

# A RESIDUE FORMULA FOR THE FUNDAMENTAL HOCHSCHILD CLASS ON THE PODLEŚ SPHERE

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ABSTRACT. The fundamental Hochschild cohomology class of the standard Podleś quantum sphere is expressed in terms of the spectral triple of Dąbrowski and Sitarz by means of a residue formula.

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## 1. INTRODUCTION

In the last decade, many contributions have enhanced the understanding of how quantum groups and their homogeneous spaces can be studied in terms of spectral triples, see e.g. [2, 4, 7, 8, 10, 15, 21, 22, 23, 25, 29] and the references therein. But still some basic questions remain untouched, e.g. in how far spectral triples generate the fundamental Hochschild cohomology class of the underlying algebra. This is what we investigate here for the standard Podleś quantum sphere [24].

Let us explain in more detail what we have in mind: the coordinate ring  $\mathcal{A} = \mathcal{O}(S_q^2)$  was shown in [16] to satisfy Poincaré duality in Hochschild (co)homology (as introduced in [28]), so we have for any  $\mathcal{A}$ -bimodule  $\mathcal{M}$  and all  $i \geq 0$  an isomorphism of  $H^0(\mathcal{A}, \mathcal{A})$ -modules

$$(1) \quad H^i(\mathcal{A}, \mathcal{M}) \simeq H_{\dim(\mathcal{A})-i}(\mathcal{A}, \omega \otimes_{\mathcal{A}} \mathcal{M}),$$

where  $\omega = H^{\dim(\mathcal{A})}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{\text{op}})$ . Here  $\dim(\mathcal{A})$  is the dimension of  $\mathcal{A}$  in the sense of [1], the ring structure on  $H^0(\mathcal{A}, \mathcal{A})$  and the  $H^0(\mathcal{A}, \mathcal{A})$ -module structure on  $H^i(\mathcal{A}, \mathcal{M})$  is given by the cup product while that on  $H_j(\mathcal{A}, \omega \otimes_{\mathcal{A}} \mathcal{M})$  is given by the cap product [1, Section XI.6]. Recall that the ring  $H^0(\mathcal{A}, \mathcal{A})$  is by definition just the centre of  $\mathcal{A}$ .

In the concrete case of the standard Podleś sphere the centre consist just of the scalars, and we have  $\dim(\mathcal{A}) = 2$  and  $\omega \simeq {}_{\sigma}\mathcal{A}$ , the bimodule obtained from  $\mathcal{A}$  by deforming the canonical left  $\mathcal{A}$ -action to  $a \triangleright b := \sigma(a)b$  for a specific automorphism  $\sigma \in \text{Aut}(\mathcal{A})$ , see [16]. Hence there is a class in  $H_2(\mathcal{A}, {}_{\sigma}\mathcal{A})$  that corresponds under (1) to  $1 \in H^0(\mathcal{A}, \mathcal{A})$  and this is what we call the fundamental Hochschild homology class. For the coordinate ring of a smooth affine algebraic

variety, this class corresponds under the algebraic Hochschild-Kostant-Rosenberg theorem to a section of the top degree Kähler differential forms, see e.g. [17, Section 3], so one should think of it as of a noncommutative generalisation of a volume form.

The connection to spectral triples arises from the residue formula [5, Theorem IV.2.γ.8] for the Hochschild cohomology class of the Chern character of the Fredholm module underlying the spectral triple. For a suitable spectral triple of spectral dimension 2 (see [ibid.]), it defines a functional  $\varphi : \mathcal{A}^{\otimes 3} \rightarrow \mathbb{C}$  which, when viewed as a functional on the standard Hochschild chain complex [20, Section 1.1.3], descends to a functional on  $H_2(\mathcal{A}, \mathcal{A})$ . For the algebra  $\mathcal{A}$  of smooth complex-valued functions on a compact smooth 2-dimensional spin manifold, this functional corresponds under the analytic Hochschild-Kostant-Rosenberg theorem [5, Proposition III.2.α.1] to the de Rham current given by integrating a top degree differential form on the manifold.

Combining this with what has been said above about the fundamental Hochschild homology class motivates asking whether Connes' formula or a variation thereof defines a nontrivial functional on  $H_2(\mathcal{A}, {}_\sigma\mathcal{A})$ , and this is what we show here:

**Theorem 1.** *Let  $q \in (0, 1)$ ,  $(\mathcal{A}, \mathcal{H}, D, \gamma)$  be the  $\mathcal{U}_q(\mathfrak{su}(2))$ -equivariant even spectral triple over the standard Podleś quantum sphere constructed by Dąbrowski and Sitarz [10],  $K$  be the standard group-like generator of  $\mathcal{U}_q(\mathfrak{su}(2))$ , and  $a_0, a_1, a_2 \in \mathcal{A}$ . Then we have*

- (1)  $\gamma a_0[D, a_1][D, a_2]K^{-2}|D|^{-z}$  is for  $\operatorname{Re} z > 2$  of trace class and  $\operatorname{tr}(\gamma a_0[D, a_1][D, a_2]K^{-2}|D|^{-z})$  has a meromorphic continuation to  $\{z \in \mathbb{C} \mid \operatorname{Re} z > 1\}$  with a pole at  $z = 2$  of order at most 1.

- (2) The functional on  $\mathcal{A}^{\otimes 3}$  given by the residue

$$\varphi(a_0, a_1, a_2) := \operatorname{Res}_{z=2} \operatorname{tr}(\gamma a_0[D, a_1][D, a_2]K^{-2}|D|^{-z})$$

descends to a nontrivial functional on  $H_2(\mathcal{A}, {}_\sigma\mathcal{A})$ .

We refer to the cohomology class of  $\varphi$  in  $(H_2(\mathcal{A}, {}_\sigma\mathcal{A}))^* \simeq H^2(\mathcal{A}, ({}_\sigma\mathcal{A})^*)$  as to the fundamental Hochschild cohomology class of  $\mathcal{A}$ .

Let us now recall in more detail the historical context of the result. The pioneering papers on the noncommutative geometry of the Podleś sphere were [21], where Masuda, Nakagami and Watanabe computed  $HH_\bullet(\mathcal{A}) = H_\bullet(\mathcal{A}, \mathcal{A})$ ,  $HC_\bullet(\mathcal{A})$  and the K-theory of the  $C^*$ -completion of  $\mathcal{A}$ , and [10], where Dąbrowski and Sitarz found the spectral triple that we use here. Schmüdgen and the second author then gave a residue formula for a cyclic cocycle [25] that looks like the one from Theorem 1, only that  $K^{-2}$  is replaced by  $K^2$ . However, Hadfield later computed

the Hochschild and cyclic homology of  $\mathcal{A}$  with coefficients in  ${}_{\sigma}\mathcal{A}$  and deduced that the cocycle from [25] is trivial as a Hochschild cocycle [12]. Finally, the first author has recently used the cup and cap products between the Hochschild (co)homology groups  $H^n(\mathcal{A}, {}_{\sigma}\mathcal{A})$  [18] to produce the surprisingly simple formula

$$(2) \quad \tilde{\varphi}(a_0, a_1, a_2) = \varepsilon(a_0)E(a_1)F(a_2)$$

for a nontrivial Hochschild 2-cocycle on  $\mathcal{A}$ . Here  $\varepsilon$  is the counit of the quantum  $\mathrm{SU}(2)$ -group  $\mathcal{B} := \mathcal{O}(\mathrm{SU}_q(2))$  in which  $\mathcal{A}$  is embedded as a subalgebra. The symbols  $E$  and  $F$  refer to the standard twisted primitive generators of  $\mathcal{U}_q(\mathfrak{su}(2))$ , and for  $X \in \mathcal{U}_q(\mathfrak{su}(2))$ ,  $a \in \mathcal{O}(\mathrm{SU}_q(2))$  we write  $X(a)$  for the pairing of  $X$  with  $a$  (cf. [14], Proposition 4.22). What Theorem 1 achieves is to express a scalar multiple of the cohomology class of this cocycle in terms of the spectral triple by means of a residue formula.

The crucial result is Proposition 1 in Section 3.4 from which it follows that

$$\tau_{\mu}(a) := \frac{\mathrm{Res}_{z=2|\mu|} \mathrm{tr}(aK^{2\mu}|D|^{-z})}{\mathrm{Res}_{z=2|\mu|} \mathrm{tr}(K^{2\mu}|D|^{-z})}$$

defines for all  $\mu \in \mathbb{R}$  a twisted trace on  $\mathcal{A}$  that we can compute explicitly. In terms of the functionals  $\int_{[1]}$  and  $\int_{[x_0]}$  defined in [18], the traces  $\tau_{\mu}$  are given by:

$\mu$	$\sigma$	$\tau_{\mu}$
$< 0$	any	$\int_{[1]} = \varepsilon$
$0$	id	$\int_{[1]} + \frac{\ln q}{2(q^{-1}-q)\ln(q^{-1}-q)} \int_{[x_0]}$
$> 0$	$A \mapsto A, B \mapsto q^{2\mu}B$	$\int_{[1]} - \frac{1-q^{-2\mu}}{q(1-q^{-2(\mu+1)})} \int_{[x_0]}$

Here  $\sigma$  is the involved twisting automorphism and  $A, B$  are certain generators of  $\mathcal{A}$  (in the notation of [18],  $A = -q^{-1}x_0$ ). From this fact it will be deduced in the final section that the multilinear functional defined in Theorem 1 is up to normalisation indeed cohomologous to the Hochschild cocycle (2) constructed in [18].

The remainder of the paper is divided into two sections: Section 2 contains background material taken mainly from [10, 25]. The subsequent section discusses the meromorphic continuation of the zeta functions  $\mathrm{tr}(aK^{2\mu}|D|^{-z})$ ,  $a \in \mathcal{A}$ , and how one can compute their residues

by replacing certain algebra elements of  $\mathcal{A}$  by simpler operators. The proof of Theorem 1 fills the final Subsection 3.5 of the paper.

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## 2. BACKGROUND

**2.1. The algebras  $\mathcal{O}(S_q^2)$ ,  $\mathcal{O}(\mathrm{SU}_q(2))$ ,  $\mathcal{U}_q(\mathfrak{su}(2))$ .** We retain all notations and conventions used in [25]. In particular, we fix a deformation parameter  $q \in (0, 1)$ , and let  $\mathcal{A} = \mathcal{O}(S_q^2)$  be the  $*$ -algebra (over  $\mathbb{C}$ ) with generators  $A = A^*$ ,  $B$  and  $B^*$  and defining relations

$$BA = q^2 AB, \quad AB^* = q^2 B^* A, \quad B^* B = A - A^2, \quad BB^* = q^2 A - q^4 A^2.$$

We consider  $\mathcal{A}$  as a subalgebra of the quantised coordinate ring  $\mathcal{B} = \mathcal{O}(\mathrm{SU}_q(2))$  which is the  $*$ -algebra generated by  $a, b, c = -q^{-1}b^*, d = a^*$  satisfying the relations given e.g. in [14, Section 4.1]. Note that it follows from the defining relations that the monomials

$$\{A^n B^m, A^n B^{*m} \mid n, m \geq 0\}$$

form a vector space basis of  $\mathcal{A}$ .

For the Hopf  $*$ -algebra  $\mathcal{U} = \mathcal{U}_q(\mathfrak{su}(2))$ , we use generators  $K, K^{-1}, E$  and  $F$  with involution  $K^* = K, E^* = F$ , defining relations

$$KE = qEK, \quad KF = q^{-1}FK, \quad EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}},$$

coproduct

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F,$$

and counit  $\varepsilon(1 - K) = \varepsilon(E) = \varepsilon(F) = 0$ .

There is a left  $\mathcal{U}$ -action on  $\mathcal{A}$  satisfying  $f \triangleright (ab) = (f_{(1)} \triangleright a)(f_{(2)} \triangleright b)$  and  $f \triangleright 1 = \varepsilon(f)1$  for  $f \in \mathcal{U}$  and  $a, b \in \mathcal{A}$ , that is,  $\mathcal{A}$  is a left  $\mathcal{U}$ -module algebra. Here and in what follows, we use Sweedler's notation  $\Delta(f) = f_{(1)} \otimes f_{(2)}$ . On the re-parametrised generators

$$(3) \quad x_{-1} = (1 + q^{-2})^{1/2} B, \quad x_0 = 1 - (1 + q^2) A, \quad x_1 = -(1 + q^2)^{1/2} B^*,$$

this action is given by

$$K \triangleright x_i = q^i x_i, \quad E \triangleright x_i = (q + q^{-1})x_{i+1}, \quad F \triangleright x_i = (q + q^{-1})x_{i-1},$$

where it is understood that  $x_2 = x_{-2} = 0$ .

**2.2. The spectral triple.** Our calculations involve the spectral triple constructed by Dąbrowski and Sitarz in [10]. For the reader's convenience and to fix notation, we recall its definition.

First of all, the  $*$ -algebra  $\mathcal{A}$  becomes represented by bounded operators on a Hilbert space  $\mathcal{H} := \mathcal{H}_- \oplus \mathcal{H}_+$  with orthonormal basis

$$v_{k,\pm}^l \in \mathcal{H}_\pm, \quad l = \frac{1}{2}, \frac{3}{2}, \dots, \quad k = -l, -l+1, \dots, l,$$

where the generators  $x_{-1}, x_0, x_1$  act by

$$(4) \quad x_i v_{k,\pm}^l = \alpha_i^-(l, k)_\pm v_{k+i,\pm}^{l-1} + \alpha_i^0(l, k)_\pm v_{k+i,\pm}^l + \alpha_i^+(l, k)_\pm v_{k+i,\pm}^{l+1}.$$

Here  $\alpha_i^\nu(l, k)_\pm \in \mathbb{R}$  are coefficients that can be found e.g. in [9], where similar conventions are used. We will only need the formulas for  $\alpha_0^\nu(l, k)_\pm$  which are given by

$$(5) \quad \alpha_0^-(l, k)_\pm = \frac{q^{k \pm 1/2} [2]_q [l-k]_q^{1/2} [l+k]_q^{1/2} [l-1/2]_q^{1/2} [l+1/2]_q^{1/2}}{[2l-1]_q^{1/2} [2l]_q [2l+1]_q^{1/2}},$$

$$(6) \quad \alpha_0^0(l, k)_\pm = [2l]_q^{-1} ([l-k+1]_q [l+k]_q - q^2 [l-k]_q [l+k+1]_q) \beta_\pm(l),$$

$$(7) \quad \alpha_0^+(l, k)_\pm = \alpha_0^-(l+1, k)_\pm$$

with

$$(8) \quad [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$$

and

$$(9) \quad \beta_\pm(l) = \frac{\pm q^{\mp 1} + (q - q^{-1})([1/2]_q [3/2]_q - [l]_q [l+1]_q)}{q[2l+2]_q}.$$

We now define

$$\text{Dom}(D) := \text{span}_{\mathbb{C}} \{v_{k,\pm}^l \mid l = \frac{1}{2}, \frac{3}{2}, \dots, \quad k = -l, -l+1, \dots, l\}$$

and on this domain an essentially self-adjoint operator  $D$  by

$$D v_{k,\pm}^l = [l+1/2]_q v_{k,\mp}^l.$$

In the sequel all operators we consider will be defined on this domain, leave it invariant, and be closable. By slight abuse of notation we will not distinguish between an operator defined on  $\text{Dom}(D)$  and its closure.

The  $v_{k,\pm}^l$  are eigenvectors of  $|D|$ :

$$|D| v_{k,\pm}^l = [l+1/2]_q v_{k,\pm}^l.$$

Furthermore, the spectral triple is even with grading  $\gamma$  given by

$$\gamma v_{k,\pm}^l = \pm v_{k,\pm}^l.$$

**2.3.  $\mathcal{U}$ -equivariance.** The spectral triple is  $\mathcal{U}$ -equivariant in the sense of [26]: on  $\text{Dom}(D)$  there is an action of  $\mathcal{U}$  given by

$$(10) \quad K v_{k,\pm}^l = q^k v_{k,\pm}^l, \quad E v_{k,\pm}^l = \alpha_k^l v_{k+1,\pm}^l, \quad F v_{k,\pm}^l = \alpha_{k-1}^l v_{k-1,\pm}^l,$$

where  $\alpha_k^l := ([l-k]_q [l+k+1]_q)^{1/2}$ , and we have on  $\text{Dom}(D)$

$$f a = (f_{(1)} \triangleright a) f_{(2)}, \quad f D = D f, \quad f \gamma = \gamma f$$

for all  $f \in \mathcal{U}$ ,  $a \in \mathcal{A}$ .

### 3. RESULTS

**3.1. A family of  $q$ -zeta functions.** Quantum group analogues of zeta functions were studied by several authors, in particular by Ueno and Nishizawa [27] and Cherednik [3]. The ones we will consider here are given on a suitable domain by

$$(11) \quad \zeta_T(z) := \text{tr}(T|D|^{-z})$$

for some possibly unbounded operator  $T$  on  $\mathcal{H}$ . The most important case we need is  $T = L^\beta K^\delta$  for  $\beta, \delta \in \mathbb{R}$ , where

$$L v_{k,\pm}^l = q^l v_{k,\pm}^l$$

and thus

$$L^\beta K^\delta v_{k,\pm}^l = q^{\beta l + \delta k} v_{k,\pm}^l.$$

The resulting zeta functions differ slightly from those considered in [3, 27], and also from the one occurring in [25]. Yet the main argument leading to a meromorphic continuation of the functions to the whole complex plane given in [27] can be applied in all these cases:

**Lemma 1.** *For all  $\beta, \delta \in \mathbb{R}$ , the function*

$$\zeta_{L^\beta K^\delta}(z) = 2 \sum_{l=\frac{1}{2}, \frac{3}{2}, \dots}^{\infty} \sum_{k=-l}^l \frac{q^{\beta l + \delta k}}{[l + 1/2]_q^z}, \quad \text{Re } z > -\beta + |\delta|,$$

*admits a meromorphic continuation to the complex plane given by*

$$\zeta_{L^\beta K^\delta}(z) = 2q^{\frac{\beta}{2}}(q^{-\frac{\delta}{2}} + q^{\frac{\delta}{2}})(1 - q^2)^z \sum_{j=0}^{\infty} \frac{\binom{z+j-1}{j} q^{2j}}{(1 - q^{\beta-\delta+2j+z})(1 - q^{\beta+\delta+2j+z})}$$

and its residue at  $z = -\beta + |\delta|$  is given by

$$\operatorname{Res}_{z=-\beta+|\delta|} \zeta_{L^\beta K^\delta}(z) = \begin{cases} \frac{2q^{\frac{\beta-|\delta|}{2}}(1-q^2)^{|\delta|-\beta}}{(q^{|\delta|-1}-1)\ln(q)}, & \delta \neq 0, \\ \frac{4q^{\frac{\beta}{2}}\ln(q^{-1}-q)}{(1-q^2)^\beta(\ln q)^2}, & \delta = 0. \end{cases}$$

*Proof.* The crucial step is to use the binomial series

$$(12) \quad (1 - q^{2(n+1)})^{-z} = \sum_{j=0}^{\infty} \binom{z+j-1}{j} q^{2(n+1)j}$$

which holds for all  $z \in \mathbb{C}$ .

First, let  $\delta = 0$ . By summing over  $k$ , replacing  $l = n + \frac{1}{2}$ , inserting (12), and interchanging the order of the summations in the absolutely convergent series, we obtain for  $\operatorname{Re} z > -\beta$

$$\begin{aligned} \zeta_{L^\beta}(z) &= 4(q^{-1} - q)^z \sum_{n=0}^{\infty} (n+1) q^{\beta(n+\frac{1}{2})} q^{(n+1)z} (1 - q^{2(n+1)})^{-z} \\ &= 4q^{-\frac{\beta}{2}} (q^{-1} - q)^z \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} (n+1) \binom{z+j-1}{j} q^{(\beta+2j+z)(n+1)}. \end{aligned}$$

Using the identity

$$\sum_{n=0}^{\infty} (n+1)t^n = \frac{d}{dt} \sum_{n=0}^{\infty} t^n = \frac{1}{(1-t)^2},$$

we can write the above sum as

$$\zeta_{L^\beta}(z) = 4q^{-\frac{\beta}{2}} (q^{-1} - q)^z \sum_{j=0}^{\infty} \binom{z+j-1}{j} \frac{q^{\beta+2j+z}}{(1 - q^{\beta+2j+z})^2},$$

and the right hand side is a meromorphic function with isolated poles of second order only at the lines  $\operatorname{Re} z = -\beta, -\beta - 2, -\beta - 4, \dots$

If  $\delta \neq 0$ , the sum over  $k$  yields

$$\sum_{k=-(n+\frac{1}{2})}^{n+\frac{1}{2}} q^{\delta k} = \frac{q^{-\delta(n+1)} - q^{\delta(n+1)}}{q^{-\frac{\delta}{2}} - q^{\frac{\delta}{2}}}.$$

Similar to the above, we get for  $\operatorname{Re} z > -\beta + |\delta|$

$$\begin{aligned} \zeta_{L^\beta K^\delta}(z) &= \\ 2q^{-\frac{\beta}{2}} \frac{(q^{-1} - q)^z}{q^{-\frac{\delta}{2}} - q^{\frac{\delta}{2}}} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \binom{z+j-1}{j} &\left( q^{(\beta-\delta+2j+z)(n+1)} - q^{(\beta+\delta+2j+z)(n+1)} \right). \end{aligned}$$

The summation over  $n$  gives

$$\sum_{n=0}^{\infty} q^{(\beta-\delta+2j+z)(n+1)} - q^{(\beta+\delta+2j+z)(n+1)} = \frac{(q^{-\delta} - q^{\delta})q^{\beta+2j+z}}{(1 - q^{\beta-\delta+2j+z})(1 - q^{\beta+\delta+2j+z})}.$$

Inserting the last equation into the previous one yields the second formula of Lemma 1 which defines a meromorphic function on the whole complex plane.

When computing the residues at the pole  $z = -\beta + |\delta|$ , we can ignore the sum over  $j > 0$  which is holomorphic in a neighbourhood of  $-\beta + |\delta|$ . Thus

$$\operatorname{Res}_{z=-\beta+|\delta|} \zeta_{L^{\beta}K^{\delta}}(z) = \operatorname{Res}_{z=-\beta+|\delta|} \frac{2q^{\frac{\beta}{2}}(1 - q^2)^z(q^{-\frac{\delta}{2}} + q^{\frac{\delta}{2}})}{(1 - q^{\beta-\delta+z})(1 - q^{\beta+\delta+z})}$$

which can be computed straightforwardly to yield the result.  $\square$

**3.2. A holomorphicity remark.** Next we need to point out that  $\zeta_{TL^{\beta}K^{\delta}}(z)$  is holomorphic for  $\operatorname{Re} z > -\beta + |\delta|$  whenever  $T$  is bounded. Let us first introduce some notation that we will use throughout the rest of the paper in order to simplify statements and proofs:

**Definition 1.** *We say that a set of complex numbers*

$$\{\nu_{l,k} \mid l \in \tfrac{1}{2}\mathbb{N}, \ k = -l, \dots, l\}$$

*is of order less than or equal to  $q^{\alpha}$ ,  $\alpha \in \mathbb{R}$ , if there exists  $C \in (0, \infty)$  such that  $|\nu_{l,k}| \leq Cq^{\alpha l}$  for all  $k, l$ . In this case we write*

$$\nu_{l,k} \lesssim q^{\alpha l}.$$

We refrain from using the notation  $\nu_{k,l} = O(q^{\alpha l})$  to avoid confusion about the fact that the second parameter  $k$  can take arbitrary values from  $\{-l, \dots, l\}$ . Note that we have for all  $\beta, \delta \in \mathbb{R}$  and  $z \in \mathbb{C}$

$$(13) \quad q^{\beta l + \delta k} \lesssim q^{(\beta - |\delta|)l}, \quad [l - k]_q [l + k]_q \lesssim q^{-2l}, \quad [\beta l + \delta]_q^{-z} \lesssim q^{|\beta| \operatorname{Re}(z) l}.$$

Now one easily observes:

**Lemma 2.** *For all bounded operators  $T$  on  $\mathcal{H}$  and for all  $\beta, \delta \in \mathbb{R}$ , the function  $\zeta_{TL^{\beta}K^{\delta}}(z)$  is holomorphic on  $\{z \in \mathbb{C} \mid \operatorname{Re} z > -\beta + |\delta|\}$ .*

*Proof.* Since  $L^{\beta}K^{\delta}|D|^{-r}$  is for  $r \in \mathbb{R}$  positive and essentially self-adjoint, the summability of its eigenvalues verified in Lemma 1 shows that it is of trace class if  $r > -\beta + |\delta|$ . Therefore  $TL^{\beta}K^{\delta}|D|^{-z} = T|D|^{-is}L^{\beta}K^{\delta}|D|^{-r}$ ,  $z = r + is$ , is a trace class operator.



If one fixes  $\epsilon > 0$ , then the infinite series defining  $\text{tr}(TL^\beta K^\delta |D|^{-z})$  converges uniformly on  $\{z \in \mathbb{C} \mid \text{Re } z \geq \epsilon - \beta + |\delta|\}$  since the geometric series

$$\sum_{l \in \mathbb{N}} q^{(\text{Re}(z) + \beta - |\delta|)l}$$

does so and we have for all bounded sequences  $t_{k,\pm}^l := \langle v_{k,\pm}^l, T v_{k,\pm}^l \rangle$

$$\frac{t_{k,\pm}^l q^{\beta l + \delta k}}{[l + 1/2]_q^z} \lesssim q^{(\text{Re}(z) + \beta - |\delta|)l}.$$

The partial sums of the series are clearly holomorphic functions and, by the above argument, converge uniformly on compact sets contained in  $\{z \in \mathbb{C} \mid \text{Re } z > -\beta + |\delta|\}$ . The result follows now from the Weierstraß convergence theorem.  $\square$

**3.3. Approximating the generator  $A$ .** It is known [22] that the spectral triple we are considering violates Connes' regularity condition, so the standard machinery of zeta functions and generalised pseudo-differential operators (see e.g. [6, 13]) can not be applied here. However, for our purposes, it suffices to show that the zeta functions from (11) have meromorphic continuations to half-planes, and this will be shown in Proposition 1 in the next section. The key step in the proof will be to approximate the generator  $A \in \mathcal{A}$  on  $\mathcal{H}$  by simpler operators, and this is what we establish here. Similar ideas have been used in [9].

**Lemma 3.** *There exists a bounded linear operator  $A_0$  on  $\mathcal{H}$  such that*

$$(14) \quad A = M + A_0 L, \quad \text{where } M := L^2 K^2.$$

*Proof.* We have to prove that  $A_0 := (A - M)L^{-1}$  extends to a bounded operator on  $\mathcal{H}$ . Inserting (3) and (4) into this definition shows that it suffices to prove that the coefficients

$$q^{-l} \alpha_0^\pm(l, k)_\pm \quad \text{and} \quad q^{-l} \left( \frac{1}{1+q^2} (1 - \alpha_0^0(l, k)_\pm) - q^{2(l+k)} \right)$$

are bounded. Applying (13) to (5) gives  $\alpha_0^\pm(l, k)_\pm \lesssim q^l$ . Therefore we have  $q^{-l} \alpha_0^\pm(l, k)_\pm \lesssim 1$  which means that these coefficients are bounded.

Using (13) and  $\frac{1}{1-q^{4l+4}} - 1 \lesssim q^{4l}$ , we get from (9)

$$\beta_\pm(l) = \frac{(q^{-1} - q)[l]_q[l+1]_q}{q[2l+2]_q} + u_l = \frac{1 - q^{2l} - q^{2l+2} + q^{4l+2}}{1 - q^{4l+4}} + u_l = 1 + v_l,$$

where  $u_l, v_l \lesssim q^{2l}$ . Similarly, we have

$$\begin{aligned} \frac{[l-k+1]_q[l+k]_q - q^2[l-k]_q[l+k+1]_q}{[2l]_q} &= \frac{1 - (1+q^2)q^{2l+2k} + (1+q^2)q^{2l}}{1 - q^{4l}} \\ &= 1 - (1+q^2)q^{2l+2k} + w_{l,k}, \end{aligned}$$

where  $w_{l,k} \lesssim q^{2l}$ . Multiplying the last two equations and comparing with (6) gives

$$\alpha_0^0(l, k)_\pm = 1 - (1+q^2)q^{2l+2k} + x_{l,k}$$

with  $x_{l,k} \lesssim q^{2l}$ . From this, we get

$$q^{-l} \left( \frac{1}{1+q^2} (1 - \alpha_0^0(l, k)_\pm) - q^{2(l+k)} \right) = q^{-l} x_{l,k} \lesssim q^l \lesssim 1$$

which finishes the proof.  $\square$

**3.4. Twisted traces as residues.** We are now ready to prove the main technical result of the paper which expresses certain twisted traces of  $\mathcal{A}$  as residues of zeta-functions:

**Proposition 1.** *The function  $\zeta_{aK^{2\mu}}(z)$ ,  $a \in \mathcal{A}$ , has a meromorphic continuation to  $\{z \in \mathbb{C} \mid \operatorname{Re} z > 2|\mu| - 1\}$ . Its residues at  $z = 2|\mu|$  are given by*

$a$	$\mu$	$\operatorname{Res}_{z=2 \mu } \zeta_{aK^{2\mu}}(z)$
$A^n B^m, A^n B^{*m}, n \geq 0, m > 0$	any	0
$A^n, n > 0$	$< 0$	0
$A^n, n > 0$	$\geq 0$	$\frac{-2q^\mu(q^{-1}-q)^{2\mu}}{(1-q^{2(n+\mu)}) \ln q}$
1	$\neq 0$	$\frac{-2q^{ \mu }(q^{-1}-q)^{2 \mu }}{(1-q^{2 \mu }) \ln(q)}$
1	0	$\frac{4 \ln(q^{-1}-q)}{(\ln q)^2}$

*Proof.* Lemma 2 implies that the traces  $\zeta_{aK^{2\mu}}$  exist and are holomorphic on  $\{z \in \mathbb{C} \mid \operatorname{Re} z > 2|\mu|\}$  for all  $a \in \mathcal{A}$ .

Furthermore,  $B$  and  $B^*$  act as shift operators in the index  $k$  of  $v_{k,\pm}^l$ . Hence the traces  $\operatorname{tr}(A^n B^m K^{2\mu} |D|^{-z})$  and  $\operatorname{tr}(A^n B^{*m} K^{2\mu} |D|^{-z})$  vanish whenever  $m > 0$ , so we can use the trivial analytic continuation here. Thus it remains to prove the claim for  $a = A^n$ .

Applying first Lemma 3 and using then the fact that, by Lemma 2,  $\text{tr}(A^{n-1}A_0M^mLK^{2\mu}|D|^{-z})$  is holomorphic on  $\{z \in \mathbb{C} \mid \text{Re } z > 2|\mu| - 1\}$ , we obtain for all  $m \geq 0, n > 0$

$$\begin{aligned} \text{Res}_{z=2|\mu|} \text{tr}(A^n M^m K^{2\mu} |D|^{-z}) &= \text{Res}_{z=2|\mu|} \text{tr}(A^{n-1}(M + A_0 L) M^m K^{2\mu} |D|^{-z}) \\ &= \text{Res}_{z=2|\mu|} \text{tr}(A^{n-1} M^{m+1} K^{2\mu} |D|^{-z}) \\ &\quad + \text{Res}_{z=2|\mu|} \text{tr}(A^{n-1} A_0 M^m L K^{2\mu} |D|^{-z}) \\ &= \text{Res}_{z=2|\mu|} \text{tr}(A^{n-1} M^{m+1} K^{2\mu} |D|^{-z}). \end{aligned}$$

Recall that  $M = L^2 K^2$ . An iterated application of the previous equation gives

$$\begin{aligned} \text{Res}_{z=2|\mu|} \text{tr}(A^n K^{2\mu} |D|^{-z}) &= \text{Res}_{z=2|\mu|} \text{tr}(M^n K^{2\mu} |D|^{-z}) \\ &= \text{Res}_{z=2|\mu|} \text{tr}(L^{2n} K^{2(\mu+n)} |D|^{-z}). \end{aligned}$$

The result now reduces to Lemma 1.  $\square$

We remark here that the table in the introduction is obtained by comparing the values of the twisted traces  $\int_{[1]}$  and  $\int_{[x_0]}$  from [18] on the basis vectors  $A^n B^m$  and  $A^n B^{*m}$  with the residues of the last proposition.

**3.5. Proof of Theorem 1.** Theorem 1 is an easy consequence of Proposition 1. As explained e.g. in [25], the operator

$$\gamma a_0 [D, a_1] [D, a_2], \quad a_0, a_1, a_2 \in \mathcal{A}$$

acts by multiplication with

$$a_0(a_1 \triangleleft E)(a_2 \triangleleft F) \in \mathcal{A}$$

on  $\mathcal{H}_+$  and by multiplication with

$$-a_0(a_1 \triangleleft F)(a_2 \triangleleft E) \in \mathcal{A}$$

on  $\mathcal{H}_-$ . Here  $\triangleleft$  denotes the standard right action of  $\mathcal{U} \subset \mathcal{B}^\circ$  on  $\mathcal{B}$  (see [14, Section 1.3.5]) given by

$$a \triangleleft f := f(a_{(1)})a_{(2)}$$

Note that unlike the left action  $f \triangleright a := a_{(1)}f(a_2)$ , this right action does not leave  $\mathcal{A} \subset \mathcal{B}$  invariant, but the products  $(a_1 \triangleleft E)(a_2 \triangleleft F)$  and  $(a_1 \triangleleft F)(a_2 \triangleleft E)$  belong to  $\mathcal{A}$  again.

By the definition of  $\triangleleft$  we have

$$\varepsilon(a_0(a_1 \triangleleft E)(a_2 \triangleleft F)) = \varepsilon(a_0)E(a_1)F(a_2)$$

and

$$\varepsilon(a_0(a_1 \triangleleft F)(a_2 \triangleleft E)) = \varepsilon(a_0)F(a_1)E(a_2),$$

and evaluation on an arbitrary cycle representing the fundamental Hochschild class in  $H_2(\mathcal{A}, {}_\sigma\mathcal{A})$  (see the proof of the nontriviality of (2) in [18]) shows that the two functionals on  $H_2(\mathcal{A}, {}_\sigma\mathcal{A})$  induced by these functionals on  $\mathcal{A}^{\otimes 3}$  coincide up to a factor of  $-q^{-2}$  (see also [19], where we carry this computation out with the help of the computer algebra system SINGULAR:PLURAL).

Thus, by Proposition 1, the cocycle

$$\begin{aligned} \varphi(a_0, a_1, a_2) &:= \operatorname{Res}_{z=2} \operatorname{tr}(\gamma a_0 [D, a_1] [D, a_2] K^{-2} |D|^{-z}) \\ &= \frac{2(q-q^{-1})}{\ln(q)} \varepsilon(a_0) (E(a_1)F(a_2) - F(a_1)E(a_2)) \end{aligned}$$

is cohomologous to  $\frac{2(q-q^{-3})}{\ln(q)} \tilde{\varphi}$ , where  $\tilde{\varphi}$  denotes the fundamental cocycle from (2). This finishes the proof of Theorem 1.  $\square$

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