Dirac Operators on Quantum Flag Manifolds

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Abstract

A Dirac operator D on quantized irreducible generalized flag manifolds is defined. This yields a Hilbert space realization of the covariant first-order differential calculi constructed by I. Heckenberger and S. Kolb. All differentials $df = i[D, f]$ are bounded operators. In the simplest case of Podleś' quantum sphere one obtains the spectral triple found by L. Dabrowski and A. Sitarz.

1 Introduction

After about 20 years of quantum groups and of noncommutative geometry in the sense of A. Connes the relation between these two theories is still not understood very well. In particular, there is no theory linking Connes' concept of spectral triple to that of finite-dimensional covariant differential calculus on quantum spaces as developed by S.L. Woronowicz [W]. It is only known that the basic examples of such calculi which are the $3D$-calculus and the $4D_\pm$-calculi on the quantum group $SU_q(2)$ itself can *not* be realized by spectral triples. This was shown by K. Schmüdgen in [S]. So the spectral triples of [CP, G] are not related to these calculi. The aim of the present paper is to show that q-deformations of irreducible generalized flag manifolds $M = G/P$ behave better in this respect.

In [HK1] I. Heckenberger and S. Kolb proved that these quantum spaces admit exactly two irreducible finite-dimensional covariant differential calculi $(\Gamma_\pm, d_\pm)$. Their direct sum (Γ, d) is a $*$-calculus whose elements are q-analogues of complex-valued differential forms on the real manifold M. The main result of this paper is that Γ can be realized by bounded operators on a Hilbert space such that $df = i[D, f]$ for a self-adjoint operator D. The latter generalizes the classical Dirac operator on M. A calculation of its spectrum seems to be a non-trivial problem, but its construction suggests that the spectrum is a smooth deformation of the classical one.

The simplest example of a generalized flag manifold is the complex projective line $\mathbb{C}P^1 \simeq S^2$. The corresponding quantum flag manifold is Podleś' quantum sphere in the so-called quantum subgroup case. For this one obtains the Dirac operator found by L. Dabrowski and A. Sitarz [DS1].

The paper is organized as follows: The first two sections are devoted to background material on quantum groups and quantum flag manifolds. In Sections 4,5 and 6 we define analogues of the tangent space, its Clifford algebra and the spinor bundle of M. With these as ingredients the Dirac operator D is constructed in Section 7. In the final section we study the associated differential calculus and prove the main results.

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2 Quantum groups

In this section we fix notations and recall some definitions and results of quantum group theory used in the sequel. We refer to the monographs [KS] and [J] for proofs and further details.
Throughout this paper $\mathfrak{g}$ is a complex simple Lie algebra, G the corresponding connected simply connected Lie group, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\{\alpha_1, \ldots, \alpha_N\}$ and $\{\omega_1, \ldots, \omega_N\}$ are a set of simple roots and the corresponding set of fundamental weights. The integral root and weight lattices are denoted by $\mathbf{Q}$ and $\mathbf{P}$, respectively. The dominant integral weights are denoted by $\mathbf{P}^+$. The Killing form induces a bilinear pairing $\langle \cdot, \cdot \rangle$ on $\mathbf{P} \times \mathbf{P}$. Then $\langle \omega_i, \alpha_j \rangle =: \delta_{ij} d_i$.
Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra corresponding to $\mathfrak{g}$ in the form denoted by $\check{U}$ in [J, 3.2.10]. We denote its generators by K_λ, E_i, F_i, $i = 1, \ldots, N$, $\lambda \in \mathbf{P}$ and set $K_i := K_{\alpha_i}$. See [KS, Section 6.1.2] for their explicit relations. For the coproduct, counit and antipode we use the conventions of [J]. In particular, the coproduct of E_i, F_i is

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i.$$

We assume $q \in (1, \infty)$ and consider $U_q(\mathfrak{g})$ as the Hopf $*$-algebra called the compact real form of $U_q(\mathfrak{g})$ in [KS, Section 6.1.7]. Its involution $*$ coincides on generators with κ' from [J, 3.3.3],

$$E_i^* = K_i F_i, \quad F_i^* = E_i K_i^{-1}, \quad K_i^* = K_i,$$

but κ' is continued to a linear map. As in [GZ] we set $\theta := * \circ S$.
There is a bilinear form $\langle \cdot, \cdot \rangle$ on $U_q(\mathfrak{g})$ called the quantum Killing form (or Rosso form) which is invariant under the adjoint action

$$\mathrm{ad}(X)Y := X \triangleright Y := X_{(1)} Y S(X_{(2)}), \quad X, Y \in U_q(\mathfrak{g})$$

of $U_q(\mathfrak{g})$ on itself in the sense that

$$\langle Z \triangleright X, Y \rangle = \langle X, S(Z) \triangleright Y \rangle \quad \forall X, Y, Z \in U_q(\mathfrak{g}).$$

Above as in the following we use Sweedler notation $\Delta(X) = X_{(1)} \otimes X_{(2)}$ for the coproduct Δ of a Hopf algebra.
The quantum Killing form satisfies [J, 3.3.3, 7.2.4]

$$\langle X^*, Y \rangle = \overline{\langle Y^*, X \rangle}, \quad \langle X^*, X \rangle > 0 \quad \forall X, Y \in U_q(\mathfrak{g}), X \neq 0.$$

and vanishes on $U_q^\lambda(\mathfrak{g}) \times U_q^\mu(\mathfrak{g})$ if $\lambda \neq -\mu$. Here $U_q^\mu(\mathfrak{g})$, $\mu \in \mathbf{Q}$, consists of the elements $X \in U_q(\mathfrak{g})$ with $K_\lambda X K_\lambda^{-1} = q^{\langle \lambda, \mu \rangle} X$ for all $\lambda \in \mathbf{P}$.
The representation theories of $U_q(\mathfrak{g})$ and $\mathfrak{g}$ are closely related. In particular, for every $\lambda \in \mathbf{P}^+$ there exists an irreducible representation $(\rho_\lambda, V_\lambda)$ of

highest weight λ and a Hermitian inner product $(\cdot,\cdot)_\lambda$ on V_λ such that ρ_λ becomes a $*$-representation. The weight structure of V_λ and the decomposition of tensor products into irreducible components remains unchanged as well [KS, Chapter 7].

Let $\mathbb{C}_q[G]$ be the coordinate algebra of the standard quantum group associated to G. This is the Hopf $*$-subalgebra of the Hopf dual $U_q(\mathfrak{g})^\circ$ generated by all matrix coefficients t^λ_{ij} of the representations V_λ, $\lambda \in \mathbf{P}^+$. We will conversely treat elements of $U_q(\mathfrak{g})$ as functionals on $\mathbb{C}_q[G]$ and write the dual pairing between $X \in U_q(\mathfrak{g})$ and $f \in \mathbb{C}_q[G]$ as $X(f)$.

This pairing turns $\mathbb{C}_q[G]$ into a $U_q(\mathfrak{g})$-bimodule with left and right action given by $X \triangleright f := X(f_{(2)})f_{(1)}$, $f \triangleleft X := X(f_{(1)})f_{(2)}$. The structure of this bimodule is given by the classical Peter-Weyl theorem. That is, the t^λ_{ij} form a vector space basis of $\mathbb{C}_q[G]$ and for fixed i (fixed j) a basis of the representation V_λ with respect to the left (right) action.

The linear functional $h : \mathbb{C}_q[G] \to \mathbb{C}$ defined by $h(t^\lambda_{ij}) := \delta_{\lambda 0}$ and called the Haar functional is biinvariant with respect to the $U_q(\mathfrak{g})$-actions [KS, Section 11.3]. The associated Hermitian inner product

$$\langle f, g\rangle_h := h(fg^*), \quad f, g \in \mathbb{C}_q[G]$$

is positive definite and is the direct sum of the $(\cdot,\cdot)_\lambda$ arising from considering $\mathbb{C}_q[G]$ as right $U_q(\mathfrak{g})$-module. That is, we have

$$\langle f \triangleleft X, g\rangle_h = \langle f, g \triangleleft X^*\rangle_h \quad \forall X \in U_q(\mathfrak{g}), f, g \in \mathbb{C}_q[G].$$

3 Quantum flag manifolds

Let P be a standard parabolic subgroup of G and $M = G/P$ the corresponding generalized flag manifold [FH, §23.3, BE]. As a real manifold M is diffeomorphic to G_0/L_0, where G_0 denotes the compact real form of G, L is the Levi factor of P and $L_0 := L \cap G_0$, cf. [BE, 6.4]. Let $\mathfrak{p}, \mathfrak{l}$ denote the Lie algebras of P, L. Throughout this paper we assume that M is irreducible, that is, that $\mathfrak{g}/\mathfrak{p}$ is irreducible with respect to the adjoint action of $\mathfrak{p}$. This implies that $\mathfrak{l}$ is the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and the root vectors E_i, F_i associated to the simple roots α_i, $i \neq r$, for a certain r, see [BE, Example 3.1.10]. For example, if $G = SL(N+1,\mathbb{C})$, then the irreducible flag manifolds exhaust the complex Grassmann manifolds $Gr(r, N+1)$, $r = 1, \ldots, N$.

Let $U_q(\mathfrak{l}) \subset U_q(\mathfrak{g})$ be the Hopf $*$-subalgebra generated by K_λ, E_i, F_i, $\lambda \in \mathbf{P}$, $i \neq r$ and define

$$\mathbb{C}_q[M] := \{f \in \mathbb{C}_q[G] \mid X \triangleright f = \varepsilon(X)f \;\forall X \in U_q(\mathfrak{l})\}.$$

This algebra is a q-deformation of a $*$-algebra of complex-valued functions on M. Completion with respect to the C^*-completion of $\mathbb{C}_q[G]$ leads to a q-deformation of the C^*-algebra associated to the compact topological space M. With slight abuse of terminology from algebraic geometry we call $\mathbb{C}_q[M]$ the coordinate algebra of the quantum flag manifold M_q. See for example [HK1, DS2] for more information about quantum flag manifolds.

The right and the left action of $U_q(\mathfrak{g})$ on $\mathbb{C}_q[G]$ commute. Hence $\mathbb{C}_q[M]$ is a right $U_q(\mathfrak{g})$-module. It decomposes into irreducible components in the same way as its classical analogue.

4 The tangent space

Let $\mathfrak{u}$ be the orthogonal complement of $\mathfrak{l}$ with respect to the Killing form of $\mathfrak{g}$. It decomposes as $\mathfrak{u} = \mathfrak{u}_+ \oplus \mathfrak{u}_-$, where $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_+$ is the Levi decomposition of $\mathfrak{p}$ and $\mathfrak{u}_-$ can be identified with the complex tangent space $\mathfrak{g}/\mathfrak{p}$ of G/P at eP. The adjoint action of $\mathfrak{l}$ on $\mathfrak{u}$ defines an embedding of $\mathfrak{l}$ into $\mathfrak{so}(2m,\mathbb{C})$, where $\dim_{\mathbb{C}} M = m$. We now introduce analogues of $\mathfrak{u}, \mathfrak{u}_\pm$ for quantum flag manifolds.
Let $\lambda = -2n \cdot \omega_r$. The number $n \in \mathbb{N} \setminus \{0\}$ is arbitrary but fixed and will play no role in the sequel. But probably it may be used to adjust the analytical properties of the Dirac operator we will derive below.
Define $X_0 := K_\lambda - 1$ and

$$X_1 := F_r \triangleright X_0 = F_r \triangleright K_\lambda = F_r K_\lambda K_r - K_\lambda F_r K_r = (1 - q^{2nd_r}) F_r K_r K_\lambda.$$

Proposition 1 *The adjoint action turns $\mathfrak{u}_- := \mathrm{ad}(U_q(\mathfrak{l}))X_1$ into the irreducible finite-dimensional representation of $U_q(\mathfrak{l})$ with highest weight $-\alpha_r$.*

Proof. Since $\Delta(K_\mu) = K_\mu \otimes K_\mu$ and $S(K_\mu) = K_\mu^{-1}$ we have

$$K_\mu \triangleright X_1 = K_\mu X_1 K_\mu^{-1} = q^{-\langle \mu, \alpha_r \rangle} X_1 \quad \forall \mu \in \mathbf{P}.$$

Furthermore, for $i \neq r$ we have

$$E_i \triangleright X_1 = E_i F_r \triangleright K_\lambda = F_r E_i \triangleright K_\lambda = 0,$$

because K_λ commutes with all E_i, F_i for $i \neq r$ and therefore

$$E_i \triangleright K_\lambda = E_i K_\lambda - K_i K_\lambda K_i^{-1} E_i = E_i K_\lambda - E_i K_\lambda = 0.$$

Since X_0 belongs to the locally finite part of $U_q(\mathfrak{g})$ [J, 7.1.3] the vector space $\mathfrak{u}_-$ is finite-dimensional. Hence the claim follows. □

Fix a basis X_i of $\mathfrak{u}_-$ consisting of weight vectors and define $X^i := X_i^*$. Then the X^i form a basis of a vector space which we denote by $\mathfrak{u}_+$. Since

$$(X \triangleright Y)^* = \theta(X) \triangleright Y^* \tag{1}$$

this vector space is $\mathrm{ad}(U_q(\mathfrak{l}))$-invariant as well. Set $\mathfrak{u} := \mathfrak{u}_+ \oplus \mathfrak{u}_-$. We also introduce $\mathfrak{u}_0 := \{X \in \mathfrak{u} \,|\, X^* = X\}$ as an analogue of the real tangent space of M. It is invariant under $U_q(\mathfrak{l}_0) := \{X \in U_q(\mathfrak{l}) \,|\, \theta(X) = X\}$. Note that $U_q(\mathfrak{l}_0)$ is a subalgebra of $U_q(\mathfrak{l})$, but not a Hopf subalgebra [GZ].
Since the highest weight representations of quantized universal enveloping algebras are for q not a root of unity of the same structure as their classical counterparts the complex dimension of $\mathfrak{u}_\pm$ equals m.
The weights of X_i are all distinct, so after an appropriate normalization we have $\langle X_i, X^j \rangle = \delta_{ij}$.

5 The Clifford algebra

We now define a quantum Clifford algebra associated to M_q. We refer to [Fr] for the appearing notions from classical spin geometry.

The Clifford algebra $\mathrm{Cl}(2m,\mathbb{C})$ is the universal algebra for which there is a vector space embedding $\gamma : \mathbb{C}^{2m} \to \mathrm{Cl}(2m,\mathbb{C})$ such that $\gamma(v)^2 = -\sum_i v_i^2$ for all $v \in \mathbb{C}^{2m}$. The spin representation σ on the space $\Sigma_{2m} := \mathbb{C}^{2^m}$ of $2m$-spinors yields an isomorphism $\mathrm{Cl}(2m,\mathbb{C}) \simeq \mathrm{End}(\Sigma_{2m})$.
The crucial point leading to a quantum analogue of $\mathrm{Cl}(2m,\mathbb{C})$ in the context of quantum flag manifolds is that γ is $\mathfrak{so}(2m,\mathbb{C})$-equivariant. Here the representation of $\mathfrak{so}(2m,\mathbb{C})$ on $\mathbb{C}^{2m}$ is the vector representation ρ and the one on $\mathrm{End}(\Sigma_{2m}) \simeq \Sigma_{2m} \otimes \Sigma_{2m}^*$ is the tensor product $\sigma \otimes \sigma^*$ of the spin representation and the dual representation. In fact, the standard vector space isomorphism $\mathrm{Cl}(2m,\mathbb{C}) \simeq \Lambda^* \mathbb{C}^{2m}$ is an isomorphism of $\mathfrak{so}(2m,\mathbb{C})$-representations and γ is the restriction to $\mathbb{C}^{2m} = \Lambda^1 \mathbb{C}^{2m}$.
Not all flag manifolds are spin manifolds [CG], but all admit spin$^{\mathbb{C}}$ structures [Fr, Section 3.4]. In any case the embedding $\mathfrak{l} \subset \mathfrak{so}(2m,\mathbb{C})$ defines the representations ρ and σ of $\mathfrak{l}$ and ρ appears in $\sigma \otimes \sigma^*$.
These representations of $\mathfrak{l}$ can be deformed to representations of $U_q(\mathfrak{l})$ which we denote by the same symbols. The decomposition of $\sigma \otimes \sigma^*$ into irreducible components remains the same. Hence we have:

Proposition 2 *There is a $U_q(\mathfrak{l})$-equivariant embedding*

$$\gamma : \mathfrak{u}_+ \oplus \mathfrak{u}_- \to \mathrm{End}(\Sigma_{2m}).$$

Without loss of generality we can assume that

$$\gamma(X^i) = \overline{\gamma(X_i)}^T =: \gamma(X_i)^*,$$

because we could embed first only $\mathfrak{u}_-$ and take the above formula as the definition of $\gamma(X^i)$.
Note that the map γ is not uniquely determined by these conditions, but it always can be assumed to be a smooth deformation of the classical one. We call the algebra generated by $\gamma(\mathfrak{u}_0)$ and $\gamma(\mathfrak{u})$ the real and complex quantum Clifford algebra associated to the quantum flag manifold M_q.

6 The spinor bundle

Next we define a spinor bundle $\mathcal{S}$ over M_q in form of a quantum homogeneous vector bundle [GZ] generalizing the homogeneous vector bundle $G_0 \times_{L_0} \Sigma_{2m}$ over M. It is defined in terms of the following vector space whose elements are interpreted as its sections:

$$\begin{aligned} \Gamma(M_q,\mathcal{S}) &:= \{\psi \in \mathbb{C}_q[G] \otimes \Sigma_{2m} \mid X \triangleright \psi = \sigma(S(X))\psi \; \forall X \in U_q(\mathfrak{l})\} \\ &\simeq \bigoplus_{\lambda \in \mathbf{P}^+} V_\lambda \otimes \mathrm{Hom}_{U_q(\mathfrak{l})}(V_\lambda, \Sigma_{2m}). \end{aligned} \tag{2}$$

The isomorphism $\simeq$ is given by Peter-Weyl decomposition of $\mathbb{C}_q[G]$.
If $\{A_i^\lambda\}$ are bases of $\mathrm{Hom}_{U_q(\mathfrak{l})}(V_\lambda, \Sigma_{2m})$ for all λ for which this space is non-trivial and if the matrix coefficients t_{ij}^λ of the Peter-Weyl basis are defined with respect to the bases $\{v_j^\lambda\}$ of V_λ, then the elements

$$\psi_{ij}^\lambda := \sum_k S(t_{kj}^\lambda) \otimes A_i^\lambda(v_k^\lambda)$$

form a basis of $\Gamma(M_q, \mathcal{S})$.
We define a Hermitian inner product $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ on $\Gamma(M_q, \mathcal{S})$ by applying $\langle\cdot,\cdot\rangle_h$ to $\mathbb{C}_q[G]$ and the invariant Hermitian inner product $(\cdot,\cdot)_\sigma$ to Σ_{2m}. We can choose the basis ψ_{ij}^λ to be orthonormal. We complete $\Gamma(M_q, \mathcal{S})$ to a Hilbert space $\mathcal{H}$ which we call the space of square-integrable spinor fields on the quantum flag manifold M_q.
The quantized universal enveloping algebra $U_q(\mathfrak{g})$ acts on $\Gamma(M_q, \mathcal{S})$ from the right (by acting from the right on $\mathbb{C}_q[G]$). The multiplication in $\mathbb{C}_q[G]$ defines a $\mathbb{C}_q[M]$-bimodule structure on $\Gamma(M_q, \mathcal{S})$ and when restricting to a one-sided action one obtains a projective module over M_q [GZ].

7 The Dirac operator

Let D_- be the linear operator acting on $\mathrm{Hom}_{U_q(\mathfrak{l})}(V_\lambda, \Sigma_{2m})$ by

$$D_- : A \mapsto -\sum_i \gamma(X^i) \circ A \circ \rho_\lambda(X_i).$$

The following proposition shows that D_- is well-defined.

Proposition 3 *We have $D_-(A) \in \mathrm{Hom}_{U_q(\mathfrak{l})}(V_\lambda, \Sigma_{2m})$.*

Proof. For $Y \in U_q(\mathfrak{l})$ we have

$$\begin{aligned}
& \sum_i \gamma(X^i) \circ A \circ \rho_\lambda(X_i)\rho_\lambda(S(Y)) \\
= \quad & \sum_i \gamma(X^i) \circ A \circ \rho_\lambda(S(Y_{(1)})Y_{(2)}X_i S(Y_{(3)})) \\
= \quad & \sum_i \gamma(X^i)\sigma(S(Y_{(1)})) \circ A \circ \rho_\lambda(Y_{(2)} \triangleright X_i) \\
= \quad & \sum_{ij} \gamma(X^i)\sigma(S(Y_{(1)})) \circ A \circ \rho_\lambda(\langle Y_{(2)} \triangleright X_i, X^j\rangle X_j) \\
= \quad & \sum_{ij} \gamma(\langle X_i, S(Y_{(2)}) \triangleright X^j\rangle X^i)\sigma(S(Y_{(1)})) \circ A \circ \rho_\lambda(X_j) \\
= \quad & \sum_j \gamma(S(Y_{(2)}) \triangleright X^j)\sigma(S(Y_{(1)})) \circ A \circ \rho_\lambda(X_j) \\
= \quad & \sum_j \sigma(S(Y_{(3)}))\gamma(X^j)\sigma(S^2(Y_{(2)}))\sigma(S(Y_{(1)})) \circ A \circ \rho_\lambda(X_j) \\
= \quad & \sigma(S(Y)) \sum_j \gamma(X^j) \circ A \circ \rho_\lambda(X_j),
\end{aligned}$$

where we used the Hopf algebra axioms and the equivariance of γ. □

The resulting operator on $\Gamma(M_q, \mathcal{S})$ which acts trivially on V_λ in (2) will be denoted by the same symbol. It can be extended to a linear operator on $\mathbb{C}_q[G] \otimes \Sigma_{2m}$ acting by

$$D_- : f \otimes v \mapsto -\sum_j (S^{-1}(X_j) \triangleright f) \otimes \gamma(X^j)v.$$

We consider D_- as densely defined operator on $\mathcal{H}$. Analogously there is an operator D_+ acting on $\mathbb{C}_q[G] \otimes \Sigma_{2m}$ by

$$D_+ : f \otimes v \mapsto -\sum_j (S^{-1}(X^j) \triangleright f) \otimes \gamma(X_j)v.$$

Finally we define the Dirac operator $D := D_+ + D_-$. Notice that for $X \in U_q(\mathfrak{g})$ and $f, g \in \mathbb{C}_q[G]$ the $U_q(\mathfrak{g})$-invariance of h and (1) imply

$$\begin{aligned} h((X \triangleright f)g^*) &= h((X_{(1)} \triangleright f)(X_{(2)}S(X_{(3)}) \triangleright g^*)) \\ &= h(X_{(1)} \triangleright (f(S(X_{(2)}) \triangleright g^*))) \\ &= h(f(S^2(X)^* \triangleright g)^*). \end{aligned}$$

Hence D is symmetric on the domain $\Gamma(M_q, \mathcal{S})$. It is the direct sum of its restrictions to the spaces $V_\lambda \otimes \mathrm{Hom}_{U_q(\mathfrak{l})}(V_\lambda, \Sigma_{2m})$ which are all finite-dimensional and pairwise orthogonal. Hence it becomes diagonal in a suitable orthonormal basis and therefore extends to a self-adjoint operator on $\mathcal{H}$ which we denote by D as well.
It seems to be a non-trivial task to generalize Parthasarathy's formula for D^2 [Pa] to the quantum case and to calculate explicitly the spectrum of D as in [CFG]. But since only finite matrices are involved which are smooth deformations of those describing the classical Dirac operator, the spectrum should as well be a smooth deformation of the classical spectrum.
In the simplest example of a generalized flag manifold $M = \mathbb{C}P^1 \simeq S^2$ the corresponding quantum flag manifold is one of Podleś' quantum spheres [Po]. In this case L. Dabrowski and A. Sitarz derived a Dirac operator starting with an ansatz and implementing the axioms for real spectral triples [DS1]. It follows from the uniqueness result [DS1, Lemma 5] that the Hilbert space representation of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ and $\mathbb{C}_q[M]$ calculated in [DS1] is the one on $\mathcal{H}$ considered here. Inserting the explicit formulae for the left action of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ on the Peter-Weyl basis one sees that the Dirac operators also coincide.

8 The differential calculus

In this section we study the covariant first-order differential calculi over $\mathbb{C}_q[M]$ induced by $D_\pm$ and D. We refer to [KS] and [HK2] for the general theory of covariant differential calculi on quantum groups and quantum homogeneous spaces.
The author's main impetus to study quantum flag manifolds was the result of [HK1] mentioned already in the introduction that on quantum flag manifolds as discussed here there exist exactly two finite-dimensional irreducible (first-order) covariant differential calculi $(\Gamma_\pm, d_\pm)$. These calculi have dimension m and their direct sum (Γ, d) is a $*$-calculus.
A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ over a $*$-algebra $\mathcal{A}$ (cf. [C]) always induces a differential $*$-calculus with $df := i[D, f]$, $f \in \mathcal{A}$. The result of this section will be that the operators $D_\pm, D$ realize the calculi $\Gamma_\pm, \Gamma$ in this way by bounded operators on $\mathcal{H}$.
We will treat only Γ_-, the analogous results for Γ_+ and Γ are immediate.

In the rest of this paper, elements of $\mathbb{C}_q[M]$ will always be treated as linear operators on $\mathcal{H}$ by considering the *right* action of $\mathbb{C}_q[M]$ on $\Gamma(M_q,\mathcal{S})$. If one rewrites this paper starting with the left coset space $P\setminus G$, the constructions will work for the left action instead.
We denote by Γ'_- the differential calculus over $\mathbb{C}_q[M]$ defined by D_-:

$$\Gamma'_- := \Big\{ i\sum_j f_j[D_-,g_j] \,\Big|\, f_j,g_j\in\mathbb{C}_q[M]\Big\} \subset \mathrm{End}(\Gamma(M_q,\mathcal{S})).$$

Then the following formula for $d'_-f := i[D_-,f]$ holds:

Proposition 4 *For all $f\in\mathbb{C}_q[M]$ we have*

$$d'_-f = -i\sum_{i=1}^m S^{-1}(X_i)\triangleright f\otimes\sigma(K_\lambda)\gamma(X^i).$$

Proof. The coproduct of $S^{-1}(X_1) = -(1-q^{2nd_r})K_\lambda^{-1}F_r$ is given by

$$S^{-1}(X_1)\otimes K_r^{-1}K_\lambda^{-1} + K_\lambda^{-1}\otimes S^{-1}(X_1).$$

Since $X_j = Y\triangleright X_1$ for some $Y\in U_q(\mathfrak{l})$ one obtains for $f\in\mathbb{C}_q[M]$ and $\sum_i g_i\otimes v_i\in\Gamma(M_q,\mathcal{S})$ the relation

$$\begin{aligned}
&\sum_i S^{-1}(X_j)\triangleright(g_if)\otimes v_i\\
=\;&\sum_i (Y_{(3)}S^{-1}(X_1)S^{-1}(Y_{(2)})\triangleright g_i)(Y_{(4)}K_r^{-1}K_\lambda^{-1}S^{-1}(Y_{(1)})\triangleright f)\otimes v_i\\
&+(Y_{(3)}K_\lambda^{-1}S^{-1}(Y_{(2)})\triangleright g_i)(Y_{(4)}S^{-1}(X_1)S^{-1}(Y_{(1)})\triangleright f)\otimes v_i\\
=\;&\sum_i (S^{-1}(X_j)\triangleright g_i)f\otimes v_i + g_i(S^{-1}(X_j)\triangleright f)\otimes K_\lambda\triangleright v_i,
\end{aligned}$$

where we used the defining properties of $\mathbb{C}_q[M],\Gamma(M_q,\mathcal{S})$ and the fact that K_λ commutes with elements of $U_q(\mathfrak{l})$. □

Since the right multiplication operators $R_g : f\mapsto fg$, $f,g\in\mathbb{C}_q[G]$ extend to bounded operators on the Hilbert space obtained by completing $\mathbb{C}_q[G]$ with respect to Haar measure Proposition 4 implies:

Corollary 1 *The elements of Γ'_- extend to bounded operators on $\mathcal{H}$.*

By [HK2, Corollary 5] there is a one-to-one correspondence between m-dimensional covariant differential calculi over $\mathbb{C}_q[M]$ and $m+1$-dimensional subspaces $\mathcal{T}$ of $\mathbb{C}_q[M]^\circ$ such that

$$\varepsilon\in\mathcal{T},\quad \Delta(\mathcal{T})\subset\mathcal{T}\otimes\mathbb{C}_q[M]^\circ,\quad U_q(\mathfrak{l})\mathcal{T}\subset\mathcal{T}. \tag{3}$$

Here $\mathbb{C}_q[M]^\circ$ denotes the dual coalgebra of $\mathbb{C}_q[M]$, see [HK2]. In view of [HK1, Theorem 6.5] it is sufficient to consider $\mathcal{T}\subset\pi(U_q(\mathfrak{g}))$, where $\pi:\mathbb{C}_q[G]^\circ\to\mathbb{C}_q[M]^\circ$ is the restriction map. Then $U_q(\mathfrak{l})\mathcal{T}\subset\mathcal{T}$ means that $\pi(XY)\in\mathcal{T}$ for all $X\in U_q(\mathfrak{l})$ and $Y\in U_q(\mathfrak{g})$ with $\pi(Y)\in\mathcal{T}$. The vector space $\mathcal{T}^0 := \{X\in\mathcal{T}\mid X(1)=0\}$ is called the quantum tangent space of the corresponding differential calculus.

Proposition 5 *The vector space $\mathcal{T}_-^0\subset\mathbb{C}_q[M]^\circ$ spanned by $\pi\circ S^{-1}(X_i)$, $i=1,\ldots,m$ coincides with the quantum tangent space of Γ_-.*

Proof. For $f \in \mathbb{C}_q[M]$ we have

$$F_r K_r K_\lambda(f) = F_r((K_r K_\lambda) \triangleright f) = F_r(f)$$

and similarly

$$(Y \triangleright X_1)(f) = Y X_1(f) \quad \forall Y \in U_q(\mathfrak{l}).$$

Hence the claim reduces to the fact that the tangent space of Γ_- is $U_q(\mathfrak{l})\pi(F_r) \subset \mathbb{C}_q[M]^\circ$, see [HK1]. $\square$

It remains to show that Γ'_- is indeed isomorphic to Γ_-. To see this we first realize Γ_- as a calculus induced by a $m+1$-dimensional covariant differential calculus over $\mathbb{C}_q[G]$. This calculus has tangent space

$$\mathcal{T}_-^{G0} := \mathbb{C}S^{-1}(X_0) \oplus S^{-1}(\mathrm{ad}(U_q(\mathfrak{l}))X_1) \subset U_q(\mathfrak{g}).$$

Using that S^{-1} is a coalgebra antihomomorphism and that K_λ commutes with all elements of $U_q(\mathfrak{l})$ one calculates that

$$\begin{aligned} & \Delta(S^{-1}(Y \triangleright X_1)) \\ = \ & K_\lambda^{-1} \otimes S^{-1}(Y \triangleright X_1) + S^{-1}(Y_{(2)} \triangleright X_1) \otimes S^{-1}(Y_{(1)} K_r K_\lambda S(Y_{(3)})) \\ \in \ & \mathcal{T}_-^G \otimes U_q(\mathfrak{g}) \end{aligned}$$

for all $Y \in U_q(\mathfrak{l})$, where $\mathcal{T}_-^G := \mathbb{C} \cdot 1 \oplus \mathcal{T}_-^{G0}$. Therefore there is indeed a differential calculus Γ_-^G over $\mathbb{C}_q[G]$ with quantum tangent space $\mathcal{T}_-^{G0}$ (the last condition in (3) becomes trivial on quantum groups). By construction we have $\pi(\mathcal{T}_-^{G0}) = \mathcal{T}_-^0$ and hence Γ_-^G induces Γ_- [HK2, Corollary 9].

The general theory of covariant differential calculi over Hopf algebras with invertible antipode (see [KS, Section 14.1]) implies that in Γ_-^G the differential can be written as

$$d_-^G f = \sum_{i=0}^{m} (S^{-1}(X_i) \triangleright f) \cdot \omega^i \quad \forall f \in \mathbb{C}_q[G], \tag{4}$$

where $\{\omega^i\}$ is a basis of Γ_-^G consisting of invariant 1-forms. Proposition 4 generalizes the above formula to differential calculi over quantum flag manifolds.

The relation (4) implies in particular that

$$\sum_i f_i d_-^G g_i = 0 \quad \Leftrightarrow \quad \sum_i f_i (S^{-1}(X_j) \triangleright g_i) = 0 \quad \forall j.$$

The matrices $\sigma(K_\lambda)\gamma(X^i)$ are linearly independent, because $\sigma(K_\lambda)$ is invertible, γ is injective and the X^i are linearly independent. Furthermore $\mathbb{C}_q[G]$ is free of zero divisors [J, 9.1.9]. Hence Proposition 4 implies

$$\sum_i f_i d'_- g_i = 0 \quad \Leftrightarrow \quad \sum_i f_i (S^{-1}(X_j) \triangleright g_i) = 0$$

and we obtain:

Proposition 6 *The map*

$$\psi : \Gamma_- \to \Gamma'_-, \quad \sum_j f_j d_- g_j \mapsto \sum_j f_j d'_- g_j$$

is an isomorphism of differential calculi.

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