ON THE PRINCIPALITY OF PIECEWISE TRIVIAL COMODULE ALGEBRAS

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Abstract. Principal G-spaces have a natural noncommutative geometry analogue in the concept of principal extensions of algebras (Galois-type extensions that are equivariantly projective). One way of constructing such extensions is to form fibre products of finitely many smash products. This leads to the notion of piecewise trivial comodule algebras. We prove that such a comodule algebra P is principal if the defining smash products can be assembled into parts of a flabby sheaf whose space of global sections is P.

1. Overview and results

Comodule algebras of Hopf algebras provide a natural noncommutative geometry generalisation of spaces equipped with group actions. Less evidently, principal extensions [5] appear to be a proper analogue of principal fibre bundles in this context (see the next section for precise definitions).

The classical concept of local triviality of G-spaces can be adapted to the setting of comodule algebras through the concept of piecewise triviality studied in this paper. The basic idea goes back to [8, 7, 10, 12, 17]. Our aim here is to clarify the interplay between piecewise triviality and principality. The first main result we prove is the following:

Theorem 1. *Let H be a Hopf algebra with bijective antipode and $\mathcal{P}$ be a flabby sheaf of H-comodule algebras over a topological space with a finite open covering $\{U_i\}$ for which $\mathcal{P}(U_i)$ is principal. Then $\mathcal{P}(U)$ is principal for any open set U.*

This means in particular that quantum principal fibre bundles in the sense of Pflaum [17] give rise to principal extensions as long as the involved sheaves are flabby. On the other hand, the underlying topological space plays for flabby sheaves only a secondary role, the main point is that the global sections of $\mathcal{P}$ form a comodule algebra which should be considered as being glued of the pieces $P_i := \mathcal{P}(U_i)$. If we consider for example a compact Hausdorff space X which is obtained by glueing together two closed subsets $X_1, X_2 \subset X$ along their intersection $X_1 \cap X_2$, then the corresponding algebras of continous complex-valued functions form a pull-back

$$\begin{array}{ccccc} & & C(X) & & \\ & \swarrow & & \searrow & \\ C(X_1) & & & & C(X_2) \\ & \searrow & & \swarrow & \\ & & C(X_{12}) & & \end{array}$$

and this can be encoded in a flabby sheaf over the auxiliary three-point space $\{0,1\}^2 \setminus \{(0,0)\}$ equipped with the topology generated by the two open sets $U_1 := \{(1,0),(1,1)\}$

and $U_2 := \{(0,1),(1,1)\}$. When we replace the algebras by arbitrary ones and pass from two to N pieces we obtain essentially the notion of a covering of an algebra from [8, 9]:

Definition 1. *Let Γ be the topological space $\{0,1\}^N \setminus \{(0,\dots,0\}$ whose topology is generated by the subsets $U_i := \{(z_1,\dots,z_N) \in \Gamma \mid z_i \neq 0\}$. A family of algebra epimorphisms $\pi_i : P \to P_i$, $i = 1,\dots,N$, is called a* covering *of P, if there exists a flabby sheaf $\mathcal{P}$ of algebras over Γ such that $P = \mathcal{P}(\Gamma)$, $P_i = \mathcal{P}(U_i)$ and the epimorphisms π_i are the corresponding restriction maps.*

In complete analogy we define coverings of comodule algebras etc. As we will explain in Section 3.3 (see Lemma 3 therein) the above means more concretely that a covering is defined by ideals $J_1,\dots,J_N \subset P$, $\bigcap_{i=1}^N J_i = \{0\}$ that generate a distributive lattice with respect to addition and intersection of ideals.

If P is now an H-comodule algebra with coaction $\Delta_P : P \to P \otimes H$ thought of as a substitute of a space with a group action, then the subalgebra $P^{\mathrm{co}H} := \{p \in P \mid \Delta_P(p) = p \otimes 1\}$ plays the role of the quotient space. Hence smash products $P = B \# H$ of an H-module algebra B by H are natural analogues of trivial principal fibre bundles. Combining this with the above idea of covering of algebras yields the following concept of a comodule algebra glued by trivial pieces:

Definition 2. *An H-comodule algebra P is called* piecewise trivial *if there exist comodule algebra epimorphisms $\pi_i : P \to P_i$, $i = 1,\dots,N$, such that:*

1. *The restrictions $\pi_i|_{P^{\mathrm{co}H}} : P^{\mathrm{co}H} \to P_i^{\mathrm{co}H}$ form a covering.*
2. *There are isomorphisms of H-comodule algebras $P_i \simeq P_i^{\mathrm{co}H} \# H$.*

Since a smash product is principal, a piecewise trivial comodule algebra is principal by Theorem 1, provided that the P_i form a covering of P. However, the distributivity condition in the notion of covering is sometimes not easy to verify for concrete examples. It is automatic for $N = 2$ and in the context of C^*-algebras where the intersection of (closed) ideals is equal to their product. In fact, we are mostly interested in P for which $P^{\mathrm{co}H}$ is a C^*-algebra which simplifies condition (1) in Definition 2.

The classical picture we have in mind is the following: Consider a principal fibre bundle X over a compact Hausdorff space M with compact structure group G. Then we can use as Hopf algebra only the regular (polynomial) functions on G in a purely algebraic way. But we can study principal extensions of $C(M)$ by this Hopf algebra, and these consist of functions on X which are continous along the base space but polynomial along the fibres, see [3] for a detailed discussion. These algebras are not C^*-algebras unless G is finite (with the discrete topology), and similarly our noncommutative extensions of C^*-algebras will typically not be C^*. Hence it is an interesting observation that conversely the principality of P implies in such cases the covering property for the P_i, that is, we have:

Theorem 2. *Let H be a Hopf algebra with bijective antipode and P be an H-comodule algebra which is piecewise trivial with respect to $\pi_i : P \to P_i$. Then the following are equivalent:*

1. *P is principal.*
2. *The $\{P_i\}_i$ form a covering of P.*

The functions continous along the base and polynomial along the fibre on a principal fibre bundle with compact structure group have an analogue in the noncommutative case: Let $\bar{H}$ be the C^*-algebra of a compact quantum group in the sense of Woronowicz [16, 24] and H the dense Hopf $*$-subalgebra spanned by the matrix coefficients of the irreducible unitary corepresentations. Let furthermore $\bar{P}$ be a unital $\bar{H}$-algebra (i.e. there is an injective C^*-algebraic coaction of $\bar{H}$ on $\bar{P}$, see e.g. [1]). Then the subalgebra $P \subset \bar{P}$ of elements which are mapped by the coaction to $\bar{P} \otimes H$ (algebraic tensor product) form an H-comodule algebra.

Definition 3. *We call P the H-comodule algebra associated to the $\bar{H}$-algebra $\bar{P}$.*

If $\bar{H} = C(U(1))$, that is, if $\bar{P}$ is a $U(1)$-C^*-algebra, and if the $U(1)$-action is principal in the sense of Ellwood [13], then P is principal [14]. We remark that the operation $\bar{P} \mapsto P$ commutes with taking fibre products.

The structure of the rest of this paper is as follows: In Section 2 we recall the needed background on principal extensions. Section 3 contains the proofs of the above theorems and some auxiliary observations. The final section is devoted to a class of examples that illustrate the theory.

2. Background

Throughout, we work over a field k and all considered algebras, coalgebras etc. are over k. An unadorned $\otimes$ denotes the tensor product of k-vector spaces.

2.1. **Principal extensions.** Let $(H, \Delta, \varepsilon, S)$ be a Hopf algebra with bijective antipode. We denote by $\mathbf{Alg}^H$ the category of (right) H-comodule algebras P, that is, (unital associative) algebras which are simultaneously right H-comodules whose coaction $\Delta_P : P \to P \otimes H$ is an algebra map. In the sequel, we will freely use Sweedler's notation for coproducts and coactions and write e.g. $p_{(0)} \otimes p_{(1)}$ for $\Delta_P(p)$, $p \in P$. For $P \in \mathbf{Alg}^H$, we call

$$P^{\mathrm{co}H} := \{p \in P \,|\, \Delta_P(p) = p \otimes 1\}$$

the subalgebra of H-invariant elements in P. Furthermore, we introduce as well a left coaction ${}_P\Delta : P \to H \otimes P$ given by $p \mapsto S^{-1}(p_{(1)}) \otimes p_{(0)}$.

Definition 4. *Let H be a Hopf algebra with bijective antipode. Then the algebra extension $B := P^{\mathrm{co}H} \subset P$ is said to be Galois if*

$$\mathrm{can} : P \otimes_B P \to P \otimes H, \quad p \otimes q \mapsto pq_{(0)} \otimes q_{(1)}$$

is bijective and principal if in addition P is equivariantly projective as left B-module.

By equivariant projectivity we here mean the existence of an H-colinear B-linear splitting s of the multiplication map $\mu : B \otimes P \to P$. This splitting can always be chosen to be unital, $\mathrm{s}(1) = 1 \otimes 1$, see [5, 6]. The map can is called the canonical map attached to the extension. In particular, a smash product $B \# H$ of an H-module algebra B by H is always principal.

2.2. **Strong connections.** If $P \in \mathbf{Alg}^H$ is a Hopf-Galois extension, then the inverse of the canonical map defines a monomorphism $H \to P \otimes_B P$, $h \mapsto \mathrm{can}^{-1}(1 \otimes h)$. It turns out that lifts of this map to $P \otimes P$ which are both right and left H-colinear yield an equivalent approach to principality [5]:

Definition 5. *Let H be a Hopf algebra with bijective antipode. Then a strong connection on $P \in \mathbf{Alg}^H$ is a linear map $\ell : H \to P \otimes P$ satisfying*

$$(\mathrm{id}_P \otimes \Delta_P) \circ \ell = (\ell \otimes \mathrm{id}_H) \circ \Delta, \quad ({}_P\Delta \otimes \mathrm{id}_P) \circ \ell = (\mathrm{id}_H \otimes \ell) \circ \Delta$$
$$\widetilde{\mathrm{can}} \circ \ell(h) = 1 \otimes h, \quad \ell(1) = 1 \otimes 1,$$

where $\widetilde{\mathrm{can}} : P \otimes P \to P \otimes H$ *is the canonical lift of* can *to* $P \otimes P$.

Thus a strong connection gives rise to a commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\ \ell\ } & P \otimes P \\ \downarrow & \swarrow{\scriptstyle \widetilde{\mathrm{can}}} & \downarrow \\ P \otimes H & \xleftarrow[\mathrm{can}]{} & P \otimes_B P \end{array} \qquad \begin{array}{ccc} h & \xrightarrow{\ \ell\ } & h^{\langle 1 \rangle} \otimes h^{\langle 2 \rangle} \\ \downarrow & \swarrow{\scriptstyle \widetilde{\mathrm{can}}} & \downarrow \\ 1 \otimes h & \xleftarrow[\mathrm{can}]{} & h^{\langle 1 \rangle} \otimes h^{\langle 2 \rangle} \end{array}$$

where we use the Sweedler-type notation $h \mapsto h^{\langle 1 \rangle} \otimes h^{\langle 2 \rangle}$ for ℓ.

It follows from the defining properties of strong connections that P is equivariantly projective with splitting

$$\mathrm{s} : p \mapsto p_{(0)}(p_{(1)})^{\langle 1 \rangle} \otimes (p_{(1)})^{\langle 2 \rangle} \in B \otimes P. \tag{1}$$

Furthermore, the map

$$P \otimes H \to P \otimes_B P, \quad p \otimes h \mapsto ph^{\langle 1 \rangle} \otimes h^{\langle 2 \rangle}$$

is an inverse of the canonical map can. Conversely, it was shown in [5, 6] that any principal extension admits a strong connection. That is, one has:

Theorem 3. *Let H be a Hopf algebra with bijective antipode. Then for $P \in \mathbf{Alg}^H$, the following are equivalent:*

1. *P is principal.*
2. *P admits a strong connection.*

3. Proofs

3.1. **On epimorphisms of principal extensions.** We first prove some remarks about quotients of principal extensions:

Lemma 1. *Let $\pi : P \to Q$ be an epimorphism in $\mathbf{Alg}^H$ and assume that P is principal. Then:*

1. *The induced map $\pi^{\mathrm{co}H} : P^{\mathrm{co}H} \to Q^{\mathrm{co}H}$ is an algebra epimorphism.*
2. *One has* $\ker \pi = \ker \pi^{\mathrm{co}H} P = P \ker \pi^{\mathrm{co}H}$.
3. *Q is principal.*
4. *π is split as a unital morphism of H-comodules.*

Proof. (1): It is obvious that $\pi(P^{\mathrm{co}H}) \subset Q^{\mathrm{co}H}$. Suppose conversely that $q \in Q^{\mathrm{co}H}$ and let $p \in \pi^{-1}(q)$ be any preimage. Since π is colinear, we have $q_{(0)} \otimes q_{(1)} = \pi(p_{(0)}) \otimes p_{(1)}$. Using furthermore that π is an algebra map and that $p_{(0)}(p_{(1)})^{\langle 1 \rangle} \otimes (p_{(1)})^{\langle 2 \rangle} \in P^{\mathrm{co}H} \otimes P$ (see equation (1)), we get

$$\begin{aligned} Q^{\mathrm{co}H} \otimes P \ni q \otimes 1 &= q\pi(1^{\langle 1 \rangle}) \otimes 1^{\langle 2 \rangle} = q_{(0)}\pi((q_{(1)})^{\langle 1 \rangle}) \otimes (q_{(1)})^{\langle 2 \rangle} \\ &= \pi(p_{(0)}(p_{(1)})^{\langle 1 \rangle}) \otimes (p_{(1)})^{\langle 2 \rangle} \in \pi(P^{\mathrm{co}H}) \otimes P. \end{aligned}$$

(2): We prove $\ker\pi = \ker\pi^{\mathrm{co}H}P$, the other equality is proven similarly. One inclusion is obvious. For the other, assume $\pi(p) = 0$. Then by equation (1)

$$0 = \pi(p)_{(0)}\pi((\pi(p)_{(1)})^{\langle 1\rangle}) \otimes (\pi(p)_{(1)})^{\langle 2\rangle} = \pi^{\mathrm{co}H}(p_{(0)}(p_{(1)})^{\langle 1\rangle}) \otimes (p_{(1)})^{\langle 2\rangle}.$$

Thus $p_{(0)}(p_{(1)})^{\langle 1\rangle} \otimes (p_{(1)})^{\langle 2\rangle} \in \ker(\pi^{\mathrm{co}H} \otimes \mathrm{id}) = (\ker\pi^{\mathrm{co}H}) \otimes P$. The claim follows since $p = p_{(0)}(p_{(1)})^{\langle 1\rangle}(p_{(1)})^{\langle 2\rangle}$ by the properties of ℓ.

(3): The map $(\pi \otimes \pi) \circ \ell$ induced by a strong connection ℓ of P is readily seen to be a strong connection on Q.

(4): Choose any k-linear unital split σ of the induced map $P^{\mathrm{co}H} \to Q^{\mathrm{co}H}$. Then $q \mapsto \sigma(q_{(0)}\pi((q_{(1)})^{\langle 1\rangle}))(q_{(1)})^{\langle 2\rangle}$ splits π. □

The following is a statement about the space of all quotients of a given principal extension that we will need below.

Lemma 2. *Let $P \in \mathbf{Alg}^H$ be principal and $B := P^{\mathrm{co}H}$. Denote by Ω_B the lattice of all ideals in B (with $+, \cap$ as operations) and by Ω_P the lattice of all ideals in P which are simultaneously subcomodules. Then the map*

$$\Omega_P \to \Omega_B, \quad J \mapsto J \cap B$$

is a monomorphism of lattices.

Proof. One obviously has $B\cap(J\cap J') = (B\cap J)\cap(B\cap J')$ and $(B\cap J)+(B\cap J') \subset B\cap(J+J')$, where $J, J' \in \Omega_P$. But if conversely $p \in J, q \in J', p+q \in B$, then applying s from (1) to $p+q$ yields

$$(p+q)\otimes 1 = p_{(0)}(p_{(1)})^{\langle 1\rangle} \otimes (p_{(1)})^{\langle 2\rangle} + q_{(0)}(q_{(1)})^{\langle 1\rangle} \otimes (q_{(1)})^{\langle 2\rangle} \in (B\cap J)\otimes P + (B\cap J')\otimes P,$$

because s is unital B-linear and J, J' are ideals and subcomodules. Applying any unital linear functional $P \to k$ to the second tensor component implies $p + q \in (B \cap J) + (B \cap J')$. The injectivity of the map is part (2) in Lemma 1. □

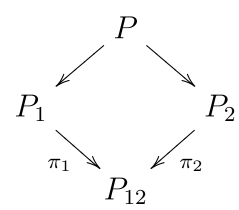
Note that the morphism is not surjective in general. A counterexample is given by the algebra P with generators u, u^{-1}, v, v^{-1} having relations

$$uu^{-1} = u^{-1}u = vv^{-1} = v^{-1}v = 1, \quad uv = qvu$$

for some $q \in k\backslash\{1\}$. This is a smash product of the Laurent polynomials $B = k[u, u^{-1}]$ by the Hopf algebra $H = k[v, v^{-1}]$ of Laurent polynomials (with $\Delta(v) = v \otimes v$), where the action of H on B is given by $v \triangleright u = qu$. Hence it is a principal H-extension of B. However, if $I \subset B$ is the two-sided ideal generated by $u - 1$, then the right ideal $IP = (u - 1)P \subset P$ is not two-sided. Hence the map can not be surjective by Lemma 1, (2).

3.2. **On sheaves of principal extensions.** The following is the main technical result in this article:

Proposition 1. *Consider a pull-back diagram in $\mathbf{Alg}^H$*



with P_1, P_2 principal and π_1, π_2 surjective. Then P is principal.

Proof. Recall that P can be constructed explicitly as

$$\{(p,q) \in P_1 \oplus P_2 \,|\, \pi_1(p) = \pi_2(q)\}.$$

Let ℓ_i, $i = 1, 2$, be strong connections on P_i which we distinguish in Sweedler notation by writing $\ell_i(h) = h^{\langle 1\rangle_i} \otimes h^{\langle 2\rangle_i}$. Let furthermore σ_i be unital H-colinear splittings of π_i as in Lemma 1 (4). For shortness, we denote by $p \mapsto \bar{p}$ both the compositions $\sigma_2 \circ \pi_1$ and $\sigma_1 \circ \pi_2$. Now it is straightforward to verify that the following defines a strong connection on P:

$$\begin{aligned}
\ell(h) \;=\;& (h^{\langle 1\rangle_1}, 0) \otimes (h^{\langle 2\rangle_1}, 0) + (h^{\langle 1\rangle_1}, 0) \otimes (0, \overline{h^{\langle 2\rangle_1}}) \\
&+ (0, \overline{h^{\langle 1\rangle_1}}) \otimes (h^{\langle 2\rangle_1}, 0) + (0, \overline{h^{\langle 1\rangle_1}}) \otimes (0, \overline{h^{\langle 2\rangle_1}}) \\
&+ (0, h^{\langle 1\rangle_2}) \otimes (0, h^{\langle 2\rangle_2}) + (0, h^{\langle 1\rangle_2}) \otimes (\overline{h^{\langle 2\rangle_2}}, 0) \\
&- (0, \overline{(h_{(1)})^{\langle 1\rangle_1}}\, \overline{(h_{(1)})^{\langle 2\rangle_1}} (h_{(2)})^{\langle 1\rangle_2}) \otimes (0, (h_{(2)})^{\langle 2\rangle_2}) \\
&- (0, \overline{(h_{(1)})^{\langle 1\rangle_1}}\, \overline{(h_{(1)})^{\langle 2\rangle_1}} (h_{(2)})^{\langle 1\rangle_2}) \otimes (\overline{(h_{(2)})^{\langle 2\rangle_2}}, 0).
\end{aligned}$$

□

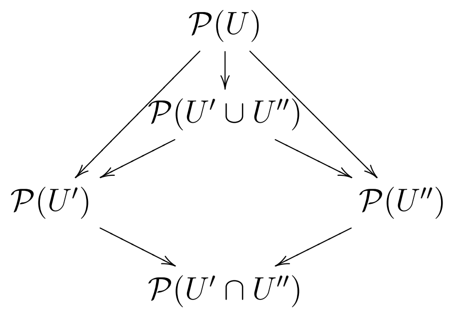
As an immediate consequence, we obtain Theorem 1: Indeed, the above proposition shows inductively that $\mathcal{P}(U_1 \cup \ldots \cup U_i)$ is principal for all i, so the global sections P of $\mathcal{P}$ are principal ($i = N$) and hence $\mathcal{P}(U)$ is principal for any open set U by Lemma 1.

3.3. **Piecewise triviality.** We now restrict to sheaves over the space Γ from Definition 1. The idea is that this is a universal space for hosting gluing data:

Lemma 3. *There is a one-to-one correspondence between:*

(1) *Flabby sheaves $\mathcal{P}$ over Γ with values in $\mathbf{Alg}^H$.*
(2) *Families $\pi_i : P \to P_i$, $i = 1, \ldots, N$ of epimorphisms in $\mathbf{Alg}^H$ whose kernels $J_i := \ker \pi_i$ generate a distributive lattice and have trivial intersection $\bigcap_{i=1}^N J_i = \{0\}$.*

Proof. For the implication (1) ⇒ (2), let $\mathcal{P}$ be a flabby sheaf over Γ, let $U', U'' \subset U \subset \Gamma$ be open subsets, and $\pi_{U,U'} : \mathcal{P}(U) \to \mathcal{P}(U')$ be the restriction maps of the sheaf. Our aim is to show that the map $U \mapsto \ker \pi_{\Gamma,U}$ which assigns ideals in P to open subsets of Γ is a morphism of lattices, that is, it transforms union and intersection of open subsets to intersection and sum of ideals. We first prove that $\ker \pi_{U,U'\cup U''} = \ker \pi_{U,U'} \cap \ker \pi_{U,U''}$. Indeed, $\mathcal{P}(U' \cup U'')$ is the pull-back of $\mathcal{P}(U')$ and $\mathcal{P}(U'')$, $\mathcal{P}(U' \cup U'') \simeq \{(p,q) \in \mathcal{P}(U') \oplus \mathcal{P}(U'') \,|\, \pi_{U',U'\cap U''}(p) = \pi_{U'',U'\cap U''}(q)\}$, so the claim follows from the commutativity of



Similarly, $\ker \pi_{U,U' \cap U''} = \ker \pi_{U,U'} + \ker \pi_{U,U''}$: There is an obvious inclusion $\supset$. For the inverse, pick $p \in \ker \pi_{U,U' \cap U''}$. Then both $p_1 := (\pi_{U,U'}(p), 0)$ and $p_2 := (0, \pi_{U,U''}(p))$ belong to $\mathcal{P}(U' \cup U'') \subset \mathcal{P}(U') \oplus \mathcal{P}(U'')$, and $p_1 + p_2 = \pi_{U,U' \cup U''}(p)$. Take any preimage $p' \in \mathcal{P}(U)$ of p_1 which exists by flabbiness. Then $p' \in \ker \pi_{U,U''}$ and $p - p' \in \ker \pi_{U,U'}$ which implies the claim.

Now (2) follows since the lattice of open subsets of any topological space is distributive. The property $\bigcap_{i=1}^N J_i = \{0\}$ follows from the sheaf condition.

(2) $\Rightarrow$ (1): Given conversely P, P_i, π_i, define $J_i := \ker \pi_i$. By definition of the topology of Γ, any open subset $U \subset \Gamma$ is a finite union of finite intersections $U_{i_1} \cap \ldots U_{i_p}$ of some of the U_i. Let J_U be the intersection of the sums of the corresponding J_i's and $\mathcal{P}(U) := P/J_U$. This defines a flabby sheaf, where the sheaf property follows from the distributivity of the ideals (cf. e.g. [19], Theorem 18 on p. 280). □

Hence Definition 1 from the introduction is indeed a reasonable notion of covering of algebras along the lines of [8, 9] and reduces in the commutative case to finite closed coverings of topological spaces.

As explained in the introduction, this notion of covering directly leads to the concept of local triviality of Hopf algebra extensions formalised in Definition 2, and as remarked after this definition, Theorem 1 implies (2) $\Rightarrow$ (1) in Theorem 2.

It remains to prove the converse direction (1) $\Rightarrow$ (2):

Proposition 2. *Let $\pi_i : P \to P_i$ be epimorphisms in $\mathbf{Alg}^H$ and assume that P is principal. Then the π_i define a covering of P if and only if the restrictions $\pi_i|_B : B \to B_i$ to $B := P^{\mathrm{co}H}, B_i := P_i^{\mathrm{co}H}$ form a covering.*

Proof. Lemma 2 implies that the sublattice of Ω_P generated by the $J_i := \ker \pi_i$ is isomorphic to the sublattice of Ω_B generated by the $J_i \cap B = \ker \pi_i|_B$. □

4. Examples

In this last section we recall from [11, 14, 2, 15] the construction of examples for the above concepts that illustrate possible areas of applications.

4.1. **The noncommutative join construction.** If G is a compact group, then the join $G*G$ becomes a G-principal fibre bundle over the unreduced suspension ΣG of G, see e.g. [4], Proposition VII.8.8 or [3]. For example, one can obtain the Hopf fibrations $S^7 \to S^4$ and $S^3 \to S^2$ in this way using $G = SU(2)$ and $G = U(1)$, respectively. Recall that $G * G$ is obtained from $[0,1] \times G \times G$ by shrinking to a point one factor G at $0 \in [0,1]$ and the other factor G at 1. Alternatively, one can shrink $G \times G$ at 0 to the diagonal. This is the picture we will generalise below. Our aim in this first part of Section 4 is to describe a noncommutative analogue of this construction that nicely fits into our general concepts and will be studied in greater detail in [11].

To this end, let H be the Hopf algebra underlying a compact quantum group $\bar{H}$ (see [24] or Chapter 11 of [16] for details). We define

$$P_1 := \{f \in C([0,1], \bar{H}) \otimes H \mid f(0) \in \Delta(H)\},$$
$$P_2 := \{f \in C([0,1], \bar{H}) \otimes H \mid f(1) \in \mathbb{C} \otimes H\}$$

which will play the roles of the two trivial pieces of the principal extension. Here we identify elements of $C([0,1], \bar{H}) \otimes H$ with functions $[0,1] \to \bar{H} \otimes H$. The P_i become

H-comodule algebras by applying the coproduct of H to H, $\Delta_{P_i} = \mathrm{id}_{C([0,1],\bar{H})} \otimes \Delta$, and the subalgebras of H-invariants can be identified with

$$B_1 := \{f \in C([0,1], \bar{H}) \mid f(0) \in \mathbb{C}\},$$
$$B_2 := \{f \in C([0,1], \bar{H}) \mid f(0) \in \mathbb{C}\}.$$

Furthermore, $P_1 \simeq B_1 \# H$, $P_2 \simeq B_2 \otimes H$, where H acts on B_1 via the adjoint action, $(a \triangleright f)(t) = a_{(1)} f(t) S(a_{(2)})$, $a \in H, f \in B_1, t \in [0,1]$, see [11]. Now one can define P as a glueing of the two pieces along $P_{12} := \bar{H} \otimes H$, that is, as the pull-back

$$P := \{(p,q) \in P_1 \oplus P_2 \mid \pi_1(p) = \pi_2(q)\}$$

of the P_i along the evaluation maps

$$\pi_1 : P_1 \to P_{12}, \quad f \mapsto f(1), \quad \pi_2 : P_2 \to P_{12}, \quad f \mapsto f(0).$$

As we remarked, a pull-back of two algebras always defines a coveirng, so Theorem 1 implies that P is principal.

4.2. **The Heegaard-type quantum 3-sphere.** Based on the idea of a Heegaard splitting of S^3 into two solid tori, a nonmommutative deformation of S^3 was proposed in [9, 14, 2]. On the level of C^*-algebras, it can be presented as a fibre product $C(S^3_{pq\theta})$ of two C^*-algebraic crossed products $\mathcal{T} \rtimes_\theta \mathbb{Z}$, $\mathcal{T} \rtimes_{-\theta} \mathbb{Z}$ of the Toeplitz algebra $\mathcal{T}$ by $\mathbb{Z}$. We denote the isometries generating $\mathcal{T}$ in the two algebras by z_1, z_2. The $\mathbb{Z}$-actions are implemented by unitaries u_1, u_2, respectively, in the following way:

$$u_1 \triangleright_\theta z_1 = u_1 z_1 u_1^{-1} := e^{2\pi i\theta} z_1, \quad u_2 \triangleright_{-\theta} z_2 = u_2 z_2 u_2^{-1} := e^{-2\pi i\theta} z_2.$$

The fibre product is taken over $C(S^1) \rtimes_\theta \mathbb{Z}$ with action $u \triangleright_\theta z := e^{2\pi i\theta} z$, where z is the generator of $C(S^1)$ and u is the unitary giving the $\mathbb{Z}$-action in this algebra. The corresponding surjections defining the fibre product are

$$\pi_1 : \mathcal{T} \rtimes_\theta \mathbb{Z} \to C(S^1) \rtimes_\theta \mathbb{Z}, \quad z_1 \mapsto z, \quad u_1 \mapsto u,$$
$$\pi_2 : \mathcal{T} \rtimes_{-\theta} \mathbb{Z} \to C(S^1) \rtimes_\theta \mathbb{Z}, \quad z_2 \mapsto u, \quad u_2 \mapsto z.$$

There is a natural $U(1)$-action on $C(S^3_{pq\theta})$ corresponding classically to the action in the Hopf fibration, see [14]. Its restriction to the two crossed products is not the canonical action of $U(1)$ viewed as the Pontrjagin dual of $\mathbb{Z}$. However, to obtain the canonical actions one can identify $C(S^3_{pq\theta})$ with a fibre product of the same crossed products, but formed with respect to the surjections

$$\hat{\pi}_1 : \mathcal{T} \rtimes_\theta \mathbb{Z} \to C(S^1) \rtimes_\theta \mathbb{Z}, \quad u_1 \mapsto u, \quad z_1 \mapsto z,$$
$$\hat{\pi}_2 : \mathcal{T} \rtimes_{-\theta} \mathbb{Z} \to C(S^1) \rtimes_\theta \mathbb{Z}, \quad u_2 \mapsto zu, \quad z_2 \mapsto z^{-1}.$$

The identification is given by

$$\varphi_{(i)} : u_{(i)} \mapsto u_{(i)}, \quad z_{(i)} \mapsto z_{(i)} u_{(i)}, \quad i = 1, 2$$

(by this we mean to define three maps $\varphi, \varphi_1, \varphi_2$), that is, one has $\hat{\pi}_i = \varphi \circ \pi_i \circ \varphi_i^{-1}$.

As mentioned in the introduction, we can pass from $C(S^3_{pq\theta})$ to the associated principal extension, and this procedure commutes with taking fibre products. In this way, we obtain a subalgebra $P \subset C(S^3_{pq\theta})$ which is a piecewise trivial $\mathbb{C}\mathbb{Z}$-comodule algebra as studied in this paper. The invariant subalgebra $P^{\mathrm{co}H}$ is the C^*-algebra of the mirror quantum 2-sphere from [15].

On the other hand, consider two copies of $\mathcal{T} \rtimes_{-\theta} \mathbb{Z}$ whose generators we will denote by u_3, z_3 and u_4, z_4, respectively, and $C(S^1) \rtimes_{-\theta} \mathbb{Z}$ with generators z', u', that is,

$$u'z'{u'}^{-1} := e^{-2\pi i\theta} z', \quad u_3 z_3 {u_3}^{-1} := e^{-2\pi i\theta} z_3, \quad u_4 z_4 {u_4}^{-1} := e^{-2\pi i\theta} z_4.$$

These can be glued using the maps

$$\check{\pi}_3 : \mathcal{T} \rtimes_{-\theta} \mathbb{Z} \to C(S^1) \rtimes_{-\theta} \mathbb{Z}, \quad u_3 \mapsto u', \quad z_3 \mapsto z',$$
$$\check{\pi}_4 : \mathcal{T} \rtimes_{-\theta} \mathbb{Z} \to C(S^1) \rtimes_{-\theta} \mathbb{Z}, \quad u_4 \mapsto z'u', \quad z_4 \mapsto z'.$$

Again, this fibre product is isomorphic to $C(S^3_{pq\theta})$, the identifying maps are now given by

$$u_1 \mapsto u_3^{-1}, \quad z_1 \mapsto z_3 u_3, \quad u_2 \mapsto u_4, \quad z_2 \mapsto u_4^{-1} z_4, \quad u \mapsto {u'}^{-1}, \quad z \mapsto z'u'.$$

However, the $U(1)$-symmetry resulting from the canonical ones on the pieces is now different, and the invariant subalgebra $P^{\mathrm{co}H}$ is the C^*-algebra of the generic Podleś quantum 2-sphere from [18], see [6]. Note that it is not possible to obtain the algebraic Podleśphere in this way by replacing $\mathcal{T} = P_i^{\mathrm{co}H}$ by the coordinate algebra of a quantum disc with generator x satisfying $x^*x - qxx^* = 1 - q$ [14].

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