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On a Maximum Principle for Inverse Monotone Matrices

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ABSTRACT

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a given vector with positive coordinates. A matrix A is said to satisfy the γ-maximum principle (γMP) if $Ax = y$, $y \geq 0$ imply $x \geq 0$ and

$$\max_{1 \leq i \leq n} \gamma_i x_i = \max_{i \in N^+(y)} \gamma_i x_i,$$

where $N^+(y)$ is the set of indices such that y is positive. For an invertible matrix A with positive inverse the γMP is characterized geometrically by means of the behavior under A^{-1} of convex boundary parts of the simplex generated in $\mathbb{R}^n_+$ by permissible multiples of the unit coordinate vectors. Some sufficient conditions and applications to M-matrices are given.

1. A MAXIMUM PRINCIPLE AND A WEIGHTED MAXIMUM PRINCIPLE

Many problems in different branches of mathematics lead or can be reduced to the solution of an equation

$$Au = f,$$

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where A is an invertible (n, n) matrix and f a given vector. Of interest then are qualitative properties of the solution u, such as its positivity, or the question at which of its components the solution u attains its maximal value. Of course, such properties of the solution depend on the matrix A as well as on the right hand side f of the equation. The matrix A in such a case is said to satisfy a maximum principle. Many papers deal with maximum principles for matrices; we refer only e.g. to [1, 3–5], where many applications are included.

In this paper, which is influenced by the theoretical parts of [5], a certain weighted maximum principle is introduced and studied. Under some natural assumptions a geometrical necessary and sufficient condition and some sufficient conditions for a matrix to satisfy the maximum principle are proved.

For matrices several maximum principles have been studied. The one under consideration in [5] is sometimes called the maximum principle for inverse column entries (see [6]).

We will use the following notation: Let n be a natural number such that $n \geq 1$. For an (n, n) matrix $A = (a_{ij})$ and a vector $x \in \mathbb{R}^n$ we write $A \geq 0$ if $a_{ij} \geq 0$ for all $i, j = 1, 2, \ldots, n$, and $x \geq 0$ or $x \in \mathbb{R}^n_+$ if $x_i \geq 0$ for all $i = 1, 2, \ldots, n$, respectively.

If $x_i > 0$ for all $i = 1, 2, \ldots, n$, i.e. $x \in \operatorname{Int} \mathbb{R}^n_+$ we write $x \gg 0$. Let N denote the set $\{1, 2, \ldots, n\}$. For $f \in \mathbb{R}^n_+$ we need the following subsets of N

$$\begin{aligned} N^+(f) &= \{j \in N : f_j > 0\}, \\ N^0(f) &= \{j \in N : f_j = 0\}. \end{aligned}$$

DEFINITION 1 (SEE [4]). A matrix A is said to satisfy the *maximum principle* (briefly, MP) if $Au = f$, $f \geq 0$ imply the conditions

(a) $u \geq 0$ and

(b) $\max_{k \in N} u_k = \max_{k \in N^+(f)} u_k$.

In [4] and [5] necessary and sufficient conditions are proved, mainly for invertible matrices with a positive inverse. In particular, simple conditions can be formulated for the class of M-matrices. An M-matrix is an invertible matrix A satisfying the conditions $A^{-1} \geq 0$ and $a_{ij} \leq 0$ for all $i, j = 1, 2, \ldots, n$, $i \neq j$ (see [2, 6]).

In order to define another maximum principle we fix some vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, $\gamma \gg 0$.

DEFINITION 2. A matrix A is said to satisfy the *weighted maximum* *principle* with respect to γ (briefly, γ-maximum principle or γMP) if $Au = f,\ f \geq 0$ imply the conditions

(a) $u \geq 0$ and

(b) $\max_{k \in N} \gamma_k u_k = \max_{k \in N^+(f)} \gamma_k u_k$.

Let A be an invertible matrix and $\gamma \gg 0$. Then A satisfies the γ-maximum principle if and only if the matrix $A\Gamma^{-1}$ satisfies the maximum principle, where Γ^{-1} denotes the diagonal matrix $\mathrm{diag}(1/\gamma_1,1/\gamma_2,\ldots,1/\gamma_n)$, i.e., the inverse to the matrix $\Gamma = \mathrm{diag}(\gamma_1,\gamma_2,\ldots,\gamma_n)$.

Indeed, if A satisfies the γMP, then $Au = f,\ f \geq 0$ imply (a) and (b) of Definition 2. Since the equation $Au = f$ can obviously be written as $(A\Gamma^{-1})(\Gamma u) = f$, then $\Gamma \geq 0$ and $u \geq 0$ yield $\Gamma u \geq 0$, and (b) means exactly $\max_{k \in N}[\Gamma u]_k = \max_{k \in N^+(f)}[\Gamma u]_k$, where $[\Gamma u]_k$ denotes the kth component of the vector Γu. Conversely, if the equation $(A\Gamma^{-1})v = f$ is considered with $f \geq 0$, then $(A\Gamma^{-1})(\Gamma u) = f$, where $u = \Gamma^{-1}v$. Since $f \geq 0$, by conditions (a) and (b) we get $\Gamma u \geq 0$ and $\max_{k \in N}[\Gamma u]_k = \max_{k \in N^+(f)}[\Gamma u]_k$, which means $v \geq 0$ and (b) from Definition 1, where u_k is replaced by v_k.

2. SOME GEOMETRIC PRELIMINARIES

Let $A = (a_{ij})_{i,j=1}^n$ be an invertible (n, n) matrix, $A^{-1} = (\alpha_{ij})_{i,j=1}^n$, and let $\beta = (\beta_1, \ldots, \beta_n)$ with $\beta_j \neq 0$ for $j = 1, \ldots, n$ be a fixed vector. We denote by α^j the jth row of the matrix A^{-1}, by e^i the ith unit vector of $\mathbb{R}^n$, by E the hyperplane through the endpoints of the vectors e^i, and by E_β the hyperplane through the endpoints of the vectors $\beta_1 e^1, \beta_2 e^2, \ldots, \beta_n e^n$. The hyperplane generated by the points $A^{-1}\beta_1 e^1 = \beta_1\alpha^1, \ldots, A^{-1}\beta_n e^n = \beta_n \alpha^n$ is denoted by $E_{A^{-1}\beta}$. Finally S, S_β, and T_β denote the intersection of $\mathbb{R}^n_+$ with E, E_β, and $E_{A^{-1}\beta}$, respectively. The representations

$$S := \left\{ x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1 \right\}$$

and

$$S_\beta := \left\{ x \in \mathbb{R}^n_+ : \sum_{i=1}^n \frac{1}{\beta_i} x_i = 1 \right\}$$

are obvious.

Due to the linear independence of the vectors $\alpha^j (j = 1,\ldots,n)$, the hyperplane $E_{A^{-1}\beta}$ never contains the origin. For simplicity, further on, we will consider only the case where the set T_β is bounded. That means we assume that the hyperplane $E_{A^{-1}\beta}$ cuts the ith coordinate axis at a nonzero distance c_i for any $i = 1, 2, \dots, n$. Therefore, only

$$T_\beta = \left\{ y \in \mathbb{R}^n_+ : \sum_{i=1}^{n} \frac{1}{c_i} y_i = 1 \right\},$$

where $c_i = c_i(\beta, A) \neq 0$, are considered. This, of course, restricts the set of all possible componentwise nonzero vectors β. For a given invertible matrix A we describe now this set exactly. Let f be the linear functional on $\mathbb{R}^n$ with the property

$$E_{A^{-1}\beta} = \{y \in \mathbb{R}^n : f(y) = 1\}.$$

Obviously one has $f(\beta_j \alpha^j) = 1$, i.e., $f(\alpha^j) = 1/\beta_j$ ($\beta_j \neq 0$) for $j = 1, \dots, n$.

LEMMA. For the hyperplane $E_{A^{-1}\beta}$ to intersect any coordinate axis at a unique point it is necessary and sufficient that

$$\beta_j = \frac{1}{k_1 \alpha_{1j} + \cdots + k_n \alpha_{nj}}, \qquad j = 1, \dots, n,$$

for some numbers $k_i \neq 0$, $i = 1, \dots, n$.

Proof. If $E_{A^{-1}\beta}$ intersects each coordinate axis, then for some $c_i \neq 0$, $i = 1, \dots, n$, there must hold $c_i e^i \in E_{A^{-1}\beta}$ for which $f(c_i e^i) = 1$ or $f(e^i) \neq 0$ can be written. From $A^{-1}A = I$ the representation $e^i = \sum_{j=1}^{n} a_{ji} \alpha^j$ follows for all $i = 1, \dots, n$ such that

$$f(e^i) = f\left(\sum_{j=1}^{n} a_{ji} \alpha^j \right) = \sum_{j=1}^{n} a_{ji} f(\alpha^j) = \sum_{j=1}^{n} a_{ji} \frac{1}{\beta_j} \neq 0.$$

Introducing the numbers $k_i = \sum_{j=1}^{n} a_{ji} 1/\beta_j$, $i = 1, \dots, n$, the last equality means

$$A^T \left(\frac{1}{\beta_1}, \dots, \frac{1}{\beta_n} \right)^T = (k_1, \dots, k_n)^T.$$

or equivalently

$$\left(\frac{1}{\beta_1}, \dots, \frac{1}{\beta_n} \right)^T = (A^{-1})^T (k_1, \dots, k_n)^T.$$

Therefore, $\beta_j(k_1\alpha_{1j} + \cdots + k_n\alpha_{nj}) = 1$ for all $j = 1, \ldots, n$. The equality $f(c_i e^i) = 1$ implies now $c_i = 1/f(e^i) = 1/k_i$ for the point at the ith coordinate axis, where the latter is intersected by the hyperplane $E_{A^{-1}\beta}$.

■

The hyperplane $E_{A^{-1}\beta}$ intersects each positive half axis if and only if all numbers k_i, $i = 1, \ldots, n$, are positive. Hence the following holds.

COROLLARY. The hyperplane $E_{A^{-1}\beta}$ has an intersection with each of the positive half axes if and only if

$$\beta_j = \frac{1}{k_1\alpha_{1j} + \cdots + k_n\alpha_{nj}}$$

for all $j = 1, \ldots, n$ with $k_1, \ldots, k_n > 0$. If, in addition to the above conditions, $A^{-1} \geq 0$, then in this case all β_j's are positive too.

A vector $\beta = (\beta_1, \ldots, \beta_n)$ with $\beta_j \neq 0$ and satisfying the condition of the lemma for all $j = 1, \ldots, n$ will be called *permissible.* Let $p(A)$ denote the collection of all permissible vectors for a given matrix A. Note that for an invertible matrix A the set $p(A)$ is not empty.

Now let A be an invertible matrix with $A^{-1} \geq 0$, and let $\beta \in p(A)$. Then $T_\beta \subset \mathbb{R}^n_+$ and $\beta \gg 0$. We consider two kinds of decomposition of T_β into subsets. The first one defines for all $i = 1, \ldots, n$ the sets

$$T_\beta^{(i)} = \{y \in T_\beta : y_i \geq y_k,\ k = 1, \ldots, n\}.$$

For the second one we fix an arbitrary vector $\gamma \gg 0$ and define

$$T_{\beta\gamma}^{(i)} = \{y \in T_\beta : \gamma_i y_i \geq \gamma_k y_k,\ k = 1, \ldots, n\}.$$

3. A NECESSARY AND SUFFICIENT CONDITION

Let A be a given invertible matrix such that $A^{-1} \geq 0$. Let $\beta \in p(A)$ and $\beta \gg 0$ hold. If $x_1, \ldots, x_n$ are n vectors, then $\mathrm{co}\{x_1, \ldots, x_n\}$ denotes their convex hull.

DEFINITION 3. The pair (A^{-1}, T_β) is said to satisfy the *condition G* if

$$A^{-1}\big(\mathrm{co}\{\beta_i e^i : i \in N'\}\big) \subset \bigcup_{i \in N'} T_\beta^{(i)}$$

for any nonvoid subset $N' \subset N = \{1, \dots, n\}$.

Let A and β be the same as before.

DEFINITION 4. Let $\gamma = (\gamma_1, \dots, \gamma_n) \gg 0$ be fixed. The pair (A^{-1}, T_β) is said to satisfy the *weighted condition* γG if

$$A^{-1}\bigl(\mathrm{co}\{\beta_i e^i : i \in N'\}\bigr) \subset \bigcup_{i \in N'} T^{(i)}_{\beta\gamma}$$

for any nonvoid subset $N' \subset N$.

Now we are able to use the geometric condition γG for the characterization of the analytic maximum principle with respect to γ.

THEOREM 1. *Let A be an invertible (n, n) matrix with $A^{-1} \geq 0$, and let $\gamma = (\gamma_1, \dots, \gamma_n) \gg 0$ be a fixed vector. Then the matrix A satisfies the weighted maximum principle with respect to γ if and only if there exists a $\widehat{\beta} \in p(A)$, $\widehat{\beta} \gg 0$ such that the pair $(A^{-1}, T_{\widehat{\beta}})$ satisfies the weighted condition γG.*

Proof. For the necessity we take an arbitrary $\beta \in p(A)$, $\beta \gg 0$, and show that for (A^{-1}, T_β) the condition γG holds. If $N' = \{i_1, \dots, i_l\}$, $l < n$ (the case $l = n$ is trivial), then any element $x \in \mathrm{co}\{\beta_{i_k} e^{i_k} : k = 1, \dots, l\}$ can be represented as $x = \mu_{i_1}\beta_{i_1}e^{i_1} + \cdots + \mu_{i_l}\beta_{i_l}e^{i_l}$ with $\mu_{i_k} \geq 0$, $\mu_{i_1} + \cdots + \mu_{i_l} = 1$. One has now

$$A^{-1}x = \mu_{i_1}\beta_{i_1}A^{-1}e^{i_1} + \cdots + \mu_{i_l}\beta_{i_l}A^{-1}e^{i_l},$$

or by using that $A^{-1}e^j$ is the jth row of the matrix A^{-1},

$$A^{-1}x = \mu_{i_1}\beta_{i_1}\alpha^{i_1} + \cdots + \mu_{i_l}\beta_{i_l}\alpha^{i_l}.$$

If the equation $Au = x$ is considered with such an x that $\mu_{i_k} > 0$, $k = 1, \dots, l$, then for $f = x$ one has $f \geq 0$ and $N^+(f) = N'$. By assumption A satisfies γMP, that is,

(a) $A^{-1}f \geq 0$ and

(b) $\max_{k \in N} \gamma_k(\mu_{i_1} \beta_{i_1} \alpha_{i_1, k} + \cdots + \mu_{i_l} \beta_{i_l} \alpha_{i_l, k}) = \max_{k \in N^+(f)} \gamma_k (\mu_{i_1} \beta_{i_1} \alpha_{i_1, k} + \cdots + \mu_{i_l} \beta_{i_l}\alpha_{i_l, k})$.

Clearly, $f \in S_\beta$ and $A^{-1}(S_\beta) \subset T_\beta$. In order to prove that γG holds, it suffices to investigate into which subsets of T_β the convex combinations of the vertices of S_β are mapped by A^{-1}. If $i_m \in N^+(f)$ such that $\gamma_{i_m} u_{i_m} \geq \gamma_k u_k$, $k = 1, \ldots, n$, then

$$u \in \tau_{\beta\gamma}^{(i_m)} \subset \bigcup_{i \in N'} \tau_{\beta\gamma}^{(i)},$$

where u_k is the kth component of the vector $u = A^{-1} f$. This shows that $A^{-1}(\mathrm{co}\{\beta_i e^i : i \in N'\}) \subset \bigcup_{i \in N'} \tau_{\beta\gamma}^{(i)}$ for any $N' \subset N$.

For the sufficiency, we argue as follows. Let for some $\widehat{\beta} \in p(A)$ the pair $(A^{-1}, T_{\widehat{\beta}})$ satisfy the condition γG. For some $f \geq 0$ consider the equation $Au = f$, where, without loss of generality,

$N \neq N^+(f) = \{i_1, i_2, \ldots, i_l\}$, i.e., $l < n$

may be assumed. $u \geq 0$ follows immediately because of $A^{-1} \geq 0$. The condition γG implies

$$A^{-1}\bigl(\mathrm{co}\{\widehat{\beta}_i e^i : i \in N'\}\bigr) \subset \bigcup_{i \in N^+(f)} \tau_{\widehat{\beta}_\gamma}^{(i)},$$

i.e., if for some element $x = \mu_{i_1} \widehat{\beta}_{i_1} e^{i_1} + \cdots + \mu_{i_l} \widehat{\beta}_{i_l} e^{i_l}$ with $\mu_{i_k} > 0$ and $\sum_{k=1}^{l} \mu_{i_k} = 1$ the vector $A^{-1}x$ is denoted by $v = (v_1, \ldots, v_n)$, then v belongs to at least one of the sets $\tau_{\widehat{\beta}_\gamma}^{(i)}$, $i \in \{i_1, \ldots, i_k\}$. That means at least one of the inequalities

$$\gamma_{i_j} v_{i_j} \geq \max_{1 \leq k \leq n} \gamma_k v_k, \qquad j = 1, \ldots, l, \tag{1}$$

holds, and that statement will not be influenced if all these inequalities (1) are multiplied by one and the same positive number α.

Since $f = f_{i_1} e^{i_1} + \cdots + f_{i_l} e^{i_l}$, there is a number $r > 0$ such that $rf \in S_{\widehat{\beta}}$. Therefore,

$$\sum_{i=1}^{n} \frac{1}{\widehat{\beta}_i} r f_i = 1 \qquad [f_i = 0 \quad \text{for } i \in N \setminus N^+(f)]$$

and

$$\frac{1}{r} = \sum_{k=1}^{l} \frac{1}{\widehat{\beta}_{i_k}} f_{i_k}.$$

On the other hand, rf is also a convex combination of the vertices $\widehat{\beta}_{i_k} e^{i_k}$ $(k = 1, 2, \ldots, l)$ of $S_{\widehat{\beta}}$, i.e., with some $\lambda_{i_k} > 0$, $\sum_{k=1}^{l} \lambda_{i_k} = 1$ one has

$$rf = \lambda_{i_1} \widehat{\beta}_{i_1} e^{i_1} + \cdots + \lambda_{i_l} \widehat{\beta}_{i_l} e^{i_l}. \tag{2}$$

For the solution u of the equation $Au = f$ with the given f we now get

$$u = A^{-1}f = \frac{1}{r}A^{-1}(rf).$$

Denote $rf = x$ and $A^{-1}(rf) = v$. Then (1) is considered with those v and x represented as (2) and with $\alpha = 1/r$. Therefore, for $u_k = (1/r)v_k$ at least one of the inequalities

$$\gamma_{i_j}u_{i_j} \geq \max_{1\leq k\leq n} \gamma_k u_k, \qquad j = 1, \ldots, l.$$

holds. From this the equality $\max_{k\in N^+(f)} \gamma_k u_k = \max_{k\in N} \gamma_k u_k$ immediately follows. ■

REMARKS.

1. As the proof of the necessity indicates, the following statement holds: If a matrix A satisfies γMP, then the pair (A^{-1}, T_β) satisfies the condition γG for any $\beta \in p(A)$, $\beta \gg 0$.

2. If $\gamma = e^{(n)} = (1, \ldots, 1)$, then under the same conditions for the matrix A to satisfy the maximum principle, it is necessary that the pair (A^{-1}, T_β) meet the condition G for any $\beta \in p(A)$, $\beta \gg 0$ and sufficient that (A^{-1}, T_β) meet G for at least one $\beta \in p(A)$, $\beta \gg 0$.

COROLLARIES

1. *Let $A^{-1} \geq 0$ and $\gamma = e^{(n)}$ hold. Assume that $(A^{-1})^T$ has the eigenvalue $\lambda = 1$ and a corresponding eigenvector $x = (x_1, \ldots, x_n) \gg 0$. Then the vector $\beta = (1/x_1, \ldots, 1/x_n)$, $\beta \gg 0$, belongs to $p(A)$ and the matrix A satisfies the maximum principle if and only if the pair (A^{-1}, S_β) satisfies G.*

Indeed, the additional condition about the eigenvalue and the eigenvector guarantees the inclusion $A^{-1}(S_\beta) \subset S_\beta$ and $\beta \in p(A)$. The statement follows now from the theorem. A further, more specific case is considered in:

2. *Let A^{-1} and γ be as in corollary 1. Assume now that $e^{(n)}$ is an eigenvector of $(A^{-1})^T$ for $\lambda = 1$. Then $A^{-1}(S) \subset S$, and A satisfies the maximum principle if and only if (A^{-1}, S) satisfies G.*

Let $D = \text{diag}(d_1, \ldots, d_n)$ be a diagonal matrix such that $d_j > 0$, $j = 1, \ldots, n$. Then D is invertible, $D^{-1} = \text{diag}(1/d_1, \ldots, 1/d_n)$, and $D^{-1} \geq 0$. If $f \geq 0$ is some vector, then $N^+(f) = N^+(Df) = N^+(D^{-1}f)$.

LEMMA. Let D be a diagonal matrix with positive diagonal elements, and let A be some matrix. Then both A and DA satisfy the maximum principle or neither of them does.

Proof. $(DA)u = f$ and $f \geq 0$ imply $Au = D^{-1}f$ and $D^{-1} \geq 0$. If now A satisfies the maximum principle, then $u \geq 0$, and

$$\max_{i \in N} u_i = \max_{i \in N^+(D^{-1}f)} u_i = \max_{N^+(f)} u_i$$

follows, i.e., DA satisfies the maximum principle. The opposite direction is proved by using the equality $N^+(Df) = N^+(f)$. ■

Now let A be an invertible matrix such that $A^{-1} = (\alpha_{ij}) \geq 0$. Let $D = \mathrm{diag}(d_1, \ldots, d_n)$, where $d_j = \sum_{i=1}^{n} \alpha_{ij}$, $j = 1, \ldots, n$. Then $d_j > 0$ for all j, and the matrix $B = DA$ is invertible with $B^{-1} \geq 0$. It is easy to see that $\lambda = 1$ and $x = e^{(n)}$ are an eigenvalue and a corresponding eigenvector for the matrix $(B^{-1})^T$, respectively. According to corollary 2 the matrix B satisfies the maximum principle if and only if the pair (B^{-1}, S) satisfies the condition G.

Since by the lemma the matrices A and B satisfy the maximum principle simultaneously, we have proved the following.

THEOREM 2. *With the notation above, the following conditions are equivalent:*

(i) *A satisfies the maximum principle;*

(ii) *B satisfies the maximum principle;*

(iii) *(B^{-1}, S) satisfies the condition G.*

The equivalence (i)–(iii) was proved by G. Stoyan in [5, p. 151].

4. SUFFICIENT CONDITIONS

For a matrix A to satisfy the maximum principle, sufficient conditions are given in [5]. It turns out that some of them can be generalized for the case of the γ-maximum principle.

For a given matrix $A = (a_{ij})$ the matrices $A^{(+)}$ and $A^{(-)}$ are defined as follows (see [4]):

$$A^{(+)} = \begin{cases} a_{ij} & \text{if } a_{ij} > 0 \text{ and } i \neq j, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad A^{(-)} = A - A^{(+)}.$$

For a diagonal matrix $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$ with $\gamma_j > 0$ for $j = 1, \ldots, n$, the matrices $(A\Gamma)^{(+)}$ and $(A\Gamma)^{(-)}$ are then $A^{(+)}\Gamma$ and $A^{(-)}\Gamma$, respectively.

THEOREM 3. *Let A be an invertible (n, n) matrix with $A^{-1} \geq 0$ and $A^{(-)}$ nonsingular. Further, let $\gamma = (\gamma_1, \ldots, \gamma_n) \gg 0$ and $A^{(-)}(1/\gamma_1, \ldots, 1/\gamma_n)^T \geq 0$. Then A satisfies the γ-maximum principle.*

Proof. The equation $Au = f$ is equivalent to $A\Gamma^{-1}\Gamma u = f$. The matrix $(A\Gamma^{-1})^{(-)}$ is nonsingular, since $A^{(-)}$ is. Moreover, $A\Gamma^{-1}$ exists and is inverse monotone. By assumption

$$A^{(-)}\left(\frac{1}{\gamma_1}, \ldots, \frac{1}{\gamma_n}\right)^T = A^{(-)}\Gamma^{-1}e^{(n)} \geq 0.$$

Therefore, using that $A^{(-)}\Gamma^{-1} = (A\Gamma^{-1})^{(-)}$, one has $(A\Gamma^{-1})^{(-)}e^{(n)} \geq 0$. By Theorem 1 from [5], the matrix $A\Gamma^{-1}$ satisfies the maximum principle. From this it immediately follows that A satisfies the γ-maximum principle. ■

The converse statement also holds in a certain sense (see also [5, p. 153]).

THEOREM 4. *Let A be an invertible matrix. If for some vector $\gamma \gg 0$ the matrix A satisfies the γ-maximum principle, then $A^{-1} \geq 0$ and*

$$A\left(\frac{1}{\gamma_1}, \ldots, \frac{1}{\gamma_n}\right)^T \geq 0.$$

Proof. Indeed, if $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$, then the matrix $A\Gamma^{-1}$ satisfies the maximum principle, $A\Gamma^{-1}e^{(n)} \geq 0$, and $\Gamma A^{-1} = (A\Gamma^{-1})^{-1} \geq 0$ [4, 5]. That is, $A(1/\gamma_1, \ldots, 1/\gamma_n)^T \geq 0$ and $A^{-1} \geq 0$. ■

5. THE WEIGHTED MAXIMUM PRINCIPLE FOR M-MATRICES

Let now A be an M-matrix, i.e., there exists $A^{-1} \geq 0$ and the nondiagonal elements of A are nonpositive, i.e., $a_{ij} \leq 0$, $i \neq j$.

THEOREM 5. *Let A be an M-matrix and $\gamma = (\gamma_1, \dots, \gamma_n) \gg 0$. Then A satisfies the γ-maximum principle if and only if* $A(1/\gamma_1, \dots, 1/\gamma_n)^T \geq 0$.

Proof. In terms of the matrices $\Gamma = \operatorname{diag}(\gamma_1, \dots, \gamma_n)$ and $B = A\Gamma^{-1}$ one has

$$A\left(\frac{1}{\gamma_1}, \dots, \frac{1}{\gamma_n}\right)^T = Be^{(n)}. \tag{3}$$

We note that B is also an M-matrix. Therefore, B satisfies the (usual) maximum principle if and only if $Be^{(n)} \geq 0$ [5]. The proof is complete on using the fact that B satisfies MP exactly when A satisfies γMP. ■

EXAMPLE Consider the M-matrices

$$A = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -2 \\ -1 & -1 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 2 & -2 \\ -1 & -1 & 4 \end{pmatrix}.$$

Then $Ae^{(n)} \geq 0$ and $Be^{(n)} \not\geq 0$, i.e., A satisfies the MP but B does not. On the other hand, for $\gamma = \left(\frac{2}{75}, \frac{1}{15}, \frac{1}{6}\right)^T$ and $\overline{\gamma} = \left(\frac{1}{7}, \frac{1}{14}, \frac{1}{9}\right)^T$ one has $A\left(\frac{75}{2}, 15, 6\right)^T \not\geq 0$ and $B(7, 14, 9) \geq 0$; therefore, A does not satisfy the γMP but B satisfies the $\overline{\gamma}$MP.

Now it is easy to show that an M-matrix A satisfies the γMP for any $\gamma \gg 0$ only if A is diagonal with positive diagonal elements.

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