The asymptotic behaviour of the Cesàro operator

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Let c be the Banach space of complex convergent sequences endowed with the supremum norm. We consider the classical *Cesàro operator* $T\colon c \to c$ defined by $Tx = (\phi_k(x))_{k\geq 0}$, where

$$\phi_k(x) = \frac{1}{k+1}\sum_{j=0}^{k} x_j, \qquad k \geq 0,$$

for $x = (x_k)_{k\geq 0} \in c$. One may ask, for which sequences $x \in c$ does the orbit $(T^n x)_{n\geq 0}$ converge to a limit with respect to the norm topology? In 2008, Galaz and Solís gave the following simple answer: the orbit converges if and only if $x_0 = \lim_{k\to\infty} x_k$. Their argument, however, is surprisingly fiddly. It relies on the powers of T being so-called *moment-difference operators* corresponding to certain measures (an observation originally due to G.H. Hardy) and on an estimate involving Stirling's formula. In this talk, I shall present an alternative and arguably more natural proof of the Galaz–Solís result based purely on ideas from operator theory. I shall further explain why, even though in general the convergence can be arbitrarily slow, sequences satisfying a slightly stronger condition give rise to orbits that converge at the rate $n^{-1/2}$ as $n \to \infty$. The talk is based on recent joint work with Andrew Pritchard (Newcastle).