \geq 4n$ we define $g_m := (f1_{K_n}) * \varphi_m \in L^p \cap C^\infty(\mathbb{R}^d)$; note that we here consider $L^p(\Omega)$ as a subspace of $L^p(\mathbb{R}^d)$ by extending functions in $L^p(\Omega)$ by 0 outside $\Omega$. However, since $g_m = 0$ outside $K_{2n}$, we find that actually $g_m \in \mathscr{D}(\Omega)$. By the first step (regularisation), there exists $m \geq 4n$ so large that $\|g_m - f1_{K_n}\|_p \leq \varepsilon$. For such $m$ we have $\|f - g_m\|_p \leq 2\varepsilon$, and the claim is proved.

**Lemma 10.4.** *Let $f \in L^1_{loc}(\Omega)$ be such that*

$$\int_\Omega f\varphi = 0 \quad \textit{for every } \varphi \in \mathscr{D}(\Omega).$$

*Then $f = 0$.*

*Proof.* We first assume that $f \in L^1(\Omega)$ is real and that $\Omega$ has finite measure. By Theorem 10.3, for every $\varepsilon > 0$ there exists $g \in \mathscr{D}(\Omega)$ such that $\|f - g\|_1 \leq \varepsilon$. By assumption, this implies

$$|\int_\Omega g\varphi| = |\int_\Omega (f-g)\varphi| \leq \varepsilon\|\varphi\|_\infty \quad \forall \varphi \in \mathscr{D}(\Omega).$$

Let $K_1 := \{x \in \Omega : g(x) \geq \varepsilon\}$ and $K_2 := \{x \in \Omega : g(x) \leq -\varepsilon\}$. Since $g$ is a test function, the sets $K_1$, $K_2$ are compact. Since they are disjoint and do not touch the boundary of $\Omega$,

$$\inf\{|x-y|, |x-z|, |y-z| : x \in K_1, y \in K_2, z \in \partial\Omega\} =: \delta > 0.$$

Let $K_i^\delta := \{x \in \Omega : \operatorname{dist}(x, K_i) \le \delta/4\}$ $(i = 1, 2)$. Then $K_1^\delta$ and $K_2^\delta$ are two compact disjoint subsets of $\Omega$. Let

$$h(x) := \begin{cases} 1 & \text{if } x \in K_1^\delta, \\ -1 & \text{if } x \in K_2^\delta, \\ 0 & \text{else,} \end{cases}$$

choose a positive test function $\varphi \in \mathscr{D}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \varphi = 1$ and $\operatorname{supp} \varphi \subseteq B(0, \delta/8)$, and let $\psi := h * \varphi$. Then $\psi \in \mathscr{D}(\Omega)$, $-1 \le \psi \le 1$, $\psi = 1$ on $K_1$ and $\psi = -1$ on $K_2$. Let $K := K_1 \cup K_2$. Then

$$\int_K |g| = \int_K g\psi \le \varepsilon + \int_{\Omega \setminus K} |g\psi| \le \varepsilon + \int_{\Omega \setminus K} |g|.$$

Hence,

$$\int_\Omega |g| = \int_K |g| + \int_{\Omega \setminus K} |g| \le \varepsilon + 2\int_{\Omega \setminus K} |g| \le \varepsilon(1 + 2|\Omega|),$$

which implies

$$\int_\Omega |f| \le \int_\Omega |f - g| + \int_\Omega |g| \le 2\varepsilon(1 + |\Omega|).$$

Since $\varepsilon > 0$ was arbitrary, we find that $f = 0$.

The general case can be obtained from the particular case ($f \in L^1$ and $|\Omega| < \infty$) by considering first real and imaginary part of $f$ separately, and then by considering $f 1_B$ for all closed (compact) balls $B \subseteq \Omega$.

## 10.2 Sobolev spaces in one dimension

Recall the fundamental rule of partial integration: if $f, g \in C^1([a,b])$ on some compact interval $[a,b]$, then

$$\int_a^b f g' = f(b)g(b) - f(a)g(a) - \int_a^b f' g.$$

In particular, for every $f \in C^1([a,b])$ and every $\varphi \in \mathscr{D}(a,b)$

$$\int_a^b f\varphi' = -\int_a^b f'\varphi, \tag{10.2}$$

since $\varphi(a) = \varphi(b) = 0$.

Let $-\infty \le a < b \le \infty$ and $1 \le p \le \infty$. We define

$$W^{1,p}(a,b) := \{u \in L^p(a,b) : \exists g \in L^p(a,b) \forall \varphi \in \mathscr{D}(a,b) : \int_a^b u\varphi' = -\int_a^b g\varphi\}.$$

The space $W^{1,p}(a,b)$ is called (first) **Sobolev space**. If $p = 2$, then we also write $H^1(a,b) := W^{1,2}(a,b)$.

By Lemma 10.4, the function $g \in L^p(a,b)$ is uniquely determined if it exists. In the following, we will write $u' := g$, in accordance with (10.2). We equip $W^{1,p}(a,b)$ with the norm

$$\|u\|_{W^{1,p}} := \|u\|_p + \|u'\|_p,$$

and if $p = 2$, then we define the inner product

$$\langle u, v\rangle_{H^1} := \int_a^b uv + \int_a^b u'v',$$

which actually yields the norm $\|u\|_{H^1} = (\|u\|_2^2 + \|u'\|_2^2)^{\frac{1}{2}}$ (which is equivalent to $\|\cdot\|_{W^{1,2}}$).

**Lemma 10.5.** *The Sobolev spaces $W^{1,p}(a,b)$ are Banach spaces, which are separable if $p \neq \infty$. The space $H^1(a,b)$ is a separable Hilbert space.*

*Proof.* The fact that the $W^{1,p}$ are Banach spaces, or that $H^1$ is a Hilbert space, is an exercise. Recall that $L^p(a,b)$ is separable (Remark 2.37). Hence, the product space $L^p(a,b) \times L^p(a,b)$ is separable, and also every subspace of this product space is separable. Now consider the linear mapping

$$T : W^{1,p}(a,b) \to L^p(a,b) \times L^p(a,b), \quad u \mapsto (u, u'),$$

which is bounded and even isometric. Hence, $W^{1,p}$ is isometrically isomomorphic to a subspace of $L^p \times L^p$ which is separable. Hence $W^{1,p}$ is separable.

**Lemma 10.6.** *Let $u \in W^{1,p}(a,b)$ be such that $u' = 0$. Then $u$ is constant.*

*Proof.* Choose $\psi \in \mathscr{D}(a,b)$ such that $\int_a^b \psi = 1$. Then, for every $\varphi \in \mathscr{D}(a,b)$, the function $\varphi - (\int_a^b \varphi)\psi$ is the derivative of a test function since $\int_a^b (\varphi - (\int_a^b \varphi)\psi) = 0$. Hence, by definition,

$$0 = \int_a^b u(\varphi - (\int_a^b \varphi)\psi),$$

or, with $c = \int_a^b u\psi = \text{const}$,

$$\int_a^b (u - c)\varphi = 0 \quad \forall \varphi \in \mathscr{D}(a,b).$$

By Lemma 10.4, $u = c$ almost everywhere.

**Lemma 10.7.** *Let $-\infty < a < b < \infty$ and let $t_0 \in [a,b]$. Let $g \in L^p(a,b)$ and define*

$$u(t) := \int_{t_0}^t g(s)\,\mathrm{d}s, \quad t \in [a,b].$$

*Then $u \in W^{1,p}(a,b)$ and $u' = g$.*

*Proof.* Let $\varphi \in \mathscr{D}(a,b)$. Then, by Fubini's theorem,

$$\begin{aligned}
\int_a^b u\varphi' &= \int_a^b \int_{t_0}^t g(s)\,\mathrm{d}s\varphi'(t)\,\mathrm{d}t \\
&= \int_a^{t_0} \int_{t_0}^t g(s)\,\mathrm{d}s\varphi'(t)\,\mathrm{d}t + \int_{t_0}^b \int_{t_0}^t g(s)\,\mathrm{d}s\varphi'(t)\,\mathrm{d}t \\
&= -\int_a^{t_0} \int_a^s \varphi'(t)\,\mathrm{d}t g(s)\,\mathrm{d}s + \int_{t_0}^b \int_s^b \varphi'(t)\,\mathrm{d}t g(s)\,\mathrm{d}s \\
&= -\int_a^{t_0} \varphi(s)g(s)\,\mathrm{d}s - \int_{t_0}^b \varphi(s)g(s)\,\mathrm{d}s \\
&= -\int_a^b g\varphi.
\end{aligned}$$

**Theorem 10.8.** *Let $u \in W^{1,p}(a,b)$ (bounded or unbounded interval). Then there exists $\tilde{u} \in C((a,b))$ which is continuous up to the boundary of $(a,b)$, which coincides with $u$ almost everywhere and such that for every $s$, $t \in (a,b)$*

$$\tilde{u}(t) - \tilde{u}(s) = \int_s^t u'(r)\,\mathrm{d}r.$$

*Proof.* Fix $t_0 \in (a,b)$ and define $v(t) := \int_{t_0}^t u'(s)\,\mathrm{d}s$ ($t \in \overline{(a,b)}$). Clearly, the function $v$ is continuous. By Lemma 10.7, $v \in W^{1,p}(c,d)$ for every bounded interval $(c,d) \subseteq (a,b)$, and $v' = u'$. By Lemma 10.6, $u - v = C$ for some constant $C$ which clearly does not depend on the choice of the interval $(c,d)$. This proves that $u$ coincides almost everywhere with the continuous function $\tilde{u} = v + C$. By Lemma 10.7,

$$\tilde{u}(t) - \tilde{u}(s) = v(t) - v(s) = \int_s^t u'(r)\,\mathrm{d}r.$$

**Remark 10.9.** By Theorem 10.8, we will identify every function $u \in W^{1,p}(a,b)$ with its continuous representant, and we say that every function in $W^{1,p}(a,b)$ is continuous.

**Lemma 10.10 (Extension lemma).** *Let $u \in W^{1,p}(a,b)$. Then there exists $\tilde{u} \in W^{1,p}(\mathbb{R})$ such that $\tilde{u} = u$ on $(a,b)$.*

*Proof.* Assume first that $a$ and $b$ are finite and define

$$g(t) := \begin{cases} u'(t) & \text{if } t \in [a,b], \\ u(a) & \text{if } t \in [a-1,a), \\ -u(b) & \text{if } t \in (b,b+1], \\ 0 & \text{else.} \end{cases}$$

Then $g \in L^p(\mathbb{R})$. Let $\tilde{u}(t) := \int_{-\infty}^t g(s)\,\mathrm{d}s$, so that $\tilde{u} = u$ on $(a,b)$. By Lemma 10.7, $\tilde{u} \in W^{1,p}(c,d)$ for every bounded interval $(c,d) \in \mathbb{R}$. However, $\tilde{u} = 0$ outside $(a-1, b+1)$ which implies that $\tilde{u} \in W^{1,p}(\mathbb{R})$.

The case of $a = -\infty$ or $b = \infty$ is treated similarly.

**Lemma 10.11.** *For every $1 \leq p < \infty$, the space $\mathscr{D}(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$.*

*Proof.* Let $u \in W^{1,p}(\mathbb{R})$.

*Regularization:* Choose a positive test function $\varphi \in \mathscr{D}(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi = 1$ and put $\varphi_n(x) = n\varphi(nx)$. Then $u_n := u * \varphi_n \in C^\infty \cap L^p(\mathbb{R})$, $u_n' = u' * \varphi_n \in L^p(\mathbb{R})$ and

$$\lim_{n\to\infty} \|u - u_n\|_p = 0 \text{ and}$$
$$\lim_{n\to\infty} \|u' - u_n'\|_p = 0,$$

so that $\lim_{n\to\infty} \|u - u_n\|_{W^{1,p}} = 0$. This proves that $W^{1,p}(\mathbb{R}) \cap C^\infty(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$.

*Truncation:* Choose a sequence $(\psi_n) \subseteq \mathscr{D}(\mathbb{R})$ such that $0 \leq \psi_n \leq 1$, $\psi_n = 1$ on $[-n,n]$ and $\|\psi_n'\|_\infty \leq C$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Choose $v \in C^\infty \cap W^{1,p}(\mathbb{R})$ such that $\|u - v\|_{W^{1,p}} \leq \varepsilon$ (regularization step). For every $n \in \mathbb{N}$, one has $v\psi_n \in \mathscr{D}(\mathbb{R})$ and it is easy to check that for all $n$ large enough, $\|v - v\psi_n\|_{W^{1,p}} \leq \varepsilon$. The claim is proved.

**Corollary 10.12.** *For every $u \in W^{1,p}(a,b)$ (bounded or unbounded interval, $1 \leq p < \infty$) and every $\varepsilon > 0$, there exists $v \in \mathscr{D}(\mathbb{R})$ such that $\|u - v|_{(a,b)}\|_{W^{1,p}} \leq \varepsilon$.*

*Proof.* Given $u \in W^{1,p}(a,b)$, we first choose an extension $\tilde{u} \in W^{1,p}(\mathbb{R})$ (extension lemma 10.10) and then a test function $v \in \mathscr{D}(\mathbb{R})$ such that $\|\tilde{u} - v\|_{W^{1,p}(\mathbb{R})} \leq \varepsilon$ (Lemma 10.11). Then $\|\tilde{u} - v\|_{W^{1,p}(a,b)} = \|u - v\|_{W^{1,p}(a,b)} \leq \varepsilon$.

**Corollary 10.13 (Sobolev embedding theorem).** *Every function $u \in W^{1,p}(a,b)$ is continuous and bounded and there exists a constant $C \geq 0$ such that*

$$\|u\|_\infty \leq C\,\|u\|_{W^{1,p}} \quad \textit{for every } u \in W^{1,p}(a,b).$$

*Proof.* If $p = \infty$, there is nothing to prove. We first prove the claim for the case $(a,b) = \mathbb{R}$.

So let $1 \leq p < \infty$ and let $v \in \mathscr{D}(\mathbb{R})$. Then $G(v) := |v|^{p-1}v \in C_c^1(\mathbb{R})$ and $G(v)' = p|v|^{p-1}v'$. By Hölder's inequality,

$$|G(v)(x)| = p\,|\int_{-\infty}^x |v|^{p-1}v'| \leq p\,\|v\|_p^{p-1}\,\|v'\|_p,$$

so that by Young's inequality ($ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'}$)

$$\|v\|_\infty = \|G(v)\|_\infty^{1/p} \leq C\|v\|_{W^{1,p}}.$$

Since $\mathscr{D}(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$ by Lemma 10.11, the claim for $(a,b) = \mathbb{R}$ follows by an approximation argument.

The case $(a,b) \neq \mathbb{R}$ is an exercise.

**Theorem 10.14 (Product rule, partial integration).** *Let $u$, $v \in W^{1,p}(a,b)$ ($1 \leq p \leq \infty$). Then:*

(i) *(Product rule). The product $uv$ belongs to $W^{1,p}(a,b)$ and*

$$(uv)' = u'v + uv'.$$

(ii) *(Partial integration). If $-\infty < a < b < \infty$, then*

$$\int_a^b u'v = u(b)v(b) - u(a)v(a) - \int_a^b uv'.$$

*Proof.* Since every function in $W^{1,p}(a,b)$ is bounded, we find that $uv$, $u'v + uv' \in L^p(a,b)$. Choose sequences $(u_n)$, $(v_n) \subseteq \mathscr{D}(\mathbb{R})$ such that $\lim_{n\to\infty} u_n|_{(a,b)} = u$ and $\lim_{n\to\infty} v_n|_{(a,b)} = v$ in $W^{1,p}(a,b)$ (Corollary 10.12). By Corollary 10.13, this implies also $\lim_{n\to\infty} \|u_n|_{(a,b)} - u\|_\infty = \lim_{n\to\infty} \|v_n|_{(a,b)} - v\|_\infty = 0$. The classical product rule implies

$$(u_n v_n)' = u_n' v_n + u_n v_n' \text{ for every } n \in \mathbb{N},$$

and the classical rule of partial integration implies

$$\int_a^b u_n' v_n = u_n(b)v_n(b) - u_n(a)v_n(a) - \int_a^b u_n v_n' \text{ for every } n \in \mathbb{N}.$$

The claim follows upon letting $n$ tend to $\infty$.

For every $1 \leq p \leq \infty$ and every $k \geq 2$ we define inductively the **Sobolev spaces**

$$W^{k,p}(a,b) := \{u \in W^{1,p}(a,b) : u' \in W^{k-1,p}(a,b)\},$$

which are Banach spaces for the norms

$$\|u\|_{W^{k,p}} := \sum_{j=0}^{k} \|u^{(j)}\|_p.$$

We denote $H^k(a,b) := W^{k,2}(a,b)$ which is a Hilbert space for the scalar product

$$\langle u, v\rangle_{H^k} := \sum_{j=0}^{k} u^{(j)} v^{(j)}{}_{L^2}.$$

Finally, we define

$$W_0^{k,p}(a,b) := \overline{\mathscr{D}(a,b)}^{\|\cdot\|_{W^{k,p}}},$$

that is, $W_0^{k,p}(a,b)$ is the closure of the test functions in $W^{k,p}(a,b)$, and we put $H_0^k(a,b) := W_0^{k,2}(a,b)$.

**Theorem 10.15.** *Let $-\infty < a < b < \infty$. A function $u \in W_0^{1,p}(a,b)$ if and only if $u \in W^{1,p}(a,b)$ and $u(a) = u(b) = 0$.*

**Theorem 10.16 (Poincaré inequality).** *Let $-\infty < a < b < \infty$ and $1 \leq p < \infty$. Then there exists a constant $\lambda > 0$ such that*

$$\lambda \int_a^b |u|^p \leq \int_a^b |u'|^p \quad \text{for every } u \in W_0^{1,p}(a,b).$$

*Proof.* Let $u \in W^{1,p}(a,b)$. Then

$$\begin{aligned} \int_a^b |u(x)|^p \, \mathrm{d}x &= \int_a^b \left| \int_a^x u'(y) \, \mathrm{d}y \right|^p \mathrm{d}x \\ &\leq \int_a^b \left( \int_a^b |u'(y)| \, \mathrm{d}y \right)^p \mathrm{d}x \\ &\leq \int_a^b (b-a)^{p-1} \int_a^b |u'(y)|^p \, \mathrm{d}y \, \mathrm{d}x \\ &= (b-a)^p \int_a^b |u'(y)|^p \, \mathrm{d}y. \end{aligned}$$

Between the first and the second line, we have used the assumption that $u(a) = 0$, while in the following inequality we applied Hölder's inequality.

**Theorem 10.17.** *Let $-\infty < a < b < \infty$. For every $f \in L^2(a,b)$ there exists a unique function $u \in H_0^1(a,b) \cap H^2(a,b)$ such that*

$$\begin{cases} u - u'' = f & \text{and} \\ u(a) = u(b) = 0 \; . \end{cases} \tag{10.3}$$

*Proof.* We first note that if $u \in H_0^1(a,b) \cap H^2(a,b)$ is a solution, then, by partial integration (Theorem 10.14), for every $v \in H_0^1(a,b)$

$$\int_a^b (uv + u'v') = (u,v)_{H_0^1} = \int_a^b fv. \tag{10.4}$$

By the Cauchy-Schwarz inequality, the linear functional $\varphi \in H_0^1(a,b)'$ defined by $\varphi(v) = \int_a^b fv$ is bounded:

$$|\varphi(v)| \leq \|f\|_2 \, \|v\|_2 \leq \|f\|_2 \, \|v\|_{H_0^1}.$$

By the theorem of Riesz-Fréchet, there exists a unique $u \in H_0^1(a,b)$ such that (10.4)holds true for all $v \in H_0^1(a,b)$. This proves uniqueness of a solution of (10.3), and if we prove that in addition $u \in H^2(a,b)$, then we prove existence, too. However, (10.4)holds in particular for all $v \in \mathscr{D}(a,b)$, i.e.

$$\int_a^b u'v' = -\int_a^b (u-f)v \quad \forall v \in \mathscr{D}(a,b)$$

and $u - f \in L^2(a,b)$ by assumption. Hence, by definition, $u' \in H^1(a,b)$, i.e. $u \in H^2(a,b)$ and $u'' = u - f$. Using also Theorem 10.15, the claim is proved.

## 10.3 Sobolev spaces in several dimensions

In order to motivate Sobolev spaces in several space dimensions, we have to recall the partial integration rule in this case.

**Theorem 10.18 (Gauß).** *Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded such that $\partial\Omega$ is of class $C^1$. Then there exists a unique Borel measure $\sigma$ on $\partial\Omega$ such that for every $u$, $v \in C^1(\bar{\Omega})$ and every $1 \leq i \leq d$*

$$\int_\Omega u \frac{\partial v}{\partial x_i} = \int_{\partial\Omega} uvn_i \,\mathrm{d}\sigma - \int_\Omega \frac{\partial u}{\partial x_i} v,$$

*where $n(x) = (n_i(x))_{1 \leq i \leq d}$ denotes the outer normal vector at a point $x \in \partial\Omega$.*

In particular, if $u \in C^1(\bar{\Omega})$ and $\varphi \in \mathscr{D}(\Omega)$, then

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega \frac{\partial u}{\partial x_i} \varphi.$$

Let $\Omega \subseteq \mathbb{R}^d$ be any open set and $1 \leq p \leq \infty$. We define

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \forall 1 \leq i \leq d \exists g_i \in L^p(\Omega) \quad \forall \varphi \in \mathscr{D}(\Omega) : \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi\}.$$

The space $W^{1,p}(\Omega)$ is called (first) **Sobolev space**. If $p = 2$, then we also write $H^1(\Omega) := W^{1,2}(\Omega)$.

Let $u \in W^{1,p}(\Omega)$. By Lemma 10.4, the functions $g_i$ are uniquely determined. We write $\frac{\partial u}{\partial x_i} := g_i$ and call $\frac{\partial u}{\partial x_i}$ the *partial derivative* of $u$ with respect to $x_i$. As in the one-dimensional case, the following holds true.

**Lemma 10.19.** *The Sobolev spaces $W^{1,p}(\Omega)$ are Banach spaces for the norms*

$$\|u\|_{W^{1,p}} := \|u\|_p + \sum_{i=1}^d \|\frac{\partial u}{\partial x_i}\|_p \quad (1 \leq p \leq \infty),$$

*and $H^1(\Omega)$ is a Hilbert space for the inner product*

$$\langle u, v \rangle_{H^1} := \langle u, v \rangle_{L^2} + \sum_{i=1}^d \langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \rangle_{L^2}.$$

*Proof.* Exercise.

Not all properties of Sobolev spaces on intervals carry over to Sobolev spaces on open sets $\Omega \subseteq \mathbb{R}^d$. For example, it is *not* true that every function $u \in W^{1,p}(\Omega)$ is continuous (without any further restrictions on $p$ and $\Omega$)!

For every open $\Omega \subseteq \mathbb{R}^d$, $1 \leq p \leq \infty$ and every $k \geq 2$ we define inductively the **Sobolev spaces**

$$W^{k,p}(\Omega) := \{u \in W^{1,p}(\Omega) : \forall 1 \leq i \leq d : \frac{\partial u}{\partial x_i} \in W^{k-1,p}(\Omega)\},$$

which are Banach spaces for the norms

$$\|u\|_{W^{k,p}} := \|u\|_p + \sum_{i=0}^{k} \|\frac{\partial u}{\partial x_i}\|_{W^{k-1,p}}.$$

We denote $H^k(\Omega) := W^{k,2}(\Omega)$ which is a Hilbert space for the inner product

$$\langle u, v \rangle_{H^k} := \langle u, v \rangle_{L^2} + \sum_{i=0}^{k} \langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \rangle_{H^{k-1}}.$$

Finally, we define

$$W_0^{k,p}(\Omega) := \overline{\mathscr{D}(\Omega)}^{\|\cdot\|_{W^{k,p}}},$$

that is, $W_0^{k,p}(\Omega)$ is the closure of the test functions in $W^{k,p}(\Omega)$, and we put $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

**Theorem 10.20 (Poincaré inequality).** *Let $\Omega \subseteq \mathbb{R}^d$ be a* bounded *domain, and let $1 \leq p < \infty$. Then there exists a constant $C \geq 0$ such that*

$$\int_\Omega |u|^p \leq C^p \int_\Omega |\nabla u|^p \quad \textit{for every } u \in W_0^{1,p}(\Omega).$$

We note that the Poincaré inequality implies that

$$\|u\| := \left( \int_\Omega |\nabla u|^p \right)^{\frac{1}{p}}$$

defines an equivalent norm on $W_0^{1,p}(\Omega)$ if $\Omega \subseteq \mathbb{R}^d$ is bounded. Clearly,

$$\|u\| \leq \|u\|_{W_0^{1,p}} \quad \text{for every } u \in W_0^{1,p},$$

by the definition of the norm in $W^{1,p}$. On the other hand,

$$\begin{aligned} \|u\|_{W_0^{1,p}} &\leq C\,(\|u\|_{L^p} + \|\nabla u\|_{L^p}) \\ &\leq C\,\|\nabla u\|_{L^p} = C\,\|u\|, \end{aligned}$$

by the Poincaré inequality.

We also state the following two theorems without proof.

**Theorem 10.21 (Sobolev embedding theorem).** *Let $\Omega \subseteq \mathbb{R}^d$ be an open set with $C^1$ boundary. Let $1 \le p \le \infty$ and define*

$$p^* := \begin{cases} \frac{dp}{d-p} & \text{if } 1 \le p < d \\ \infty & \text{if } d < p, \end{cases}$$

*and if $p = d$, then $p^* \in [1, \infty)$. Then, for every $p \le q \le p^*$ we have*

$$W^{1,p}(\Omega) \subseteq L^q(\Omega)$$

*with continuous embedding, that is, there exists $C = C(p,q) \ge 0$ such that*

$$\|u\|_{L^q} \le C \|u\|_{W^{1,p}} \quad \text{for every } u \in W^{1,p}(\Omega).$$

**Theorem 10.22 (Rellich-Kondrachov).** *Let $\Omega \subseteq \mathbb{R}^d$ be an open and bounded set with $C^1$ boundary. Let $1 \le p \le \infty$ and define $p^*$ as in the Sobolev embedding theorem. Then, for every $p \le q < \infty$ the embedding*

$$W^{1,p}(\Omega) \subseteq L^q(\Omega)$$

*is compact, that is, every bounded sequence in $W^{1,p}(\Omega)$ has a subsequence which converges in $L^q(\Omega)$.*

## 10.4 * Elliptic partial differential equations

Let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set, $f \in L^2(\Omega)$, and consider the elliptic partial differential equation

$$\begin{cases} u - \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{10.5}$$

where

$$\Delta u(x) := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} u(x)$$

stands for the **Laplace operator**.

*If* $u \in H_0^1(\Omega) \cap H^2(\Omega)$ is a solution of (10.5), then, by definition of the Sobolev spaces, for every $v \in \mathscr{D}(a,b)$

$$\begin{aligned}
\langle u,v\rangle_{H_0^1} &= \int_\Omega \big(uv + \sum_{i=1}^d \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\big) \\
&= \int_\Omega \big(uv - \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2} v\big) \\
&= \int_\Omega (u - \Delta u)v \\
&= \int_\Omega fv.
\end{aligned}$$

By density of the test functions in $H_0^1(\Omega)$, this equality holds actually for all $v \in H_0^1(\Omega)$. This may justify the following definition of a weak solution. A function $u \in H_0^1(\Omega)$ is called a **weak solution** of (10.5) if for every $v \in H_0^1(\Omega)$

$$\langle u,v\rangle_{H_0^1} = \int_\Omega uv + \int_\Omega \nabla u \nabla v = \int_\Omega fv, \tag{10.6}$$

where $\nabla u$ is the usual, euclidean gradient of $u$.

**Theorem 10.23.** *Let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set. Then, for every $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ of the problem* (10.5).

*Proof.* By the Cauchy-Schwarz inequality, the linear functional $\varphi \in H_0^1(\Omega)'$ defined by $\varphi(v) = \int_\Omega fv$ is bounded:

$$|\varphi(v)| \le \|f\|_2 \, \|v\|_2 \le \|f\|_2 \, \|v\|_{H_0^1}.$$

By the theorem of Riesz-Fréchet, there exists a unique $u \in H_0^1(\Omega)$ such that (10.6) holds true for all $v \in H_0^1(a,b)$. The claim is proved.

# Chapter 11
# Bochner-Lebesgue and Bochner-Sobolev spaces

## 11.1 The Bochner integral

Let $X$ and $Y$ be Banach spaces with norms denoted by $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. If the norm is clear from the context, we simply write $\|\cdot\|$. The space of all bounded, linear operators from $X$ into $Y$ is denoted by $\mathscr{L}(X,Y)$. Let $(\Omega,\mathscr{A},\mu)$ be a measure space. A function $f:\Omega\to X$ is called **step function**, if there exists a sequence $(A_n)\subseteq\mathscr{A}$ of mutually disjoint measurable sets and a sequence $(x_n)\subseteq X$ such that $f=\sum_n 1_{A_n}x_n$. A function $f:\Omega\to X$ is called **measurable**, if there exists a sequence $(f_n)$ of step functions $f_n:\Omega\to X$ such that $f_n\to f$ pointwise $\mu$-almost everywhere.

**Remark 11.1.** Note that there may be a difference to the definition of mesurability of scalar valued functions. On the one hand, measurability of a function is here depending on the measure $\mu$. However, if the measure space $(\Omega,\mathscr{A},\mu)$ is **complete** in the sense that $\mu(A)=0$ and $B\subseteq A$ implies $B\in\mathscr{A}$, then the above definition of measurability and the classical definition of measurability coincide. Note that one may always consider complete measure spaces. On the other hand, measurability of a function between two measurable spaces is defined via the property that preimages of measurable sets are measurable. Although one may always equip a Banach space with the Borel-$\sigma$-algebra, this definition via preimages is not appropriate for the following purposes.

**Lemma 11.2.** *If $f:\Omega\to X$ is measurable, then $\|f\|_X:\Omega\to\mathbb{R}$ is measurable. More generally, if $f:\Omega\to X$ is measurable and if $g:X\to Y$ is continuous, then $g\circ f:\Omega\to Y$ is measurable.*

*Proof.* This is an easy consequence of the definition of measurability and the continuity of $g$. Note that in particular the norm $\|\cdot\|_X:X\to\mathbb{R}$ is continuous.

**Lemma 11.3.** *If $f:\Omega\to X$ and $g:\Omega\to\mathbb{K}$ are measurable, then $fg:\Omega\to X$ is measurable. Similarly, if $f:\Omega\to X$ and $g:\Omega\to X'$ are measurable, then $\langle g,f\rangle_{X',X}:\Omega\to\mathbb{K}$ is measurable.*

*Proof.* For the proof it suffices to use the definition of measurability and to show that the (duality) product of two step functions is again a step function. This is, however, straightforward.

**Theorem 11.4 (Pettis).** *A function $f : \Omega \to X$ is measurable if and only if $\langle x', f\rangle$ is measurable for every $x' \in X'$ (we say that $f$ is* **weakly measurable***) and if there exists a $\mu$-null set $N \in \mathscr{A}$ such that $f(\Omega \setminus N)$ is separable (we say that $f$ is* **almost separably valued***).*

For the following proof of Pettis' theorem, see HILLE & PHILLIPS [**?**].

*Proof. Sufficiency.* Assume that $f$ is measurable. Then $f$ is weakly measurable by Lemma 11.2. Moreover, by definition, there exists a sequence $(f_n)$ of step functions and a $\mu$-null set $N \in \mathscr{A}$ such that

$$f_n(t) \to f(t) \text{ for all } t \in \Omega \setminus N.$$

Hence,

$$f(\Omega \setminus N) \subseteq \overline{\bigcup_n f_n(\Omega)}.$$

Since for every step function $f_n$ the range is countable, the set on the right-hand side of this inclusion is separable, and hence $f$ is almost separably valued.

*Necessity.* Assume that $f$ is weakly measurable and almost separably valued. We first show that $\|f\|_X$ is measurable. By assumption, there exists a $\mu$-null set and a sequence $(x_n)$ in $X$ such that $D := \{x_n : n \in \mathbb{N}\}$ is dense in $f(\Omega \setminus N)$. By the Hahn-Banach theorem, there exists a sequence $(x_n')$ in $X'$ such that $\|x_n'\|_{X'} = 1$ and $\langle x_n', x_n\rangle = \|x_n\|_X$. Since $f$ is weakly measurable, $|\langle x_n', f\rangle|$ is measurable for every $n$. As a consequence, $\sup_n |\langle x_n', f\rangle|$ is measurable. But $\sup_n |\langle x_n', f\rangle| = \|f\|_X$ on $\Omega \setminus N$ by the choice of the sequence $(x_n')$ and the density of $D$ in the $f(\Omega \setminus N)$. Since our measure space $(\Omega, \mathscr{A}, \mu)$ is supposed to be complete, we obtain that $\|f\|_X$ is measurable. In a similar way, one shows that $|f - x|_X$ is measurable for every $x \in X$, and in particular for $x = x_n$.

Now fix $m \in \mathbb{N}$ and define

$$\begin{aligned}
A_{m1} &:= \{\|f - x_1\|_X \le \inf_{1\le k\le m} \|f - x_k\|_X\}, \\
A_{m2} &:= \{\|f - x_2\|_X \le \inf_{1\le k\le m} \|f - x_k\|_X\} \setminus A_{m1}, \\
A_{m3} &:= \{\|f - x_3\|_X \le \inf_{1\le k\le m} \|f - x_k\|_X\} \setminus (A_{m1} \cup A_{m2}), \\
&\vdots \quad \vdots \\
A_{mm} &:= \{\|f - x_m\|_X \le \inf_{1\le k\le m} \|f - x_k\|_X\} \setminus (\bigcup_{k=1}^{m-1} A_{mk}).
\end{aligned}$$

Then $(A_{mn})_{1\leq n\leq m}$ is a family of measurable, mutually disjoint sets such that $\bigcup_{n=1}^{m} A_{mn} = \Omega$. Define[1]

$$f_m := \sum_{n=1}^{m} 1_{A_{mn}} x_n.$$

Then $(f_m)$ is a sequence of step functions, $(\|f_m - f\|_X)_m$ is decreasing pointwise everywhere, and since $D$ is dense in $f(\Omega \setminus N)$,

$$\lim_{m\to\infty} \|f_m(t) - f(t)\|_X = 0 \text{ for every } t \in \Omega \setminus N.$$

that is, $f_m \to f$ $\mu$-almost everywhere. As a consequence, $f$ is measurable.

**Remark 11.5.** The above proof of Pettis' theorem shows that a measurable function can always be approximated almost everywhere by a sequence of *finite* step functions. The proof in [**?**] is slightly different and shows that a measurable, separably valued function can always be approximated *uniformly* by a sequence of step functions.

**Corollary 11.6.** *If $(f_n)$ is a sequence of measurable functions $\Omega \to X$ such that $f_n \to f$ pointwise $\mu$-almost everywhere, then $f$ is measurable.*

*Proof.* We assume that this corollary is known in the scalar case, that is, when $X = \mathbb{K}$.

By Pettis's theorem (Theorem 11.4), for all $n$ there exists a $\mu$-null set $N_n \in \mathscr{A}$ such that $f_n(\Omega \setminus N_n)$ is separable. Moreover there exists a $\mu$-null set $N_0 \in \Omega$ such that $f_n(t) \to f(t)$ for all $t \in \Omega \setminus N_0$. Let $N := \bigcup_{n\geq 0} N_n$; as a countable union of $\mu$-null sets, $N$ is a $\mu$-null set.

Then $f$ (restricted to $\Omega \setminus N$) is the pointwise limit everywhere of the sequence $(f_n)$. In particular $f$ is weakly measurable. Moreover, $f(\Omega \setminus N)$ is separable since

$$f(\Omega \setminus N) \subseteq \overline{\bigcup_n f_n(\Omega \setminus N)},$$

and since $f_n(\Omega \setminus N)$ is separable. The claim follows from Pettis' theorem.

A measurable function $f : \Omega \to X$ is called **integrable** if $\int_\Omega \|f\|_X \,\mathrm{d}\mu < \infty$.

**Lemma 11.7.** *For every integrable step function $f : \Omega \to X$, $f = \sum_n 1_{A_n} x_n$ the series $\sum_n x_n \mu(A_n)$ converges absolutely and its limit is independent of the representation of $f$.*

*Proof.* Let $f = \sum_n 1_{A_n} x_n$ be an integrable step function. The sets $(A_n) \subseteq \mathscr{A}$ are mutually disjoint and $(x_n) \subseteq X$. Then

$$\sum_n \|x_n\|_X \, \mu(A_n) = \int_\Omega \|f\|_X \,\mathrm{d}\mu < \infty.$$

[1] We are grateful to Anton Claußnitzer for the idea of the definition of the sets $A_{mn}$ and the functions $f_m$.

Let $f : \Omega \to X$ be an integrable step function, $f = \sum_n 1_{A_n} x_n$. We define the **Bochner integral** (for integrable step functions) by

$$\int_\Omega f \,\mathrm{d}\mu := \sum_n x_n \,\mu(A_n).$$

**Lemma 11.8.** (a) *For every measurable function $f : \Omega \to X$ there exists a sequence $(f_n)$ of step functions $\Omega \to X$ such that $\|f_n\|_X \le \|f\|_X$ and $f_n \to f$ pointwise $\mu$-almost everywhere.*

(b) *Let $f : \Omega \to X$ be integrable. Let $(f_n)$ be a sequence of integrable step functions such that $\|f_n\|_X \le \|f\|_X$ and $f_n \to f$ pointwise $\mu$-almost everywhere. Then*

$$x := \lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu \text{ exists}$$

*and*

$$\|x\|_X \le \int_\Omega \|f\|_X \,\mathrm{d}\mu.$$

*Proof.* (a) Let $f : \Omega \to X$ be measurable. Then $\|f\|_X : \Omega \to \mathbb{R}$ is measurable. Therefore there exists a sequence $(g_n)$ of real step functions such that $0 \le g_n \le \|f\|_X$ and $g_n \to \|f\|_X$ pointwise $\mu$-almost everywhere.

Since $f$ is measurable, there exists a sequence $(\tilde{f}_n)$ of step functions $\Omega \to X$ such that $\tilde{f}_n \to f$ pointwise $\mu$-almost everywhere. Put

$$f_n := \frac{\tilde{f}_n \, g_n}{\|\tilde{f}_n\|_X + \frac{1}{n}}.$$

(b) For every integrable step function $g : \Omega \to X$ one has

$$\left\| \int_\Omega g \,\mathrm{d}\mu \right\|_X \le \int_\Omega \|g\|_X \,\mathrm{d}\mu.$$

Hence, for every $n$, $m$

$$\left\| \int_\Omega f_n - f_m \,\mathrm{d}\mu \right\|_X \le \int_\Omega \|f_n - f_m\|_X \,\mathrm{d}\mu,$$

and by Lebesgue's dominated convergence theorem the sequence $(\int_\Omega f_n \,\mathrm{d}\mu)$ is a Cauchy sequence. When we put $x = \lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu$ then

$$\|x\|_X \le \liminf_{n\to\infty} \int_\Omega \|f_n\|_X \,\mathrm{d}\mu = \int_\Omega \|f\|_X \,\mathrm{d}\mu.$$

Let $f : \Omega \to X$ be integrable. We define the **Bochner integral**

$$\int_\Omega f \,\mathrm{d}\mu := \lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu,$$

where $(f_n)$ is a sequence of (integrable) step functions $\Omega \to X$ such that $\|f_n\|_X \leq \|f\|_X$ and $f_n \to f$ pointwise $\mu$-almost everywhere (Lemma 11.8 (a)). The definition of the Bochner integral for integrable functions is independent of the choice of the sequence $(f_n)$ of step functions, by Lemma 11.8 (b). Moreover, if $f$ is a step function, then this definition of the Bochner integral and the previous definition coincide. Finally, by Lemma 11.8 (b),

$$\left\| \int_\Omega f \,\mathrm{d}\mu \right\|_X \leq \int_\Omega \|f\|_X \,\mathrm{d}\mu \quad \text{(triangle inequality).} \tag{11.1}$$

**Remark 11.9.** We will also use the following notation for the Bochner integral:

$$\int_\Omega f \text{ or } \int_\Omega f(t)\,\mathrm{d}\mu(t),$$

and if $\Omega = (a,b)$ is an interval in $\mathbb{R}$:

$$\int_a^b f \text{ or } \int_a^b f(t)\,\mathrm{d}\mu(t).$$

If $\mu = \lambda$ is the Lebesgue measure then we also write

$$\int_\Omega f(t)\,\mathrm{d}t \text{ or } \int_a^b f(t)\,\mathrm{d}t.$$

**Lemma 11.10.** *Let $f : \Omega \to X$ be integrable and $T \in \mathscr{L}(X,Y)$. Then $Tf : \Omega \to Y$ is integrable and*

$$\int_\Omega Tf \,\mathrm{d}\mu = T \int_\Omega f \,\mathrm{d}\mu.$$

*Proof.* Exercise.

**Theorem 11.11 (Lebesgue, dominated convergence).** *Let $(f_n)$ be a sequence of integrable functions. Suppose there exists an integrable function $g : \Omega \to \mathbb{R}$ and an (integrable) measurable function $f : \Omega \to X$ such that $\|f_n\| \leq g$ and $f_n \to f$ pointwise $\mu$-almost everywhere. Then*

$$\int_\Omega f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu.$$

*Proof.* By the triangle inequality and the classical Lebesgue dominated convergence theorem,

$$\left\| \int_\Omega f \,\mathrm{d}\mu - \int_\Omega f_n \,\mathrm{d}\mu \right\|_X \leq \int_\Omega \|f - f_n\|_X \,\mathrm{d}\mu \to 0 \text{ as } n \to \infty.$$

## 11.2 Bochner-Lebesgue spaces

Given a measure space $(\Omega, \mathscr{A}, \mu)$ and a Banach space $X$, we define

$$\mathscr{L}^p(\Omega;X) := \{f : \Omega \to X \text{ measurable} : \int_\Omega \|f\|_X^p \, \mathrm{d}\mu < \infty\} \quad \text{if } 1 \le p < \infty, \text{ and}$$
$$\mathscr{L}^\infty(\Omega;X) := \{f : \Omega \to X \text{ measurable} : \exists C \ge 0 \text{ such that } \mu(\{\|f\|_X \ge C\}) = 0\}.$$

Similarly as in the scalar case one shows that these sets a linear spaces and that

$$\|f\|_p := \big(\int_\Omega \|f\|_X^p \, \mathrm{d}\mu\big)^{1/p} \quad (1 \le p < \infty), \text{ resp.}$$
$$\|f\|_\infty := \inf\{C \ge 0 : \mu(\{\|f\|_X \ge C\}) = 0\},$$

are seminorms. Starting with these definitions, and building on the following general principle, the proof of which is left as an exercise, we define the Bochner-Lebesgue $L^p$ spaces.

**Lemma 11.12.** *If $\|\cdot\|_{\mathscr{X}}$ is a seminorm on the vector space $\mathscr{X}$, then*

$$\mathscr{N} := \{x \in \mathscr{X} : \|x\|_{\mathscr{X}} = 0\}$$

*is a linear subspace. Moreover, the quotient space*

$$X := \mathscr{X} / \mathscr{N}$$

*becomes a normed space for the norm*

$$\|[x]\|_X := \|x\|_{\mathscr{X}} \quad ([x] = x + \mathscr{N} \in X).$$

By Lemma 11.12, for every $1 \le p \le \infty$

$$\mathscr{N}_p := \{f \in \mathscr{L}^p(\Omega;X) : \|f\|_p = 0\}$$
$$= \{f \in \mathscr{L}^p(\Omega;X) : f = 0 \; \mu\text{-almost everywhere}\}$$

is a linear subspace of $\mathscr{L}^p(\Omega;X)$. The **Bochner-Lebesgue** $\mathrm{L}^p$ **space** is then defined to be the quotient space

$$\mathrm{L}^p(\Omega;X) := \mathscr{L}^p(\Omega;X)/\mathscr{N}_p,$$

which is the space of all equivalence classes

$$[f] := f + \mathscr{N}_p, \quad f \in \mathscr{L}^p(\Omega;X).$$

By Lemma 11.12, it is a normed space for the norm

$$\|[f]\|_p := \|f\|_p.$$

**Remark 11.13.** As in the scalar case we will in the following identify *functions* $f \in \mathscr{L}^p(\Omega;X)$ with their *equivalence classes* $[f] \in \mathrm{L}^p(\Omega;X)$, and we say that $\mathrm{L}^p$ is a *function space* although we should be aware that it is only a space of equivalence classes of functions.

**Remark 11.14.** For $\Omega = (a,b)$ an interval in $\mathbb{R}$ and for $\mu = \lambda$ the Lebesgue measure we simply write

$$\mathrm{L}^p(a,b;X) := \mathrm{L}^p((a,b);X).$$

We can do so since the spaces $\mathrm{L}^p([a,b];X)$ and $\mathrm{L}^p((a,b);X)$ coincide since the end points $\{a\}$ and $\{b\}$ have Lebesgue measure zero and there is no danger of confusion.

**Theorem 11.15 (Fischer-Riesz).** *For every $1 \le p \le \infty$, the space $\mathrm{L}^p(\Omega;X)$ is a Banach space.*

*Proof.* The proof follows the same lines as in the classical case, that is, when $X = \mathbb{K}$.

**Lemma 11.16.** *For every $1 \le p < \infty$, the set of all p-integrable step functions $\Omega \to X$ is dense in $L^p(\Omega;X)$.*

*Proof.*

**Lemma 11.17.** *Let the measure space $(\Omega,\mathscr{A},\mu)$ be such that $\mathrm{L}^p(\Omega)$ is separable for $1 \le p < \infty$ (for example, $\Omega \subset \mathbb{R}^d$ be an open set with the Lebesgue measure). Let X be separable. Then $\mathrm{L}^p(\Omega;X)$ is separable for $1 \le p < \infty$.*

*Proof.* By assumption the spaces $\mathrm{L}^p(\Omega)$ and $X$ are separable. Let $(h_n) \subseteq \mathrm{L}^p(\Omega;X)$ and $(x_n) \subseteq X$ be two dense sequences. Then the set

$$\mathscr{F} := \{f : \Omega \to X : f = h_n x_m\}$$

is countable. It suffices to shows that $\mathscr{F} \subseteq \mathrm{L}^p(\Omega;X)$ is total, that is, $\operatorname{span}\mathscr{F}$ is dense in $\mathrm{L}^p(\Omega;X)$. This is an exercise.

**Lemma 11.18.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Then $\mathrm{C}(\bar{\Omega};X) \subseteq \mathrm{L}^p(\Omega;X)$ for every $1 \le p \le \infty$.*

*Proof.* Actually, for finite measure spaces, we have the more general inclusions

$$\mathrm{L}^\infty(\Omega;X) \subseteq \mathrm{L}^p(\Omega;X) \subseteq \mathrm{L}^q(\Omega;X) \subseteq \mathrm{L}^1(\Omega;X)$$

if $1 \le q \le p \le \infty$.

**Theorem 11.19.** *Let $\Omega$ be as in Lemma 11.17. Let $1 < p < \infty$ and assume that X is reflexive. Then the space $\mathrm{L}^p(\Omega;X)$ is reflexive and*

$$\mathrm{L}^p(\Omega;X)' \cong \mathrm{L}^{p'}(\Omega;X').$$

*Proof.* Without proof.

## 11.3 The convolution

**Theorem 11.20 (Young's inequality).** *Let* $T \in \mathrm{L}^1(\mathbb{R}^N;\mathscr{L}(X,Y))$ *and* $f \in \mathrm{L}^p(\mathbb{R}^N;X)$ *(*$1 \le p \le \infty$*). Then for almost every* $x \in \mathbb{R}^N$ *the integral*

$$T * f(x) := \int_{\mathbb{R}^N} T(x-y)f(y)\,\mathrm{d}y$$

*converges absolutely, and for the function* $T*f$ *thus defined one has*

$$\begin{aligned}&T*f \in \mathrm{L}^p(\mathbb{R}^N;Y) \text{ and}\\ &\|T*f\|_{\mathrm{L}^p} \le \|T\|_{\mathrm{L}^1}\,\|f\|_{\mathrm{L}^p}\,.\end{aligned}$$

*Proof.* The case $p=\infty$ is almost trivial. Actually, the strong continuity of the shift semigroup on $\mathrm{L}^1$ yields continuity (and thus measurability) of $T*f$ while the boundedness of $T*f$ and Young's inequality are immediate from the triangle inequality.

Assume now that $p=1$. By Tonnelli's theorem, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \|T(x-y)\|_{\mathscr{L}(X,Y)}\,\|f(y)\|_X\,\mathrm{d}y\,\mathrm{d}x\\ &=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \|T(x-y)\|_{\mathscr{L}(X,Y)}\,\|f(y)\|_X\,\mathrm{d}x\,\mathrm{d}y\\ &=\|T\|_{\mathrm{L}^1}\,\|f\|_{\mathrm{L}^1}\,,\end{aligned}$$

and from this equality follows the claim.

Assume now $1<p<\infty$. From the previous case we deduce that for almost all $x \in \mathbb{R}^N$

$$\|T(x-\cdot)\|_{\mathscr{L}(X,Y)}\,\|f(\cdot)\|_X^p \in \mathrm{L}^1(\mathbb{R}^N),$$

and thus

$$\|T(x-\cdot)\|_{\mathscr{L}(X,Y)}^{\frac{1}{p}}\,\|f(\cdot)\|_X \in \mathrm{L}^p(\mathbb{R}^N).$$

On the other hand, $\|T(x-\cdot)\|_{\mathscr{L}(X,Y)}^{\frac{1}{p'}} \in \mathrm{L}^{p'}(\mathbb{R}^N)$ for every $x \in \mathbb{R}^N$. By Hölder's inequality, for almost every $x \in \mathbb{R}^N$,

$$\|T(x-\cdot)\|_{\mathscr{L}(X,Y)}\,\|f(\cdot)\|_X \in \mathrm{L}^1(\mathbb{R}^N),$$

and

$$\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \|T(x-y)\|_{\mathscr{L}(X,Y)} \, \|f(y)\|_X \, \mathrm{d}y \right)^p \mathrm{d}x$$
$$\leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \|T(x-y)\|_{\mathscr{L}(X,Y)} \, \mathrm{d}y \right)^{\frac{p}{p'}} \int_{\mathbb{R}^N} \|T(x-y)\|_{\mathscr{L}(X,Y)} \, \|f(y)\|_X^p \, \mathrm{d}y \, \mathrm{d}x$$
$$= \|T\|_{\mathrm{L}^1}^{p-1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \|T(x-y)\|_{\mathscr{L}(X,Y)} \, \|f(y)\|_X^p \, \mathrm{d}x \, \mathrm{d}y$$
$$= \|T\|_{\mathrm{L}^1}^p \, \|f\|_{\mathrm{L}^p}^p$$
$$< \infty.$$

For every $T \in \mathrm{L}^1(\mathbb{R}^N;\mathscr{L}(X,Y))$ and every $f \in \mathrm{L}^1(\mathbb{R}^N;X)$ we call the function $T * f \in \mathrm{L}^p(\mathbb{R}^N;Y)$ the **convolution** of $T$ and $f$. It is a fundamental tool in harmonic analysis and the theory of partial differential equations. One first property is the following regularizing effect of the convolution. We recall that we adopt multi-index notation. For example, for every **multi-index** $\alpha \in \mathbb{N}_0^N$ we define

$$|\alpha| := \sum_{k=1}^N \alpha_k,$$
$$\alpha! := \prod_{k=1}^N \alpha_k!, \text{ and}$$
$$x^\alpha := \prod_{k=1}^N x_k^{\alpha_k} \quad (x \in \mathbb{C}^N).$$

Moreover, we denote by $\partial_k$ the partial derivative operator with respect to the $k$-th variable, and define the **$\alpha$-th partial derivative**

$$\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_N^{\alpha_N}.$$

Let $\Omega \subseteq \mathbb{R}^N$ be an open set. For every function $f \in \mathrm{C}(\Omega;X)$ we define the **support**

$$\operatorname{supp} f := \overline{\{x \in \Omega : f(x) \neq 0\}},$$

where the closure has to be taken in $\Omega$! We then define for $k \in \mathbb{N}_0 \cup \{\infty\}$

$$\mathrm{C}_\mathrm{c}^k(\Omega;X) := \{f \in \mathrm{C}^k(\Omega;X) : \operatorname{supp} f \text{ is compact}\},$$

the space of compactly supported $\mathrm{C}^k$-functions. In the special case $X = \mathbb{K}$ we define

$$\mathscr{D}(\Omega) := \mathrm{C}_\mathrm{c}^\infty(\Omega).$$

Elements of $\mathscr{D}(\Omega)$ are called **test functions**.

**Lemma 11.21 (Regularization).** *For every $f \in \mathrm{L}^1_{loc}(\mathbb{R}^N;X)$ and every $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^N)$ one has $f * \varphi \in \mathrm{C}^\infty(\mathbb{R}^N;X)$ and*

$$\partial^\alpha (f * \varphi) = f * \partial^\alpha \varphi.$$

*Proof.* Let $e_i \in \mathbb{R}^d$ be the $i$-th unit vector. Then

$$\lim_{h\to 0} \frac{1}{h}(\varphi(x+he_i) - \varphi(x)) = \frac{\partial \varphi}{\partial x_i}(x)$$

uniformly in $x \in \mathbb{R}^d$ (note that $\varphi$ has compact support). Hence, for every $x \in \mathbb{R}^d$

$$\begin{aligned} &\frac{1}{h}(f*\varphi(x+he_i) - f*\varphi(x)) \\ =\ &\frac{1}{h}\int_{\mathbb{R}^d} f(y)(\varphi(x+he_i-y) - \varphi(x-y))\,\mathrm{d}y \\ \to\ &\int_{\mathbb{R}^d} f(y)\frac{\partial \varphi}{\partial x_i}(x-y)\,\mathrm{d}y. \end{aligned}$$

*Proof.* Let $i \in \{1,\dots,N\}$ and let $e_i \in \mathbb{R}^N$ be the $i$-th canonical unit basis vector.

**Lemma 11.22 (Strong continuity of the shift-group).** *For every $x \in \mathbb{R}^N$ and every $1 \le p \le \infty$ we define the* **shift operator** $S(x) \in \mathscr{L}(\mathrm{L}^p(\mathbb{R}^N;X))$ *by*

$$(S(x)f)(y) := f(x+y) \quad (f \in \mathrm{L}^p(\mathbb{R}^N;X),\, y \in \mathbb{R}^N).$$

*Then $S(x)$ is an isometric isomorphism and, if $p < \infty$,*

$$\lim_{x\to 0} \|S(x)f - f\|_{\mathrm{L}^p} = 0 \textit{ for every } f \in \mathrm{L}^p(\mathbb{R}^N;X).$$

*Proof.* The first statement about $S(x)$ being an isometric isomorphism is easy (with $S(x)^{-1} = S(-x)$). Next, for every simple step function $f = 1_Q \otimes x$ with a cube $Q \subseteq \mathbb{R}^N$, the second statement follows easily from Lebesgue's dominated convergence theorem. By linearity, the second statement holds for every $f$ in the dense subspace

$$D := \operatorname{span}\{1_Q \otimes x : Q \subseteq \mathbb{R}^N \text{ a cube}, x \in X\}.$$

Now fix $f \in \mathrm{L}^p(\mathbb{R}^N;X)$ and let $\varepsilon > 0$. Then there exists $g \in D$ such that $\|f-g\|_{\mathrm{L}^p} < \varepsilon$. Moreover, there exists $\delta > 0$ such that $\|S(x)g - g\|_{\mathrm{L}^p} < \varepsilon$ for every $x \in \mathbb{R}^N$ with $\|x\|_X < \delta$. Hence, for every $x \in \mathbb{R}^N$ with $\|x\|_X < \delta$

$$\begin{aligned} \|S(x)f - f\|_{\mathrm{L}^p} &\le \|S(x)f - S(x)g\|_{\mathrm{L}^p} + \|S(x)g - g\|_{\mathrm{L}^p} + \|g-f\|_{\mathrm{L}^p} \\ &\le 2\,\|g-f\|_{\mathrm{L}^p} + \|S(x)g - g\|_{\mathrm{L}^p} \\ &< 3\varepsilon. \end{aligned}$$

If $\varphi \in \mathrm{L}^1(\mathbb{R}^N)$ is such that $\int_{\mathbb{R}^N} \varphi = 1$, then we call the sequence $(\varphi_n)_n$ given by

$$\varphi_n(x) := n^N \varphi(nx) \quad (x \in \mathbb{R}^N,\, n \in \mathbb{N})$$

an **approximate identity** or an **approximate unit**. The reason for this notion follows from the following lemma.

**Lemma 11.23 (Property of an approximate identity).** *Let $f \in \mathrm{L}^p(\mathbb{R}^N;X)$ ($1 \le p < \infty$) and let $(\varphi_n)_n$ be an approximate identity. Then*

$$\lim_{n\to\infty} f * \varphi_n = f \text{ in } \mathrm{L}^p(\mathbb{R}^N;X).$$

*Proof.* By Tonnelli's theorem, the Hölder inequality, by the strong continuity of the shift-group and by Lebesgue's dominated convergence theorem we have

$$\begin{aligned}
\|f * \varphi_n - f\|_{\mathrm{L}^p}^p &= \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} f(x-y)\varphi_n(y)\,\mathrm{d}y - f(x) \right|^p \mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \|f(x-y) - f(x)\|\, |\varphi_n(y)|\,\mathrm{d}y \right)^p \mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \|f(x-y) - f(x)\|^p\, |\varphi_n(y)|\,\mathrm{d}y\, \|\varphi_n\|_{\mathrm{L}^1}^{p-1}\,\mathrm{d}x \\
&= \|\varphi_n\|_{\mathrm{L}^1}^{p-1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \|f(x-y) - f(x)\|^p\,\mathrm{d}x\, |\varphi_n(y)|\,\mathrm{d}y \\
&= \|\varphi_n\|_{\mathrm{L}^1}^{p-1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left\| f(x - \frac{y}{n}) - f(x) \right\| \mathrm{d}x\, \varphi(y)\,\mathrm{d}y \\
&\to 0 \quad (n \to \infty).
\end{aligned}$$

**Corollary 11.24.** *For every $1 \le p < \infty$ the space $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^N;X)$ is dense in $\mathrm{L}^p(\mathbb{R}^N;X)$.*

*Proof (by regularization and truncation).* Let $f \in \mathrm{L}^p(\mathbb{R}^N;X)$. In the first step, the regularization step, we choose an approximate identity $(\varphi_n)$ starting with a test function $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^N)$. By Young's inequality, $f * \varphi_n \in \mathrm{L}^p(\mathbb{R}^N;X)$, by Lemma 11.21, $f * \varphi_n \in \mathrm{C}^\infty(\mathbb{R}^N)$, and by Lemma 11.23,

$$\lim_{n\to\infty} \|f * \varphi_n - f\|_{\mathrm{L}^p} = 0.$$

In the second step, the truncation step, we choose a sequence $(\psi_m)_m$ of test functions satisfying $0 \le \psi_m \le 1$ and $\psi_m = 1$ on the ball $B(0,m)$ (such functions can be obtained by convolving characteristic functions $\chi_{B(0,2m)}$ with appropriate positive test functions, relying on Lemma 11.21). It is clear from Lebesgue's dominated convergence theorem, that for every $g \in \mathrm{L}^p(\mathbb{R}^N;X)$ one has

$$\lim_{m\to\infty} \|g\psi_m - g\|_{\mathrm{L}^p} = 0.$$

Combining the preceding two equalities, we find a sequence $(m_n)_n$ in $\mathbb{N}$ such that

$$\lim_{n\to\infty} \|(f * \varphi_n)\,\psi_{m_n} - f\|_{\mathrm{L}^p} = 0,$$

and since $(f * \varphi_n)\,\psi_{m_n} \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^N;X)$, the claim is proved.

**Corollary 11.25.** *Let $f \in \mathrm{L}^1_{loc}(\mathbb{R}^N;X)$ be such that*

$$\int_{\mathbb{R}^N} f\varphi = 0 \textit{ for every } \varphi \in \mathscr{D}(\mathbb{R}^N).$$

*Then* $f = 0$*.*

*Proof.* The assumption implies that

$$f * \varphi(x) = \int_{\mathbb{R}^N} f(y)\varphi(x-y)\,\mathrm{d}y = 0 \text{ for every } x \in \mathbb{R}^N,\ \varphi \in \mathscr{D}(\mathbb{R}^N),$$

which just means that

$$f * \varphi = 0 \text{ for every } \varphi \in \mathscr{D}(\mathbb{R}^N).$$

The claim now follows upon choosing an approximate identity $(\varphi_n)$ out of a test function $\varphi$ and by applying Lemma 11.23.

## 11.4 Bochner-Sobolev spaces

Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. We define the **Bochner-Sobolev space**

$$W^{k,p}(\Omega;X) := \{u \in \mathrm{L}^p(\Omega;X) : \forall \alpha \in \mathbb{N}_0^N \exists v_\alpha \in \mathrm{L}^p(\Omega;X) \forall \varphi \in \mathscr{D}(\Omega) \\ \int_\Omega u\partial^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi\}$$

The functions $v_\alpha$ in this definition of the space $W^{k,p}(\Omega;X)$ are uniquely determined. We write $v_\alpha =: \partial^\alpha u$ and we call the function $\partial^\alpha u$ the **weak $\alpha$-th partial derivative** of $u$. The space $W^{k,p}(\Omega;X)$ becomes a Banach space for the norm

$$\|u\|_{W^{k,p}} := \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq k}} \|\partial^\alpha u\|_{\mathrm{L}^p}.$$

Similarly as in the case of the $\mathrm{L}^p$-spaces we write $W^{k,p}(a,b;X)$ instead of $W^{k,p}((a,b);X)$. In the special case when $p = 2$ and $X = H$ is a Hilbert space, we also write

$$H^k(\Omega;H) := W^{k,2}(\Omega;H).$$

This space is a Hilbert space for the inner product

$$\langle u,v \rangle_{H^k} := \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq k}} \langle \partial^\alpha u, \partial^\alpha v \rangle_{\mathrm{L}^2}.$$

The resulting norm $\|\cdot\|_{H^k}$ is equivalent to the norm $\|\cdot\|_{W^{k,2}}$ defined above.

The main results about Sobolev spaces of scalar-valued functions remain true for Sobolev spaces of Banach space valued functions if interpreted properly. In particular, the Sobolev embedding theorem, a version of the product rule, the integration by parts formula and Poincaré's inequality remain true. Even a version of the Rellich-Kondrachev theorem remains true.

**Lemma 11.26.** *For every $-\infty < a < b < \infty$ and every $1 \leq p \leq \infty$ one has $W^{1,p}(a,b;X) \subseteq \mathrm{C^b}(\overline{(a,b)};X)$. For every $u \in W^{1,p}(a,b;X)$ and every $s$, $t \in (a,b)$ one has*

$$u(t) - u(s) = \int_s^t u'(r)\,\mathrm{d}r.$$

**Lemma 11.27.** *Assume that the embedding $V \hookrightarrow H$ is continuous and let $u \in W^{1,2}(0,T;H) \cap \mathrm{L}^\infty(0,T;V)$. Then $u$ is weakly continuous with values in $V$, that is, for every $v \in V'$ the function $t \mapsto \langle v, u(t)\rangle_{V',V}$ is continuous on $[0,T]$.*

*Proof.* Since every function $u \in W^{1,2}(0,T;H)$ is continuous (and hence weakly continuous) with values in $H$, the claim follows from [**?**, Lemma 1.4, page 263] .

**Lemma 11.28.** *Assume that the embedding $V \hookrightarrow H$ is continuous and let $(u_n)$ be a sequence such that*

$$u_n \rightharpoonup u \ \text{ in } \ W^{1,2}(0,T;H) \text{ and}$$
$$u_n \overset{\mathrm{w}*}{\rightarrow} u \ \text{ in } \ L^\infty(0,T;V).$$

*Then there exists a subsequence of $(u_n)$ (which we denote again by $(u_n)$) such that*

$$u_n(t) \rightharpoonup u(t) \ \text{ in } \ V \text{ for every } t \in [0,T].$$

*Proof.* Using the fact that the point evaluation in $t \in [0,T]$ from $W^{1,2}(0,T;H)$ into $H$ is bounded and linear, and maps weakly convergent sequences into weakly convergent sequences, the assumption implies that for every $t \in [0,T]$

$$u_n(t) \rightharpoonup u(t) \ \text{ in } \ H.$$

Let now $w \in H'$ and $t \in [0,T]$. Then one has

$$\langle w, u_n(t) - u(t)\rangle_{V',V} = \langle w, u_n(t) - u(t)\rangle_{H',H} \longrightarrow 0.$$

Using the fact that $H'$ is dense in $V'$ and that the sequence $(u_n(t))$ is bounded in $V$, the claim follows from Lemma ??.

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