DIMENSIONS OF MATHEMATICS TEACHING AND THEIR IMPLICATIONS FOR MATHEMATICS TEACHER EDUCATION

Andrea Hoffkamp and Elke Warmuth

Humboldt-Universität zu Berlin

Describing and developing the professional competence of math teachers we need to consider different dimensions, e.g. the content knowledge, the pedagogical content knowledge, and the pedagogical knowledge. Empirical studies in this area seem to provoke certain "trends" regarding the conclusions about the "most effective" characteristics of a good (math) teacher. Our own findings with future math teachers show that our students over-emphasize the pedagogical dimension in a certain way while (almost) neglecting the importance of content knowledge. We analyse how the different dimensions interdepend and present exemplary learning scenarios for the education of future math teachers focussing on the content knowledge dimension but - at the same time - combining it with pedagogical intentions derived from the special nature of our subject.

INTRODUCTION AND RATIONALE

The professionalization of mathematics teachers is still in the focus of politics and empirical studies like the international TEDS-M study (König & Blömeke 2012). In order to investigate and develop the professional competence of mathematics teachers different dimensions need to be considered. Describing the professional competence of teachers, Bromme (1997) distinguishes between *general pedagogical knowledge, content knowledge, curricular knowledge, the philosophy of the subject,* and *pedagogical content knowledge*. However the description of the complex structure of interweaving conditions between these dimensions is often missing or not addressed.

Especially since the study of Hattie (2009) there is a strong focus on the teacher personality:

> "[..] the differences between high-effect and low-effect teachers are primarily related to the attitudes and expectations that teachers have when they decide on the key issues of teaching – that is, what they teach and at what level of difficulty, and their understandings of progress and of the effects of their teaching. This brings me to the first set of attributes [..]: passionate and inspired teachers." (Hattie, 2009, p. 26)

This led to a certain trend: the attempt to promote the pedagogical dimension within the teacher personality by describing and recommending only certain factors. For example Anthony & Walshaw (2009) describe "characteristics of effective teaching of mathematics". In our opinion some of the descriptions are too little connected to the subject mathematics. Also the special (complex) relationship between the characteristics is not a subject of discussion, which implicitly implies that they can be treated independently.

Hattie also stated that the *content knowledge* dimension seems to have little effect on the quality of student outcomes. This can be similarily found in the study of Bromme (1997). By misinterpreting these results one might underestimate the dimension of content knowledge. Indeed Hattie concluded that "experts possess knowledge that is more integrated" and – of course – the content knowledge is an integral part of it (Hattie, 2009, p. 28). In fact Hattie (ibid.) emphasizes the importance of formative assessment and feedback. Of course this requires a strong mathematical background of the teachers since the effect size of feedback referring to the subject or content appears to be high.

An opposite standpoint to the over-emphasis of pedagogical dimensions is the contribution of Wu (2005) – a professor of mathematics in Berkeley. He criticizes a "mathematics avoidance syndrome" at school and analyses how content "opens up the world of pedagogy and offers many more effective pedagogical possibilities".

Helmke (2012) criticizes that there are hardly any empirical studies investigating the professional quality concerning the content knowledge dimension of teaching at schools. Two very important works in this context originate from Ball et al. (2008) and Wittmann et al. (2001). Both refer to primary school mathematics. Based on Shulman's (1987) categories of teacher knowledge Ball et al. (2008) analyse the *content-specific dimension* detached from the *general dimensions* (like the pedagogical one). They characterise the subject matter knowledge, which is specific for mathematics teaching and differentiate it from the pedagogical content knowledge. With a different focus we find a similar approach in the work of Wittmann et al. (2001). They describe the *content knowledge as the core of mathematics teaching*. Moreover they develop the pedagogical dimension and the teaching methodology on the basis of mathematical ideas or content. In contrast to Ball et al. they particularly emphasize the role of metacognition ("consciousness"): teachers need to encourage the children to perceive the specifics of mathematics as a subject. This helps children to establish self-regulation mechanisms with regard to the subject.

In a deep theoretical analysis the educational scientist Gruschka (2008) also underlines the importance of content knowledge for teaching processes: "Teaching at school suffers by the shrinkage of content" (Gruschka, p. 73). In his work Gruschka often refers to mathematics teaching and reflects the role of content in a systemic way. According to Gruschka the professional competence concerning the content knowledge needs to be regarded within the complex of curriculum, pedagogy and philosophy of the subject. In more sophisticated words Gruschka (2008, p. 49) states that the unity of content knowledge and philosophy of the subject manifests itself in convictions about the pedagogical content dimension; whereas the pedagogical content dimension is determined by the anthropology of the students (as the core of pedagogical knowledge) and the attitude of the teacher towards the curriculum.

In our contribution we adopt the positions of Wittmann et al. (2001) and Gruschka (2008) for secondary mathematics teachers. We consider the whole complex of dimensions based on the specifics of our subject mathematics in an integrated way. By presenting examples we illustrate that the general dimensions, like the pedagogical or educational ones, are strongly connected to the content knowledge and philosophy of our subject mathematics. Our article is meant as a theoretical contribution to this topic containing illustrating examples. We also draw conclusions for the design of learning scenarios for mathematics teacher education at university. Following the above theoretical considerations we start with an analysis of some statements of future math teachers in this context. These statements show that the beliefs about the dimensions of teacher competence have been shifted in a disadvantageous way.

DIMENSIONS OF TEACHER COMPETENCE FROM THE STUDENTS' POINT OF VIEW

To understand the context we shortly describe the situation of teacher education at Humboldt-Universität zu Berlin. At this university the future math teachers learn their subject mathematics by attending math lectures given by the mathematicians of the institute. Although these lectures are established for the future math teachers only they are usually not practically oriented with regard to their future profession. As we will see later this is a dilemma. The pedagogical knowledge is acquired separately at the department of educational sciences. The pedagogical content knowledge is taught in seminars and lectures of the math education group. The only courses in which the dimensions mentioned in the rationale are explicitly combined are some courses of the math education group called "Stochastics and its pedagogy" or "Algebra and number theory and its pedagogy". Within these courses the content knowledge and pedagogical content knowledge dimensions are combined. Apart from these courses the dimensions are *not* taught in an integrated way. Especially the pedagogical dimension is nearly completely separated from the content knowledge.

The following statements and reflections originate from future math teachers within a seminar in 2014 preparing the educational practical training phase (a 4-weeks period at school) in the master study program. Our students have to complete two practical training phases – one for each studied subject. Most of the students of this course will become secondary school teachers.

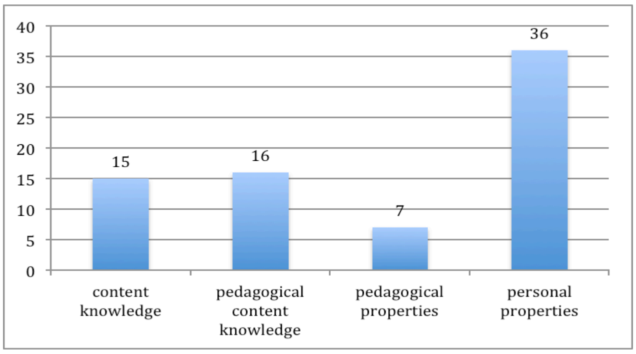

Fig. 1: Students' answers to „Name the three most important characteristics of a math teacher": Absolute frequencies sorted by four categories.

In the second session of the seminar we asked the students to write down the three most important characteristics of a good math teacher. In the written answers of 24 students four categories could be identified: *content knowledge, pedagogical content knowledge, pedagogical properties and abilities,* and *personal properties*. The absolute frequencies of the answers are presented in Figure 1. Unexpectedly half of the students' answers could be attributed to the personal dimension. These were answers like *personality, fairness, patience, empathy, being not unforgiving, authenticity,* or *spontaneity*. Since the personal dimension is closely related to the pedagogical dimension, we were wondering about the students' beliefs about the connections and interrelationships between the categories.

Students' reflections about mathematics and pedagogy

Considering the mathematical content as the core of teaching we tried to find out which interrelations the students describe between the pedagogical and the content knowledge dimension. We were especially interested if the students deduce any pedagogical or personal factors from the content dimension. Therefore the students were faced with the results of the first questioning in the following way:

> "Comparing and clustering the characteristics named by the students of the course we see that most of the characteristics refer to personal and pedagogical abilities resp. qualities (see list of characteristics attached). On the other hand certain educational goals could be deduced from the list of characteristics.
>
> Please draw up your opinion to the following statements and write it down:
>
> 1. Without a firm mastery of the mathematical content, good pedagogy is impossible.
> 2. A firm mastery of the mathematical content opens up the scope for pedagogical actions and reinforces the pedagogical effectiveness."

The statements are adapted from Wu (2005, p. 7). Different from Wu the term "pedagogy" addresses mainly the pedagogical and personal dimension ("Pädagogik" in German) and not (only) the pedagogical content dimension.

The results of this questioning lead to more differentiated categories and allow a deeper insight. The following categories are specified by typical formulations of the students.

- *Content knowledge as the core of teaching:*

A firm mastery of the mathematical content leads to charisma, respect, self-consciousness and self-confidence of the teacher. Without a strong mathematical background a teacher cannot manage a class or will provoke discipline problems. Credibility and authenticity need a high level of mathematical competence.

"Without a firm mastery in mathematics good pedagogy does not make sense, because the mathematical content and its mediation is the core of teaching and it has to be mediated by using pedagogical abilities."

In the statements within this category the students connect personal properties (self-confidence, charisma) and pedagogical factors (discipline, classroom management) directly with a high competence in the content knowledge dimension. In the students' view a strong mathematical background seems to strengthen personal and pedagogical skills that are important for teaching at school. However none of the students deduced personal or pedagogical properties or goals that are specific for the subject mathematics.

- *Mathematical content and pedagogy are separated areas:*

"No, a firm mastery of the mathematical content and the scope for pedagogical actions are two totally disconnected 'construction sites'."

- *Good Pedagogy is possible with low or no content knowledge:*

"I am convinced of the thesis that a teacher with low professional content knowledge can cope well or better than teachers with high mathematical competence. I compare this with the study where an actor without any mathematical content knowledge gives a convincing talk about game theory in front of experts in this field."

This illustrates the dilemma mentioned at the beginning of this section. Mathematics lectures at university are not considered to be relevant for the teaching at school.

- *High mathematical competence could hinder good teaching and pedagogy.*

"I observed cases in which a high competence in mathematics hindered the establishment of empathy for the children, because these teachers were not able to imagine why and wherewith the children have problems."

The statements in these categories reflect the students' strong need for the human and emotional component of their profession. The implicitly mentioned aim of the students is: They want to be well received and appear likeable when teaching at school. In this sense the pedagogical dimension is over-emphasized by separating it from the content knowledge. Usually the students refer to their own experienced math classes and want to do better. Therefore the students need to build their own pedagogical framework and system of values. We are confident that by considering the subject mathematics as the core for education in an integrated way, we could and have to support the development of the students in the above sense.

Analysis of a teaching professional's introduction of a new math concept

With the following example we want to show how a teacher loses pedagogical effectiveness because of insufficient mathematical competence. We chose this example because it illustrates in an impressive way the interrelationships between the different dimensions concerning the professionalism of math teachers. Besides the discussion of the interrelations we also show that examples like the following offer a valuable pedagogical potential for the education of future math teachers.

One of our master students completed her 4-weeks practical training phase in which she had to teach and observe mathematics classes at school and write a training report. These reports consist of different parts. Aside from two lesson protocols and their reflection, the students describe their own lesson planning and resource development. The planning section especially contains an analysis of the taught subject by the student – as part of the content knowledge. About 30% of our students do *not* succeed in this part in their first attempt.

In her training report she documented the observation of a "basic math course" in grade 13 (last year of secondary school). In the following protocol of the lesson the student focussed on the methods of the math teacher to introduce a new concept to the class – the concept of "expected value" in the case of the binomial distribution.

The teacher started with an extrinsic motivation: "What I will do next, will also be important for the next written exam." Having said this the teacher immediately moderates his statement by "But it is not that hard." The teacher uses an inductive approach to the concept and solely uses examples of the following type: "If I throw a fair dice 720 times. What do you think, how often will I get a 4?" The pupils answer "120 times" together with the reasoning "Well, there are six possible results. Thus 720 divided by 6 is 120." is accepted by the teacher with the words "Good, now the example with the tetrahedron. Whose example was that?". After four more examples of exactly this kind the teacher explains "Let's write this down in a mathematical way. $\mu = E(X)$ denotes what we expect. The number we receive is not a probability any more. The number usually doesn't lie between 0 and 1. We can also receive integers, for example if n is very large." After another example, which does not really fit to the binomial distribution, the teacher writes down the formula for the standard deviation by saying: "That's not difficult, you can just learn the formula." After that the class ends after 45 minutes.

This example reveals several dimensions of our subject matter. First the *content knowledge dimension*: The teacher does not have a conceptual understanding of the concept of "expected value" (*low content knowledge*). Therefore he *cannot act didactically*. He is not able to address the previous knowledge of his students and cannot use it for the development of the new concept. Therefore the teacher is *methodologically restricted* to direct instruction, since the content is not presented logically coherent and does not allow for pupil-centred methods. Also the teacher does not refer to the relevance of the concept for everyday life. Therefore he *cannot act educationally resp. pedagogically*. *Education to critical use of reason* would mean to discuss the significance and the misinterpretation of mean values as well as random fluctuations specific for stochastic phenomena. Expected values are a mathematical means for structuring and communicating. Since they reduce information they are supplemented by standard deviations. Their legitimacy as a teaching subject is only given if their relevance and limits are experienced (*curriculum dimension*). This is the prerequisite to educate mature people in an intellectually honest way.

Coming back to the education of future math teachers we take a look at the student's reflection of her protocol: The student criticized some methodological details and the abrupt termination of the class. From a mathematical perspective she did not have any objections. This example and the fact that about 30% of our students fail when working out a subject analysis is of course an alarming feedback for our education at university. It shows that we have to put more emphasis on the linkage between the content, the pedagogical content, and the pedagogical knowledge dimensions, and enable the future math teachers to reflect on these linkages.

LEARNING SCENARIOS INTEGRATING DIFFERENT DIMENSIONS

As we analysed in the last section the different dimensions of teaching and learning mathematics depend on each other in a complex way. Since we cannot assume that the students achieve this view on their own, we need to offer substantial learning scenarios at university that allow them to actively deal with the dimensions in an integrated way. We want to present two exemplary learning scenarios following our theoretical considerations.

Using authentic material to educate reflective practitioners

As Gruschka (2008, p. 59 & 49) writes "if you want to understand teaching you have to understand the content dimension of the subject", and the core of pedagogical knowledge lies in the anthropology of the pupils. Therefore we start from the subject mathematics and use authentic material for the design of the learning scenario. The basis of the following activity is the protocol (presented in the last section) which is used as authentic material. The students receive the whole protocol as working material. The tasks should be worked out in small groups and afterwards discussed and reflected with the whole group.

Give-your-opinion!-task

a) Work out the definition of the "expected value E(X)" of a discrete random variable X with finite range of values. Which information of the distribution of X contains E(X), which information gets lost? Illustrate three different examples by using a graphical representation of the distribution of X.

b) Let E(X)=3. Interpret this value by switching from the level of mathematical model to the real world level. To which previous knowledge do you have to connect to?

c) Assess the approach of the teacher to introduce the new concept of "expected value". Do you agree with the teacher's given view of mathematics? Give reasons for your answer.

d) Give a sketch of your ideas for the introduction of the concept of "expected value".

e) How is the concept of "expected value" connected to the education to the critical use of reason? Where are the limits of the concept and its necessity to complement it by further concepts?

The above task combines the content knowledge dimension (a) as a necessary condition and basis for the following subtasks containing the pedagogical content and learning psychological dimension (b and d). Afterwards (c) the reflection of the situation and of the own view of the subject mathematics is required (philosophical dimension). The last subtask (e) refers to the pedagogical dimension and its connection to the mathematical subject matter. Particularly the pedagogical dimension in this scenario has the potential to develop – by *criticising the authority* of the teacher – *autonomy* and *critical faculties*, which are worthwhile pedagogical aims when teaching mathematics.

Including the metacognitive dimension

With the next scenario the content knowledge is combined with the philosophical dimension. This activity aims at the reflection of the special nature of the subject, since – as we analysed in the rationale – metacognition does play an important role when teaching mathematics and acting pedagogically. In a sense the following can be seen as a continuation of the first scenario, as it makes the *role of definitions within mathematics* a subject of metacognitive and philosophical discussion. The next scenario builds on the work of Hoffkamp et al. (2013). We already integrated it in a university course at Humboldt University and will briefly refer to our experiences.

> **Task 1:** (An exercise in defining in number theory)
>
> Define the concept "even number". Also consider how you would define this concept at school and at university on different levels: primary school, secondary level and at the transition from school to university. Discuss the validity of the given definitions.

This seemingly simple task led the students to definitions like "the number 0,2,4,6 and so on", "all twosome numbers" (primary level), "all numbers that can be divided by two without remainder" (secondary level), or "the definition of divisibility leads to the description of the set $2\mathbf{Z}$" (university level). Then a lively discussion about the validity of the different definitions arose. Especially the definitions at primary level were not accepted by everybody as "being mathematical". With this task the students realized, that a definition is not necessarily unique, but depends on the mathematical context and purpose.

> **Input phase:** The students are confronted with the definitions and propositions of Euclid in the "Elements" (Book VII and IX): *An even number is that which is divisible into two equal parts. An odd number is that which is not divisible into two equal parts, or that which differs by a unit from an even number.*
>
> Using these definitions the following (simple) propositions of Euclid were deduced together with the students: *If as many odd numbers as we please are added together, and their multitude is even, then the sum is even. If an odd number is subtracted from an odd number, then the remainder is even.*

What the students experience in this part of the activity is that definitions change under historical conditions. They perceive the work of Euclid as the beginning of the axiomatic method and realize that Euclid's definitions are descriptive. They also realize that the proofs of the propositions "differ" from each other when using different definitions (like Euclid's or the modern university definition).

Task 2: (An example from geometry) Is it possible to decompose a square in two congruent parts? (from Fischer & Malle 1985)

By discussing this task the students realize that the answer to the above question depends on our predefined concepts of "square", "decomposition" and "congruency". One can show that it is actually impossible to decompose a square into two congruent (and disjoint) parts. In fact a square could be mathematically described as a set of points in the plane. Then we have to ask: If we "cut" the square at the "center line", to which part do the "dots of the line" or the midpoint belong? Certainly mathematical definitions abstract from reality (of course we can cut a quadratic sheet of paper with a pair of paper scissors into two equal parts) and create ideal (mathematical) objects. Because of the idealization we need to proof our statements within our theory. The first two tasks lead to a sort of cognitive conflict: Both mathematical objects (even number and square) are familiar terms and the above difficulties are unexpected. This opens the way to discuss the nature and role of definitions from a metatheoretical point of view.

Task 3: Give your opinion to the following statement: *Definitions are abstractions from reality following certain interests/purposes and change under historical conditions.*

Based on the previously made experiences the students discussed this statement philosophically in an explicit way. They started to emancipate from absolute truths and to reveal convictions about their subject. They especially realized that – if each definition follows a certain purpose – this purpose has to be made explicit at school. This is strongly connected with the pedagogical dimension: as teachers we should take the pupils seriously as partners in a dialogue about mathematics and enable them to decide reasonably in a self-determined way.

CONCLUSION

In our article we analysed the dependency of the different professional dimensions of teaching mathematics forming an integrative entity. Based on our findings with future math teachers we reasoned that the content knowledge dimension should be the core of mathematics teaching. We also derived pedagogical aims connected to our subject and its philosophy: a serious and genuine dialogue with the students and the education of the students at school (and university) to act and reason autonomously and rationally. In other words we developed the dimensions of the teacher competence based on the content and philosophy of mathematics – which defines the special nature of our subject. We claim that by offering learning scenarios (like the described ones) at university we help the students to create their professional system of values concerning educational aims. This could enable the students to build their own pedagogical framework based on the specifics of the subject mathematics. In this sense this is a very important point in the professionalization of future math teachers.

Our present and future work is and will be guided by this approach and more learning scenarios will be developed and evaluated.

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