A residue formula for the fundamental Hochschild 3-cocycle for $SU_q(2)$

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Abstract

An analogue of a spectral triple over $SU_q(2)$ is constructed for which the usual assumption of bounded commutators with the Dirac operator fails. An analytic expression analogous to that for the Hochschild class of the Chern character for spectral triples yields a non-trivial twisted Hochschild 3-cocycle. The problems arising from the unbounded commutators are overcome by defining a residue functional using projections to cut down the Hilbert space.

1 Introduction

This paper studies the homological dimension of the quantum group $SU_q(2)$ from the perspective of Connes' spectral triples. We use an analogue of a spectral triple to construct, by a residue formula, a nontrivial Hochschild 3-cocycle. Thus we obtain finer dimension information than is provided by the nontriviality of a K-homology class, which is sensitive only to dimension modulo 2.

The position of quantum groups within noncommutative geometry has been studied intensively over the last 15 years. In particular, Chakraborty and Pal [ChP1] introduced a spectral triple for $SU_q(2)$, and this construction was subsequently refined in [DLSSV] and generalised by Neshveyev and Tuset in [NT2] to all compact Lie groups G. These spectral triples have analytic dimension $\dim G$ and nontrivial K-homology class. However, when Connes computed the Chern character for Chakraborty and Pal's spectral triple [C1], he found that it had cohomological dimension 1 in the sense that the degree $\dim SU(2) = 3$ term in the local index formula is a Hochschild coboundary. Analogous results for the spectral triple from [DLSSV] were obtained in [DLSSV2].

Contrasting these 'dimension drop' results, Hadfield and the first author [HK1, HK2] showed that $SU_q(2)$ is a twisted Calabi-Yau algebra of dimension 3 whose twist is the inverse of the modular automorphism for the Haar state on this compact quantum group, cf. Section 2. They also computed a cocycle representing a generator of the nontrivial degree 3 Hochschild cohomology groups (which we call the fundamental cocycle), and a dual degree 3 Hochschild cycle which we denote $dvol$.

The starting point of the present paper is the concept of a 'modular' spectral triple [CNNR]. These are analogous to ordinary spectral triples except for the use of twisted traces. The examples considered

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in [CNNR] arise from KMS states of circle actions on C^*-algebras, and yield nontrivial KK-classes with 1-dimensional Chern characters in twisted cyclic cohomology. In [KW] it was then shown that they also can be used to obtain the fundamental cocycle of the standard Podleś quantum 2-sphere. Motivated by this, our construction here extends the modular spectral triple on the Podleś sphere to all of $SU_q(2)$. This extension is not a modular spectral triple, but as our main theorem shows, still captures the homological dimension 3: we give a residue formula for a twisted Hochschild 3-cocycle which is a nonzero multiple of the fundamental cocycle. We obtain this formula by analogy with Connes' formula for the Hochschild class of the Chern character of spectral triples, [C, Theorem 8, IV.2.γ] and [BeF, CPRS1]. A natural next question that arises is whether our constructions provide a representative of a nontrivial K-homological class.

The organisation of the paper is as follows. In Section 2 we recall the definitions of $SU_q(2)$, the Haar state on $SU_q(2)$ and the associated GNS representation, and finally the modular theory of the Haar state. In Section 3 we recall the homological constructions of [HK1, HK2], and prove some elementary results we will need when we come to show that our residue cocycle does indeed recover the class of the fundamental cocycle.

Section 4 contains all the key analytic results on meromorphic extensions of certain functions that allow us to prove novel summability type results for operators whose eigenvalues have mixed polynomial and exponential growth, see Lemma 4.2.

Section 5 constructs an analogue of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ over the algebra $\mathcal{A}$ of polynomials in the standard generators of the C^*-algebra $SU_q(2)$. The key requirement of bounded commutators fails, and this 'spectral triple' fails to be finitely summable in the usual sense (however, it is θ-summable). Using an ultraviolet cutoff we can recover finite summability of the operator $\mathcal{D}$ on a subspace of $\mathcal{H}$ with respect to a suitable twisted trace. However, our representation of $\mathcal{A}$ does not restrict to this subspace, and so we are prevented from obtaining a genuine spectral triple.

In Section 6 we define a residue functional τ. Heuristically, for an operator T,

$$\tau(T) = \mathrm{Res}_{s=3}\mathrm{Trace}(\Delta^{-1} Q T (1 + \mathcal{D}^2)^{-s/2}).$$

Here Δ implements the modular automorphism of the Haar state, $\mathcal{D}$ is our Dirac operator and Q is a suitable projection that implements the cutoff. The existence, first of the trace, and then the residue, are both nontrivial matters.

The main properties of τ are described in Theorem 6.3, and in particular we show that the domain of τ contains the products of commutators $a_0[\mathcal{D}, a_1][\mathcal{D}, a_2][\mathcal{D}, a_3]$ for $a_i \in \mathcal{A}$. In addition, τ is a twisted trace on a suitable subalgebra of the domain containing these products. The main result, Theorem 6.5, proves that the map $a_0, \ldots, a_3 \mapsto \tau(a_0[\mathcal{D}, a_1][\mathcal{D}, a_2][\mathcal{D}, a_3])$ is a twisted Hochschild 3-cocycle, whose cohomology class is non-trivial and coincides with (a multiple of) the fundamental class.

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2 Background on $SU_q(2)$

The notations and conventions of [KS] will be used throughout for consistency. We recall that $\mathcal{A} := \mathcal{O}(SU_q(2))$, for $q \in (0, 1)$, is the unital Hopf $*$-algebra with generators a, b, c, d satisfying the relations

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb$$
$$ad = 1 + qbc, \quad da = 1 + q^{-1}bc$$

and carrying the usual Hopf structure, as in e.g. [KS]. The involution is given by

$$a^* = d, \quad b^* = -qc, \quad c^* = -q^{-1}b, \quad d^* = a.$$

We choose to view $\mathcal{A}$ as being generated by a, b, c, d explicitly, rather than just a, b, in order to make formulae more readable.

Proposition 2.1 ([KS, Proposition 4.4])**.** *The set $\{a^n b^m c^r,\ b^m c^r d^s \mid m, r, s \in \mathbb{N}_0,\ n \in \mathbb{N}\}$ is a vector space basis of $\mathcal{A}$. These monomials will be referred to as the polynomial basis.*

Recall that for each $l \in \frac{1}{2}\mathbb{N}_0$, there is a unique (up to unitary equivalence) irreducible corepresentation V_l of the coalgebra $\mathcal{A}$ of dimension $2l+1$, and that $\mathcal{A}$ is cosemisimple. That is, if we fix a vector space basis in each of the V_l and denote by $t^l_{i,j} \in \mathcal{A}$ the corresponding matrix coefficients, then we have the following analogue of the Peter-Weyl theorem.

Theorem 2.2 ([KS, Theorem 4.13])**.** *Let $I_l := \{-l, -l+1, \ldots, l-1, l\}$. Then the set $\{t^l_{i,j} \mid l \in \frac{1}{2}\mathbb{N}_0,\ i, j \in I_l\}$ is a vector space basis of $\mathcal{A}$.*

This will be referred to as the Peter-Weyl basis. With a suitable choice of basis in $V_{\frac{1}{2}}$, one has

$$a = t^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}}, \qquad b = t^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}, \qquad c = t^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}, \qquad d = t^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}.$$

The expressions for the Peter-Weyl basis elements as linear combinations of the polynomial basis elements can be found in [KS, Section 4.2.4].
The quantized universal enveloping algebra $U_q(\mathfrak{sl}(2))$ is a Hopf algebra which is generated by k, k^{-1}, e, f with relations

$$kk^{-1} = k^{-1}k = 1, \quad kek^{-1} = qe, \quad kfk^{-1} = q^{-1}f, \quad [e, f] = \frac{k^2 - k^{-2}}{q - q^{-1}}.$$

Note that in [KS] this algebra is denoted by $\breve{U}_q(\mathrm{sl}_2)$ and $U_q^{\mathrm{ext}}(\mathrm{sl}_2)$. The algebra $U_q(\mathfrak{sl}(2))$ carries the following Hopf structure

$$\Delta(k) = k \otimes k, \quad \Delta(e) = e \otimes k + k^{-1} \otimes e, \quad \Delta(f) = f \otimes k + k^{-1} \otimes f$$
$$S(k) = k^{-1}, \quad S(e) = -qe, \quad S(f) = -q^{-1}f$$
$$\varepsilon(k) = 1, \quad \varepsilon(e) = \varepsilon(f) = 0.$$

Adding the following involution

$$k^* = k, \quad e^* = f, \quad f^* = e$$

we obtain a Hopf $*$-algebra which we denote by $U_q(\mathfrak{su}(2))$.

Theorem 2.3 ([KS, Theorem 4.21]). *There exists a unique dual pairing $\langle\cdot,\cdot\rangle$ of the Hopf algebras $U_q(\mathfrak{sl}(2))$ and $\mathcal{A}$ such that*

$$\langle k,a\rangle = q^{-\frac{1}{2}},\quad \langle k,d\rangle = q^{\frac{1}{2}},\quad \langle e,c\rangle = \langle f,b\rangle = 1$$
$$\langle k,b\rangle = \langle k,c\rangle = \langle e,a\rangle = \langle e,b\rangle = \langle e,d\rangle = \langle f,a\rangle = \langle f,c\rangle = \langle f,d\rangle = 0.$$

This pairing is compatible with the $*$-structures on $U_q(\mathfrak{sl}(2))$ and $\mathcal{A}$, [KS, Chapter 1].

The dual pairing between the Hopf algebras $\langle\cdot,\cdot\rangle : U_q(\mathfrak{sl}(2)) \times \mathcal{A} \to \mathbb{C}$ defines left and right actions of $U_q(\mathfrak{sl}(2))$ on $\mathcal{A}$. Using Sweedler notation ($\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$) these actions are given by

$$g \triangleright x := \sum x_{(1)} \left\langle g, x_{(2)} \right\rangle \qquad x \triangleleft g := \sum x_{(2)} \left\langle g, x_{(1)} \right\rangle, \qquad \text{for all } \; x \in \mathcal{A},\; g \in U_q(\mathfrak{sl}(2)).$$

The left and right actions make $\mathcal{A}$ a $U_q(\mathfrak{sl}(2))$-bimodule [KS, Proposition 1.16].
Our definition of the q-numbers is

$$[a]_q := \frac{q^{-a} - q^a}{q^{-1} - q} = Q(q^{-a} - q^a) \qquad \text{for any } a \in \mathbb{C},$$

where we abbreviated $Q := (q^{-1} - q)^{-1} \in (0,\infty)$. The following lemma recalls the explicit formulas for the action of the generators on the Peter-Weyl basis.

Lemma 2.4. *For all $n \in \mathbb{Z}$,*

$$k^n \triangleright t^l_{i,j} = q^{nj} t^l_{i,j} \qquad\qquad t^l_{i,j} \triangleleft k^n = q^{ni} t^l_{i,j}$$
$$e \triangleright t^l_{i,j} = \sqrt{\left[l+\tfrac{1}{2}\right]_q^2 - \left[j+\tfrac{1}{2}\right]_q^2}\; t^l_{i,j+1} \qquad\qquad f \triangleright t^l_{i,j} = \sqrt{\left[l+\tfrac{1}{2}\right]_q^2 - \left[j-\tfrac{1}{2}\right]_q^2}\; t^l_{i,j-1}.$$

Later we will use the notation

$$\partial_k := k \triangleright \cdot\,, \qquad \partial_e := e \triangleright \cdot\,, \qquad \partial_f := f \triangleright \cdot\,,$$

especially when we extend these operators from $\mathcal{A}$ to suitable completions. Also observe that $\Delta(k^n) = k^n \otimes k^n$ for all $n \in \mathbb{Z}$, hence $k^n \triangleright \cdot$ and $\cdot \triangleleft k^n$ are algebra automorphisms on $\mathcal{A}$. They are not $*$-algebra automorphisms since for $\alpha \in \mathcal{A}$ we have $(k \triangleright \alpha)^* = k^{-1} \triangleright \alpha^*$, $(\alpha \triangleleft k)^* = \alpha^* \triangleleft k^{-1}$. Finally, we introduce

$$\partial_H(t^l_{i,j}) = j t^l_{i,j},$$

and we note that formally $\partial_k = q^{\partial_H}$.

2.1 The GNS representation for the Haar state

We denote by $A := C^*(SU_q(2))$ the universal C^*-completion of the $*$-algebra $\mathcal{A}$ [KS, Section 4.3.4]. Let h be the Haar state of A whose values on basis elements are

$$h(a^i b^j c^k) = h(d^i b^j c^k) = \delta_{i,0}\delta_{j,k}(-1)^k [k+1]_q^{-1}, \quad h(t^l_{i,j}) = \delta_{l0}.$$

Let $\mathcal{H}_h$ denote the GNS space $L^2(A,h)$, where the inner product $\langle x,y\rangle = h(x^*y)$ is conjugate linear in the first variable. The representation of A on $\mathcal{H}_h$ is is induced by left multiplication in A. The set $\{t^l_{i,j} \mid l \in \frac{1}{2}\mathbb{N}_0,\; i,j \in I_l\}$ is an orthogonal basis for $\mathcal{H}_h$ with

$$\left\langle t^l_{i,j}, t^{l'}_{i',j'} \right\rangle = \delta_{l,l'}\delta_{i,i'}\delta_{j,j'} q^{-2i} [2l+1]_q^{-1}.$$

2.2 Modular Theory

Following Woronowicz, we call the automorphism

$$\vartheta(\alpha) := k^{-2} \triangleright \alpha \triangleleft k^{-2}, \quad \alpha \in \mathcal{A}$$

the modular automorphism of $\mathcal{A}$. The action of ϑ on the generators of $\mathcal{A}$ and the Peter-Weyl basis is given by

$$\vartheta(a) = q^2 a, \;\; \vartheta(b) = b, \;\; \vartheta(c) = c, \;\; \vartheta(d) = q^{-2} d, \;\; \vartheta(t^l_{r,s}) = q^{-2(r+s)} t^l_{r,s}.$$

The modular automorphism is a (non $*$-) algebra automorphism; more precisely for any $\alpha \in \mathcal{A}$

$$\vartheta(\alpha)^* = \vartheta^{-1}(\alpha^*).$$

The Haar state is related to the modular automorphism by the following proposition.

Proposition 2.5 ([KS, Proposition 4.15])**.** *For α, $\beta \in \mathcal{A}$, we have $h(\alpha\beta) = h(\vartheta(\beta)\alpha)$.*

In fact, h extends to a KMS state on A for the strongly continuous one-parameter group ϑ_t, $t \in \mathbb{R}$, of $*$-automorphisms of A which is given on the generators by

$$\vartheta_t(a) := q^{-2it} a\,, \qquad \vartheta_t(b) := b\,, \qquad \vartheta_t(c) := c\,, \qquad \vartheta_t(d) := q^{2it} d\,.$$

We extend this to an action $\vartheta_\cdot : \mathbb{C} \times \mathcal{A} \to \mathcal{A}$ by algebra (not $*$-) automorphisms that is defined on generators by

$$\vartheta_z(a) := q^{-2iz} a\,, \qquad \vartheta_z(b) := b\,, \qquad \vartheta_z(c) := c\,, \qquad \vartheta_z(d) := q^{2iz} d\,,$$

so that the modular automorphism ϑ is ϑ_i.
We can implement ϑ_t in the GNS representation on $\mathcal{H}_h$. To do this, we define an unbounded linear operator Δ_F on $\mathcal{A} \subset \mathcal{H}_h$ by

$$\Delta_F(t^l_{i,j}) := q^{2i+2j} t^l_{i,j}$$

and call this the full modular operator. Then we have

$$\vartheta_t(x)\xi = \Delta_F^{it} x \Delta_F^{-it} \xi\,, \qquad \text{for all } x \in A \text{ and } \xi \in \mathcal{H}_h.$$

The subscript F denotes that this operator is associated to the full modular automorphism ϑ. In addition, we define the left and the right modular operators on $\mathcal{A} \subset \mathcal{H}_h$ by

$$\Delta_L(t^l_{i,j}) := q^{2j} t^l_{i,j}, \qquad \Delta_R(t^l_{i,j}) := q^{2i} t^l_{i,j},$$

so $\Delta_F = \Delta_L \Delta_R = \Delta_R \Delta_L$. Just as Δ_F implements the modular automorphism group, the left and right modular operators implement one-parameter groups of automorphisms of A:

$$\sigma_{L,t}(t^l_{r,s}) = q^{2its} t^l_{r,s} = \Delta_L^{it} t^l_{r,s} \Delta_L^{-it}\,, \qquad \sigma_{R,t}(t^l_{r,s}) = q^{2itr} t^l_{r,s} = \Delta_R^{it} t^l_{r,s} \Delta_R^{-it}.$$

As with the full action, the left and right actions are periodic and hence give rise to actions of $\mathbb{T}$ on A. These may be extended to a complex action on the $*$-subalgebra $\mathcal{A}$ which we will denote $\sigma_{L,z}$ and $\sigma_{R,z}$. In particular, we obtain for $z = i$ the algebra automorphisms

$$\sigma_L := k^{-2} \triangleright \cdot \qquad \sigma_R := \cdot \triangleleft k^{-2} \qquad \vartheta = \sigma_L\sigma_R = \sigma_R\sigma_L$$
$$\sigma_L(t^l_{r,s}) = q^{-2s}t^l_{r,s} \qquad \sigma_R(t^l_{r,s}) = q^{-2r}t^l_{r,s}$$
$$\vartheta(\alpha)\xi = \Delta_F^{-1}\alpha\Delta_F\xi \qquad \sigma_L(\alpha)\xi = \Delta_L^{-1}\alpha\Delta_L\xi \qquad \sigma_R(\alpha)\xi = \Delta_R^{-1}\alpha\Delta_R\xi.$$

The fixed point algebra for the left action on $\mathcal{A}$ is isomorphic to the standard Podleś quantum 2–sphere $\mathcal{O}(S^2_q)$. We will denote its C^*-completion by B. As the left action is periodic, we may define a positive faithful expectation $\Phi\colon A \to B$ by

$$\Phi(x) = \frac{\ln(q^{-2})}{2\pi}\int_0^{2\pi/\ln(q^{-2})} \sigma_{L,t}(x)dt.$$

More generally, given $n \in \mathbb{Z}$ and $x \in A$ we define

$$\Phi_n(x) = \frac{\ln(q^{-2})}{2\pi}\int_0^{2\pi/\ln(q^{-2})} t^{-n}\sigma_{L,t}(x)dt.$$

Since $\sigma_{L,t}$ is a strongly continuous action on A, the Φ_n are continuous maps on A. Observe that $\Phi = \Phi_0$ and

$$\Phi_n(t^l_{i,j}) = \delta_{n,2j}t^l_{i,j}$$

Hence the Φ_n can be extended to bounded operators on the GNS space $\mathcal{H}_h$, and in fact the Φ_n are projections onto the spectral subspaces of the left circle action. So we make explicit the decomposition of A into the left spectral subspaces by defining

$$B_n := \Phi_n(A) = \{\alpha \in A \mid \sigma_{L,t}(\alpha) = q^{2int}\alpha\} \quad \text{and} \quad \mathcal{H}_n := L^2(B_n, h)$$

where h is the Haar state (restricted to B_n). This leads to the following decomposition for the GNS space

$$\mathcal{H}_h = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n.$$

The commutation relations for the projections Φ_n and the operators ∂_k, ∂_e and ∂_f are found from the definitions on the Peter-Weyl basis to be

$$\partial_k\Phi_n = \Phi_n\partial_k = q^{\frac{n}{2}}\Phi_n \qquad \partial_H\Phi_n = \Phi_n\partial_H = \frac{n}{2}\Phi_n \qquad \Delta_L\Phi_n = \Phi_n\Delta_L = q^n\Phi_n$$
$$\partial_e\Phi_n = \Phi_{n+2}\partial_e \qquad \partial_f\Phi_n = \Phi_{n-2}\partial_f\,.$$

The left actions of e and f are twisted derivations in the sense that for α, $\beta \in \mathcal{A}$

$$\partial_e(\alpha\beta) = \partial_e(\alpha)\partial_k(\beta) + \partial_k^{-1}(\alpha)\partial_e(\beta)$$
$$\partial_f(\alpha\beta) = \partial_f(\alpha)\partial_k(\beta) + \partial_k^{-1}(\alpha)\partial_f(\beta)\,.$$

More generally, given $\alpha \in A$ and $\xi \in \mathcal{H}_h$

$$\begin{aligned}\partial_e(\alpha\xi) &= \partial_e(\alpha)\Delta_L^{\frac{1}{2}}\xi + \sigma_L^{\frac{1}{2}}(\alpha)\partial_e(\xi) \\ &= \partial_e(\alpha)\Delta_L^{\frac{1}{2}}\xi + \Delta_L^{-\frac{1}{2}}\alpha\Delta_L^{\frac{1}{2}}\partial_e(\xi)\end{aligned} \qquad \begin{aligned}\partial_f(\alpha\xi) &= \partial_f(\alpha)\Delta_L^{\frac{1}{2}}\xi + \sigma_L^{\frac{1}{2}}(\alpha)\partial_f(\xi) \\ &= \partial_f(\alpha)\Delta_L^{\frac{1}{2}}\xi + \Delta_L^{-\frac{1}{2}}\alpha\Delta_L^{\frac{1}{2}}\partial_f(\xi).\end{aligned} \tag{2.1}$$

See e.g. [BHMS] and the references therein for background on the generalisation of this setting in terms of Hopf-Galois extensions.

3 Twisted homology and cohomology

We recall that the algebra $\mathcal{A}$ is a ϑ^{-1}-twisted Calabi-Yau algebra of dimension 3, see [HK2] and the references therein for this result and some background. Since the centre of $\mathcal{A}$ consists only of the scalar multiples of $1_{\mathcal{A}}$, this means in particular that the cochain complex $C^\bullet := \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}^{\otimes_{\mathbb{C}}\bullet+1}, \mathbb{C})$, with differential $b_{\vartheta^{-1}} : C^n \to C^{n+1}$ given by

$$\begin{aligned}(b_{\vartheta^{-1}}\varphi)(a_0, \dots, a_n, a_{n+1}) = &\sum_{i=0}^{n} (-1)^n \varphi(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ &+ (-1)^{n+1}\varphi(\vartheta^{-1}(a_{n+1})a_0, a_1, \dots, a_n),\end{aligned}$$

is exact in degrees $n > 3$ and has third cohomology $H^3(C, b_{\vartheta^{-1}}) \simeq \mathbb{C}$. An explicit cocycle whose cohomology class generates $H^3(C, b_{\vartheta^{-1}})$ can be constructed using the following incarnation of the cup product $\smile$ in Hochschild cohomology:

Lemma 3.1. *Let $\sigma_0, \dots, \sigma_3$ be automorphisms of $\mathcal{A}$, $\int : \mathcal{A} \to \mathbb{C}$ be a $\sigma_0 \circ \vartheta^{-1} \circ \sigma_3^{-1}$-twisted trace, that is,*

$$\int \alpha\beta = \int \sigma_0(\vartheta^{-1}(\sigma_3^{-1}(\beta)))\alpha,$$

and $\partial_i : \mathcal{A} \to \mathcal{A}$, $i = 1, 2, 3$, be σ_{i-1}-σ_i-twisted derivations, that is,

$$\partial_i(\alpha\beta) = \sigma_{i-1}(\alpha)\partial_i(\beta) + \partial_i(\alpha)\sigma_i(\beta).$$

Then the functional defined via the cup product by

$$\left(\int \smile \partial_1 \smile \partial_2 \smile \partial_3\right)(a_0, a_1, a_2, a_3) := \int \sigma_0(a_0)\partial_1(a_1)\partial_2(a_2)\partial_3(a_3)$$

is a ϑ^{-1}-twisted cocycle, $b_{\vartheta^{-1}}(\int \smile \partial_1 \smile \partial_2 \smile \partial_3) = 0$.

Proof. This is a straightforward computation:

$$\begin{aligned}&\left(b_{\vartheta^{-1}} \int \smile \partial_1 \smile \partial_2 \smile \partial_3\right)(a_0, a_1, a_2, a_3, a_4) \\ = \ &\int \sigma_0(a_0 a_1)\partial_1(a_2)\partial_2(a_3)\partial_3(a_4) - \int \sigma_0(a_0)\partial_1(a_1 a_2)\partial_2(a_3)\partial_3(a_4) \\ &+ \int \sigma_0(a_0)\partial_1(a_1)\partial_2(a_2 a_3)\partial_3(a_4) - \int \sigma_0(a_0)\partial_1(a_1)\partial_2(a_2)\partial_3(a_3 a_4) \\ &+ \int \sigma_0(\vartheta^{-1}(a_4)a_0)\partial_1(a_1)\partial_2(a_2)\partial_3(a_3) \\ = \ &-\int \sigma_0(a_0)\partial_1(a_1)\partial_2(a_2)\partial_3(a_3)\sigma_3(a_4) + \int \sigma_0(\vartheta^{-1}(a_4))\sigma_0(a_0)\partial_1(a_1)\partial_2(a_2)\partial_3(a_3) \\ = \ &0.\end{aligned}$$

□

Less straightforward is that when applying the above result with

$$\sigma_0 = \sigma_1 = k^{-4} \triangleright \cdot, \quad \sigma_2 = k^{-2} \triangleright \cdot, \quad \sigma_3 = \mathrm{id},$$

$$\partial_1 = (k^{-4} \triangleright \cdot) \circ \partial_H, \quad \partial_2 = (k^{-3} \triangleright \cdot) \circ \partial_e, \quad \partial_3 = (k^{-1} \triangleright \cdot) \circ \partial_f$$

and a suitable twisted trace, one obtains a cohomologically nontrivial ϑ^{-1}-twisted cocycle.

Lemma 3.2 ([HK2, Corollary 3.8])**.** *Define a linear functional $\int_{[1]} : \mathcal{A} \to \mathbb{C}$ by*

$$\int_{[1]} a^n b^m c^r := \delta_{n,0}\delta_{m,0}\delta_{r,0}, \qquad \int_{[1]} b^m c^r d^s := \delta_{m,0}\delta_{r,0}\delta_{s,0}.$$

Then $\int_{[1]}$ is a $\sigma_L^2 \circ \vartheta^{-1}$-twisted trace, and the cochain $\varphi \in C^3$ given by

$$\varphi(a_0, \dots, a_3) = \int_{[1]} \left(k^{-4} \triangleright (a_0\, \partial_H(a_1))\right) \left(k^{-3} \triangleright \partial_e(a_2)\right) \left(k^{-1} \triangleright \partial_f(a_3)\right)$$

is a cocycle, $b_{\vartheta^{-1}}\varphi = 0$, whose cohomology class is nontrivial, $b_{\vartheta^{-1}}\psi \neq \varphi$ for all $\psi \in C^2$.

Later, we will also have to consider the cocycles that are obtained by using the (twisted) derivations $\partial_H, \partial_e, \partial_f$ in a different order. Explicitly, this is handled by the following result.

Lemma 3.3. *In the situation of Lemma 3.1, define*

$$\tilde{\partial}_3 = \sigma_1 \circ \sigma_2^{-1} \circ \partial_3, \quad \tilde{\partial}_2 := \partial_2 \circ \sigma_2^{-1} \circ \sigma_3, \quad \hat{\partial}_2 := \sigma_0 \circ \sigma_1^{-1} \circ \partial_2, \quad \hat{\partial}_1 := \partial_1 \circ \sigma_1^{-1} \circ \sigma_2.$$

Then we have

$$\int \smile \partial_1 \smile \partial_2 \smile \partial_3 + \int \smile \partial_1 \smile \tilde{\partial}_3 \smile \tilde{\partial}_2 = b_{\vartheta^{-1}}\psi_{132},$$
$$\int \smile \partial_1 \smile \partial_2 \smile \partial_3 + \int \smile \hat{\partial}_2 \smile \hat{\partial}_1 \smile \partial_3 = b_{\vartheta^{-1}}\psi_{213},$$

where

$$\psi_{132}(a_0, a_1, a_2) := \int \sigma_0(a_0)\partial_1(a_1)\partial_2(\sigma_2^{-1}(\partial_3(a_2))),$$
$$\psi_{213}(a_0, a_1, a_2) := -\int \sigma_0(a_0)\partial_1(\sigma_1^{-1}(\partial_2(a_1)))\partial_3(a_2).$$

Proof. Straightforward computation. □

Applying Lemma 3.3 repeatedly to the cocycle φ from Lemma 3.2 gives cohomologous cocycles.

Corollary 3.4. *The cocycle φ from Lemma 3.2 is cohomologous to each of*

$$\varphi_{132}(a_0, a_1, a_2, a_3) := -q^{-2} \int_{[1]} \left(k^{-4} \triangleright (a_0\, \partial_H(a_1))\right) \left(k^{-3} \triangleright \partial_f(a_2)\right) \left(k^{-1} \triangleright \partial_e(a_3)\right),$$

$$\varphi_{213}(a_0, a_1, a_2, a_3) := -\int_{[1]} \left(k^{-4} \triangleright a_0\right) \left(k^{-3} \triangleright \partial_e(a_1)\right) \left(k^{-2} \triangleright \partial_H(a_2)\right) \left(k^{-1} \triangleright \partial_f(a_3)\right),$$

$$\varphi_{312}(a_0,a_1,a_2,a_3) := q^{-2}\int_{[1]} \left(k^{-4}\triangleright a_0\right)\left(k^{-3}\triangleright \partial_f(a_1)\right)\left(k^{-2}\triangleright \partial_H(a_2)\right)\left(k^{-1}\triangleright \partial_e(a_3)\right),$$

$$\varphi_{231}(a_0,a_1,a_2,a_3) := \int_{[1]} \left(k^{-4}\triangleright a_0\right)\left(k^{-3}\triangleright \partial_e(a_1)\right)\left(k^{-1}\triangleright \partial_f(a_2)\right)\left(\partial_H(a_3)\right)$$

and

$$\varphi_{321}(a_0,a_1,a_2,a_3) := -q^{-2}\int_{[1]} \left(k^{-4}\triangleright a_0\right)\left(k^{-3}\triangleright \partial_f(a_1)\right)\left(k^{-1}\triangleright \partial_e(a_2)\right)\left(\partial_H(a_3)\right).$$

Proof. To begin, one applies Lemma 3.3 to φ with

$$\tilde\partial_3 = (k^{-3}\triangleright\cdot)\circ\partial_f,\quad \tilde\partial_2 = (k^{-3}\triangleright\cdot)\circ\partial_e\circ(k^2\triangleright\cdot),\quad \hat\partial_2 = (k^{-3}\triangleright\cdot)\circ\partial_e,\quad \hat\partial_1 := (k^{-4}\triangleright\cdot)\circ\partial_H(\cdot)\circ(k^2\triangleright\cdot).$$

The formulae for these derivations can be simplified by commuting ∂_e and $k\triangleright$ to obtain

$$\tilde\partial_3 = (k^{-3}\triangleright\cdot)\circ\partial_f,\quad \tilde\partial_2 = q^{-2}(k^{-1}\triangleright\cdot)\circ\partial_e,\quad \hat\partial_2 = (k^{-3}\triangleright\cdot)\circ\partial_e,\quad \hat\partial_1 := (k^{-2}\triangleright\cdot)\circ\partial_H(\cdot).$$

This gives φ_{132} and φ_{213}. Then we can apply Lemma 3.3 again to φ_{213}. Going from φ_{213} to φ_{312} is easy, since it only involves exchanging e and f. Next we obtain φ_{231} from φ_{213} by applying Lemma 3.3 with

$$\sigma_0 = k^{-4}\triangleright\cdot,\quad \sigma_1=\sigma_2 = k^{-2}\triangleright\cdot,\quad \sigma_3 = \mathrm{id},$$
$$\partial_1 = (k^{-3}\triangleright\cdot)\circ\partial_e,\quad \partial_2 = (k^{-2}\triangleright\cdot)\circ\partial_H,\quad \partial_3 = (k^{-1}\triangleright\cdot)\circ\partial_f$$

which gives

$$\tilde\partial_3 = \sigma_1\circ\sigma_2^{-1}\circ\partial_3 = \partial_3 = (k^{-1}\triangleright\cdot)\circ\partial_f,$$
$$\tilde\partial_2 = \partial_2\circ\sigma_2^{-1}\circ\sigma_3 = (k^{-2}\triangleright\cdot)\circ\partial_H\circ(k^2\triangleright\cdot) = \partial_H.$$

The last cocycle is obtained analogously from φ_{312}. □

A homologically nontrivial 3-cycle *dvol* in the (pre)dual chain complex $C_\bullet := \mathcal{A}^{\otimes_{\mathbb{C}}\bullet+1}$ (with differential dual to $b_{\vartheta^{-1}}$) has been computed in [HK1, HK2]:

$$\begin{aligned} dvol := {} & d\otimes a\otimes b\otimes c - d\otimes a\otimes c\otimes b + q\,d\otimes c\otimes a\otimes b \\ & - q^2\,d\otimes c\otimes b\otimes a + q^2\,d\otimes b\otimes c\otimes a - q\,d\otimes b\otimes a\otimes c \\ & + c\otimes b\otimes a\otimes d - c\otimes b\otimes d\otimes a + q\,c\otimes d\otimes b\otimes a \\ & - c\otimes d\otimes a\otimes b + c\otimes a\otimes d\otimes b - q^{-1}\,c\otimes a\otimes b\otimes d \\ & + (q^{-1}-q)\,c\otimes b\otimes c\otimes b \end{aligned} \tag{3.1}$$

With this normalisation, we have $\varphi(dvol) = 1$.

4 Some meromorphic functions

In this section we demonstrate that certain functions have meromorphic continuations. These functions arise in the residue formula for the Hochschild cocycle in the next two sections. We require the following notation. For any $l\in\frac{1}{2}\mathbb{N}_0$ and $-(2l+1)\le n\le(2l+1)$ define

$$\lambda_{l,n} := \sqrt{\left(\frac{n}{2}\right)^2 + q^n\left(\left[l+\frac{1}{2}\right]_q^2 - \left[\frac{n}{2}\right]_q^2\right)}. \tag{4.1}$$

We also define the finite sets

$$\mathcal{J}_l := \begin{cases} \{0, 2, \dots, 2l-1\} & l \in (\mathbb{N}_0 + \frac{1}{2}) \\ \{1, 3, \dots, 2l-1\} & l \in \mathbb{N} \end{cases}.$$

Lemma 4.1. *The formulas*

$$z \mapsto f_1(z) := \sum_{2l=1}^{\infty} \sum_{i=-l}^{l} \sum_{n \in \mathcal{J}_l} \frac{q^{2l-2i}}{(1+\lambda_{l,n}^2)^{z/2}}$$

$$z \mapsto f_2(z) := \sum_{2l=1}^{\infty} \sum_{i=-l}^{l} \sum_{n \in \mathcal{J}_l} \frac{q^{2l-n}}{(1+\lambda_{l,n}^2)^{z/2}}$$

define holomorphic functions on Dom_2*, where we abbreviate*

$$\mathrm{Dom}_t := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > t\}, \quad t \in \mathbb{R}.$$

Proof. We will show that the sums converge uniformly on compacta. To begin with, we take $z = t \in (2, \infty)$, and compute the summation over the i parameter for f_1 and f_2 giving

$$f_1(t) = \sum_{2l=1}^{\infty} \sum_{n \in \mathcal{J}_l} \frac{q^{2l}[2l+1]_q}{(1+\lambda_{l,n}^2)^{t/2}}, \qquad f_2(t) = \sum_{2l=1}^{\infty} \sum_{n \in \mathcal{J}_l} \frac{(2l+1)q^{2l-n}}{(1+\lambda_{l,n}^2)^{t/2}}. \tag{4.2}$$

For $l \in \frac{1}{2}\mathbb{N}_0$ and $n \in \mathcal{J}_l$ we have the inequality

$$\left[l + \frac{1}{2}\right]_q^2 - \left[\frac{n}{2}\right]_q^2 \geq [2l]_q$$

with equality attained for $n = 2l - 1$. This inequality implies

$$1 + \lambda_{l,n}^2 \geq 1 + \left(\frac{n}{2}\right)^2 + q^n[2l]_q \geq 1 + \left(\frac{n}{2}\right)^2 + q^{n-2l+1}. \tag{4.3}$$

Since the summands in Equation (4.2) are positive, we may invoke Tonelli's theorem to rearrange the order of summation

$$\sum_{2l=1}^{\infty} \sum_{n \in \mathcal{J}_l} \to \sum_{n=0}^{\infty} \sum_{l=(n+1)/2}^{\infty}.$$

Combining the elementary inequality $q^{2l}[2l+1]_q \leq q^{-1}Q$ with Equation (4.3) gives the inequalities

$$f_1(t) \leq q^{-1}Q \sum_{n=0}^{\infty} \sum_{l=\frac{n+1}{2}}^{\infty} \frac{1}{(1 + \left(\frac{n}{2}\right)^2 + q^{n-2l+1})^{t/2}}, \qquad f_2(t) \leq \sum_{n=0}^{\infty} \sum_{l=\frac{n+1}{2}}^{\infty} \frac{(2l+1)q^{2l-n}}{(1 + \left(\frac{n}{2}\right)^2 + q^{n-2l+1})^{t/2}}.$$

We reparameterise the sums defining f_1 and f_2 using $y = 2l - 1 - n$ with summation range $y = 0$ to $y = \infty$. This yields

$$f_1(t) \leq q^{-1}Q \sum_{n=0}^{\infty} \sum_{y=0}^{\infty} \frac{1}{(1 + \left(\frac{n}{2}\right)^2 + q^{-y})^{t/2}}, \qquad f_2(t) \leq \sum_{n=0}^{\infty} \sum_{y=0}^{\infty} \frac{(y+n+2)q^{y+1}}{(1 + \left(\frac{n}{2}\right)^2 + q^{-y})^{t/2}}. \tag{4.4}$$

Next we employ the inequality $\alpha^2+\beta^2 \geq \alpha\beta$, valid for any positive real numbers α and β, to $f_1(t)$. This yields

$$f_1(t) \leq q^{-1} Q \sum_{n=0}^{\infty} \sum_{y=0}^{\infty} q^{yt/4} \left(1+\left(\frac{n}{2}\right)^2\right)^{-t/4} < \infty \qquad \text{for all } t > 2.$$

For the function $f_2(t)$, we evaluate the sums over y on the right hand side to obtain, for some positive constants C_1 and C_2,

$$f_2(t) \leq \sum_{n=0}^{\infty} \sum_{y=0}^{\infty} \frac{(y+n+2)q^{y+1}}{\left(1+\left(\frac{n}{2}\right)^2\right)^{t/2}} = \sum_{n=0}^{\infty} \frac{C_1 + C_2 n}{\left(1+\left(\frac{n}{2}\right)^2\right)^{t/2}}.$$

This last sum is finite for all $t > 2$, and bounded uniformly for $t \geq 2+\epsilon$ for any $\epsilon > 0$. This establishes that f_1, f_2 are finite for all $\mathrm{Re}(z) > 2$, and the sums defining them converge uniformly on vertical strips, and so on compacta. Finally, to show that f_1, f_2 are holomorphic in the half-plane $\mathrm{Re}(z) > 2$, we invoke the Weierstrass convergence theorem. $\square$

Lemma 4.2. *For any positive reals x, y, $r > 0$, $w \in \mathbb{N}$, and $z \in \mathrm{Dom}_3$, define*

$$h(z) := \sum_{n=1}^{\infty} \sum_{m=w}^{\infty} \frac{e^{rm}}{(x^2 n^2 + y^2 e^{rm})^{z/2}}$$

Then we have:

1. *h is a holomorphic function on Dom_3;*
2. *h has a meromorphic continuation to Dom_2 with a simple pole at $z = 3$;*
3. *This continuation can be written as*

$$h(z) = \frac{\sqrt{\pi}}{2xy^{z-1}} \frac{\Gamma(\frac{z-1}{2})}{\Gamma(\frac{z}{2})} \frac{e^{-rw(z-3)/2}}{1-e^{-r(z-3)/2}} - \frac{1}{2y^z} \frac{e^{-rw(z-2)/2}}{1-e^{-r(z-2)/2}} + err(z)$$

where err is a holomorphic function on Dom_2 that satisfies

$$|err(z)| \leq \frac{1}{2y^{\mathrm{Re}(z)}} \frac{e^{-rw(\mathrm{Re}(z)-2)/2}}{1-e^{-r(\mathrm{Re}(z)-2)/2}}.$$

Proof. Until further notice, we take z real and positive. Later we will extend our results to complex z as in Lemma 4.1. Inserting the Mellin transform of $f(t) = e^{-(x^2n^2+y^2e^{rm})t}$ gives

$$h(z) = \sum_{n=1}^{\infty} \sum_{m=w}^{\infty} \frac{e^{rm}}{\Gamma(\frac{z}{2})} \int_0^{\infty} t^{\frac{z}{2}-1} e^{-tx^2n^2} e^{-ty^2e^{rm}} dt.$$

For z real, all terms above are positive. Therefore we can apply Tonelli's theorem to exchange the order of integration with summation. Having done this, we consider the sum $\sum_{n=1}^{\infty} e^{-tx^2n^2}$. The Poisson summation formula provides the identity

$$\sum_{n=1}^{\infty} e^{-tx^2n^2} = \frac{1}{2}\left(\sqrt{\frac{\pi}{tx^2}}\left(1+2\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2}{tx^2}}\right) - 1\right).$$

Substituting this identity into the expression for $h(z)$ we find

$$h(z) = \frac{1}{2}\sum_{m=w}^{\infty}\frac{e^{rm}}{(y^2e^{rm})^{\frac{z}{2}}}\left(\frac{\sqrt{\pi}}{x}\frac{\Gamma(\frac{z-1}{2})}{\Gamma(\frac{z}{2})}(y^2e^{rm})^{\frac{1}{2}} - 1\right)$$
$$+\frac{\sqrt{\pi}}{x}\sum_{n=1}^{\infty}\sum_{m=w}^{\infty}\frac{e^{rm}}{\Gamma(\frac{z}{2})}\int_0^\infty t^{\frac{z-1}{2}-1}e^{-\frac{n^2\pi^2}{tx^2}}e^{-ty^2e^{rm}}dt.$$

To explore the convergence of the double sum we denote

$$g_n(s) := \int_0^\infty t^{\frac{z-1}{2}-1}e^{-\frac{n^2\pi^2}{tx^2}}e^{-ts}dt.$$

Later we will set $s = y^2e^{rm} > 0$, so we consider only positive, real s, making $g_n(s)$ a positive real function. Using [OS, Section 26:14] to evaluate this Laplace transform gives

$$g_n(s) = 2\left(\frac{n\pi}{x\sqrt{s}}\right)^{\frac{z-1}{2}}K_{\frac{z-1}{2}}\left(\frac{2n\pi\sqrt{s}}{x}\right)$$

where $u \mapsto K_\nu(u)$ is the modified Bessel function of the second kind. For $u > 0$ and real $\nu > 1/2$, $u^\nu K_\nu(u)$ is positive, as both u^ν and $K_\nu(u)$ are positive. Also, the derivative (referring again to [OS]) is given by

$$\frac{\partial}{\partial u}\left(u^\nu K_\nu(u)\right) = -u^\nu K_{\nu-1}(u) \le 0 \qquad \text{for all } u \ge 0.$$

Thus the function $u \mapsto u^\nu K_\nu(u)$ is positive and monotonically decreasing for all $u > 0$. Hence for all $\epsilon > 0$ we have the bound

$$\epsilon\sum_{n=1}^{\infty}(\epsilon n)^\nu K_\nu(\epsilon n) \le \int_0^\infty u^\nu K_\nu(u)du. \tag{4.5}$$

Evaluating the integral (using [OS, Chapter 51]) yields

$$\sum_{n=1}^{\infty}(\epsilon n)^\nu K_\nu(\epsilon n) \le \frac{1}{\epsilon}2^{\nu-1}\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2}).$$

If we now set $s = y^2e^{rm}$, we obtain the bound

$$\sum_{n=1}^{\infty}\sum_{m=w}^{\infty}\frac{e^{rm}}{\Gamma(\frac{z}{2})}\int_0^\infty t^{\frac{z-1}{2}-1}e^{-\frac{n^2\pi^2}{tx^2}}e^{-ty^2e^{rm}}dt \le 2\sum_{n=1}^{\infty}\sum_{m=w}^{\infty}\frac{e^{rm}}{\Gamma(\frac{z}{2})}\left(\frac{n\pi}{xye^{rm/2}}\right)^{\frac{z-1}{2}}K_{\frac{z-1}{2}}\left(\frac{2n\pi ye^{rm/2}}{x}\right).$$

Now estimating the sum over n on the right using Equation (4.5) gives us

$$2\sum_{n=1}^{\infty}\left(\frac{n\pi}{x\sqrt{s}}\right)^{\frac{z-1}{2}}K_{\frac{z-1}{2}}\left(\frac{2n\pi\sqrt{s}}{x}\right) = 2\left(\frac{1}{2s}\right)^{\frac{z-1}{2}}\sum_{n=1}^{\infty}\left(\frac{2n\pi\sqrt{s}}{x}\right)^{\frac{z-1}{2}}K_{\frac{z-1}{2}}\left(\frac{2n\pi\sqrt{s}}{x}\right)$$
$$\le 2\left(\frac{1}{2s}\right)^{\frac{z-1}{2}}\frac{x}{2\pi\sqrt{s}}2^{\frac{z-1}{2}-1}\Gamma(\tfrac{z}{2})\Gamma(\tfrac{1}{2})$$
$$= \frac{x\Gamma(\frac{1}{2})\Gamma(\frac{z}{2})}{2\pi}\frac{1}{s^{z/2}} = \frac{x\Gamma(\frac{1}{2})\Gamma(\frac{z}{2})}{2\pi}\frac{1}{y^ze^{zrm/2}}.$$

Hence by summing the remaining geometric series in m we obtain the bound

$$\sum_{n=1}^{\infty}\sum_{m=w}^{\infty}\frac{e^{rm}}{\Gamma(\frac{z}{2})}\int_0^\infty t^{\frac{z-1}{2}-1}e^{-\frac{n^2\pi^2}{tx^2}}e^{-ty^2e^{rm}}dt \le \frac{\Gamma\left(\frac{z}{2}\right)}{\Gamma(\frac{z}{2})}\frac{x\Gamma(\frac{1}{2})}{y^z 2\pi}\sum_{m=w}^{\infty}\frac{e^{rm}}{e^{rmz/2}}$$
$$\le \frac{x\Gamma(\frac{1}{2})}{y^z 2\pi}\frac{e^{-rw(z-2)/2}}{1-e^{-r(z-2)/2}}.$$

Evaluating the remaining geometric series in $h(z)$ as above, we arrive at

$$h(z) = \frac{\sqrt{\pi}}{2xy^{z-1}}\frac{\Gamma(\frac{z-1}{2})}{\Gamma(\frac{z}{2})}\frac{e^{-rw(z-3)/2}}{1-e^{-r(z-3)/2}} - \frac{1}{2y^z}\frac{e^{-rw(z-2)/2}}{1-e^{-r(z-2)/2}} + err(z) \tag{4.6}$$

where

$$err(z) := \frac{\sqrt{\pi}}{x}\sum_{n=1}^{\infty}\sum_{m=w}^{\infty}\frac{e^{rm}}{\Gamma(\frac{z}{2})}\int_0^\infty t^{\frac{z-1}{2}-1}e^{-\frac{n^2\pi^2}{tx^2}}e^{-ty^2e^{rm}}dt,$$
$$err(z) \le \frac{1}{2y^z}\frac{e^{-rw(z-2)/2}}{1-e^{-r(z-2)/2}}.$$

Thus the sum defining the function err converges for all $z > 2$, and this convergence is uniform on compact intervals. Now we observe that for $z \in \mathbb{C}$ we have $|h(z)| \le h(|z|)$ and similarly $|err(z)| \le err(|z|)$. Hence the sums defining h converge uniformly on closed vertical strips in the half-plane Dom_3, and so on compacta. Similarly the sums and integral defining err converge uniformly on compact subsets of the half-plane Dom_2.
Hence the Weierstrass convergence theorem implies that err is holomorphic on the half-plane Dom_2 and that h is holomorphic on Dom_3. Moreover the formula for h, Equation (4.6), provides a meromorphic continuation of h to the half-plane Dom_2. □

Lemma 4.3. *The formula*

$$f(z) := \sum_{n=0}^{\infty}\sum_{l=\frac{n+1}{2}}^{\infty}\frac{q^{n-2l}}{(1+\lambda_{l,n}^2)^{z/2}}$$

defines a holomorphic function on Dom_3*. Moreover f has a meromorphic continuation to* Dom_2*, a simple pole at $z = 3$ with residue* $4qQ^{-2}/\ln(q^{-1})$*.*

Proof. First we write

$$1+\lambda_{l,n}^2 = 1+\tfrac{n^2}{4}+q^n\left(\left[l+\tfrac{1}{2}\right]^2-\left[\tfrac{n}{2}\right]^2\right) = \tfrac{1}{4}n^2+Q^2q^{-1}q^{n-2l}+C_{n,l}$$

where $C_{n,l}$ is uniformly bounded in n, l, and is given by

$$C_{n,l} = 1+Q^2q^n(q^{2l+1}-2)-q^n\left[\tfrac{n}{2}\right]^2, \qquad |C_{n,l}| \le 1+3Q^2.$$

Now we reparametrise the summation by letting $m = 2l - n$, yielding

$$f(z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{q^{-m}}{(\frac{1}{4}n^2+Q^2q^{-1}q^{-m}+C_{n,m})^{z/2}}$$

where we understand $C_{n,m} = C_{n,l=(n+m)/2}$. The function

$$z \mapsto \sum_{m=1}^{\infty} \frac{q^{-m}}{(Q^2 q^{-1} q^{-m} + C_{0,m})^{z/2}} = \sum_{m=1}^{\infty} \frac{q^{m(\frac{z}{2}-1)}}{(Q^2 q^{-1} + q^m C_{0,m})^{z/2}}$$

has summands with absolute value bounded by $Mq^{m(\frac{z}{2}-1)}$, $M > 0$ constant, and so by the Weierstrass convergence theorem is holomorphic for $\mathrm{Re}(z) > 2$. Hence for some holomorphic function *holo* on Dom_2 we have

$$\begin{aligned} f(z) &= \sum_{n,m=1}^{\infty} \frac{q^{-m}}{(\frac{1}{4}n^2 + Q^2 q^{-1} q^{-m} + C_{n,m})^{z/2}} + holo(z) \\ &= \sum_{n,m=1}^{\infty} \frac{q^{-m}}{(\frac{1}{4}n^2 + Q^2 q^{-1} q^{-m})^{z/2}} \left(1 + \frac{C_{n,m}}{\frac{1}{4}n^2 + Q^2 q^{-1} q^{-m}}\right)^{-z/2} + holo(z). \end{aligned} \tag{4.7}$$

The strategy now is to perform a binomial expansion on

$$\left(1 + \frac{C_{n,m}}{\frac{1}{4}n^2 + Q^2 q^{-1} q^{-m}}\right)^{-z/2}$$

ending up with a new sum of functions $\sum_{n,m,k} D_{n,m,k}\, h(z + 2k)$ where h is as in Lemma 4.2. The binomial expansion requires the inequality

$$\frac{C_{n,m}}{\frac{1}{4}n^2 + Q^2 q^{-1} q^{-m}} < 1$$

which holds for sufficiently large m. Recall that $|C_{n,m}| \leq 1 + 3Q^2 =: C$ uniformly in n, m, and so we may choose $p \in \mathbb{N}$ such that

$$q^{-p} > qQ^{-2}C \qquad \Longrightarrow \qquad \frac{|C_{n,m}|}{\frac{1}{4}n^2 + Q^2 q^{-1} q^{-m}} < 1 \quad \forall n \geq 1,\ m \geq p.$$

Now, for any fixed p, sums of the form

$$\sum_{n=1}^{\infty} \sum_{m=1}^{p-1} \frac{q^{-m}}{(\frac{1}{4}n^2 + Q^2 q^{-1} q^{-m} + C_{n,m})^{z/2}}$$

can immediately be seen to be holomorphic for $\mathrm{Re}(z) > 2$ as the sum can be bounded by a constant multiple of the Riemann zeta function. Hence for such a choice of $p \in \mathbb{N}$ and for some holomorphic function *holo* on Dom_2 we have

$$f(z) = \sum_{n=1}^{\infty} \sum_{m=p}^{\infty} \frac{q^{-m}}{(\frac{1}{4}n^2 + Q^2 q^{-1} q^{-m})^{z/2}} \left(1 + \frac{C_{n,m}}{\frac{1}{4}n^2 + Q^2 q^{-1} q^{-m}}\right)^{-z/2} + holo(z).$$

Now we perform the binomial expansion, separating the resulting infinite sum $\sum_{k=0}^{\infty}$ into the $k = 0$ term and $\sum_{k=1}^{\infty}$. This gives

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}\sum_{m=p}^{\infty} \frac{q^{-m}}{(\frac{1}{4}n^2+Q^2q^{-1}q^{-m})^{z/2}} + \sum_{k=1}^{\infty}\begin{pmatrix} -\frac{z}{2} \\ k \end{pmatrix}\sum_{n=1}^{\infty}\sum_{m=p}^{\infty}\frac{q^{-m}(C_{n,m})^k}{(\frac{1}{4}n^2+Q^2q^{-1}q^{-m})^{\frac{z+2k}{2}}} + holo(z) \\ &= h(z) + \sum_{k=1}^{\infty}\begin{pmatrix} -\frac{z}{2} \\ k \end{pmatrix}\sum_{n=1}^{\infty}\sum_{m=p}^{\infty}\frac{q^{-m}(C_{n,m})^k}{(\frac{1}{4}n^2+Q^2q^{-1}q^{-m})^{\frac{z+2k}{2}}} + holo(z), \end{aligned}$$

where h is as in Lemma 4.2, with $x = 1/2$, $y = q^{-1/2}Q$, $r = \ln(q^{-1})$ and $w = p$. Our aim now is to show that $f - h$ is a holomorphic function on Dom_2. We need to show that the remaining summation converges to such a function. This remaining sum is bounded by

$$\begin{aligned} &\left|\sum_{k=1}^{\infty}\begin{pmatrix} -\frac{z}{2} \\ k \end{pmatrix}\sum_{n=1}^{\infty}\sum_{m=p}^{\infty}\frac{q^{-m}(C_{n,m})^k}{(\frac{1}{4}n^2+Q^2q^{-1}q^{-m})^{\frac{z+2k}{2}}}\right| \\ &\quad\leq \sum_{k=1}^{\infty}\left|\begin{pmatrix} -\frac{z}{2} \\ k \end{pmatrix}\right| C^k \sum_{n=1}^{\infty}\sum_{m=p}^{\infty}\frac{q^{-m}}{(\frac{1}{4}n^2+Q^2q^{-1}q^{-m})^{\frac{\mathrm{Re}(z)+2k}{2}}} \\ &\quad= \sum_{k=1}^{\infty}\left|\begin{pmatrix} -\frac{z}{2} \\ k \end{pmatrix}\right| C^k h(\mathrm{Re}(z)+2k). \end{aligned}$$

To estimate this sum of functions, we infer from Lemma 4.2 that there exists a positive function M which is defined for $\mathrm{Re}(z) > 3$ and such that

$$|h(z)| \leq M(z)\frac{e^{-\mathrm{Re}(z)rp/2}}{y^{\mathrm{Re}(z)}} = M(z)(q^{\frac{1}{2}(p+1)}Q^{-1})^{\mathrm{Re}(z)}.$$

Hence

$$\left|\sum_{k=1}^{\infty}\begin{pmatrix} -\frac{z}{2} \\ k \end{pmatrix}\sum_{n=1}^{\infty}\sum_{m=p}^{\infty}\frac{q^{-m}(C_{n,m})^k}{(\frac{1}{4}n^2+Q^2q^{-1}q^{-m})^{\frac{z+2k}{2}}}\right| \leq \sum_{k=1}^{\infty}\left|\begin{pmatrix} -\frac{z}{2} \\ k \end{pmatrix}\right| C^k M(z+2k)(q^{\frac{1}{2}(p+1)}Q^{-1})^{\mathrm{Re}(z)+2k}.$$

Recall that p was chosen such that $q^{-p} > qQ^{-2}C$. Also the function $z \mapsto M(z)$ is uniformly bounded for $\mathrm{Re}(z) \geq 4$. Hence, for all z with $\mathrm{Re}(z) \geq 2$, the function $k \mapsto M(z+2k)$ is uniformly bounded in k, by $\mathbf{M}$ say. It thus follows that the sum

$$\sum_{k=1}^{\infty}\left|\begin{pmatrix} -\frac{z}{2} \\ k \end{pmatrix}\right| C^k M(z+2k)(q^{\frac{1}{2}(p+1)}Q^{-1})^{\mathrm{Re}(z)+2k} \leq \mathbf{M}\sum_{k=1}^{\infty}\left|\begin{pmatrix} -\frac{z}{2} \\ k \end{pmatrix}\right| (q^{p+1}Q^{-2}C)^k$$

converges for $\mathrm{Re}(z) > 2$, by comparing with the binomial expansion on the right hand side. The convergence is again uniform on compacta, so invoking Weierstrass' convergence theorem we conclude that $f(z) - h(z)$ is holomorphic for $\mathrm{Re}(z) > 2$. Hence there exists a function *holo* which is defined and holomorphic for $\mathrm{Re}(z) > 2$ such that

$$f(z) = \frac{\sqrt{\pi}}{(q^{-\frac{1}{2}}Q)^{z-1}}\frac{\Gamma(\frac{z-1}{2})}{\Gamma(\frac{z}{2})}\frac{q^{p(z-3)/2}}{1-q^{(z-3)/2}} + holo(z)$$

So we see $f(z)$ is holomorphic for $\mathrm{Re}(z) > 3$, meromorphic for $\mathrm{Re}(z) > 2$ and has a a simple pole at $z = 3$ with residue $4qQ^{-2}/\ln(q^{-1})$. □

5 An analogue of a spectral triple

We now introduce an analogue of a spectral triple over $\mathcal{A}$. Let $\mathcal{H} := \mathcal{H}_h \oplus \mathcal{H}_h$ be the Hilbert space given by two copies of the GNS space $\mathcal{H}_h = L^2(A, h)$. We define a grading on $\mathcal{H}$ by $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For any operator ω on $\mathcal{H}$ we abbreviate

$$\omega^+ := \frac{1+\Gamma}{2}\omega\frac{1+\Gamma}{2}, \quad \omega^- := \frac{1-\Gamma}{2}\omega\frac{1-\Gamma}{2}. \tag{5.1}$$

The algebra $\mathcal{A}$ is represented on $\mathcal{H}$ by

$$\alpha \mapsto \begin{pmatrix} \pi_h(\alpha) & 0 \\ 0 & \pi_h(\alpha) \end{pmatrix}$$

for $\alpha \in \mathcal{A}$. Here π_h denotes the GNS representation by left multiplication on each copy of the space. In the sequel we will omit the symbol π_h. We now introduce some unbounded operators and projections

$$\hat{\Delta}_R = \begin{pmatrix} \Delta_R & 0 \\ 0 & \Delta_R \end{pmatrix} \qquad \hat{\Delta}_L = \begin{pmatrix} q^{-1}\Delta_L & 0 \\ 0 & q\Delta_L \end{pmatrix} \qquad \Psi_n = \begin{pmatrix} \Phi_{n+1} & 0 \\ 0 & \Phi_{n-1} \end{pmatrix}$$

on $\mathcal{A} \oplus \mathcal{A} \subset \mathcal{H}$ and use them to define (on the same domain)

$$\mathcal{D} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \Psi_n \begin{pmatrix} n & 0 \\ 0 & -n \end{pmatrix} + \hat{\Delta}_L^{\frac{1}{2}} \begin{pmatrix} 0 & \partial_e \\ \partial_f & 0 \end{pmatrix}.$$

We will see in the following lemma that the commutators $[\mathcal{D}, \alpha]$ of $\mathcal{D}$ with algebra elements are not necessarily bounded, yet unbounded in a very controlled manner. Even though $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ thus fails to be a spectral triple, we will still be able to construct an analytic expression for a residue Hochschild cocycle from the commutators.

Lemma 5.1. *The triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has the following properties:*

1. *The unbounded operator $\mathcal{D}$ is essentially self-adjoint.*
2. *The commutator $[\mathcal{D}, \alpha]$ is given by $\tilde{S}(\alpha) + \tilde{T}(\alpha)\hat{\Delta}_L$, where the linear maps $\tilde{S}, \tilde{T} \colon \mathcal{A} \to B(\mathcal{H})$ are given by*

$$\tilde{S}(\alpha) = \partial_H(\alpha)\Gamma \qquad \tilde{T}(\alpha) = \begin{pmatrix} 0 & q^{-\frac{1}{2}}\partial_e(\sigma_L^{-\frac{1}{2}}(\alpha)) \\ q^{\frac{1}{2}}\partial_f(\sigma_L^{-\frac{1}{2}}(\alpha)) & 0 \end{pmatrix}.$$

Proof. First we recall from Section 4 the numbers

$$\lambda_{l,n} := \sqrt{\left(\frac{n}{2}\right)^2 + q^n\left(\left[l+\frac{1}{2}\right]_q^2 - \left[\frac{n}{2}\right]_q^2\right)}, \tag{5.2}$$

where $l \in \frac{1}{2}\mathbb{N}_0$ and $-(2l+1) \le n \le (2l+1)$. Also recall $I_l := \{-l, -l+1, \dots, l-1, l\}$. Then the set

$$\left\{ \begin{pmatrix} 0 \\ t^l_{i,l} \end{pmatrix}, \begin{pmatrix} t^l_{i,-l} \\ 0 \end{pmatrix}, \begin{pmatrix} t^l_{i,j} \\ C^l_{j,\pm}\, t^l_{i,j-1} \end{pmatrix} : l \in \frac{1}{2}\mathbb{N}_0,\ i \in I_l,\ j \in I_l \backslash \{-l\} \right\},$$

$$where\ \ C^l_{j,\pm} = \frac{\pm\lambda_{l,2j-1} - (j - \frac{1}{2})}{q^{j-\frac{1}{2}} \sqrt{\left[l + \frac{1}{2}\right]_q^2 - \left[j - \frac{1}{2}\right]_q^2}}$$

is an orthogonal basis for $\mathcal{H}$ comprised of eigenvectors of $\mathcal{D}$. The corresponding eigenvalues are $-(l + \frac{1}{2}), -(l + \frac{1}{2})$ and $\pm\lambda_{l,2j-1}$ respectively. This spectral representation establishes that $\mathcal{D}$ is essentially self-adjoint.

Next, the commutator of $\mathcal{D}$ with a homogeneous algebra element $\alpha = \Phi_p(\alpha)$, for some $p \in \mathbb{Z}$, is computed directly. It is sufficient to consider just this case, because $\mathcal{A}$ consists of finite linear combinations of homogeneous elements (the generators are homogeneous). For such an element α we have

$$\begin{aligned}
[\mathcal{D}, \alpha] = {} & \frac{1}{2} \sum_{n=-\infty}^{\infty} \Psi_n \alpha \begin{pmatrix} n & 0 \\ 0 & -n \end{pmatrix} + \hat{\Delta}_L^{\frac{1}{2}} \begin{pmatrix} 0 & \partial_e \\ \partial_f & 0 \end{pmatrix} \alpha \\
& - \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha \Psi_n \begin{pmatrix} n & 0 \\ 0 & -n \end{pmatrix} - \alpha \hat{\Delta}_L^{\frac{1}{2}} \begin{pmatrix} 0 & \partial_e \\ \partial_f & 0 \end{pmatrix}.
\end{aligned}$$

It follows from the definition of the projections Φ_n, now regarded as a linear operator on $\mathcal{H}_h$, that $\alpha\Phi_n = \Phi_{n+p}\alpha$ for any $n \in \mathbb{Z}$. Using this, together with the definition of the derivations ∂_e and ∂_f in Equation 2.1, the commutator simplifies to

$$\begin{aligned}
[\mathcal{D}, \alpha] = {} & \frac{1}{2} \alpha \sum_{n=-\infty}^{\infty} \Psi_n \left(\begin{pmatrix} n+p & 0 \\ 0 & -n-p \end{pmatrix} - \begin{pmatrix} n & 0 \\ 0 & -n \end{pmatrix} \right) \\
& + \hat{\Delta}_L^{\frac{1}{2}} \begin{pmatrix} 0 & (\partial_e(\alpha)\Delta_L^{\frac{1}{2}} + \sigma_L^{\frac{1}{2}}(\alpha)\partial_e) \\ (\partial_f(\alpha)\Delta_L^{\frac{1}{2}} + \sigma_L^{\frac{1}{2}}(\alpha)\partial_f) & 0 \end{pmatrix} - \alpha \hat{\Delta}_L^{\frac{1}{2}} \begin{pmatrix} 0 & \partial_e \\ \partial_f & 0 \end{pmatrix}.
\end{aligned}$$

Since $\sigma_L^{\frac{1}{2}}(\alpha) = \hat{\Delta}_L^{-\frac{1}{2}} \alpha \hat{\Delta}_L^{\frac{1}{2}}$ as operators on $\mathcal{A} \oplus \mathcal{A} \subset \mathcal{H}$, the last expression for the commutator simplifies to

$$\hat{\Delta}_L^{\frac{1}{2}} \begin{pmatrix} 0 & \sigma_L^{\frac{1}{2}}(\alpha)\partial_e \\ \sigma_L^{\frac{1}{2}}(\alpha)\partial_f & 0 \end{pmatrix} = \alpha \hat{\Delta}_L^{\frac{1}{2}} \begin{pmatrix} 0 & \partial_e \\ \partial_f & 0 \end{pmatrix},$$

and hence

$$\begin{aligned}
[\mathcal{D}, \alpha] & = \frac{p}{2} \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \hat{\Delta}_L^{\frac{1}{2}} \begin{pmatrix} 0 & \partial_e(\alpha)\Delta_L^{\frac{1}{2}} \\ \partial_f(\alpha)\Delta_L^{\frac{1}{2}} & 0 \end{pmatrix} \\
& = \partial_H(\alpha)\Gamma + \begin{pmatrix} 0 & q^{-\frac{1}{2}} \partial_e(\sigma_L^{-\frac{1}{2}}(\alpha)) \\ q^{\frac{1}{2}} \partial_f(\sigma_L^{-\frac{1}{2}}(\alpha)) & 0 \end{pmatrix} \hat{\Delta}_L.
\end{aligned}$$

□

6 The residue Hochschild cocycle

The main step in the definition of the residue Hochschild cocycle is the construction of a functional that plays the role of an integral. In the situations considered in the literature thus far, [C, BeF, GVF, CNNR, CPRS1, KW], functionals of the form

$$T \mapsto \tau(T(1+\mathcal{D}^2)^{-z/2})$$

were used, where $z \in \mathbb{C}$ and τ is a faithful normal semifinite trace, or at worst a weight, on a von Neumann algebra containing the algebra of interest. Often, the von Neumann algebra is just $\mathcal{B}(\mathcal{H})$, and the functional τ is the operator trace.
In this example, we need to apply our functional to products of commutators $[\mathcal{D}, \alpha] \sim \hat{\Delta}_L$ with $\alpha \in \mathcal{A}$, so it has to be defined on an algebra of unbounded operators. We will deal with this using a cutoff that is defined by the projections

$$L_k := \tilde{L}_k \oplus \tilde{L}_k, \quad \tilde{L}_k(t^l_{i,j}) := \begin{cases} t^l_{i,j} & l \leq k \\ 0 & \text{otherwise} \end{cases}$$

and

$$P_1 = \sum_{n=0}^{\infty} \Psi_n \qquad P_2 \begin{pmatrix} t^l_{i,j} \\ 0 \end{pmatrix} = (1-\delta_{j,-l}) \begin{pmatrix} t^l_{i,j} \\ 0 \end{pmatrix} \qquad P_2 \begin{pmatrix} 0 \\ t^l_{i,j} \end{pmatrix} = (1-\delta_{j,l}) \begin{pmatrix} 0 \\ t^l_{i,j} \end{pmatrix}.$$

Observe P_2 is the projection onto $\left(\ker \begin{pmatrix} 0 & \partial_e \\ \partial_f & 0 \end{pmatrix}\right)^{\perp}$, and that the projections L_k converge strongly to the identity in $\mathcal{B}(\mathcal{H})$.
For $s \in \mathbb{R}^+$ we now define a functional Υ_s on positive operators $\omega \in \mathcal{B}(\mathcal{H})$ in the following way:

$$\Upsilon_s(\omega) := \sup_{k \in \mathbb{N}} \mathrm{Tr}\left(P_1 P_2 L_k (1+\mathcal{D}^2)^{-s/4} \hat{\Delta}_F^{-\frac{1}{2}} \omega \hat{\Delta}_F^{-\frac{1}{2}} (1+\mathcal{D}^2)^{-s/4} P_1 P_2 L_k\right), \quad \hat{\Delta}_F = \hat{\Delta}_R \hat{\Delta}_L$$

where Tr is the operator trace on $\mathcal{B}(\mathcal{H})$. This expression continues to make sense for possibly unbounded positive operators defined on and preserving the subspace $\mathcal{A} \oplus \mathcal{A} \subset \mathcal{H}$.

Lemma 6.1. *For each $s \in \mathbb{R}_+$ the functional Υ_s is positive and normal on $\mathcal{B}(\mathcal{H})_+$. It is faithful and semifinite on $P_1 P_2 \mathcal{B}(\mathcal{H})_+ P_1 P_2$.*

Proof. We will compute the operator trace using the Peter-Weyl basis $\left\{ \begin{pmatrix} t^l_{i,j} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t^l_{i,j} \end{pmatrix} \right\}$ for $\mathcal{H}$.
The operators $(1+\mathcal{D}^2)$, $\hat{\Delta}_F$, P_1, P_2 and L_k are all positive and diagonal in this basis. By using the definition of the operator trace, the value of the operators $\hat{\Delta}_F^{-1}$ and $(1+\mathcal{D}^2)^{-s/4}$ on this basis, and the symmetry property for self-adjoint operators, we compute $\Upsilon_s(\omega)$ for $\omega \in \mathcal{B}(\mathcal{H})_+$ (or even $\omega \geq 0$ and affiliated to $\mathcal{B}(\mathcal{H})$) by

$$\mathrm{Tr}\left(P_1P_2L_k(1+\mathcal{D}^2)^{-s/4}\hat{\Delta}_F^{-\frac{1}{2}}\omega\hat{\Delta}_F^{-\frac{1}{2}}(1+\mathcal{D}^2)^{-s/4}P_1P_2L_k\right)=$$
$$=\sum_{2l=0}^{\infty}\sum_{i=-l}^{l}\sum_{j=-l}^{l}\frac{q^{-2i-(2j-1)}}{(1+\lambda_{l,2j-1}^2)^{s/2}}\frac{\left\langle P_1^+P_2^+L_kt_{i,j}^l,\omega^+P_1^+P_2^+L_kt_{i,j}^l\right\rangle}{\left\langle t_{i,j}^l,t_{i,j}^l\right\rangle}$$
$$+\sum_{2l=0}^{\infty}\sum_{i=-l}^{l}\sum_{j=-l}^{l}\frac{q^{-2i-(2j+1)}}{(1+\lambda_{l,2j+1}^2)^{s/2}}\frac{\left\langle P_1^-P_2^-L_kt_{i,j}^l,\omega^-P_1^-P_2^-L_kt_{i,j}^l\right\rangle}{\left\langle t_{i,j}^l,t_{i,j}^l\right\rangle},$$

where ω^+ and ω^- are as in Equation (5.1). Now,

$$P_1^+P_2^+L_kt_{i,j}^l=\begin{cases}t_{i,j}^l & \frac{1}{2}\le j\le l,\ \frac{1}{2}\le l\le k\\ 0 & \text{otherwise}\end{cases}$$
$$P_1^-P_2^-L_kt_{i,j}^l=\begin{cases}t_{i,j}^l & -\frac{1}{2}\le j\le l-1,\ \frac{1}{2}\le l\le k\\ 0 & \text{otherwise.}\end{cases}$$

So if we set $n=2j\pm1$ and recall the sets

$$\mathcal{J}_l:=\begin{cases}\{0,2,\dots,2l-1\} & l\in(\mathbb{N}_0+\frac{1}{2})\\ \{1,3,\dots,2l-1\} & l\in\mathbb{N}\end{cases}$$

we may express the trace as

$$\mathrm{Tr}\left(P_1P_2L_k(1+\mathcal{D}^2)^{-s/4}\hat{\Delta}_F^{-\frac{1}{2}}\omega\hat{\Delta}_F^{-\frac{1}{2}}(1+\mathcal{D}^2)^{-s/4}P_1P_2L_k\right)=$$
$$=\sum_{2l=1}^{2k}\sum_{i=-l}^{l}\sum_{n\in\mathcal{J}_l}\frac{q^{-2i-n}}{(1+\lambda_{l,n}^2)^{s/2}}\left(\frac{\left\langle t_{i,\frac{n+1}{2}}^l,\omega^+t_{i,\frac{n+1}{2}}^l\right\rangle}{\left\langle t_{i,\frac{n+1}{2}}^l,t_{i,\frac{n+1}{2}}^l\right\rangle}+\frac{\left\langle t_{i,\frac{n-1}{2}}^l,\omega^-t_{i,\frac{n-1}{2}}^l\right\rangle}{\left\langle t_{i,\frac{n-1}{2}}^l,t_{i,\frac{n-1}{2}}^l\right\rangle}\right). \tag{6.1}$$

This shows that Υ_s is a supremum of a sum of positive vector states and so automatically positive and normal. To see that it is faithful on $P_1P_2\mathcal{B}(\mathcal{H})_+P_1P_2$ we observe that the operator trace is faithful and that $P_1P_2\hat{\Delta}_F^{-1/2}(1+\mathcal{D}^2)^{-s/4}$ is injective on $P_1P_2\mathcal{H}$. The semifiniteness comes from the fact that finite rank operators are in the domain of Υ_s. □

We extend Υ_s to an unbounded positive normal linear functional on $\mathcal{B}(\mathcal{H})$ as usual. In fact, we extend it also to unbounded operators ω defined on and preserving $\mathcal{A}\oplus\mathcal{A}$ by decomposing $L_k\omega L_k$ for each k into a linear combination of positive bounded operators.
If for an operator ω (not necessarily bounded) the function $s\mapsto\Upsilon_s(\omega)$ has a meromorphic continuation to $\mathrm{Dom}_{3-\delta}$ for some $\delta>0$, then we define

$$\tau(\omega):=\mathrm{Res}_{z=3}\Upsilon_z(\omega).$$

Lemma 6.2. *The functional τ is defined on the positive operator c^*c, and $\tau(c^*c) = 0$. Indeed, for all $m \geq 1$,*

$$\tau\left(\begin{pmatrix} (c^*c)^m & 0 \\ 0 & 0 \end{pmatrix}\right) = \tau\left(\begin{pmatrix} 0 & 0 \\ 0 & (c^*c)^m \end{pmatrix}\right) = 0.$$

Proof. The action of the operator $c = c^+ + c^-$ may be described using the Clebsch-Gordan coefficients (see for example [DLSSV], [KS]): we have

$$c^+ t^l_{i,j} = c^{l+}_{ij} t^{l+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}} \qquad c^- t^l_{i,j} = c^{l-}_{ij} t^{l-\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2}},$$

where

$$c^{l+}_{ij} = q^{(i+j)/2} \frac{([l+i+1]_q [l-j+1]_q)^{1/2}}{[2l+1]_q}, \qquad c^{l-}_{ij} = -q^{(i+j)/2} \frac{([l-i]_q [l+j]_q)^{1/2}}{[2l+1]_q}.$$

Using this description of c to compute the action of c^*c, we find

$$\begin{aligned}
(c^*c) t^l_{i,j} = {} & q^{i+j-1} \left(\frac{[l+i+1]_q [l-j+1]_q}{[2l+1]_q [2l+2]_q} + \frac{[l-i]_q [l+j]_q}{[2l]_q [2l+1]_q} \right) t^l_{i,j} \\
& - q^{i+j-1} \left(\frac{([l+i+1]_q [l-i+1]_q [l+j+1]_q [l-j+1]_q)^{\frac{1}{2}}}{[2l+1]_q [2l+2]_q} t^{l+1}_{i,j} \right. \\
& \left. + \frac{([l+i]_q [l-i]_q [l+j]_q [l-j]_q)^{\frac{1}{2}}}{[2l]_q [2l+1]_q} t^{l-1}_{i,j} \right)
\end{aligned}$$

Let $\epsilon_k = Q(1-q^{2k})$, so that $[k]_q = q^{-k}\epsilon_k$. Then the above expression can be written as

$$\begin{aligned}
(c^*c) t^l_{i,j} = {} & q^{2l} \left(q^{2j} \frac{\epsilon_{l+i+1}\epsilon_{l-j+1}}{\epsilon_{2l+1}\epsilon_{2l+2}} + q^{2i} \frac{\epsilon_{l-i}\epsilon_{l+j}}{\epsilon_{2l}\epsilon_{2l+1}} \right) t^l_{i,j} \\
& - q^{2l+i+j} \left(\frac{(\epsilon_{l+i+1}\epsilon_{l-i+1}\epsilon_{l+j+1}\epsilon_{l-j+1})^{\frac{1}{2}}}{\epsilon_{2l+1}\epsilon_{2l+2}} t^{l+1}_{i,j} + \frac{(\epsilon_{l+i}\epsilon_{l-i}\epsilon_{l+j}\epsilon_{l-j})^{\frac{1}{2}}}{\epsilon_{2l}\epsilon_{2l+1}} t^{l-1}_{i,j} \right).
\end{aligned}$$

Define the scalars $C_1(l,i,j)$ and $C_2(l,i,j)$ to be

$$C_1(l,i,j) := \frac{\epsilon_{l+i+1}\epsilon_{l-j+1}}{\epsilon_{2l+1}\epsilon_{2l+2}} \qquad C_2(l,i,j) := \frac{\epsilon_{l-i}\epsilon_{l+j}}{\epsilon_{2l}\epsilon_{2l+1}}.$$

The definition of ϵ_k implies that C_1 and C_2 are uniformly bounded for all l, i, j appearing in the formula for $\Upsilon_z(c^*c)$.
As in the proof of Lemma 6.1 we compute for $z \in \mathbb{R}$

$$\begin{aligned}
& \mathrm{Tr}\left(P_1 P_2 L_k (1+\mathcal{D}^2)^{-z/4} \hat{\Delta}_F^{-\frac{1}{2}} \begin{pmatrix} c^*c & 0 \\ 0 & 0 \end{pmatrix} \hat{\Delta}_F^{-\frac{1}{2}} (1+\mathcal{D}^2)^{-z/4} P_1 P_2 L_k \right) \\
& \quad = \sum_{2l=1}^{2k} \sum_{i=-l}^{l} \sum_{n \in \mathcal{J}_l} \frac{q^{-2i-n}}{(1+\lambda^2_{l,n})^{z/2}} \left(q^{2l+n+1} C_1(l,i,\tfrac{n+1}{2}) + q^{2l+2i} C_2(l,i,\tfrac{n+1}{2}) \right),
\end{aligned}$$

$$\operatorname{Tr}\left(P_1P_2L_k(1+\mathcal{D}^2)^{-z/4}\hat{\Delta}_F^{-\frac{1}{2}}\begin{pmatrix}0&0\\0&c^*c\end{pmatrix}\hat{\Delta}_F^{-\frac{1}{2}}(1+\mathcal{D}^2)^{-z/4}P_1P_2L_k\right)$$
$$=\sum_{2l=1}^{2k}\sum_{i=-l}^{l}\sum_{n\in\mathcal{J}_l}\frac{q^{-2i-n}}{(1+\lambda_{l,n}^2)^{z/2}}\left(q^{2l+n+1}C_1(l,i,\tfrac{n-1}{2})+q^{2l+2i}C_2(l,i,\tfrac{n-1}{2})\right).$$

The uniform boundedness of C_1 and C_2, together with Lemma 4.1, demonstrate that the limits as $k\to\infty$ of the two sums above exist for $z>2$. Hence

$$z\mapsto\Upsilon_z\left(\begin{pmatrix}c^*c&0\\0&0\end{pmatrix}\right),\quad z\mapsto\Upsilon_z\left(\begin{pmatrix}0&0\\0&c^*c\end{pmatrix}\right)$$

are well-defined functions for $z>2$. Indeed the arguments of Lemma 4.1, together with the Weierstrass convergence theorem, show that these functions extend to holomorphic functions on Dom_2. In particular, these functions are holomorphic at $z=3$ and hence

$$\tau\left(\begin{pmatrix}c^*c&0\\0&0\end{pmatrix}\right)=\tau\left(\begin{pmatrix}0&0\\0&c^*c\end{pmatrix}\right)=0.$$

By linearity it follows that $\tau(c^*c)=0$ also. Using the normality of c, for any operator X we have the operator inequality

$$X^*(c^*c)^mX\le\|c^*c\|^{m-1}X^*c^*cX,$$

and so for $z>2$ real, we have $\Upsilon_z((c^*c)^m)\le\|c\|^{2m-2}\Upsilon_z(c^*c)$. Thus for $z>2$, the sum defining $\Upsilon_z((c^*c)^m)$ converges. Once more invoking the Weierstrass convergence theorem shows that $z\mapsto\Upsilon_z((c^*c)^m)$ extends to a holomorphic function for $\mathrm{Re}(z)>2$. Similar estimates now show that

$$\tau\left(\begin{pmatrix}(c^*c)^m&0\\0&0\end{pmatrix}\right)=\tau\left(\begin{pmatrix}0&0\\0&(c^*c)^m\end{pmatrix}\right)=0.\qquad\square$$

Theorem 6.3. *Let $\alpha\in\mathcal{A}$ and X,Y be any closed linear operators on $\mathcal{H}_h$ which are defined on and preserve $\mathcal{A}$. Then we have the following well-defined evaluations of τ:*

1. $\tau\left(\begin{pmatrix}0&X\\0&0\end{pmatrix}\right)=\tau\left(\begin{pmatrix}0&0\\Y&0\end{pmatrix}\right)=0$
2. $\tau(\alpha\Gamma)=0$
3. $\tau\left(\hat{\Delta}_L^2\begin{pmatrix}\alpha&0\\0&0\end{pmatrix}\right)=\tau\left(\hat{\Delta}_L^2\begin{pmatrix}0&0\\0&\alpha\end{pmatrix}\right)=R\int_{[1]}\alpha$

where $\int_{[1]}\colon\mathcal{A}\to\mathbb{C}$ is the functional defined in Lemma 3.2 and $R=4(q^{-1}-q)/\ln(q^{-1})$.

Proof. Throughout this proof we assume without loss of generality that any element of $\mathcal{A}$ is homogeneous with respect to both the left and right actions (that is $\sigma_L(\alpha)=q^p\alpha$, $\sigma_R(\alpha)=q^{p'}\alpha$ for some p,p'). This is because finite linear combinations of homogeneous elements span $\mathcal{A}$ (cf. Theorem 2.2). Indeed, if $\alpha\in\mathcal{A}$ is homogeneous of a non-zero degree for either the left or right action, then $\langle t^l_{i,j},\alpha\, t^l_{i,j}\rangle=0$ and so for any linear operator C that is diagonal in the Peter-Weyl basis, $\Upsilon_s(C\alpha)=0$ for all $s\in\mathbb{R}_+$. Hence, we need only consider those elements of $\mathcal{A}$ that are homogeneous of degree zero for the left and right actions. A convenient spanning set for these algebra elements is $\{1_{\mathcal{A}},(c^*c)^m\colon m\in\mathbb{N}\}$.

1. By definition $\Upsilon_s\left(\left(\begin{array}{cc} 0 & X \\ 0 & 0 \end{array}\right)\right) = 0$ for all $s > 0$, and similarly for $\left(\begin{array}{cc} 0 & 0 \\ Y & 0 \end{array}\right)$.
2. Lemma 6.2 has established that for all $m \geq 1$,

$$\tau\left(\left(\begin{array}{cc} (c^*c)^m & 0 \\ 0 & 0 \end{array}\right)\right) = \tau\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & (c^*c)^m \end{array}\right)\right) = 0.$$

By linearity we can extend this to conclude that $\tau((c^*c)^m\Gamma) = 0$. Finally, for z large and real we compute $\Upsilon_z(\Gamma)$ using the proof of Lemma 6.1. Now

$$\begin{aligned}\mathrm{Tr}&\left(P_1P_2L_k(1+\mathcal{D}^2)^{-z/4}\hat{\Delta}_F^{-\frac{1}{2}}\Gamma\hat{\Delta}_F^{-\frac{1}{2}}(1+\mathcal{D}^2)^{-z/4}P_1P_2L_k\right)\\ &= \sum_{2l=1}^{2k}\sum_{i=-l}^{l}\sum_{n\in\mathcal{J}_l}\frac{q^{-2i-n}}{(1+\lambda_{l,n}^2)^{z/2}} - \sum_{2l=1}^{2k}\sum_{i=-l}^{l}\sum_{n\in\mathcal{J}_l}\frac{q^{-2i-n}}{(1+\lambda_{l,n}^2)^{z/2}},\end{aligned}$$

and for each k the summands above are finite and hence subtract to give zero. Hence $\Upsilon_z(\Gamma) = 0$ for all z and so $\tau(\Gamma) = 0$.
3. For z large and real, the evaluation of Υ_z as sums of positive real numbers (as in the proof of Lemma 6.1) implies the numerical inequality

$$\Upsilon_z(\hat{\Delta}_L^2(c^*c)^m) \leq \Upsilon_z((c^*c)^m).$$

This is because the introduction of $\hat{\Delta}_L^2$ multiplies each summand by $q^{2n} \leq 1$ (cf. Equation (6.1)). Lemma 6.2 demonstrates that $\Upsilon_z((c^*c)^m)$ extends to a function that is holomorphic in a neighbourhood of $z = 3$, and together with the Weierstrass convergence theorem the result follows.
Finally we analyse $\Upsilon_z\left(\hat{\Delta}_L^2\left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right)\right)$ and $\Upsilon_z\left(\hat{\Delta}_L^2\left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right)\right)$. Again using the proof of Lemma 6.1 we find

$$\begin{aligned}\mathrm{Tr}&\left(P_1P_2L_k(1+\mathcal{D}^2)^{-z/4}\hat{\Delta}_F^{-\frac{1}{2}}\hat{\Delta}_L^2\left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right)\hat{\Delta}_F^{-\frac{1}{2}}(1+\mathcal{D}^2)^{-z/4}P_1P_2L_k\right)\\ &= \mathrm{Tr}\left(P_1P_2L_k(1+\mathcal{D}^2)^{-z/4}\hat{\Delta}_F^{-\frac{1}{2}}\hat{\Delta}_L^2\left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right)\hat{\Delta}_F^{-\frac{1}{2}}(1+\mathcal{D}^2)^{-z/4}P_1P_2L_k\right)\\ &= \sum_{2l=1}^{2k}\sum_{i=-l}^{l}\sum_{n\in\mathcal{J}_l}\frac{q^{-2i+n}}{(1+\lambda_{l,n}^2)^{z/2}}\\ &\qquad = Qq^{-1}\sum_{2l=1}^{2k}\sum_{n\in\mathcal{J}_l}\frac{q^{n-2l}}{(1+\lambda_{l,n}^2)^{z/2}} - Qq\sum_{2l=1}^{2k}\sum_{n\in\mathcal{J}_l}\frac{q^{n+2l}}{(1+\lambda_{l,n}^2)^{z/2}}\end{aligned}$$

For z real, the sum $\sum_{2l=1}^{2k}\sum_{n\in\mathcal{J}_l} q^{n+2l}/(1+\lambda_{l,n}^2)^{z/2}$ is bounded above by $f_2(z)$ from Lemma 4.1 for all k. By the Weierstrass convergence theorem we conclude that as $k \to \infty$, this sum converges to a function with a holomorphic extension about $z = 3$. Next, when considering the sum $\sum_{2l=1}^{2k}\sum_{n\in\mathcal{J}_l} q^{n-2l}/(1+\lambda_{l,n}^2)^{z/2}$, observe by rearranging the order of summation

$$\sum_{2l=1}^{2k}\sum_{n\in\mathcal{J}_l} \to \sum_{n=0}^{2k}\sum_{l=(n+1)/2}^{k},$$

that Lemma 4.3 proves that the sum has a limit as $k \to \infty$ and the corresponding function of z extends to a meromorphic function with a simple pole at $z = 3$. The residue at $z = 3$ is $4qQ^{-2}/\ln(q^{-1})$ and from the definition of τ we conclude that for $R = 4(q^{-1} - q)/\ln(q^{-1})$,

$$\tau\left(\hat{\Delta}_L^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \tau\left(\hat{\Delta}_L^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = R.$$

Finally, we compare the definition of $R\int_{[1]}$ in Lemma 3.2 to the evaluation of τ on $\mathcal{A}$ derived here and observe that they agree on $\mathcal{A}$. □

Lemma 6.4. *Given any matrix $M \in \mathcal{M}_2(\mathcal{A})$ and any $\alpha \in \mathcal{A}$ then $\tau(M\hat{\Delta}_L^2\alpha) = \tau(\vartheta^{-1}(\alpha)M\hat{\Delta}_L^2)$.*

Proof. From Lemma 3.2, the linear functional $\int_{[1]}$ is a $\sigma_L^2 \circ \vartheta^{-1}$-twisted trace. That is, given any $\alpha, \beta \in \mathcal{A}$

$$\int_{[1]} \alpha\beta = \int_{[1]} \sigma_L^2(\vartheta^{-1}(\beta))\alpha.$$

Now we separate the matrix $M = M_d + M_o$ into diagonal and off-diagonal matrices respectively. Then by Theorem 6.3, $\tau(M_d\hat{\Delta}_L^2\alpha)$ and $\tau(M_o\hat{\Delta}_L^2\alpha)$ are both well-defined, so by linearity

$$\tau(M\hat{\Delta}_L^2\alpha) = \tau(M_d\hat{\Delta}_L^2\alpha) + \tau(M_o\hat{\Delta}_L^2\alpha) = \tau(M_d\hat{\Delta}_L^2\alpha) + 0.$$

Since M_d is diagonal, we may write

$$M_d\hat{\Delta}_L^2 = \hat{\Delta}_L^2\sigma_L^2(M_d)$$

where σ_L acts componentwise on the matrix. Using the value of $\tau(\hat{\Delta}_L^2\sigma_L^2(M_d)\alpha)$ from Theorem 6.3, we have

$$\begin{aligned}\tau(M\hat{\Delta}_L^2\alpha) = \tau(\hat{\Delta}_L^2\sigma_L^2(M_d)\alpha) &= R\int_{[1]} \sigma_L^2(M_d^+)\alpha + R\int_{[1]} \sigma_L^2(M_d^-)\alpha \\ &= R\int_{[1]} \sigma_L^2(\vartheta^{-1}(\alpha))\sigma_L^2(M_d^+) + R\int_{[1]} \sigma_L^2(\vartheta^{-1}(\alpha))\sigma_L^2(M_d^-),\end{aligned}$$

by the twisted trace property of $\int_{[1]}$. Recombining these two terms yields

$$\tau(\hat{\Delta}_L^2\sigma_L^2(M_d)\alpha) = \tau(\hat{\Delta}_L^2\sigma_L^2(\vartheta^{-1}(\alpha))\sigma_L^2(M_d)) = \tau(\vartheta^{-1}(\alpha)\hat{\Delta}_L^2\sigma_L^2(M_d)) = \tau(\vartheta^{-1}(\alpha)M_d\hat{\Delta}_L^2).$$

Now, $\tau(\vartheta^{-1}(\alpha)M_o\hat{\Delta}_L^2)$ is well defined and has value zero, so we can write

$$\tau(M\hat{\Delta}_L^2\alpha) = \tau(\vartheta^{-1}(\alpha)M_d\hat{\Delta}_L^2) + \tau(\vartheta^{-1}(\alpha)M_o\hat{\Delta}_L^2) = \tau(\vartheta^{-1}(\alpha)M\hat{\Delta}_L^2).$$ □

Theorem 6.5. *Given any $a_0, \ldots, a_3 \in \mathcal{A}$, the map $\phi_{\text{res}} : a_0, \ldots, a_3 \mapsto \tau(a_0[\mathcal{D}, a_1][\mathcal{D}, a_2][\mathcal{D}, a_3])$ is a ϑ^{-1}-twisted Hochschild 3-cocycle, whose cohomology class is non-trivial. The cocycle ϕ_{res} has non-zero pairing with the ϑ^{-1}-twisted 3-cycle dvol defined in* (3.1)*, giving*

$$\langle \phi_{\text{res}}, dvol \rangle = 3R(q^{-1} + q) = 4!\,\frac{q^{-1} + q}{2}\,\frac{q^{-1} - q}{\ln(q^{-1})}.$$

The cocycle may be written as

$$\phi_{\rm res} = q^2 R(\varphi + \varphi_{213} + \varphi_{231}) + R(\varphi_{132} + \varphi_{312} + \varphi_{321})$$

where φ and φ_{ijk} are the cocycles described in Lemma 3.2 and Corollary 3.4.

Proof. First consider $\pi_{\mathcal{D}}(a_0, a_1, a_2, a_3) = a_0[\mathcal{D}, a_1][\mathcal{D}, a_2][\mathcal{D}, a_3]$ as an unbounded operator on $\mathcal{A}\oplus\mathcal{A} \subset \mathcal{H}$. Using the equality $[\mathcal{D}, \alpha] = \tilde{S}(\alpha) + \tilde{T}(\alpha)\hat{\Delta}_L$, we see that $\pi_{\mathcal{D}}(a_0, a_1, a_2, a_3)$ can be expanded into 8 terms. Recall that by Theorem 6.3 the functional τ vanishes on off-diagonal operators. Four of the eight terms in the expansion of $\pi_{\mathcal{D}}(a_0, a_1, a_2, a_3)$ are off-diagonal since, for all $\alpha \in \mathcal{A}$, $\tilde{S}(\alpha)$ is diagonal and $\tilde{T}(\alpha)$ is off-diagonal. Thus

$$\tau\Big(a_0\Big(\tilde{T}(a_1)\hat{\Delta}_L\tilde{S}(a_2)\tilde{S}(a_3) + \tilde{S}(a_1)\tilde{T}(a_2)\hat{\Delta}_L\tilde{S}(a_3) \\ + \tilde{S}(a_1)\tilde{S}(a_2)\tilde{T}(a_3)\hat{\Delta}_L + \tilde{T}(a_1)\hat{\Delta}_L\tilde{T}(a_2)\hat{\Delta}_L\tilde{T}(a_3)\hat{\Delta}_L\Big)\Big) = 0.$$

Therefore, $\phi_{\rm res}(a_0, a_1, a_2, a_3)$ reduces to

$$\phi_{\rm res}(a_0, a_1, a_2, a_3) = \tau\Big(a_0\Big(\tilde{S}(a_1)\tilde{S}(a_2)\tilde{S}(a_3) + \tilde{S}(a_1)\tilde{T}(a_2)\hat{\Delta}_L\tilde{T}(a_3)\hat{\Delta}_L \\ + \tilde{T}(a_1)\hat{\Delta}_L\tilde{S}(a_2)\tilde{T}(a_3)\hat{\Delta}_L + \tilde{T}(a_1)\hat{\Delta}_L\tilde{T}(a_2)\hat{\Delta}_L\tilde{S}(a_3)\Big)\Big). \tag{6.2}$$

From Lemma 5.1 it follows that

$$a_0\tilde{S}(a_1)\tilde{S}(a_2)\tilde{S}(a_3) = a_0\partial_H(a_1)\partial_H(a_2)\partial_H(a_3)\Gamma$$

and recall that from Theorem 6.3, $\tau(\alpha\Gamma) = 0$ for all $\alpha \in \mathcal{A}$. Since $a_0\partial_H(a_1)\partial_H(a_2)\partial_H(a_3) \in \mathcal{A}$ we have

$$\tau(a_0\tilde{S}(a_1)\tilde{S}(a_2)\tilde{S}(a_3)) = 0.$$

We now move all the $\hat{\Delta}_L$'s to the right in the remaining terms in Equation (6.2). For $\alpha \in \mathcal{A}$, we use $\hat{\Delta}_L\tilde{S}(\alpha) = \tilde{S}(\sigma_L^{-1}(\alpha))\hat{\Delta}_L$, and

$$\begin{aligned}
\hat{\Delta}_L\tilde{T}(\alpha) &= \begin{pmatrix} 0 & q^{-1}\Delta_L q^{-\frac{1}{2}}\partial_e(\sigma_L^{-\frac{1}{2}}(\alpha)) \\ q\Delta_L q^{\frac{1}{2}}\partial_f(\sigma_L^{-\frac{1}{2}}(\alpha)) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & q^{-2}q^{-\frac{1}{2}}\sigma_L^{-1}(\partial_e(\sigma_L^{-\frac{1}{2}}(\alpha))) \\ q^2 q^{\frac{1}{2}}\sigma_L^{-1}(\partial_f(\sigma_L^{-\frac{1}{2}}(\alpha))) & 0 \end{pmatrix}\hat{\Delta}_L \\
&= \begin{pmatrix} 0 & q^{-\frac{1}{2}}\partial_e(\sigma_L^{-\frac{3}{2}}(\alpha)) \\ q^{\frac{1}{2}}\partial_f(\sigma_L^{-\frac{3}{2}}(\alpha)) & 0 \end{pmatrix}\hat{\Delta}_L \\
&= \tilde{T}(\sigma_L^{-1}(\alpha))\hat{\Delta}_L.
\end{aligned}$$

This yields

$$\phi_{\rm res}(a_0, a_1, a_2, a_3) = \tau\Big(a_0\Big(\tilde{S}(a_1)\tilde{T}(a_2)\tilde{T}(\sigma_L^{-1}(a_3)) \\ + \tilde{T}(a_1)\tilde{S}(\sigma_L^{-1}(a_2))\tilde{T}(\sigma_L^{-1}(a_3)) + \tilde{T}(a_1)\tilde{T}(\sigma_L^{-1}(a_2))\tilde{S}(\sigma_L^{-2}(a_3))\Big)\hat{\Delta}_L^2\Big).$$

In this form Theorem 6.3 tells us that $\phi_{\rm res}$ is a well defined, multilinear functional on $\mathcal{A}^{\otimes 4}$. In order to demonstrate that this cochain is indeed a twisted Hochschild cocycle, it remains only to show that the boundary operator maps the cochain to zero. This result follows from the Leibniz property of the commutators together with Lemma 6.4. Explicitly,

$$\begin{aligned}(b_3^{\vartheta^{-1}}\phi_{\rm res})(a_0,\dots,a_4) &= \tau(a_0a_1[\mathcal{D},a_2][\mathcal{D},a_3][\mathcal{D},a_4]) - \tau(a_0[\mathcal{D},a_1a_2][\mathcal{D},a_3][\mathcal{D},a_4])\\ &\quad + \tau(a_0[\mathcal{D},a_1][\mathcal{D},a_2a_3][\mathcal{D},a_4]) - \tau(a_0[\mathcal{D},a_1][\mathcal{D},a_2][\mathcal{D},a_3a_4]) + \tau(\vartheta^{-1}(a_4)a_0[\mathcal{D},a_1][\mathcal{D},a_2][\mathcal{D},a_3])\\ &= -\tau(a_0[\mathcal{D},a_1][\mathcal{D},a_2][\mathcal{D},a_3]a_4) + \tau(\vartheta^{-1}(a_4)a_0[\mathcal{D},a_1][\mathcal{D},a_2][\mathcal{D},a_3]) = 0,\end{aligned}$$

where the last equality follows from Lemma 6.4. In order to identify $\phi_{\rm res}$, we use Lemma 5.1 to write, for $a_0,\dots,a_3\in\mathcal{A}$,

$$\begin{aligned}a_0\Big(&\tilde S(a_1)\tilde T(a_2)\tilde T(\sigma_L^{-1}(a_3)) + \tilde T(a_1)\tilde S(\sigma_L^{-1}(a_2))\tilde T(\sigma_L^{-1}(a_3))\\ &+\tilde T(a_1)\tilde T(\sigma_L^{-1}(a_2))\tilde S(\sigma_L^{-2}(a_3))\Big) = \begin{pmatrix}\pi_1(a_0,\dots,a_3) & 0\\ 0 & \pi_2(a_0,\dots,a_3)\end{pmatrix}\end{aligned}$$

for some multi-linear maps $\pi_1,\pi_2\colon\mathcal{A}^{\otimes 4}\to\mathcal{A}$. Again using Lemma 5.1, we have

$$\begin{aligned}\pi_1(a_0,\dots,a_3) &= a_0\partial_H(a_1)\partial_e(\sigma_L^{-\frac{1}{2}}(a_2))\partial_f(\sigma_L^{-\frac{3}{2}}(a_3)) - a_0\partial_e(\sigma_L^{-\frac{1}{2}}(a_1))\partial_H(\sigma_L^{-1}(a_2))\partial_f(\sigma_L^{-\frac{3}{2}}(a_3))\\ &\quad + a_0\partial_e(\sigma_L^{-\frac{1}{2}}(a_1))\partial_f(\sigma_L^{-\frac{3}{2}}(a_2))\partial_H(\sigma_L^{-2}(a_3)),\end{aligned}\tag{6.3}$$

$$\begin{aligned}\pi_2(a_0,\dots,a_3) &= -a_0\partial_H(a_1)\partial_f(\sigma_L^{-\frac{1}{2}}(a_2))\partial_e(\sigma_L^{-\frac{3}{2}}(a_3)) + a_0\partial_f(\sigma_L^{-\frac{1}{2}}(a_1))\partial_H(\sigma_L^{-1}(a_2))\partial_e(\sigma_L^{-\frac{3}{2}}(a_3))\\ &\quad - a_0\partial_f(\sigma_L^{-\frac{1}{2}}(a_1))\partial_e(\sigma_L^{-\frac{3}{2}}(a_2))\partial_H(\sigma_L^{-2}(a_3)).\end{aligned}\tag{6.4}$$

Then by Theorem 6.3, and the σ_L invariance of $\int_{[1]}$, we have

$$\phi_{\rm res}(a_0,a_1,a_2,a_3) = R\int_{[1]}\pi_1(a_0,\dots,a_3) + R\int_{[1]}\pi_2(a_0,\dots,a_3).\tag{6.5}$$

Comparing Equations (6.3), (6.4), (6.5) with the expressions for the cocycles identified in Lemma 3.2 and Corollary 3.4 we find

$$\phi_{\rm res} = q^2R(\varphi+\varphi_{213}+\varphi_{231}) + R(\varphi_{132}+\varphi_{312}+\varphi_{321}).$$

The evaluation of this cocycle on the cycle *dvol* (see Equation (3.1)) is a straightforward computation using the explicit expressions obtained. The result is

$$\langle\phi_{\rm res}, dvol\rangle = 3R(q^{-1}+q).$$

□

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