MATHEMATICAL ENCULTURATION – ARGUMENTATION AND PROOF AT THE TRANSITION FROM SCHOOL TO UNIVERSITY

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University students, especially in their first semesters, often lack specific mathematical learning and working techniques that are necessary to develop and apply mathematical notions, definitions, theorems and proofs. We consider this to be a key factor for problems arising in the secondary-tertiary transition in mathematics. Usually the foundations of mathematical proofs as well as questions of validity and justification are not an explicit aim of university teaching. In our approach we invoke philosophical aspects of epistemology. We show how our theoretical considerations lead to the development of a teaching design for the teaching and learning of mathematical reasoning, argumentation and proof, making the methodological foundations of mathematics explicit and accessible for students.

INTRODUCTION

Courses in mathematics at university level are often considered to be difficult. Unsatisfactory exam results and high drop-out rates seem to confirm this. Of course there are many reasons for this and the measures and projects to deal with it is also manifold. In Germany, for instance, a notable example is the council the "German Centre for Higher Mathematics Education" (http://www.khdm.de) that was founded in 2011, but there are also many projects like the already completed project SAiL-M (http://www.sail-m.de), in which one of the authors took part as a team member.

Many students fail because of their insufficient prerequisites, some just do not have the ability to work autonomously or do not have enough self-discipline to prepare and review lectures and exercise meetings efficiently. In fact "a shift between two institutional cultures happens when entering university" (Gueudet, 2008, p. 245). To attenuate the ruptures occurring at the transition to university most of the German universities have established so-called math-bridging courses for students who are about to begin first semester.

A lot of math-bridging courses at the secondary-tertiary transition are organized as blended learning scenarios aiming at the repetition of mathematical content and training exercises (e.g. the project http://www.math-bridge.org, Biehler et al., 2012). Within such courses mathematical learning and working strategies are mainly taught implicitly.

Talking about the secondary-tertiary transition, Gueudet (2008) named several related perspectives and issues. In our work we take an epistemological and didactical perspective, and we present a teaching design within our bridging course that focuses on argumentation and mathematical proof. Unlike many other bridging courses at German universities, we do not (only) aim at the repetition of mathematical skills and knowledge mentioning superior concepts and working strategies more or less explicitly. But we have a strong focus on uncovering, discussing and training the methodological foundations of argumentation and proof. In our approach mathematical skills and contents are used as occasions and examples for the development and reflection of comprehensive strategies in an explicit and general way.

The secondary-tertiary transition concerning mathematical argumentation and proof is characterized as follows:

Mathematical argumentation and especially mathematical proving at school is only taught in an exemplary way (Douek, 1999). Bringing the meta-theoretical aspects of mathematical proofs that legitimate the employed methodological means up for discussion is not envisaged or reflected explicitly. We think that this is a key factor for transition from school to university, since

> […] proofs provided during lectures at university play a new role: they are central in the building of the university mathematical culture, because they indicate methods, and also what requires justification or what does not. (Gueudet, 2008, p. 247)

The present article is meant as a theoretical contribution to the described topic although we present a concrete teaching design illustrating and underpinning our theoretical approach. In fact, we consider the science of mathematics education to be a ‚design science' that

> […] presupposes a specific didactic approach that integrates different aspects into a coherent and comprehensive picture of mathematics teaching and learning and then transposing it to practical use in a constructive way. (Wittmann, 1995, p. 356)

In our article we focus on a learning activity exploring the epistemological aspects of mathematical proofs. In the next section we describe our *theoretical position* and give a *didactical analysis of the issue*. After that we shortly describe the *target audience, objectives and context* of our course. Following this, we present a learning scenario together with didactical and methodological comments (*mathematical reasoning and justification – learning scenario and didactical comments*). We close the article with a *summary and outlook on research questions guiding our future work*.

THEORETICAL POSITION AND DIDACTICAL PERSPECTIVE

One of the most important methodological principles of mathematics – and in fact of any science – is the principle of trans-subjective comprehensibility of its results. This means that there is a demand for the comprehensibility of the terminology and the notions, but also the duty to explain and justify its assertions. In particular, everybody

– provided that she/he has the adequate prerequisites – should be able to check and understand mathematical assertions (at least this should be possible in principle) (Janich, 2002; Gatzemeier, 2005).

In other words, the main target of mathematical arguments and proofs is to convince ourselves or others of the truth of (mathematical) propositions that seem questionable or are uncertain (Thiel, 1973). In this sense the methods of mathematical proof could be considered as tools or as a guideline for the development of mathematical argumentations, and thus ensure the trans-subjective comprehensibility of scientific findings (Lorenzen, 1968).

Following the above epistemological and normative perspective of mathematical argumentation we are – at this point – only concerned with the minimum requirements for effective mathematical reasoning. Therefore, we abstract from special didactical differentiations between proofs as processes resp. proofs as products, or between (fully) formalized proofs resp. proofs "really performed" in textbooks (Duval, 1991; Douek, 1999). Moreover, we assume that our approach could lead to fruitful suggestions to close the gaps between the above differentiations in a theoretical and practical way.

What is our understanding of mathematical argumentation? Mathematical argumentation (as any argumentation) can be structured into the classical premise-inference-conclusion pattern written in linear form (Tetens, 2004). The verifiability of the conclusion is derived from certain propositions (premises). These do not require further justification or have to be accepted as having been justified already. In particular, the rules of inference are stated and justified explicitly. A mathematical argumentation that is modelled after this pattern can neither be accused of running into fruitless circular reasoning, nor of culminating in an infinite argumentative regress, nor of aborting the argumentation by appealing to a dogma, i.e., it does not lead into the notorious "Münchhausen-Trilemma" formulated by Hans Albert (1991).

The dialogical character of argumentation is a central idea. In some sense, it could be seen as a basis for the interpretation of (logical) implications. For a short overview from a didactical perspective we refer to Durand-Guerrier and Barrier (2007). Following this idea, logic does not deal with (everlasting) laws of verity (Frege, 2003), but logic is a means to create new knowledge from existing knowledge (Thiel, 1980).

Therefore, we want the students to critically discuss the ideal of trans-subjective comprehensibility as a minimum condition for mathematics as a science. The premises-inference-conclusion pattern should be recognized as a tool to reach this ideal. For this reason, we do not only want to teach some basic theoretical knowledge of argumentation and train the students to read and produce proofs basing on this pattern. In addition, we want the students to discuss these tools on a normative basis considering their usefulness to achieve the ideal of trans-subjective comprehensibility. The above logical pattern presented in a linear form has some didactical advantages in the sense of a didactical reduction.

The linear form is much easier to understand than the common Toulmin-scheme (as didactically analysed in Barrier, Mathé & Durand-Guerrier, 2009) or game theoretical concepts (as presented in Vernant, 2007; Marion, 2006; Durand-Guerrier & Barrier, 2008), since we reduce the number of notions (warrants, backings etc.) and the number of dialogical rules. However, the main advantage is that, beginning with Euclid's "Elements", proofs in mathematical literature and lectures have a monologist form of argumentation by presenting a proof as a step-by-step construction. Their dialogical character weighing the pros and cons and showing the decisions taken is usually not obvious. In this sense, mathematical texts emphasize the product-aspect more than the process-aspect of proofs.

We believe that the normative discussion could enable the students to critically reflect general objectives of mathematics as a science and to recognize methodological decisions as being appropriate. At least, we hope that this could lead to higher motivation and autonomous learning following an ideal of continuous rationality as part of the "mathematical culture".

TARGET AUDIENCE, CONTEXT AND OBJECTIVES

The math-bridging course is designed for first semester students with mathematics as a major subject, but also for future math teachers. A first course integrating the presented learning scenario was held at the University of Education Ludwigsburg in October 2012. In Ludwigsburg, future teachers (primary school, secondary school up to grade 10 and special school) are educated and have to take a variety of math courses depending on their choice of study programme. Especially here we hope that our approach has a positive impact on the future teachers' *idea of mathematics* and hence on their teaching at school.

We expect the target audience to be very heterogeneous and differ widely in their mathematical competencies, learning motivation and general academic ability. Nevertheless, we want to achieve some common learning goals.

At school the students usually come in contact with mathematical proofs in an exemplary form - for instance when applying geometrical theorems about the congruence of triangles in their reasoning or when proving the irrationality of $\sqrt{2}$. In the "educational standards" for mathematics in Germany (KMK 2003) *mathematical reasoning* is one out of six mentioned competencies within mathematics education at school. With our learning scenario we want to build on this basic knowledge of the students and achieve the following learning objectives:

The students should

- analyze mathematical proofs considering the example of the *proof by contradiction* and understand as well as describe their deductive structure, and

- realize that completeness and deductive derivation are necessary criteria for mathematical proofs and adopt the ideal of the premises-inference-conclusion pattern as a reasonable means for mathematical argumentation.

MATHEMATICAL REASONING AND JUSTIFICATION – TEACHING DESIGN

The presented teaching design consists of three phases:
First, a lecture (*Phase 1: Information*) in which the students are provided with information about the basics of "naïve logic" and the method of "proof by contradiction". Second, the first part of the exercise session in which mathematical proofs are analyzed and compared (*Phase 2: Cognition*). Third, the second part of the exercise session that provides an activity to think about argumentation and proof from a philosophical point of view (*Phase 3: Metacognition*). We will shortly describe each phase including didactical and methodological comments referring to our theoretical position.

Phase 1 (Lecture): In the first part of the lecture we introduce the usual logical operators including the quantifiers. Although we want formal interrelations to become transparent by discussing some logical riddles and parallels to everyday language, this part is organized in a stringent way and mainly directed and executed by the lecturer.

The second part of the lecture includes considerably more interaction with the audience. Here we analyze the "model" of a *proof by contradiction* by comparing two examples. First, we present an example of Cohors-Fresenborg and Kaune (2010, p. 31). Here, a judge reasons why a defendant is proved to be innocent following the scheme of a proof by contradiction in the following way:

> Assuming that the defendant *is guilty* (here: robbed the bank). Then he would have been in A-town at 16:00 h. This means he could have been in B-town no earlier than 17:30 h, since you need at least 90 minutes for this distance. Since the bank in B-town was robbed at 17:00 h, this contradicts the given facts. Hence, the defendant has to be innocent.

Together with the audience we develop the following scheme of the proof:

Logical opposite of the claim → conclusion from the first line → conclusion from the second line → fact → conclusion ("a contradiction appears") → conclusion ("the claim is true").

To obtain the pattern of a chain of deductions the students need to identify the premises and the implication steps. Moreover, this example allows resp. forces us to reflect on the character of mathematical propositions and leads us to the *law of the excluded middle* and the *law of non-contradiction* (Russell, 1912).

The second example is a proof of the infinity of the prime numbers as given by Euclid. The lecturer has cut the steps of the proof in lines and put the lines in a wrong order (*proof-jigsaw*). The audience is asked to find the right order and figure out the correspondences between the two examples. The results are collected by the lecturer.

Unlike many other bridging courses do, we do not aim at completeness in the sense that we try to communicate the complete range of methods of mathematical proofs. However, we have chosen the „proof by contradiction“ as being an exemplary method. By choosing the above approach we hope to implicitly suggest that formal (naïve) logic is a technical means for the proper formulation of mathematical propositions and everyday statements.

Phase 2 (Cognition): The above scenario is taken up again in the exercise session in the afternoon, where *two more proof-jigsaws* of the proof of the infinity of primes are given, differing widely in their level of detail. One proof-jigsaw is a very brief version of the proof similar to the one in Aigner and Ziegler (2009). The other one is a very long and detailed version including a high level of formalization. The students form groups and are again asked to put the proof-lines into a meaningful order, while at the same time comparing and discussing all three given proofs for the infinity of the prime numbers. We hope that this creates some sort of provocation resp. irritation in the following sense: at school, mathematics is usually done by learning and applying “recipes”. Usually there is only one way of execution and the result can be right or wrong. The above scenario allows the discussion of three different forms of argumentations, providing an action-oriented and suggestive access to the main objective of the learning scenario: The students discover and reflect the structure of mathematical proofs as chains of deductions. But they also implicitly discover the dialogical character of mathematics as a science through considering a proof to be an attempt to convince another person of the truth of my own statement. This attempt depends on the mathematical background or context of the reader/producer of a proof.

The learning activity leads directly to a higher level of abstraction through the thought experiment “What if a mathematical proof could never be formulated completely?” In the end, the question “What are the basic rules a proof must follow to be convincing?” – a question of validity – could arise, leading to the last phase of our learning scenario.

Phase 3 (Metacognition): The last phase is directed by the questions: *How do I know that it is true?* or *What is a perfect proof?* For this activity it is important to create an atmosphere of discussion and reflection to gain various ideas of the participants. Therefore, we put the students together in groups providing them with only few – but accentuated – impulses. The students are provided with some further material leading them to the question of justification and validity as follows:

> *The Münchhausen-Trilemma*
>
> If we ask of any knowledge: "How do I know that it's true?", we may provide proof; yet that same question can be asked of the proof, and any subsequent proof. The

Münchhausen Trilemma is that we have only three options when providing proof in this situation:

- The circular argument, in which theory and proof support each other (i.e. we repeat ourselves at some point)

- The regressive argument, in which each proof requires a further proof, ad infinitum (i.e. we just keep giving proofs, presumably forever)

- The axiomatic argument, which rests on accepted precepts (i.e. we reach some bedrock assumption or certainty)

The first two methods of reasoning are fundamentally weak, and because the Greek sceptics advocated deep questioning of all accepted values they refused to accept proofs of the third sort. The trilemma, then, is the decision among the three equally unsatisfying options.[1]

The students are asked to work in groups on the following tasks: *Formulate connections between the proof-jigsaws and the Münchhausen-Trilemma text. Name characteristics of "a perfect proof".*

The results of the groups are collected in the end. Using this method we hope to get a larger variety of ideas as well as to involve all the students in the discussion process by generating commitment for everybody. But especially we want to allow a first step towards the establishment of socio-mathematical norms "which means in this context, criteria shared by students and teachers to decide whether a proof is valid or not, what is a satisfactory explanation, etc." (Gueudet, 2008, p. 243).

The didactical potential of this learning activity lies in the following:

Since most of the students have not thought about justification-theoretical aspects of mathematical proofs so far, this activity could create the awareness of the problem area. Therefore, we provide the students with three proofs that differ widely in the given details, which can be seen as an analogue to the "infinite regress" in the Münchhausen-Trilemma. We want the students to reflect their implicit convictions resulting from their previous mathematical experiences, and lead them to the question *"What is a perfect proof?"* in a suggestive way.

The reflection about the three options given in the Münchhausen-Trilemma could lead to the conviction that the premises-inference-conclusion pattern is a reasonable and adequate means for mathematical argumentation. But it also allows the recognition of "mathematical work": Defined properties can be used without proof, whereas all other properties must be proved by only using the definitions. This is also mentioned by Duval (1991) who pointed out the importance of the awareness of the

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[1] The Münchhausen-Trilemma after H. Albert (1991). English formulation as in http://en.wikipedia.org/wiki/Münchhausen_Trilemma (14.9.2012)

logical status of propositions when trying to understand a mathematical proof – especially the difference between premise, theorem, conclusion etc.

SUMMARY AND OUTLOOK

The aims of the math-bridging course we present in this paper are twofold: the development of basic mathematical skills and the explicit training of strategies for the learning of mathematics as a science. Our central method for teaching these learning strategies is to construct a teaching scenario that encourages the students to do two things. First, to implicitly apply these strategies in the course of an exercise. Second, to become aware of and to discuss their methodological status as norms or ideals that one should follow in order to justify knowledge as scientifically objective. As a concrete example, we outline a teaching scenario, including some comments on our didactical reasoning and teaching methods, that aims at clarifying the central inner- and meta-mathematical significance of deductive reasoning. This way, the students should not only value the importance of the concept of deductive reasoning in scientific mathematics, but also discuss the legitimacy of this concept as a helpful tool in reaching objective scientific knowledge in science in general.

In October 2012 a first bridging course integrating the above learning scenario was realized at the University of Education Ludwigsburg. This was meant as a first test of our approach and led us to further considerations and the formulation of research questions. In Ludwigsburg about 300 students took part in the bridging course. 150 of the students are future primary school and special school teachers who did not choose mathematics as a major subject, but will have to attend several mathematics courses. These students visited the lecture of one of the authors and a small part of these students took part in the exercise sessions conducted by her. Their attitude towards mathematics was mainly characterized by a procedural view of mathematics and by the fear of not managing the subject. Our experiences let us hope that our approach is practical and useful even with such an audience. Although the first realization is far from being empirically solid, we present some of the students' comments to give an impression of the students' "view of mathematics" after the bridging course:

> "I got another view of mathematics in the sense that things are not just as they are, but that there is a reason for everything and one could ask for the reason. The children will also ask for reasons at school."

> "I found it good to get motivated and to get another view of mathematics (questioning calculation rules: Why is it like that?)"

Therefore, we are confident that our approach could give the students "a new idea of mathematics" by initializing mathematical enculturation at the secondary-tertiary transition.

In the future we plan to develop further learning scenarios and a large pool of differentiated exercises for subsequent courses to allow the students to group themselves according to their individual needs. Another bridging course integrating our approach will be held at Humboldt-Universität zu Berlin in October 2013. For this course we will also implement assessment methods. Our future work will be guided by the following research questions:

- Will the "metacognitive" considerations lead to more comprehension concerning mathematical argumentation and proof and their relation to everyday language?
- Will our approach help the students not only to understand a proof as a product but to produce their own proofs?
- In which way is the students' view of mathematics as a science changed, and how does this change influence their learning of mathematics? E.g. will the students be enabled to work more autonomously on mathematical problems (including tasks that aim at argumentation, but also tasks to train skills)?

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