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Duality and products in algebraic (co)homology theories

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ABSTRACT

The origin and interplay of products and dualities in algebraic (co)homology theories is ascribed to a $\times_A$-Hopf algebra structure on the relevant universal enveloping algebra. This provides a unified treatment for example of results by Van den Bergh about Hochschild (co)homology and by Huebschmann about Lie–Rinehart (co)homology.

1. Introduction

Most classical (co)homology theories of algebraic objects such as groups or Lie, Lie–Rinehart or associative algebras can be realised as

$$H^\bullet(X, M) := \mathrm{Ext}^\bullet_U(A, M), \qquad H_\bullet(X, N) := \mathrm{Tor}^U_\bullet(N, A) \tag{1}$$

for an augmented ring $X = (U, A)$ (a ring U with a distinguished left module A) that is functorially attached to a given object. The cohomology coefficients are left U-modules M and those in homology are right U-modules N.

Our aim here is to clarify the origin and interplay of products and dualities between such (co)homology groups, and to provide a unified treatment of results by Van den Bergh on Hochschild (co)homology [25] and by Huebschmann on Lie–Rinehart (co)homology [7]. The key concept involved is that of a $\times_A$-Hopf algebra introduced by Schauenburg [21].

The main results can be summarised as follows:

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Theorem 1. *For any A-biprojective $\times_A$-Hopf algebra U there is a functor*

$$\otimes : U\text{-}\mathbf{Mod} \times U^{\mathrm{op}}\text{-}\mathbf{Mod} \to U^{\mathrm{op}}\text{-}\mathbf{Mod}$$

*that induces for $M \in U$-**Mod**, $N \in U^{\mathrm{op}}$-**Mod** and $m, n \geqslant 0$ natural products*

$$\frown : \mathrm{Ext}^m_U(A, M) \times \mathrm{Tor}^U_n(N, A) \to \mathrm{Tor}^U_{n-m}(M \otimes N, A).$$

*If $A \in U$-**Mod** admits a finitely generated projective resolution of finite length and there exists $d \geqslant 0$ with $\mathrm{Ext}^m_U(A, U) = 0$ for $m \neq d$, then there is a canonical element*

$$[\omega] \in \mathrm{Tor}^U_d\big(A^*, A\big), \quad A^* := \mathrm{Ext}^d_U(A, U)$$

*such that for $m \geqslant 0$ and $M \in U$-**Mod** with $\mathrm{Tor}^A_q(M, A^*) = 0$ for $q > 0$ the map*

$$\cdot \frown [\omega] : \mathrm{Ext}^m_U(A, M) \to \mathrm{Tor}^U_{d-m}\big(M \otimes A^*, A\big)$$

is an isomorphism.

As we will recall below, $\times_A$-bialgebras and $\times_A$-Hopf algebras generalise bialgebras and Hopf algebras towards noncommutative base algebras A. Besides Hopf algebras, both the universal enveloping algebra $U(A, L)$ of a Lie–Rinehart algebra (A, L) and the enveloping algebra $A^e = A \otimes_k A^{\mathrm{op}}$ of an associative algebra A are $\times_A$-Hopf algebras.

For any $\times_A$-bialgebra U, the base algebra A carries a left U-action and the category U-**Mod** of left U-modules is monoidal with unit object A. But only for $\times_A$-Hopf algebras one has a canonical operation $\otimes$ as in Theorem 1 which turns U^{op}-**Mod** into a module category over (U-**Mod**, $\otimes$, A) (Lemma 3).

Any $\times_A$-Hopf algebra U carries two left and two right actions of the base algebra A, all commuting with each other. The biprojectivity assumed in Theorem 1 refers to the projectivity of two of these, see Section 2.1. Under this condition, we can use the elegant formalism of suspended monoidal categories from [23] to define for $M, N \in U$-**Mod** and $P \in U^{\mathrm{op}}$-**Mod** products

$$\smile : H^m(X, M) \times H^n(X, N) \to H^{m+n}(X, M \otimes N),$$

$$\frown : H^n(X, N) \times H_p(X, P) \to H_{p-n}(X, N \otimes P),$$

where we again use the abbreviations from Eq. (1) above (cf. Sections 3.2 and 3.5).

In the last part of Theorem 1, $A^* = H^d(X, U) = \mathrm{Ext}^d_U(A, U)$ is a right U-module via right multiplication in U, and if we define the functor

$$\hat{} : U\text{-}\mathbf{Mod} \to U^{\mathrm{op}}\text{-}\mathbf{Mod}, \quad M \mapsto \hat{M} := M \otimes A^*,$$

then the statement can be rewritten as an isomorphism

$$H^m(X, M) \simeq H_{\dim(X)-m}(X, \hat{M}), \quad \dim(X) := \mathrm{proj.dim}_U(A)$$

that is given by the cap product with the fundamental class $[\omega] \in H_{\dim(X)}(X, \hat{A})$ which corresponds under the duality to $\mathrm{id}_A \in H^0(X, A) = \mathrm{Hom}_U(A, A)$. For $M = A$ this simply means that the $H^\bullet(X, A)$-module $H_\bullet(X, A^*)$ is free with generator $[\omega]$.

Theorem 1 is well known in group and Lie algebra (co)homology. For $U = A \otimes_k A^{\mathrm{op}}$ it reduces to Van den Bergh's result [25] that stimulated a lot of recent research, see e.g. [2,4,5,13]. Note that we do not need Van den Bergh's invertibility assumption about A^* which says that $\hat{}$ is an equivalence. However, it is satisfied for many well-behaved algebras [2,4,5,13] and implies the condition $\mathrm{Tor}_q^A(M, A^*) = 0$ for arbitrary A-bimodules M (since invertible bimodules are finitely generated projective as one-sided modules from either side). For Lie–Rinehart algebras (A, L), Theorem 1 is due to Huebschmann [7], and we find the general setting helpful for example to understand the different roles of left and right modules that he has observed. As Huebschmann has showed, the conditions of Theorem 1 are satisfied whenever L is finitely generated projective over A, and A^* coincides as an A-module with $\Lambda_A^d L$ and is in particular projective, so also here we have $\mathrm{Tor}_q^A(M, A^*) = 0$ for arbitrary (A, L)-modules M.

We were also motivated by the current discussion of the numerous bialgebroid generalisations of Hopf algebras, see [1]. Several authors have raised the question where Lie–Rinehart algebras fit in. They were shown in [27,17] to be $\times_A$-bialgebras, see also [11,8]; here we add the observation that they are in fact always $\times_A$-Hopf algebras. So both these examples and the applications in homological algebra clearly demonstrate the relevance of the concept of a $\times_A$-Hopf algebra.

Theorem 1 could be generalised to differentially graded $\times_A$-Hopf algebras, sheaves of such, or suitable abstract monoidal categories. One can also drop the condition $\mathrm{Ext}_U^n(A, U) = 0$ for $n \neq d$ and the assumption that $\mathrm{Tor}_q^A(M, A^*) = 0$. Then one obtains for a bounded below chain complex M over U-**Mod** an isomorphism $\mathrm{RHom}_U(A, M) \simeq (M \otimes_A^{\mathrm{L}} \mathrm{RHom}_U(A, U)) \otimes_U^{\mathrm{L}} A$.

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2. Preliminaries on $\times_A$-Hopf algebras

2.1. Some conventions

Throughout this paper, "ring" means "unital and associative ring", and we fix a commutative ring k. All other algebras, modules, etc., will have an underlying structure of a k-module. Secondly, we fix a k-algebra A, i.e., a ring with a ring homomorphism $\eta_A : k \to Z(A)$ to its centre. We denote by A-**Mod** the category of left A-modules, by A^{op} the opposite and by $A^e := A \otimes_k A^{\mathrm{op}}$ the enveloping algebra of A. Thus left A^e-modules are A-bimodules with symmetric action of k.

Our main object is finally an algebra U over A^e, where we now refer to the less standard notion of an algebra over a possibly noncommutative base algebra: U is a k-algebra with a k-algebra homomorphism $\eta = \eta_U : A^e \to U$. This gives rise to a forgetful functor U-**Mod** $\to A^e$-**Mod** using which we consider every U-module M also as an A-bimodule with actions

$$a \triangleright m \triangleleft b := \eta(a \otimes_k b)m, \quad a, b \in A,\ m \in M. \tag{2}$$

Similarly, every right U-module N is also an A-bimodule via

$$a \blacktriangleright m \blacktriangleleft b := n\eta(b \otimes_k a), \quad a, b \in A,\ n \in N. \tag{3}$$

In particular, U itself carries two left and two right A-actions all commuting with each other. Usually we consider U as an A^e-module using $a \triangleright u \triangleleft b$, and otherwise we write e.g. ${}_\blacktriangleright U_\triangleleft$ to denote which actions are considered. Since this will be repeatedly a necessary technical condition, we define:

Definition 1. For an A^e-algebra U we call $M \in U$-**Mod** A-biprojective if both ${}_\triangleright M \in A$-**Mod** and $M_\triangleleft \in A^{\mathrm{op}}$-**Mod** are projective modules.

2.2. $\times_A$-bialgebras [24]

Consider an A^e-algebra U as above which is also a coalgebra in the monoidal category A^e-**Mod**. That is, there are maps

$$\Delta : U \to U \otimes_A U, \qquad \varepsilon : U \to A$$

satisfying the usual coalgebra axioms (see e.g. [1] for the details), where

$$U \otimes_A U = U \otimes_k U / \operatorname{span}_k\{u \lhd a \otimes_k v - u \otimes_k a \rhd v \mid a \in A,\ u, v \in U\}. \tag{4}$$

For $A = k$ one calls U a bialgebra if Δ and ε are algebra homomorphisms, but in general there is no natural algebra structure on $U \otimes_A U$. The way out of this problem was found by Takeuchi [24] and involves the embedding

$$\iota : U \times_A U \to U \otimes_A U, \tag{5}$$

where $U \times_A U$ is the centre of the A-bimodule ${}_\blacktriangleright U_\lhd \otimes_A {}_\rhd U_\blacktriangleleft$:

$$U \times_A U := \left\{ \sum_i u_i \otimes_A v_i \in U \otimes_A U \;\middle|\; \sum_i a \blacktriangleright u_i \otimes_A v_i = \sum_i u_i \otimes_A v_i \blacktriangleleft a \right\}.$$

The product of U turns this into an algebra over A^e, with

$$\eta_{U \times_A U} : A^e \to U \times_A U, \quad a \otimes_k b \mapsto \eta(a \otimes_k 1) \otimes_A \eta(1 \otimes_k b).$$

Similarly, A is an algebra over k, but not over A^e in general. To handle this one needs the canonical map

$$\pi : \operatorname{End}_k(A) \to A, \quad \varphi \mapsto \varphi(1), \tag{6}$$

and the fact that $\operatorname{End}_k(A)$ is an algebra over A^e, with

$$\eta_{\operatorname{End}_k(A)} : A^e \to \operatorname{End}_k(A), \qquad \big(\eta_{\operatorname{End}_k(A)}(a \otimes b)\big)(c) := acb.$$

Now it makes sense to require Δ and ε to factor through ι and π:

Definition 2. A (left) $\times_A$-bialgebra is an algebra U over A^e together with two homomorphisms $\hat{\Delta} : U \to U \times_A U$ and $\hat{\varepsilon} : U \to \operatorname{End}_k(A)$ of algebras over A^e such that U is a coalgebra in A^e-**Mod** via $\Delta = \iota \circ \hat{\Delta}$ and $\varepsilon = \pi \circ \hat{\varepsilon}$.

So one has for example for any $\times_A$-bialgebra

$$\Delta(a \rhd u \lhd b) = a \rhd u_{(1)} \otimes_A u_{(2)} \lhd b, \qquad \Delta(a \blacktriangleright u \blacktriangleleft b) = u_{(1)} \blacktriangleleft b \otimes_A a \blacktriangleright u_{(2)},$$

where we started to use Sweedler's shorthand notation $u_{(1)} \otimes_A u_{(2)}$ for $\Delta(u)$.

Be aware that the four A-actions are not the only feature that disappears for $A = k$. Another crucial one is that the counit $\varepsilon : U \to A$ is not necessarily a ring homomorphism. Note also that many authors write $s(a) := \eta(a \otimes 1)$ and $t(a) := \eta(1 \otimes a)$ and formulate the theory using these so-called source and target maps rather than η.

2.3. The monoidal category U-**Mod** [20]

Definition 2 might appear complicated, but is the correct concept from several points of view. For example, there is the following result of Schauenburg [20, Theorem 5.1]:

Theorem 2. *The $\times_A$-bialgebra structures on an algebra $\eta : A^e \to U$ over A^e correspond bijectively to monoidal structures on U-**Mod** for which the forgetful functor U-**Mod** $\to A^e$-**Mod** induced by η is strictly monoidal.*

Given a $\times_A$-bialgebra structure on U, the monoidal structure on U-**Mod** is defined as for bialgebras: one takes the tensor product $M \otimes_A N$ of the A-bimodules underlying $M, N \in U$-**Mod** and defines a left U-action via Δ,

$$u(m \otimes_A n) := u_{(1)}m \otimes_A u_{(2)}n, \quad u \in U,\ m \in M,\ n \in N. \tag{7}$$

Definition 3. If U is a $\times_A$-bialgebra and $M, N \in U$-**Mod** are left U-modules, we denote the left U-module $M \otimes_A N$ with U-action (7) by $M \otimes N$.

The unit object in U-**Mod** is A on which U acts via

$$\hat{\varepsilon}(u)(a) = \varepsilon(a \blacktriangleright u) = \varepsilon(u \blacktriangleleft a),$$

where the last equality is a consequence of the definition of a $\times_A$-bialgebra.

There is an analogous notion of right $\times_A$-bialgebra for which U^{op}-**Mod** is monoidal. However, for a left $\times_A$-bialgebra there is no canonical monoidal structure on U^{op}-**Mod** or even only right action of U on A.

2.4. $\times_A$-Hopf algebras [21]

Let U be a $\times_A$-bialgebra and define

$$\beta : {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} \to U_{\triangleleft} \otimes_A {}_{\triangleright}U, \quad u \otimes_{A^{\mathrm{op}}} v \mapsto u_{(1)} \otimes_A u_{(2)}v, \tag{8}$$

the so-called Galois map of U, where

$${}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} = U \otimes_k U / \operatorname{span}\{a \blacktriangleright u \otimes_k v - u \otimes_k v \triangleleft a \mid u, v \in U,\ a \in A\}.$$

One could flip the tensor components in order to avoid taking the tensor product over A^{op}, but we found it more convenient to keep β in the form which is standard for bialgebras over fields. For the latter it is easily seen that β is bijective if and only if U is a Hopf algebra with $\beta^{-1}(u \otimes_k v) := u_{(1)} \otimes S(u_{(2)})v$, where S is the antipode of U. This motivates the following definition due to Schauenburg [21]:

Definition 4. A $\times_A$-bialgebra U is a $\times_A$-Hopf algebra if β is a bijection.

Following Schauenburg, we adopt a Sweedler-type notation

$$u_+ \otimes_{A^{\mathrm{op}}} u_- := \beta^{-1}(u \otimes_A 1) \tag{9}$$

for the so-called translation map

$$\beta^{-1}(\cdot \otimes_A 1) : U \to {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft}.$$

Since substantial for the subsequent calculations, we list some properties of β^{-1} as proven in [21, Proposition 3.7]: one has for all $u, v \in U$, $a, b \in A$:

$$u_{+(1)} \otimes_A u_{+(2)} u_- = u \otimes_A 1 \in U_{\triangleleft} \otimes_A {}_{\triangleright} U, \tag{10}$$

$$u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-} u_{(2)} = u \otimes_{A^{\mathrm{op}}} 1 \in {}_{\blacktriangleright} U \otimes_{A^{\mathrm{op}}} U_{\triangleleft}, \tag{11}$$

$$u_+ \otimes_{A^{\mathrm{op}}} u_- \in U \times_{A^{\mathrm{op}}} U, \tag{12}$$

$$u_+ \otimes_{A^{\mathrm{op}}} u_{-(1)} \otimes_A u_{-(2)} = u_{++} \otimes_{A^{\mathrm{op}}} u_- \otimes_A u_{+-}, \tag{13}$$

$$(uv)_+ \otimes_{A^{\mathrm{op}}} (uv)_- = u_+ v_+ \otimes_{A^{\mathrm{op}}} v_- u_-, \tag{14}$$

$$\eta(a \otimes b)_+ \otimes_{A^{\mathrm{op}}} \eta(a \otimes b)_- = \eta(a \otimes 1) \otimes_{A^{\mathrm{op}}} \eta(b \otimes 1), \tag{15}$$

where in (12) we abbreviated

$$U \times_{A^{\mathrm{op}}} U := \left\{ \sum_i u_i \otimes_{A^{\mathrm{op}}} v_i \in {}_{\blacktriangleright} U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} \;\middle|\; \sum_i u_i \triangleleft a \otimes_{A^{\mathrm{op}}} v_i = \sum_i u_i \otimes_{A^{\mathrm{op}}} a \blacktriangleright v_i \right\}$$

and in (13) the tensor product over A^{op} links the first and third tensor component (cf. [21, Eq. (3.7)]). By (10) and (12) one can write

$$\beta^{-1}(u \otimes_A v) = u_+ \otimes_{A^{\mathrm{op}}} u_- v \tag{16}$$

which is easily checked to be well defined over A with (14) and (15).

2.5. Examples

Clearly, Hopf algebras over k such as universal enveloping algebras of Lie algebras or group algebras are $\times_k$-Hopf algebras. But also the enveloping algebra of an associative algebra that governs Hochschild (co)homology is an example as pointed out by Schauenburg [21]:

Example 1. The enveloping algebra $U := A^e$ of any k-algebra A is a $\times_A$-bialgebra with $\eta = \mathrm{id}_{A^e}$ and coproduct and counit

$$\Delta : U \to U \otimes U, \quad a \otimes_k b \mapsto (a \otimes_k 1) \otimes_A (1 \otimes_k b), \qquad \varepsilon : U \to A, \quad a \otimes_k b \mapsto ab.$$

As for the $\times_A$-Hopf algebra structure, the tensor product in question reads

$${}_{\blacktriangleright} U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} = U \otimes_k U / \mathrm{span}_k \big\{ (a \otimes_k cb) \otimes_k \big(a' \otimes_k b'\big) - (a \otimes_k b) \otimes_k \big(a' \otimes_k b'c\big) \big\},$$

where cb and $b'c$ is understood to be the product in A. One then easily verifies that

$$(a \otimes_k b)_+ \otimes_{A^{\mathrm{op}}} (a \otimes_k b)_- := (a \otimes_k 1) \otimes_{A^{\mathrm{op}}} (b \otimes_k 1)$$

yields an inverse of the Galois map defined as in (16).

Finally we discuss Lie–Rinehart algebras which define for example Poisson (co)homology. Several authors [27,11,17] have shown that their enveloping algebras are $\times_A$-bialgebras, but they are in fact $\times_A$-Hopf algebras:

Example 2. Let (A, L) be a Lie–Rinehart algebra over k [19,6]. We denote by $(a, X) \mapsto aX$ the A-module structure on L and by $(X, a) \mapsto X(a)$ the L-action on A given by the anchor $\hat{\varepsilon} : L \to \mathrm{Der}_k(A)$. Its universal enveloping algebra $U = U(A, L)$ is the universal k-algebra equipped with two maps

$$\iota_A : A \to U, \qquad \iota_L : L \to U$$

of k-algebras and of k-Lie algebras, respectively, and subject to the identities

$$\iota_A(a)\iota_L(X) = \iota_L(aX), \qquad \iota_L(X)\iota_A(a) - \iota_A(a)\iota_L(X) = \iota_A\big(X(a)\big)$$

for $a \in A$, $X \in L$; confer [19] for the precise construction. The map ι_A is injective, so we refrain from further mentioning it. We will also merely write X when we mean $\iota_L(X)$ (if L is A-projective, then ι_L is injective as well).

Recall now from e.g. [27,17] that U carries the structure of a $\times_A$-bialgebra: the maps $\eta(- \otimes 1)$ and $\eta(1 \otimes -)$ are equal and given by ι_A. The prescriptions

$$\Delta(X) = 1 \otimes_A X + X \otimes_A 1, \qquad \Delta(a) = a \otimes_A 1 \tag{17}$$

which map $X \in L$ and $a \in A$ into $U \times_A U$ can be extended by the universal property to a coproduct $\hat{\Delta} : U \to U \times_A U$. The counit is similarly given by the extension of the anchor $\hat{\varepsilon}$ to U. The bijectivity of the Galois map is seen in the same way: the translation map is given on generators as

$$a_+ \otimes_{A^{\mathrm{op}}} a_- := a \otimes_{A^{\mathrm{op}}} 1, \qquad X_+ \otimes_{A^{\mathrm{op}}} X_- := X \otimes_{A^{\mathrm{op}}} 1 - 1 \otimes_{A^{\mathrm{op}}} X. \tag{18}$$

These maps stay in $U \times_{A^{\mathrm{op}}} U$ which is an algebra through the product of U in the first and its opposite in the second tensor factor. By universality we obtain a map $U \to U \times_{A^{\mathrm{op}}} U$, and then β^{-1} is defined using (16).

3. Multiplicative structures

3.1. $\mathcal{D}^-(U)$ as a suspended monoidal category [23]

For any ring U, we denote by $\mathcal{D}^-(U)$ the derived category of bounded above cochain complexes of left U-modules. As usual, we identify any $M \in U$-**Mod** with a complex in $\mathcal{D}^-(U)$ concentrated in degree 0, and any bounded below chain complex $P_\bullet$ with a bounded above cochain complex by putting $P^n := P_{-n}$.

If U is an A-biprojective $\times_A$-bialgebra, then any projective $P \in U$-**Mod** is A-biprojective. Hence the monoidal structure of U-**Mod** extends to a monoidal structure on $\mathcal{D}^-(U)$ with unit object given by A and product being the total tensor product $\otimes^{\mathrm{L}} = \otimes^{\mathrm{L}}_A$ (the A-biprojectivity of U-projectives is needed for example to have [26, Lemma 10.6.2]).

Together with the shift functor $T : \mathcal{D}^-(U) \to \mathcal{D}^-(U)$, $(TC)^n = C^{n+1}$, $\mathcal{D}^-(U)$ becomes what is called a suspended monoidal category in [23]. This just means that for all $C, D \in \mathcal{D}^-(U)$, the canonical isomorphisms

$$TC \otimes^{\mathrm{L}} D \simeq T(C \otimes^{\mathrm{L}} D) \simeq C \otimes^{\mathrm{L}} TD$$

given by the obvious renumbering make the diagrams

$$\begin{tikzcd} A \otimes^{\mathrm{L}} TC \arrow[r] \arrow[d] & TC \arrow[dl] \\ T(A \otimes^{\mathrm{L}} C) & \end{tikzcd} \qquad \begin{tikzcd} TC \otimes^{\mathrm{L}} A \arrow[r] \arrow[d] & TC \arrow[dl] \\ T(C \otimes^{\mathrm{L}} A) & \end{tikzcd}$$

commutative and the diagram

$$\begin{tikzcd} TC \otimes^{\mathrm{L}} TD \arrow[r] \arrow[d] & T(C \otimes^{\mathrm{L}} TD) \arrow[d] \\ T(TC \otimes^{\mathrm{L}} D) \arrow[r] & T^2(C \otimes^{\mathrm{L}} D) \end{tikzcd}$$

anticommutative (commutative up to a sign -1).

3.2. $\smile$ and $\circ$ [23]

As a special case of the constructions from [23], we define for any A-biprojective $\times_A$-bialgebra U and $L, M, N \in U$-**Mod** the cup product

$$\smile : \mathrm{Ext}^m_U(A, M) \times \mathrm{Ext}^n_U(A, N) \to \mathrm{Ext}^{m+n}_U(A, M \otimes N)$$

and the classical Yoneda product

$$\circ : \mathrm{Ext}^m_U(N, M) \times \mathrm{Ext}^n_U(L, N) \to \mathrm{Ext}^{m+n}_U(L, M).$$

The latter is just the composition of morphisms in $\mathcal{D}^-(U)$ if one identifies

$$\mathrm{Ext}^n_U(L, N) \simeq \mathrm{Hom}_{\mathcal{D}^-(U)}\big(L, T^n N\big),$$

and

$$\mathrm{Ext}^m_U(N, M) \simeq \mathrm{Hom}_{\mathcal{D}^-(U)}\big(N, T^m M\big) \simeq \mathrm{Hom}_{\mathcal{D}^-(U)}\big(T^n N, T^{m+n} M\big).$$

The former is obtained as follows: given

$$\varphi \in \mathrm{Ext}^m_U(A, M) \simeq \mathrm{Hom}_{\mathcal{D}^-(U)}\big(A, T^m M\big),$$
$$\psi \in \mathrm{Ext}^n_U(A, N) \simeq \mathrm{Hom}_{\mathcal{D}^-(U)}\big(A, T^n N\big),$$

one defines $\varphi \smile \psi$ as the composition

$$\begin{aligned} A \simeq A \otimes^{\mathrm{L}} A \quad &\xrightarrow{\varphi\otimes\psi} \quad T^m M \otimes^{\mathrm{L}} T^n N \simeq T^m\big(M \otimes^{\mathrm{L}} T^n N\big) \simeq T^{m+n}(M \otimes^{\mathrm{L}} N) \\ &\longrightarrow \quad T^{m+n}(M \otimes N), \end{aligned}$$

where the last map is the augmentation $M \otimes^{\mathrm{L}} N \to H^0(M \otimes^{\mathrm{L}} N) \simeq \mathrm{Tor}^A_0(M, N) \simeq M \otimes N$, or rather T^{m+n} applied to this morphism in $\mathcal{D}^-(U)$.

A straightforward extension of Theorem 1.7 from [23] now gives:

Theorem 3. *If U is an A-biprojective $\times_A$-bialgebra, then we have*

$$\psi \circ \varphi = \varphi \smile \psi = (-1)^{mn} \psi \smile \varphi, \quad \varphi \in \mathrm{Ext}^m_U(A, A),\ \psi \in \mathrm{Ext}^n_U(A, M),$$

as elements of $\mathrm{Ext}^{m+n}_U(A, M) \simeq \mathrm{Ext}^{m+n}_U(A, A \otimes M) \simeq \mathrm{Ext}^{m+n}_U(A, M \otimes A)$.

In particular, $\mathrm{Ext}_U(A, A)$ *becomes through either of the products a graded commutative algebra over the commutative subring* $\mathrm{Hom}_U(A, A)$.

Proof. This is proven exactly as in [23]. For the reader's convenience we include one of the diagrams involved. The unlabelled arrows are canonical maps coming from the suspended monoidal structure.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\varphi"'] & A \otimes A \arrow[d, "\mathrm{id}\otimes\varphi"'] \arrow[dr, "\psi\otimes\mathrm{id}"] & \\
T^m A \arrow[r] \arrow[dr] \arrow[dd, "\mathrm{id}"'] & A \otimes T^m A \arrow[d] \arrow[dr, "\psi\otimes\mathrm{id}"] & T^n M \otimes^{\mathrm{L}} A \arrow[d, "\mathrm{id}\otimes\varphi"] \\
 & T^m(A \otimes A) \arrow[dl] \arrow[d, "T^m(\psi\otimes\mathrm{id})"'] & T^n M \otimes^{\mathrm{L}} T^m A \arrow[dl] \arrow[d] \\
T^m A \arrow[d, "T^m(\psi)"'] & T^m(T^n M \otimes^{\mathrm{L}} A) \arrow[dl] \arrow[d] & T^n(M \otimes^{\mathrm{L}} T^m A) \arrow[dl] \\
T^{m+n} M & T^{m+n}(M \otimes^{\mathrm{L}} A) \arrow[l] &
\end{tikzcd}$$

The morphism $\psi \circ \varphi \in \mathrm{Hom}_{\mathcal{D}^-(U)}(A, T^{m+n}M)$ is the path going straight down from A to $T^{m+n}M$, and $\psi \smile \varphi$ is the one which goes clockwise round the whole diagram. All faces of the diagram commute except the lower right square which introduces a sign $(-1)^{mn}$, so we get $\psi \circ \varphi = (-1)^{mn} \psi \smile \varphi$. The other identity is shown with a similar diagram. □

3.3. *Tensoring projectives*

This paragraph is a small excursus about the projectivity of the tensor product of two projective objects of a monoidal category. For example, $U \otimes U \in U$-**Mod** is not necessarily projective even for a bialgebra U over a field $A = k$ (so the A-projectivity of U or the exactness of $\otimes$ does not help). Here is a simple example (for a detailed study of examples of categories of Mackey functors see [14]):

Example 3. Consider the bialgebra $U = \mathbb{C}[a, b, c]$ over $A = k = \mathbb{C}$ with

$$\Delta(a) = a \otimes a, \qquad \Delta(b) = a \otimes b + b \otimes c, \qquad \Delta(c) = c \otimes c,$$

$$\varepsilon(a) = 1, \qquad \varepsilon(b) = 0, \qquad \varepsilon(c) = 1.$$

Geometrically, U is the coordinate ring of the complex algebraic semigroup G of upper triangular 2×2-matrices, and Δ and ε are dual to the semigroup law $G \times G \to G$ and the embedding of the identity matrix into G.

We prove that $U \otimes U \in U$-**Mod** is not projective by considering the fibres of the semigroup law $G \times G \to G$. The fibre over a generic and hence invertible element is 3-dimensional, but over 0 it is 4-dimensional, and this will imply our claim. We can use for example [15, Theorem 19 on p. 79]:

Theorem 4. *Let $U \subset V$ be a flat extension of commutative Noetherian rings, $\mathfrak{p} \subset V$ be a prime ideal and $\mathfrak{q} := U \cap \mathfrak{p}$. Then*

$$\dim(V_\mathfrak{p}) = \dim(U_\mathfrak{q}) + \dim\big(V_\mathfrak{p} \otimes_U U(\mathfrak{q})\big),$$

where dim *denotes the Krull dimension of a ring, $V_\mathfrak{p}$ is the localisation of V at $\mathfrak{p}$ and $U(\mathfrak{q}) := U_\mathfrak{q}/\mathfrak{q}U_\mathfrak{q}$ is the residue field of the localisation $U_\mathfrak{q}$.*

Apply this to our example $U \simeq \Delta(U) \subset V := U \otimes U$: let $\mathfrak{p}$ be the ideal of V generated by $a \otimes_\mathbb{C} 1$, $1 \otimes_\mathbb{C} a$, $b \otimes_\mathbb{C} 1$, $1 \otimes_\mathbb{C} b$, $c \otimes_\mathbb{C} 1$, $1 \otimes_\mathbb{C} c$. Geometrically, V is the coordinate ring of $\mathbb{C}^6$ and $V_\mathfrak{p}$ is the local ring in 0, so $\dim(V_\mathfrak{p}) = 6$. Since $1 \notin \mathfrak{p}$, $\mathfrak{q} = U \cap \mathfrak{p}$ is proper, and it contains the ideal generated by $\Delta(a) = a \otimes_\mathbb{C} a$, $\Delta(b) = a \otimes_\mathbb{C} b + b \otimes_\mathbb{C} c$, $\Delta(c) = c \otimes_\mathbb{C} c$ which is maximal in U, so $\mathfrak{q} \subset U$ is the ideal generated by a, b, c, and $U_\mathfrak{q}$ is the local ring of $\mathbb{C}^3$ at 0 with $\dim(U_\mathfrak{q}) = 3$. The field $U(\mathfrak{q})$ is obviously $\mathbb{C}$, and we can write $V_\mathfrak{p} \otimes_U U(\mathfrak{q})$ also as $V_\mathfrak{p}/\Delta(\mathfrak{q})V_\mathfrak{p}$. Since $\Delta(\mathfrak{q})V_\mathfrak{p}$ is contained in the ideal $\mathfrak{r}$ generated in $V_\mathfrak{p}$ by the elements $a \otimes_\mathbb{C} 1$, $1 \otimes_\mathbb{C} c$, we have $\dim(V_\mathfrak{p}/\Delta(\mathfrak{q})V_\mathfrak{p}) \geqslant \dim(V_\mathfrak{p}/\mathfrak{r})$. Now $V_\mathfrak{p}/\mathfrak{r}$ is the local ring of $\mathbb{C}^4 \subset \mathbb{C}^6$ at 0 and hence $\dim(V_\mathfrak{p}/\mathfrak{r}) = 4$. In total, we get the strict inequality $3 + \dim(V_\mathfrak{p}/\Delta(\mathfrak{q})V_\mathfrak{p}) \geqslant 3 + 4 = 7 > 6$, and hence V is not flat over U and in particular not projective.

For $\times_A$-Hopf algebras the situation is, however, much simpler: notice that

$${}_\blacktriangleright U \otimes_{A^{\mathrm{op}}} M_\triangleleft := U \otimes_k M/\mathrm{span}\{a \blacktriangleright u \otimes_k m - u \otimes_k m \triangleleft a \mid u \in U,\ a \in A,\ m \in M\}$$

is for any $\times_A$-bialgebra U and $M \in U$-**Mod** a left U-module by left multiplication on the first factor. Just as for $M = U$, there is a Galois map

$$\beta_M : {}_\blacktriangleright U \otimes_{A^{\mathrm{op}}} M_\triangleleft \to U \otimes M, \quad u \otimes_{A^{\mathrm{op}}} m \mapsto u_{(1)} \otimes_A u_{(2)}m,$$

and we have:

Lemma 1. *For any $\times_A$-bialgebra U, the generalised Galois map β_M is a morphism of U-modules. If U is a $\times_A$-Hopf algebra, then β_M is bijective.*

Proof. The U-linearity of β_M follows immediately from the fact that $\hat{\Delta} : U \to U \times_A U$ is a homomorphism of algebras over A^e. Furthermore, if β is a bijection, then β_M is so as well since we can identify β_M with $\beta \otimes_U \mathrm{id}_M$, and then the inverse is simply given by $\beta_M^{-1}(u \otimes_A m) = u_+ \otimes_{A^{\mathrm{op}}} u_- m$. □

Using this one now gets:

Theorem 5. *If U is a $\times_A$-Hopf algebra and $U_\triangleleft \in A^{\mathrm{op}}$-**Mod** is projective, then $P \otimes Q \in U$-**Mod** is projective for all projectives $P, Q \in U$-**Mod**.*

Proof. By assumption, a projective U-module is also projective over A^{op}, and if $\varphi : R \to S$ is any ring map, then $S \otimes_R \cdot : R$-**Mod** $\to S$-**Mod** maps projectives to projectives. This shows that ${}_\blacktriangleright U \otimes_{A^{\mathrm{op}}} U_\triangleleft$ and hence (Lemma 1) $U \otimes U$ is projective. Since $\otimes = \otimes_A$ commutes with arbitrary direct sums, $P \otimes Q$ is projective for all projectives P, Q. □

Corollary 1. *If U is as in Theorem 5 and $P \in \mathcal{D}^{-}(U)$ is a projective resolution of $A \in U$-**Mod**, then so is $P \otimes P := \mathrm{Tot}(P_\bullet \otimes P_\bullet) = P \otimes^{\mathrm{L}} P$.*

This leads to the traditional construction of $\smile$ given for $A = k$ in [3, Chapter XI]: one fixes a projective resolution P of A, and by the above, $\mathrm{Ext}_U(A, M \otimes N)$ is the total (co)homology of the double (cochain) complex

$$C^2_{mn} := \mathrm{Hom}_U(P_m \otimes P_n, M \otimes N).$$

Then $\smile$ is given as the composition of the canonical map

$$\begin{aligned}
&\bigoplus_{m+n=p} \mathrm{Ext}^m_U(A, M) \otimes_k \mathrm{Ext}^n_U(A, N) \\
&\quad \simeq \bigoplus_{m+n=p} H^m\big(\mathrm{Hom}_A(P_\bullet, M)\big) \otimes_k H^n\big(\mathrm{Hom}_A(P_\bullet, N)\big) \\
&\quad \to H^p\left(\bigoplus_{m+n=\bullet} \mathrm{Hom}_A(P_m, M) \otimes_k \mathrm{Hom}_A(P_n, N)\right) = H^p\big(\mathrm{Tot}\big(C^1_{\bullet\bullet}\big)\big)
\end{aligned}$$

where $C^1_{mn} := \mathrm{Hom}_U(P_m, M) \otimes_k \mathrm{Hom}_U(P_n, N)$, with the map

$$H\big(\mathrm{Tot}\big(C^1_{\bullet\bullet}\big)\big) \to H\big(\mathrm{Tot}\big(C^2_{\bullet\bullet}\big)\big) \simeq \mathrm{Ext}_U(A, M \otimes N)$$

that is induced by the morphism of double complexes

$$C^1_{mn} \ni \varphi \otimes_k \psi \mapsto \big\{x \otimes y \mapsto \varphi(x) \otimes \psi(y)\big\} \in C^2_{mn}.$$

For the sake of completeness let us finally remark that as for $A = k$ one can in particular use the bar construction to obtain a canonical resolution:

Lemma 2. *For any $\times_A$-bialgebra U, the complex of left U-modules*

$$C^{\mathrm{bar}}_n := ({}_\blacktriangleright U_\lhd)^{\otimes_{A^{\mathrm{op}}} n+1}, \qquad u(v_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} v_n) := uv_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} v_n$$

whose boundary map is given by

$$\begin{aligned}
b' : u_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_n \mapsto &\sum_{i=0}^{n-1} (-1)^i u_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_i u_{i+1} \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_n \\
&+ (-1)^n u_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} \varepsilon(u_n) \blacktriangleright u_{n-1}
\end{aligned}$$

*is a contractible resolution of $A \in U$-**Mod** with augmentation*

$$\varepsilon : C^{\mathrm{bar}}_0 = U \to A =: C^{\mathrm{bar}}_{-1},$$

*and if $U_\lhd \in A^{\mathrm{op}}$-**Mod** is projective, then $C^{\mathrm{bar}}_n \in U$-**Mod** is projective.*

Proof. All claims are straightforward: there is a contracting homotopy

$$s : C_n^{\mathrm{bar}} \to C_{n+1}^{\mathrm{bar}}, \quad u_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_n \mapsto 1 \otimes_{A^{\mathrm{op}}} u_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_n, \quad n \geqslant 0,$$

$$s : A = C_{-1}^{\mathrm{bar}} \to U = C_0^{\mathrm{bar}}, \quad a \mapsto \eta(a \otimes 1),$$

and the projectivity of C_n^{bar} follows as in the proof of Theorem 5. □

3.4. *The functor $\otimes : U$-**Mod** $\times\, U^{\mathrm{op}}$-**Mod** $\to U^{\mathrm{op}}$-**Mod***

Now we introduce the functor $\otimes$ mentioned in Theorem 1.

Lemma 3. *Let U be a $\times_A$-Hopf algebra and $M \in U$-**Mod**, $P \in U^{\mathrm{op}}$-**Mod** be left and right U-modules, respectively. Then the formula*

$$(m \otimes_A p)u := u_- m \otimes_A p u_+, \quad u \in U,\ m \in M,\ p \in P, \tag{19}$$

defines a right U-module structure on the tensor product

$$M \otimes_A P := M \otimes_k P / \operatorname{span}\{m \triangleleft a \otimes_k p - m \otimes_k a \blacktriangleright p \mid a \in A\}. \tag{20}$$

If N is any other (left) U-module, then the canonical isomorphism

$$(M \otimes N) \otimes_A P \simeq M \otimes_A (N \otimes_A P) \tag{21}$$

*of A-bimodules is also an isomorphism in U^{op}-**Mod**. Finally, the tensor flip*

$$(M \otimes_A P) \otimes_U N \to P \otimes_U (N \otimes_A M), \quad m \otimes_A p \otimes_U n \mapsto p \otimes_U n \otimes_A m$$

is an isomorphism of k-modules.

Proof. To show firstly that (19) is well defined over A, we compute

$$\begin{aligned}\big(m \otimes_A (a \blacktriangleright p)\big)u &= u_- m \otimes_A p\eta(1 \otimes a)u_+ = u_- m \otimes_A p(u_+ \triangleleft a)\\ &= (a \blacktriangleright u_-)m \otimes_A p u_+ = u_-\big(\eta(1 \otimes a)m\big) \otimes_A p u_+\\ &= \big((m \triangleleft a) \otimes_A p\big)u,\end{aligned}$$

where (12) and the action properties were used. Together with (20) this also proves the well-definedness with respect to the presentation of $u_+ \otimes_{A^{\mathrm{op}}} u_-$. With the help of (14) one sees immediately that for $u, v \in U$ we have

$$(m \otimes_A p)(uv) = (uv)_- m \otimes_A p(uv)_+ = v_- u_- m \otimes_A p u_+ v_+ = \big((m \otimes_A p)u\big)v,$$

since P and M were right and left U-modules, respectively. As a conclusion, $M \otimes_A P \in U^{\mathrm{op}}$-**Mod**. Eq. (21) is a direct consequence of the associativity of the tensor product of A-bimodules and of (13).

For the last part one has to check that the flip is well defined: we have

$$\begin{aligned}\eta(1\otimes a)m\otimes_A p\otimes_U n \mapsto p\otimes_U n\otimes_A \eta(1\otimes a)m &= p\otimes_U \eta(1\otimes a)(n\otimes_A m)\\ &= p\eta(1\otimes a)\otimes_U (n\otimes_A m),\end{aligned}$$

which is what $m\otimes_A p\eta(1\otimes a)\otimes_U n$ gets mapped to. And secondly, we have

$$\begin{aligned}m\otimes_A p\otimes_U un \mapsto p\otimes_U un\otimes_A m &= p\otimes_U (u_+)_{(1)}n\otimes_A (u_+)_{(2)}u_-m\\ &= p\otimes_U u_+(n\otimes_A u_-m) = pu_+\otimes_U n\otimes_A u_-m,\end{aligned}$$

which is what $u_-m\otimes_A pu_+\otimes_U n = (m\otimes_A p)u\otimes_U n$ gets mapped to. □

Definition 5. We denote the above constructed U^{op}-module by $M\otimes P$.

Thus an unadorned $\otimes$ refers from now on either to the monoidal product on U-**Mod** or to the just defined action of U-**Mod** on U^{op}-**Mod**. For example, (21) would now simply be written as

$$(M\otimes N)\otimes P \simeq M\otimes (N\otimes P).$$

Example 4. Let (A, L) be a Lie–Rinehart algebra and M be a left and N a right $U(A, L)$-module, respectively (or, in the terminology of [6,7], left and right (A, L)-modules). Using (18), one gets the right $U(A, L)$-module structure on $M\otimes_A N$ from formula (2.4) in [7, p. 112]:

$$(m\otimes_A n)X = m\otimes_A nX - Xm\otimes_A n, \quad m\in M,\ n\in N,\ X\in L.$$

If we assume again that U is A-biprojective, then the above results extend directly to the derived category $\mathcal{D}^-(U^{\mathrm{op}})$: we obtain a functor

$$\otimes^{\mathrm{L}} = \otimes_A^{\mathrm{L}} : \mathcal{D}^-(U)\times\mathcal{D}^-\big(U^{\mathrm{op}}\big)\to\mathcal{D}^-\big(U^{\mathrm{op}}\big)$$

and we have for all $M, N\in\mathcal{D}^-(U)$, $P\in\mathcal{D}^-(U^{\mathrm{op}})$ canonical isomorphisms

$$(M\otimes^{\mathrm{L}} N)\otimes^{\mathrm{L}} P\simeq M\otimes^{\mathrm{L}}(N\otimes^{\mathrm{L}} P), \qquad (M\otimes^{\mathrm{L}} P)\otimes_U^{\mathrm{L}} N\simeq P\otimes_U^{\mathrm{L}}(N\otimes^{\mathrm{L}} M). \tag{22}$$

3.5. $\frown$ *and* $\bullet$

These products are dual to $\smile$ and $\circ$. The first one is

$$\bullet : \mathrm{Ext}_U^m(L, M)\times\mathrm{Tor}_n^U(N, L)\to\mathrm{Tor}_{n-m}^U(N, M),$$

which exists for a ring U and $L, M\in U$-**Mod**, $N\in U^{\mathrm{op}}$-**Mod**: an element

$$\varphi\in\mathrm{Ext}_U^m(L, M)\simeq\mathrm{Hom}_{\mathcal{D}^-(U)}\big(L, T^m M\big)$$

defines a morphism in $\mathcal{D}^-(\mathbb{Z})$,

$$N\otimes_U^{\mathrm{L}} L\to N\otimes_U^{\mathrm{L}} T^m M, \quad x\otimes_U y\mapsto x\otimes_U \varphi(y),$$

and $\varphi \bullet \cdot$ is the induced map in (co)homology

$$\begin{aligned}
&\mathrm{Tor}_n^U(N, L) \simeq H^{-n}(N \otimes_U^{\mathrm{L}} L) \\
&\xrightarrow{H^{-n}(\mathrm{id}\otimes\varphi)} H^{-n}\big(N \otimes_U^{\mathrm{L}} T^m M\big) \simeq H^{m-n}(N \otimes_U^{\mathrm{L}} M) \simeq \mathrm{Tor}_{n-m}^U(N, M).
\end{aligned}$$

For $M \in U$-**Mod**, $N \in U^{\mathrm{op}}$-**Mod** as before, the cap product

$$\frown : \mathrm{Ext}_U^m(A, M) \times \mathrm{Tor}_n^U(N, A) \to \mathrm{Tor}_{n-m}^U(M \otimes N, A)$$

involves the functor $\otimes$ from the previous paragraph, so for this we want U to be an A-biprojective $\times_A$-Hopf algebra again. Similarly as for $\bullet$,

$$\varphi \in \mathrm{Ext}_U^m(A, M) \simeq \mathrm{Hom}_{\mathcal{D}^-(U)}\big(A, T^m M\big)$$

defines a morphism in $\mathcal{D}^-(k)$,

$$\begin{aligned}
&N \otimes_U^{\mathrm{L}} A \simeq N \otimes_U^{\mathrm{L}} (A \otimes A) \\
&\xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\varphi} N \otimes_U^{\mathrm{L}} \big(A \otimes^{\mathrm{L}} T^m M\big) \simeq N \otimes_U^{\mathrm{L}} \big(T^m A \otimes^{\mathrm{L}} M\big) \simeq (M \otimes^{\mathrm{L}} N) \otimes_U^{\mathrm{L}} T^m A \\
&\longrightarrow (M \otimes N) \otimes_U^{\mathrm{L}} T^m A,
\end{aligned}$$

where the last $\simeq$ in the second line is induced by the tensor flip as in the derived version (22) of Lemma 3, and the morphism from the second to the third line is similarly as in the definition of $\smile$ induced by the morphism $M \otimes^{\mathrm{L}} N \to M \otimes N$ in $\mathcal{D}^-(U^{\mathrm{op}})$ that takes zeroth cohomology. Passing to cohomology we get $\varphi \frown \cdot : \mathrm{Tor}_n^U(N, A) \to \mathrm{Tor}_{n-m}(M \otimes N, A)$.

Explicitly, if $P \in \mathcal{D}^-(U)$ is a projective resolution of A, then $\frown$ is induced by

$$B_{ij}^1 \ni n \otimes_U (x \otimes_A y) \mapsto \big\{\varphi \mapsto \big(\varphi(y) \otimes_A n\big) \otimes_U x\big\} \in B_{ij}^2$$

which is a morphism of double complexes between

$$B_{ij}^1 := N \otimes_U (P_i \otimes_A P_j)$$

whose total homology is $\mathrm{Tor}^U(N, A)$ and the double complex

$$B_{ij}^2 := \mathrm{Hom}_k\big(\mathrm{Hom}_U(P_j, M), (M \otimes N) \otimes_U P_i\big)$$

whose homology has a natural map to $\mathrm{Hom}_k(\mathrm{Ext}_U(A, M), \mathrm{Tor}^U(M \otimes N, A))$.

In direct analogy with Theorem 3 we get:

Theorem 6. *If U is an A-biprojective $\times_A$-Hopf algebra, then we have*

$$\varphi \bullet (x \otimes_U y) = \varphi \frown (x \otimes_U y), \quad \varphi \in \mathrm{Ext}_U^m(A, A),\ x \otimes_U y \in N \otimes_U^{\mathrm{L}} A,$$

as elements of $N \otimes_U^{\mathrm{L}} A \simeq (A \otimes N) \otimes_U^{\mathrm{L}} A$.

4. Duality and the proof of Theorem 1

4.1. The underived case

In the special case that A is finitely generated projective itself, Theorem 1 reduces to standard linear algebra. We go through this case first since it is both instructive and used in the proof of the general case. For the reader's convenience we include full proofs.

Lemma 4. *Let U be a ring, $A \in U$-**Mod** be finitely generated projective, and A^* be $\mathrm{Hom}_U(A,U)$ with its canonical U^{op}-module structure.*

1. *A^* is finitely generated projective, and if $e_1,\dots,e_n$ are generators of A, then there exist generators $e^1,\dots,e^n \in A^*$ with*

$$\sum_i e^i(a)e_i = a, \qquad \sum_i e^i\alpha(e_i) = \alpha$$

for all $a \in A$ and $\alpha \in A^*$. The element

$$\omega := \sum_i e^i \otimes e_i \in A^* \otimes_U A$$

is independent of the choice of the generators e_i, e^j.

2. *For all U^{op}-modules M, the assignment*

$$\delta(m \otimes a)(\alpha) := m\alpha(a), \quad m \in M,\ a \in A,\ \alpha \in A^*,$$

extends uniquely to an isomorphism of abelian groups

$$\delta : M \otimes_U A \to \mathrm{Hom}_{U^{\mathrm{op}}}\big(A^*, M\big).$$

3. *One has $(A^*)^* \simeq A$ and $A^* \otimes_U M \simeq \mathrm{Hom}_U(A,M)$ for $M \in U$-**Mod**.*

Proof. Since A is projective, there is a splitting $\iota : A \to U^n$ of

$$\pi : U^n \to A, \quad (u_1,\dots,u_n) \mapsto \sum_i u_i e_i.$$

Hence $U^n \simeq A \oplus A_\perp$ for some $A_\perp \in U$-**Mod**. Dually this gives $A^* \oplus (A_\perp)^* = (U^n)^* \simeq U^n$, whence A^* is finitely generated projective. The e^i can be defined as the composition of ι with the projection of U^n on the i-th summand. This proves the first parts of 1. For 2 just note that

$$\mathrm{Hom}_{U^{\mathrm{op}}}\big(A^*, M\big) \ni \varphi \mapsto \sum_i \varphi\big(e^i\big) \otimes e_i \in M \otimes_U A$$

inverts δ. Since $\omega = \delta^{-1}(\mathrm{id}_{A^*})$, it does not depend on the choice of generators. 3 now follows from 1 and 2. $\square$

As in the introduction, let us abbreviate in the situation of this theorem

$$H^0(X,M) := \mathrm{Hom}_U(A,M), \qquad H_0(X,N) := N \otimes_U A$$

for $M \in U$-**Mod**, $N \in U^{\mathrm{op}}$-**Mod**, and call $\omega \in H_0(X, A^*)$ the fundamental class of (U, A). Then 3 says for $M = A$ that we have an isomorphism

$$\cdot \bullet \omega : H^0(X, A) \to H_0\big(X, A^*\big), \quad \varphi \mapsto \sum_i e^i \otimes \varphi(e_i). \tag{23}$$

Using Lemma 3 we can upgrade this to the underived case of Theorem 1:

Lemma 5. *Let U be a $\times_A$-Hopf algebra and assume A is finitely generated projective as a U-module. Then the cap product with the fundamental class $\omega \in H_0(X, A^*) = A^* \otimes_U A$ defines for all $M \in U$-**Mod** an isomorphism*

$$\cdot \frown \omega : H^0(X, M) \to H_0\big(X, M \otimes A^*\big), \quad X := (U, A).$$

Proof. We have $\varphi \frown \omega = \sum_i (\varphi(1) \otimes_A e^i) \otimes_U e_i$, and Lemma 3 identifies

$$H_0\big(X, M \otimes A^*\big) = \big(M \otimes A^*\big) \otimes_U A \simeq A^* \otimes_U (A \otimes M) \simeq A^* \otimes_U M.$$

In this chain of identifications, $\varphi \frown \omega$ is mapped to

$$\varphi \frown \omega \mapsto \sum_i e^i \otimes_U \big(e_i \otimes_A \varphi(1)\big) \mapsto \sum_i e^i \otimes_U \big(e_i\varphi(1)\big) = \sum_i e^i \otimes_U \varphi(e_i)$$

which is identified with φ under the isomorphism $\mathrm{Hom}_U(A, M) \simeq A^* \otimes_U M$ given by $\varphi \mapsto \sum_i e^i \otimes_U \varphi(e_i)$ as in (23). The claim follows. $\square$

4.2. The derived case

It remains to throw in some homological algebra to obtain Theorem 1 in general. To shorten the presentation, we use the following terminology:

Definition 6. A module A over a ring U is perfect if it admits a finite resolution by finitely generated projectives. We call such a module a duality module if there exists $d \geqslant 0$ such that $\mathrm{Ext}^n_U(A, U) = 0$ for all $n \neq d$. We abbreviate in this case $A^* := \mathrm{Ext}^d_U(A, U)$ and call d the dimension of A.

The main remaining step is to prove a derived version of Lemma 4. One could use a result of Neeman by which $A \in U$-**Mod** is perfect if and only if $\mathrm{Hom}_U(A, \cdot)$ commutes with direct sums [10,18], or the Ischebeck spectral sequence which degenerates at E^2 if A is a duality module [9,12,22]. However, we include a more elementary and self-contained proof.

Theorem 7. *Let $A \in U$-**Mod** be a duality module of dimension d.*

1. *The projective dimension of $A \in U$-**Mod** is d.*
2. *A^* is a duality module of the same dimension d.*
3. *If $P_\bullet \to A$ is a finitely generated projective resolution of length d, then $P^*_{d-\bullet} = \mathrm{Hom}_U(P_{d-\bullet}, U)$ is a finitely generated projective resolution of A^* and the canonical isomorphism*

$$\delta : M \otimes_U P_i \to \mathrm{Hom}_U\big(P^*_i, M\big), \quad m \otimes_U p \mapsto \big\{\alpha \mapsto m\alpha(p)\big\},$$

induces for all U^{op}-modules M a canonical isomorphism

$$\mathrm{Tor}_i^U(M, A) \to \mathrm{Ext}_{U^{\mathrm{op}}}^{d-i}\big(A^*, M\big).$$

4. *There is a canonical isomorphism $(A^*)^* \simeq A$.*

Proof. Let $P_\bullet \to A$ be a finitely generated projective resolution of finite length $m \geqslant 0$ (which exists since A is perfect). Then the (co)homology of

$$0 \to P_0^* \to \cdots \to P_m^* \to 0, \quad P_n^* = \mathrm{Hom}_U(P_n, U),$$

is $\mathrm{Ext}_U^\bullet(A, U)$, so by assumption we have $m \geqslant d$ and the above complex is exact except at P_d^* where the homology is $A^* = \mathrm{Ext}_U^d(A, U)$. Furthermore, all the P_n^* are finitely generated projective since the P_n are so (Lemma 4).

Let π_i be the map $P_i^* \to P_{i+1}^*$ and put $K := \ker \pi_{d+1}$. By construction,

$$0 \to K \to P_{d+1}^* \to \cdots \to P_m^* \to 0 \tag{24}$$

is exact. If one compares this exact sequence with the sequence

$$\ldots \to 0 \to 0 \to P_m^* \to P_m^* \to 0$$

using Schanuel's lemma (see [16, 7.1.2]), one obtains that K is projective.

The exactness of $P_\bullet^*$ at P_{d+1}^* gives $K = \mathrm{im}\,\pi_d$, and by the projectivity of K, the map $\pi_d : P_d^* \to K \subset P_{d+1}^*$ splits so that $P_d^* \simeq K \oplus K_\perp$, $K_\perp := \ker \pi_d$. In particular, both K and $K_\perp$ are finitely generated.

It follows from all this that the complex

$$0 \to P_0^* \to \cdots \to P_{d-1}^* \to K_\perp \to 0 \tag{25}$$

is a finitely generated projective resolution of A^*: since $\mathrm{im}\,\pi_{d-1} \subset P_d^*$ is contained in $\ker \pi_d = K_\perp$, the above complex is still exact at P_{d-1}^*, and the homology at $K_\perp$ is the homology of $P_\bullet^*$ at P_d^*, that is, A^*.

Since (24) is a finitely generated projective resolution of 0 and $P_{d-\bullet}^*$ is as a complex a direct sum of (25) and (a shift of) (24) we also know that $\mathrm{Ext}_{U^{\mathrm{op}}}^\bullet(A^*, M)$ is for any $M \in U^{\mathrm{op}}$-**Mod** the (co)homology of $\mathrm{Hom}_U(P_{d-\bullet}^*, M)$. By Lemma 4, this is isomorphic as a chain complex to $M \otimes_U P_{d-\bullet}$ via the isomorphism given in 3, and the homology of this complex is $\mathrm{Tor}_{d-\bullet}^U(M, A)$. This proves 3. The special case $M = U$ implies the remaining claims. □

Assume finally that in the situation of the above theorem, U is an A-biprojective $\times_A$-Hopf algebra. Since P is a finitely generated projective resolution, we have $M \otimes_U P \simeq M \otimes_U^{\mathrm{L}} P$ and $\mathrm{Hom}_U(P^*, M) \simeq \mathrm{RHom}_U(P^*, M)$, and δ gives an isomorphism between these two complexes. The fundamental class is defined to be

$$\omega := \delta^{-1}(\mathrm{id}_{A^*}) \in A^* \otimes_U^{\mathrm{L}} A \simeq P^* \otimes_U A \simeq A^* \otimes_U P,$$

and Theorem 7 gives immediately:

Corollary 2. *If $e_1, \dots, e_n$ and $\tilde{e}^1, \dots, \tilde{e}^n$ are generators of A and of A^*, respectively, then there are $e^1, \dots, e^n \in P_0^*$ and $\tilde{e}_1, \dots, \tilde{e}_n \in P_d$ such that*

$$\omega = \sum_i e^i \otimes_U e_i = \sum_i \tilde{e}^i \otimes_U \tilde{e}_i,$$

and δ is given by the Yoneda product $\cdot \bullet \omega$.

Theorem 1 follows as in the underived case (Lemma 5) working with $\mathrm{RHom}_U(A, M)$ and $(M \otimes^{\mathrm{L}} A^*) \otimes^{\mathrm{L}}_U A$ instead of $H^0(X, M) = \mathrm{Hom}_U(A, M)$ and $H_0(X, M \otimes A^*) = (M \otimes A^*) \otimes_U A$: using Theorem 6 and (22) one gets

$$\begin{aligned}\big(M \otimes^{\mathrm{L}} A^*\big) \otimes^{\mathrm{L}}_U A &\simeq A^* \otimes^{\mathrm{L}}_U (A \otimes^{\mathrm{L}} M) \simeq A^* \otimes^{\mathrm{L}}_U M \\ &\simeq P^* \otimes^{\mathrm{L}}_U M \simeq \mathrm{RHom}_U(P, M) \\ &\simeq \mathrm{RHom}_U(A, M),\end{aligned}$$

where we hide the reindexing of the complexes for the sake of better readability (so P^* stands for $P^*_{d-\bullet}$, and $\mathrm{RHom}_U(P, M)$ and $\mathrm{RHom}_U(A, M)$ are reindexed in the same way). This leads to a convergent spectral sequence

$$\mathrm{Tor}^U_p\big(\mathrm{Tor}^A_q\big(M, A^*\big), A\big) \quad \Rightarrow \quad \mathrm{Ext}^{d-p-q}_U(A, M),$$

and under the last assumption of Theorem 1 ($\mathrm{Tor}^A_q(M, A^*) = 0$ for $q > 0$) this spectral sequence degenerates to the claimed isomorphism.

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