CYCLIC VS MIXED HOMOLOGY

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Abstract. The spectral theory of the Karoubi operator due to Cuntz and Quillen is extended to general mixed (duchain) complexes, that is, chain complexes which are simultaneously cochain complexes. Connes' coboundary map B can be viewed as a perturbation of the noncommutative De Rham differential by a polynomial in the Karoubi operator. The homological impact of such perturbations is expressed in terms of two short exact sequences.

Contents

1. Mixed complexes

Inspired by Connes' work on cyclic homology [2, 3], Dwyer and Kan [7, 8] initiated the study of general chain complexes which simultaneously are cochain complexes:

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Definition 1. A *mixed complex* of R-modules is a triple (Ω, b, d) where (Ω, b) and (Ω, d) are a chain respectively a cochain complex:

$$\ldots \underset{d}{\overset{b}{\rightleftarrows}} \Omega_2 \underset{d}{\overset{b}{\rightleftarrows}} \Omega_1 \underset{d}{\overset{b}{\rightleftarrows}} \Omega_0 \underset{0}{\overset{0}{\rightleftarrows}} 0\ , \qquad d^2 = b^2 = 0.$$

The *mixed homology* $\mathsf{HM}(\Omega)$ is the homology of $(\mathsf{T}(\Omega), b + d)$, where

$$\mathsf{T}_n(\Omega) := \bigoplus_{i \geq 0} \hat{\Omega}_{n-2i}, \qquad \hat{\Omega}_i := \Omega_i / \mathsf{im}\, \xi, \qquad \xi := bd + db.$$

Dwyer and Kan used the term *duchain* rather than *mixed* complex, but the latter (introduced by Kassel [11]) is now the standard terminology, although it is mostly associated with the special case $\xi = 0$.

The motivating examples are the noncommutative differential forms over an associative algebra with the De Rham differential d and the Hochschild boundary map b, see Example 5 below and [14, Section 2.6] for a detailed account. However, mixed complexes appear in a wide range of contexts, e.g. Poisson manifolds [1, 12], Lie-Rinehart algebras (Lie algebroids) [10], and Hopf algebras [4, 5].

2. The spectral decomposition

Our aim here is to revisit the construction of cyclic homology from the perspective of general mixed complexes. To this end, we view Ω as a $k[x]$-module, where k is the centre of R and x acts by ξ. Thus Ω defines a sheaf of mixed complexes over the affine line k; this generalises the spectral decomposition of Ω considered by Cuntz and Quillen [6].

The localisation $S^{-1}\Omega := k[x, x^{-1}] \otimes_{k[x]} \Omega$ is contractible as a chain and cochain complex, for if ξ is invertible, then we have

$$b(\xi^{-1} d) + (\xi^{-1} d) b = \mathrm{id}, \quad d(\xi^{-1} b) + (\xi^{-1} b) d = \mathrm{id}.$$

Thus the only stalk of Ω supporting (co)homology is $\hat{\Omega} = \Omega / \mathsf{im}\, \xi$ at $x = 0$. A particularly well-behaved class of mixed complexes is therefore formed by those which are globally contractible to $\hat{\Omega}$:

Definition 2. We call (Ω, b, d) a *(co)homological skyscraper* if

$$\Omega \to \hat{\Omega} = \Omega / \mathsf{im}\, \xi$$

is a quasiisomorphism of (co)chain complexes.

This holds for example when $\Omega = \mathsf{ker}\, \xi \oplus \mathsf{im}\, \xi$ so that $\mathsf{im}\, \xi \cong S^{-1}\Omega$, and in particular when k is a field and ξ is diagonalisable over k.

Example 1. For an example of a non-skyscraper, define

$$\Omega_n := \begin{cases} R \oplus R & n = 0, 1, \\ 0 & n > 1, \end{cases} \qquad \begin{array}{ll} d\colon \Omega_0 \to \Omega_1, & (r, s) \mapsto (r, s), \\ b\colon \Omega_1 \to \Omega_0, & (u, v) \mapsto (0, u). \end{array}$$

The homology of Ω is R in both degrees and so is that of $\hat{\Omega}$, but while the map induced on homology by the quotient $\Omega \to \hat{\Omega}$ is the identity in degree 0 it vanishes in degree 1, so Ω is not a homological skyscraper.

Example 2. Consider the De Rham complex (Ω, d) of a compact Riemannian manifold, and let b be the adjoint of d with respect to the Riemannian volume form. Then ξ is the Laplace operator and the spectral decomposition of this elliptic (essentially) self-adiont operator yields $\Omega = \mathsf{ker}\xi \oplus \mathsf{im}\xi$, so Ω is a skyscraper and is contractible to $\mathsf{ker}\xi$, the space of harmonic forms. The results of this paper can therefore also be viewed as an abstraction of the Hodge theorem.

3. Statement of the main results

The noncommutative differential forms over an algebra are not a skyscraper with respect to the De Rham differential d, but they are with respect to the coboundary map B that defines cyclic homology (cf. Section 4 below). Our goal is to compare cyclic and mixed homology, and we will do so for more general deformations of d by polynomials in ξ:

Definition 3. Given any mixed complex (Ω, b, d) and a sequence of polynomials $c_n \in k[x]$, we define a new coboundary map

$$B_n := c_n d_n$$

and a new spectral parameter

$$\upsilon_n := b_{n+1} B_n + B_{n-1} b_n.$$

We denote the mixed homology of (Ω, b, B) by $\mathsf{HC}(\Omega)$.

The new spectral parameter turns Ω into a $k[x, y]$-module, y acting by υ, that is, a sheaf over the affine plane, and we denote by $(\tilde{\Omega}, \tilde{b}, \tilde{B})$ the stalk at $x = y = 0$:

Definition 4. We abbreviate

$$\tilde{\Omega}_n := \Omega_n / (\mathsf{im}\, \xi_n + \mathsf{im}\, \upsilon_n), \qquad \mathsf{HS}(\Omega) := \mathsf{HC}(\tilde{\Omega}),$$

$$\mathsf{HB}(\Omega) := (\mathsf{ker}\, \tilde{B} \cap \mathsf{im}\, \tilde{b}) / (\mathsf{im}\, (\tilde{b} + \tilde{B}) \cap \mathsf{im}\, \tilde{b}) \subset \mathsf{HS}(\Omega),$$

where we view $\tilde{b}, \tilde{B}$ as maps on $\mathsf{T}(\tilde{\Omega})$.

With these notations introduced, we can state our main result:

Theorem 1. *If all $c_n \in k[x]$ are invertible in $k[\![x]\!]$ and (Ω, b, B) is a homological skyscraper, then there are canonical short exact sequences*

$$0 \to \mathsf{HB}_n(\Omega) \to \mathsf{HM}_n(\Omega) \to \mathsf{HS}_n(\Omega)/\mathsf{HB}_n(\Omega) \to 0, \tag{1}$$

$$0 \to \mathsf{HC}_n(\Omega) \to \mathsf{HS}_n(\Omega) \to \mathsf{HC}_{n-1}(\operatorname{im}\xi) \to 0. \tag{2}$$

The maps in (2) are induced by the embedding $\operatorname{im}\xi \to \Omega$ and the quotient $\Omega \to \tilde{\Omega}$, those in (1) will be described in Section 10.

Thus if the two short exact sequences split, then choosing a split for both yields an isomorphism

$$\mathsf{HM}_n(\Omega) \cong \mathsf{HC}_n(\Omega) \oplus \mathsf{HC}_{n-1}(\operatorname{im}\xi).$$

Examples 7 and 8 at the end of the paper illustrate the nontriviality of Theorem 1 by exhibiting mixed complexes for which $\mathsf{HC}(\operatorname{im}\xi) \neq 0$ respectively $\mathsf{HB}(\Omega) \neq 0$.

A key step in the proof is the following computation that relates the two spectral parameters; as we will explain below, this extends a result of Cuntz and Quillen.

Proposition 1. *We have*

$$\upsilon_n = \xi_n c_n - d_{n-1} b_n f_n = b_{n+1} d_n f_n + \xi_n c_{n-1}, \tag{3}$$

where $f_n := c_n - c_{n-1}$, and

$$(\upsilon_n - \xi_n c_n)(\upsilon_n - \xi_n c_{n-1}) = 0. \tag{4}$$

4. Cyclic homology

The result and the notation used are motivated by the definition of cyclic homology:

Definition 5. If

$$c_n = \sum_{i=0}^{n} (1-x)^i = \frac{1-(1-x)^{n+1}}{x} = \sum_{i=0}^{n} \binom{n+1}{i+1} x^i,$$

we call $B_n = d_n \sum_{i=0}^{n} (\mathrm{id} - \xi_n)^i$ the *Connes coboundary map* and $\mathsf{HC}(\Omega)$ the *cyclic homology* of Ω, and we say Ω is a *cyclic complex* if $\upsilon = 0$.

Theorem 1 relates, in particular, the mixed homology of a cyclic complex to its cyclic homology, as long as the constant coefficients $n+1$ of c_n are invertible in the ground ring k. If $\upsilon = 0$, we in fact obtain an isomorphism $\mathsf{HC}_n(\operatorname{im}\xi) = \bigoplus_{i \geq 0} \ker \xi_{n-2i} \cap \operatorname{im} \xi_{n-2i}$:

Corollary 1. *If (Ω, b, d) is a cyclic complex of $\mathbb{Q}$-vector spaces, then there are (noncanonical) isomorphisms of vector spaces*

$$\mathsf{HM}_n(\Omega) \cong \mathsf{HC}_n(\Omega) \oplus \bigoplus_{i \geq 0} \mathsf{ker}\, \xi_{n-1-2i} \cap \mathsf{im}\, \xi_{n-1-2i}.$$

The formulas from Proposition 1 reduce in this case to

$$T_n = (\mathrm{id} - b_{n+1} d_n)\kappa_n^n, \quad \kappa^{n+1} = T_n(\mathrm{id} - d_{n-1} b_n), \tag{5}$$

$$(T_n - \kappa_n^{n+1})(T_n - \kappa_n^n) = 0, \tag{6}$$

where

$$\kappa_n := \mathrm{id} - \xi_n, \qquad T_n := \mathrm{id} - \upsilon_n \tag{7}$$

are the *Karoubi operators* of the two mixed complexes (Ω, b, d) and (Ω, b, B), respectively. This generalises [6, Proposition 3.1] to arbitrary mixed complexes and in particular to all cyclic ones (Cuntz and Quillen only considered the example of noncommutative differential forms).

5. Quasiisomorphisms

Before beginning the proofs of the main results, we remark that what one should call a quasiisomorphism (or weak equivalence) of mixed complexes is a subtle question that depends on one's aims (see e.g. [14, Section 2.5.14] and [8] for two different choices). We will, however, only encounter the simple case that is covered by the following proposition which is a straightforward generalisation of [14, Corollary 2.2.3]:

Proposition 2. *A morphism $\varphi : (\Omega, b, d) \to (\Omega', b', d')$ of mixed complexes with $bd+db = b'd'+d'b' = 0$ induces an isomorphism on homology if and only if it induces an isomorphism on mixed homology.*

Example 3. Observe that the analogue of the proposition for cohomological quasiisomorphisms fails: consider for example the two mixed complexes

$$\Omega_n := \begin{cases} \mathbb{C} & n = 0, \\ 0 & n > 0, \end{cases} \qquad \Omega'_n := \mathbb{C},$$

with $b_n = d_n = b'_n = 0$ and

$$d'_n := \begin{cases} 0 & n = 2k, \\ \mathrm{id} & n = 2k+1, \end{cases} \qquad k \in \mathbb{N}.$$

We have

$$\mathsf{H}^n(\Omega) = \Omega_n \cong \mathsf{H}^n(\Omega'),$$

so the map

$$\varphi_n := \begin{cases} \mathrm{id} & n = 0, \\ 0 & n > 0 \end{cases}$$

is a quasiisomorphism of cochain complexes:

$$\begin{tikzcd}
\cdots \arrow[r, shift left, "0"] & 0 \arrow[l, shift left, "0"] \arrow[r, shift left, "0"] \arrow[d, "0"] & 0 \arrow[l, shift left, "0"] \arrow[r, shift left, "0"] \arrow[d, "0"] & 0 \arrow[l, shift left, "0"] \arrow[r, shift left, "0"] \arrow[d, "0"] & 0 \arrow[l, shift left, "0"] \arrow[r, shift left, "0"] \arrow[d, "0"] & \mathbb{C} \arrow[l, shift left, "0"] \arrow[r, shift left, "0"] \arrow[d, "\mathrm{id}"] & 0 \arrow[l, shift left, "0"] \\
\cdots \arrow[r, shift left, "0"] & \mathbb{C} \arrow[l, shift left, "0"] \arrow[r, shift left, "0"] & \mathbb{C} \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "0"] & \mathbb{C} \arrow[l, shift left, "0"] \arrow[r, shift left, "0"] & \mathbb{C} \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "0"] & \mathbb{C} \arrow[l, shift left, "0"] \arrow[r, shift left, "0"] & 0 \arrow[l, shift left, "0"]
\end{tikzcd}$$

However, one obtains by direct inspection

$$\mathsf{HM}_n(\Omega') \cong \mathbb{C}, \qquad \mathsf{HM}_n(\Omega) \cong \begin{cases} \mathbb{C} & n = 2k, \\ 0 & n = 2k+1, \end{cases} \qquad k \in \mathbb{N}.$$

Remark 1. The moral is that, although the rôles of d and b are entirely symmetric in Ω, this symmetry is broken in the definition of mixed homology, as the action of d is somewhat artificially cut off on $\hat{\Omega}_n \subset \mathsf{T}_n(\Omega)$. This changes when one considers the $\mathbb{Z}_2$-graded periodic homology theories; however, then there are two variants:

$$\mathsf{T}_s^{\mathrm{per},\Pi}(\Omega) := \prod_{j \in \mathbb{N}} \hat{\Omega}_{s+2j}, \quad \mathsf{T}_s^{\mathrm{per},\oplus}(\Omega) := \bigoplus_{j \in \mathbb{N}} \hat{\Omega}_{s+2j}, \quad s \in \mathbb{Z}_2.$$

Proposition 2 holds in the same way for $\mathsf{HM}^{\mathrm{per},\Pi}$, but it is cohomological quasiisomorphisms rather than homological ones that induce isomorphisms in $\mathsf{HM}^{\mathrm{per},\oplus}$.

6. The proof of Proposition 1

From now on, let (Ω, b, d) be a mixed complex and assume $c_n \in k[x]$ satisfy the conditions in Theorem 1. We now develop the theory that leads to its proof. The steps are illustrated using the example of cyclic homology, and the first one is the proof of Proposition 1 in which we relate the maps ξ and υ:

Proof of Proposition 1. The first equation is obtained by straightforward computation:

$$\begin{aligned} \upsilon_n &= b_{n+1}B_n + B_{n-1}b_n = b_{n+1}c_nd_n + c_{n-1}d_{n-1}b_n \\ &= b_{n+1}d_nc_n + d_{n-1}b_nc_{n-1} = (\xi_n - d_{n-1}b_n)c_n + d_{n-1}b_nc_{n-1} \\ &= b_{n+1}d_nc_n + (\xi_n - b_{n+1}d_n)c_{n-1}. \end{aligned}$$

Thus the first factor in (4) equals $-d_{n-1}b_nf_{n+1}$ and the second one $b_{n+1}d_nf_n$, so their product equals 0 as $b_nb_{n+1} = 0$. $\square$

Remark 2. If one perturbs not just d_n to $B_n = c_n d_n$ but also b_n to $D_n := a_n b_n$ for some polynomials $a_n \in k[x]$, then one has

$$B_{n-1}D_n + D_{n+1}B_n = \xi_n a_{n+1}c_n - d_{n-1}b_n f_n = b_{n+1}d_n f_n + \xi_n a_n c_{n-1}$$

with $f_n = a_{n+1}c_n - a_n c_{n-1}$. That is, one obtains υ but for d perturbed by the polynomials $a_{n+1}c_n$ and in this sense it is sufficient to focus on deformations of d alone.

Example 4. In the case of cyclic homology (cf. Section 4), we obtain

$$c_n = \frac{1-y^{n+1}}{1-y}, \quad f_n = y^n, \quad y := 1-x.$$

Inserting this into the formulas in Proposition 1 yields the formulas (5)-(7) from Section 4.

7. The quasiisomorphism $\Omega \to \bar{\Omega}$

As part of the assumptions of Theorem 1, (Ω, b, B) is a homological skyscraper, so $(\mathsf{im}\,\upsilon, b)$ has trivial homology. We now use this to relate the mixed homology of Ω to that of the quotients

$$\bar{\Omega} := \Omega/\mathsf{im}\,\upsilon, \qquad \tilde{\Omega} := \Omega/(\mathsf{im}\,\xi + \mathsf{im}\,\upsilon).$$

In the sequel, $\bar{d}, \bar{b}, \bar{\xi}$ and $\tilde{b}, \tilde{d}, \tilde{\xi}$ refer to the structure maps in $\bar{\Omega}$ respectively $\tilde{\Omega}$.

Lemma 1. $(\mathsf{im}\,\xi \cap \mathsf{im}\,\upsilon, b)$ *has trivial homology.*

Proof. If $x \in (\mathsf{im}\,\upsilon \cap \mathsf{im}\,\xi)_n$ and $b_n x = 0$, then as $(\mathsf{im}\,\upsilon, b)$ has no homology, there is $y \in \Omega_{n+1}$ with $x = b_{n+1}\upsilon_{n+1}y$. By Proposition 1, this equals $b_{n+1}\xi_{n+1}c_n y$, so $x \in b(\mathsf{im}\,\upsilon \cap \mathsf{im}\,\xi)_{n+1}$. □

Lemma 2. *The canonical quotient* $\hat{\Omega} \to \tilde{\Omega}$ *is a quasiisomorphism of chain complexes. In particular, the quotient* $(\Omega, b, d) \to (\bar{\Omega}, \bar{b}, \bar{d})$ *induces isomorphisms* $\mathsf{HM}(\Omega) \cong \mathsf{HM}(\bar{\Omega})$.

Proof. We need to show that the kernel

$$\mathsf{im}\,\upsilon/(\mathsf{im}\,\xi \cap \mathsf{im}\,\upsilon)$$

has trivial homology. However, this follows from the fact that $\mathsf{im}\,\upsilon$ and $\mathsf{im}\,\xi \cap \mathsf{im}\,\upsilon$ have trivial homology ($\mathsf{im}\,\upsilon$ has trivial homology by the assumption that (Ω, b, B) is a homological skyscraper, and $\mathsf{im}\,\xi \cap \mathsf{im}\,\upsilon$ has trivial homology by Lemma 1). The second claim now follows from Proposition 2. □

Example 5. If Ω is a cyclic complex and HC defines its cyclic homology as in Definition 5, then $\upsilon = 0$ and $\Omega = \bar{\Omega}$, so the above lemma becomes trivial. However, consider an associative algebra A and the A-bimodule A_σ which is A as a left A-module but whose right action is given by $x \triangleleft y := x\sigma(y)$ for some algebra endomorphism σ. The boundary map b in the mixed complex formed by the noncommutative differential forms over A can be "twisted" by σ so that it computes the Hochschild homology of A with coefficients in A_σ. Explicitly, we have

$$\Omega_n := A \otimes_k (A/k)^{\otimes_k n}$$

where k is embedded into A as scalar multiples of the unit element, b_n is induced by the map

$$\begin{aligned} a_0 \otimes_k a_1 \otimes_k \cdots \otimes_k a_n \mapsto\ & a_0\sigma(a_1) \otimes_k a_2 \otimes_k \cdots \otimes_k a_n \\ & - a_0 \otimes_k a_1a_2 \otimes_k \cdots \otimes_k a_n + \ldots \\ & + (-1)^{n-1} a_0 \otimes_k a_1 \otimes_k \cdots \otimes_k a_{n-1}a_n \\ & + (-1)^n a_na_0 \otimes_k a_1 \otimes_k \cdots \otimes_k a_{n-1} \end{aligned}$$

and d_n is induced by

$$d_n(\omega) := 1 \otimes_k \omega.$$

In this case, $\mathsf{HC}(\Omega)$ is the *twisted cyclic homology* of A that was first considered by Kustermans, Murphy and Tuset [13]. To generalise the theory of Cuntz and Quillen (which concerns the case where $\sigma = \mathrm{id}$) to this setting was one of our original aims, motivated in particular by Shapiro's extension [15] of Karoubi's noncommutative De Rham theory.

8. The quasiisomorphism $\ker \bar{\xi}^2 \to \bar{\Omega}$

From now on, we will study the mixed complex $\bar{\Omega}$ in further detail.

Lemma 3. *We have*

$$\bar{\Omega} = \ker \bar{\xi}^2 \oplus \operatorname{im} \bar{\xi}^2. \tag{8}$$

Proof. That all c_n are invertible in $k[\![x]\!]$ means their constant coefficients are invertible in k. Hence also $c_{n-1}c_n$ has an invertible constant coefficient $\varepsilon_n \in k$. Let $\delta_n, \gamma_n \in k$ be its linear and quadratic coefficient,

$$c_{n-1}c_n = \varepsilon_n + \delta_n x + \gamma_n x^2 + \ldots$$

and define

$$\bar{p}_n := \varepsilon_n^{-2}(\varepsilon_n - \delta_n\bar{\xi})\bar{c}_{n-1}\bar{c}_n = 1 + \left(\frac{\gamma_n}{\varepsilon_n} - \frac{\delta_n^2}{\varepsilon_n^2}\right)\bar{\xi}^2 + \ldots,$$

where $\bar{c}_n : \bar{\Omega}_n \to \bar{\Omega}_n$ is the map obtained by inserting $\bar{\xi}$ into c_n.

Since υ induces the trivial map on $\bar{\Omega} = \Omega/\mathsf{im}\,\upsilon$, Proposition 1 implies

$$\bar{\xi}^2 \bar{c}_{n-1}\bar{c}_n = 0, \tag{9}$$

so we get

$$\mathsf{im}\,\bar{p}_n \subset \mathsf{im}\,\bar{c}_{n-1}\bar{c}_n \subset \mathsf{ker}\,\bar{\xi}^2, \quad \mathsf{im}\,\bar{\xi}^2 \subset \mathsf{ker}\,\bar{c}_{n-1}\bar{c}_n \subset \mathsf{ker}\,\bar{p}_n.$$

Conversely, $\bar{p}_n$ acts by definition as the identity on $\mathsf{ker}\,\bar{\xi}^2$, so we also have $\mathsf{ker}\,\bar{\xi}^2 \subset \mathsf{im}\,\bar{p}_n$, and on $\mathsf{ker}\,\bar{p}_n$ we have $1 = \bar{\xi}^2\left(\frac{\delta_n^2}{\varepsilon_n^2} - \frac{\gamma_n}{\varepsilon_n}\right) + \ldots$, so $\mathsf{ker}\,\bar{p}_n \subset \mathsf{im}\,\bar{\xi}^2$. It follows that $\mathsf{ker}\,\bar{\xi}^2 = \mathsf{im}\,\bar{p}_n$ and $\mathsf{im}\,\bar{\xi}^2 = \mathsf{ker}\,\bar{p}_n$, and also that $\bar{p}_n^2 = \bar{p}_n$, so $\Omega/\mathsf{im}\,\upsilon = \mathsf{im}\,\bar{p}_n \oplus \mathsf{ker}\,\bar{p}_n$. □

Lemma 4. *The inclusion* $\mathsf{ker}\,\bar{\xi}^2 \to \bar{\Omega}$ *induces isomorphisms*

$$\mathsf{HM}(\bar{\Omega}) \cong \mathsf{HM}(\mathsf{ker}\,\bar{\xi}^2), \quad \mathsf{HC}(\bar{\Omega}) \cong \mathsf{HC}(\mathsf{ker}\,\bar{\xi}^2).$$

Proof. As $\bar{\xi}$ is a morphism of mixed complexes, (8) is a decomposition of mixed complexes. Since we have

$$\mathsf{ker}\,\bar{\xi} \subset \mathsf{ker}\,\bar{\xi}^2, \quad \mathsf{im}\,\bar{\xi}^2 \subset \mathsf{im}\,\bar{\xi},$$

we conclude

$$\tilde{\Omega} = \Omega/(\mathsf{im}\,\upsilon + \mathsf{im}\,\xi) \cong \bar{\Omega}/\mathsf{im}\,\bar{\xi} \cong \mathsf{ker}\,\bar{\xi}^2/\mathsf{im}\,\bar{\xi},$$

so the first isomorphism is obvious. Equation (8) also implies that $\bar{\xi}^2$ and hence $\bar{\xi}$ is invertible on $\mathsf{im}\,\bar{\xi}^2$, so $\mathsf{im}\,\bar{\xi}^2$ is contractible as explained in Section 2, which means the inclusion is a quasiisomorphism with respect to $\bar{b}$. Hence the second isomorphism follows from Proposition 2. □

For later use, we record here another elementary consequence of Lemma 3:

Corollary 2. *We have* $\mathsf{im}\,\bar{\xi} \cap \mathsf{ker}\,\bar{\xi}^2 = \mathsf{im}\,\bar{\xi} \cap \mathsf{ker}\,\bar{\xi}$.

Proof. Given $y = \bar{\xi}(x) \in \mathsf{ker}\,\bar{\xi}^2$, decompose x as $x = v + w$ with $v \in \mathsf{ker}\,\bar{\xi}^2$ and $w \in \mathsf{im}\,\bar{\xi}^2$. Then $\bar{\xi}^2(y) = 0$ means $\bar{\xi}^3(v) + \bar{\xi}^3(w) = 0$; so $\bar{\xi}^2(v) = 0$ yields $\bar{\xi}^3(w) = 0$. However, $\bar{\xi}$ is injective on $\mathsf{im}\,\xi^2$ as already remarked in the previous proof, so $w = 0$, hence $x = v$, so $\bar{\xi}(y) = \bar{\xi}^2(x) = \bar{\xi}^2(v) = 0$. □

Remark 3. All the above computations are abstractions of those made by Cuntz and Quillen for the noncommutative differential forms over an associative algebra [6]. Informally speaking, the message of Lemma 4 can be stated as follows: the "best" mixed complexes are those where $\xi = 0$, as one can compute $\mathsf{HM}(\Omega)$ straight from Ω using a spectral sequence. The second best case is $\Omega = \ker \xi \oplus \operatorname{im} \xi$; as mentioned after Definition 2 this means ξ vanishes in a strong homotopical sense. Lemma 4 tells us that in general ξ^2 vanishes in this homotopical sense, so ξ is homotopically infinitesimal.

9. The second exact sequence

We now will derive the second of the two short exact sequences in Theorem 1.

First, we need the following computation:

Lemma 5. *On* $\ker \bar{\xi}^2$*, we have* $\bar{b}\bar{\xi} = \bar{d}\bar{\xi} = 0$, $\bar{B}_n := \bar{c}_n \bar{d}_n = \beta_n \bar{d}_n$*, where* $\beta_n \in k$ *is the constant coefficient of* c_n*, and we have*

$$\bar{\xi}_n = (1 - \frac{\beta_{n-1}}{\beta_n})\bar{d}_{n-1}\bar{b}_n = (1 - \frac{\beta_n}{\beta_{n-1}})\bar{b}_{n+1}\bar{d}_n.$$

Proof. Multiplying the second expression for $\upsilon = 0$ in (3) in Proposition 1 on the left by b_n and using $\bar{\xi}^2 = 0$ gives

$$\bar{b}_n \bar{\xi} \bar{c}_{n-1} = \beta_{n-1} \bar{b}_n \bar{\xi} = 0,$$

so $\bar{b}\bar{\xi} = 0$ as all β_n are invertible. Similarly, one obtains $\bar{d}\bar{\xi} = 0$. The fact that $\bar{B}_n = \beta_n d_n$ is an immediate consequence, and the formulas for $\bar{\xi}_n$ are obtained by direct computation:

$$\begin{aligned}\bar{\xi}_n &= \bar{d}_{n-1}\bar{b}_n + \bar{b}_{n+1}\bar{d}_n = \bar{d}_{n-1}\bar{b}_n + \beta_n^{-1}\bar{b}_{n+1}\bar{B}_n \\ &= \bar{d}_{n-1}\bar{b}_n - \beta_n^{-1}\bar{B}_{n-1}\bar{b}_n = (1 - \frac{\beta_{n-1}}{\beta_n})\bar{d}_{n-1}\bar{b}_n\end{aligned}$$

and similarly $\bar{\xi}_n = (1 - \frac{\beta_n}{\beta_{n-1}})\bar{b}_{n+1}\bar{d}_n$. □

Additionally, we will utilise the following general statement (recall $\hat{\Omega} = \Omega / \operatorname{im} \xi$):

Lemma 6. *If* (Ω, b, d) *is a mixed complex with* $\xi^2 = \upsilon = 0$*, there are short exact sequences*

$$0 \longrightarrow \mathsf{HC}_n(\Omega) \longrightarrow \mathsf{HC}_n(\hat{\Omega}) \longrightarrow \bigoplus_{i \geq 0} \operatorname{im} \xi_{n-1-2i} \longrightarrow 0$$

Proof. The short exact sequence

$$0 \longrightarrow (\operatorname{im} \xi, b, B) \longrightarrow (\Omega, b, B) \longrightarrow (\hat{\Omega}, \hat{b}, \hat{B}) \longrightarrow 0$$

of mixed complexes induces short exact sequences of the total complexes

$$0 \longrightarrow \mathsf{T}(\mathsf{im}\,\xi) \longrightarrow \mathsf{T}(\Omega) \longrightarrow \mathsf{T}(\hat{\Omega}) \longrightarrow 0 \tag{10}$$

whose differential is $b + B$ (recall that $\upsilon = bB + Bb = 0$ so that $\mathsf{T}_n(\Omega) = \Omega_n \oplus \Omega_{n-2} \ldots$ here). However, by Lemma 5, $b + B$ vanishes on $\mathsf{im}\,\xi$, so $\mathsf{T}(\mathsf{im}\,\xi)$ is its own homology. Furthermore, the inclusion $\mathsf{im}\,\xi \to \Omega$ induces the trivial map on homology, as Lemma 5 implies

$$\begin{aligned}&(\xi_n x_n, \xi_{n-2}x_{n-2}, \ldots)\\ &= (b+B)((1-\frac{\beta_n}{\beta_{n-1}})d_n x_n, (1-\frac{\beta_{n-2}}{\beta_{n-3}})d_{n-2}x_{n-2}, \ldots),\end{aligned}$$

so indeed, the homology class of an element in $\mathsf{T}(\mathsf{im}\,\xi)$ becomes trivial in $\mathsf{HC}(\Omega)$. Therefore, the long exact homology sequence induced by (10) splits up into the short exact sequences stated in the lemma. $\square$

Proof of Theorem 1 (2). We apply Lemma 6 to $\ker\bar{\xi}^2 \subset \bar{\Omega}$. This yields short exact sequences

$$0 \longrightarrow \mathsf{HC}_n(\ker\bar{\xi}^2) \longrightarrow \mathsf{HC}_n(\ker\bar{\xi}^2/\mathsf{I}) \longrightarrow \bigoplus_{i\geq 0} \mathsf{I}_{n-1-2i} \longrightarrow 0 \tag{11}$$

where we abbreviate

$$\mathsf{I} := \mathsf{im}\,\bar{\xi} \cap \ker\bar{\xi}^2 = \mathsf{im}\,\bar{\xi} \cap \ker\bar{\xi},$$

the second equality having been proved in Corollary 3.

Note that in view of the decomposition $\bar{\Omega} = \ker\bar{\xi}^2 \oplus \mathsf{im}\,\bar{\xi}^2$ we have

$$\ker\bar{\xi}^2/\mathsf{I} \cong \bar{\Omega}/\mathsf{im}\,\bar{\xi} \cong \tilde{\Omega} = \Omega/(\mathsf{im}\xi + \mathsf{im}\upsilon).$$

In other words, we have a commutative diagram

$$\begin{tikzcd} \ker\bar{\xi}^2 \arrow[r] \arrow[d, hook] & \ker\bar{\xi}^2/\mathsf{I} \cong \bar{\Omega}/\mathsf{im}\,\bar{\xi} \arrow[d, "\cong"] \\ \bar{\Omega} = \Omega/\mathsf{im}\,\upsilon \arrow[r] & \tilde{\Omega} = \Omega/(\mathsf{im}\,\upsilon + \mathsf{im}\,\xi), \end{tikzcd}$$

where the horizontal maps are the canonical projections, the left vertical map is the inclusion, and the right vertical map is an isomorphism induced by this inclusion.

By Lemma 4, the left vertical arrow induces an isomorphism on HC,

$$\mathsf{HC}(\ker\bar{\xi}^2) \cong \mathsf{HC}(\bar{\Omega}) = \mathsf{HC}(\Omega).$$

Similarly, the right vertical isomorphism yields an isomorphism

$$\mathsf{HC}(\ker \bar{\xi}^2/\mathsf{I}) \cong \mathsf{HS}(\Omega) = \mathsf{HC}(\hat{\Omega}).$$

These isomorphisms are compatible with the horizontal quotient maps in the diagram. In other words, the injectivity of the embedding $\mathsf{HC}(\ker \bar{\xi}^2) \to \mathsf{HC}(\ker \bar{\xi}^2/\mathsf{I})$ established in (11) transfers to injectivity of the map $\mathsf{HC}(\Omega) \to \mathsf{HC}(\hat{\Omega})$ induced by the quotient $\Omega \to \hat{\Omega}$.

Since the canonical map $\mathsf{HC}(\Omega) \to \mathsf{HC}(\hat{\Omega})$ is injective, the long exact homology sequence resulting from the short exact sequence

$$0 \to \mathsf{im}\,\xi \to \Omega \to \hat{\Omega} \to 0$$

splits into the short exact sequences stated in the theorem. □

Example 6. When considering the cyclic homology of a cyclic complex, we have $v = 0$, hence $\bar{\xi} = \xi$ and we obtain

$$\mathsf{HC}_n(\mathsf{im}\,\xi) = \bigoplus_{i \geq 0} \mathsf{I}_{n-2i} = \bigoplus_{i \geq 0} \ker \xi_{n-2i} \cap \mathsf{im}\,\xi_{n-2i}.$$

10. The first exact sequence

We begin by pointing out that, without loss of generality, we can work with $\tilde{\Omega}$:

Lemma 7. *The canonical quotient maps $\Omega \to \tilde{\Omega}$ respectively $\hat{\Omega} \to \tilde{\Omega}$ induce isomorphisms $\mathsf{HM}(\Omega) \cong \mathsf{HM}(\tilde{\Omega})$ and $\mathsf{HC}(\hat{\Omega}) \cong \mathsf{HS}(\Omega)$.*

Proof. By definition we have $\mathsf{HM}(\Omega) = \mathsf{HM}(\hat{\Omega})$ and $\mathsf{HC}(\hat{\Omega}) = \mathsf{HC}(\tilde{\Omega})$. It remains to verify that $\mathsf{HM}(\hat{\Omega}) \cong \mathsf{HM}(\tilde{\Omega})$. To this end, note that by the assumption that Ω is a homological skyscraper with respect to B, the quotient map $\Omega \to \bar{\Omega}$ is a quasiisomorphism of chain complexes, and in the proof of Lemma 2 we noted that the quotient map $\bar{\Omega} \to \tilde{\Omega}$ is, also; that is to say, the composition $\Omega \to \tilde{\Omega}$ is a quasiisomorphism of chain complexes and, since this factors through $\hat{\Omega}$, the two quotient maps $\Omega \to \hat{\Omega}, \hat{\Omega} \to \tilde{\Omega}$ are quasiisomorphisms of chain complexes. The claim now follows from Proposition 2. □

The bulk of the remaining computations needed to prove Theorem 1 are performed in the following lemma:

Lemma 8. *The map $\varphi_n : \mathsf{T}(\tilde{\Omega}) \to \mathsf{T}(\tilde{\Omega})$ given by*

$$(x_n, x_{n-2}, \ldots) \mapsto (u_n, u_{n-2}, \ldots) := (x_n, \beta_{n-2}^{-1} x_{n-2}, \beta_{n-2}^{-1}\beta_{n-4}^{-1} x_{n-4}, \ldots)$$

induces isomorphisms

$$\mathsf{HM}(\tilde\Omega/\mathsf{im}\,\tilde b) \cong \mathsf{HC}(\tilde\Omega/\mathsf{im}\,\tilde b), \qquad \mathsf{HM}(\mathsf{im}\,\tilde b) \cong \mathsf{HC}(\mathsf{im}\,\tilde b),$$

$$\mathsf{im}(\mathsf{HM}(\tilde\Omega) \to \mathsf{HM}(\tilde\Omega/\mathsf{im}\,\tilde b)) \cong \mathsf{im}(\mathsf{HS}(\Omega) \to \mathsf{HC}(\tilde\Omega/\mathsf{im}\,\tilde b)), \tag{12}$$

$$\ker(\mathsf{HM}_n(\tilde\Omega) \to \mathsf{HM}_n(\tilde\Omega/\mathsf{im}\,\tilde b)) \cong \ker(\mathsf{HC}_n(\tilde\Omega) \to \mathsf{HC}_n(\tilde\Omega/\mathsf{im}\,\tilde b)) \tag{13}$$

Proof. Explicitly, a class in $\mathsf{HM}_n(\tilde\Omega/\mathsf{im}\,\tilde b)$ is represented by an element $x = (x_n, x_{n-2}, \ldots) \in \mathsf{T}_n(\tilde\Omega)$ such that there exists $y \in \mathsf{T}_n(\tilde\Omega)$ with

$$\tilde b x_n + \tilde d x_{n-2} = \tilde b y_n, \quad \tilde b x_{n-2} + \tilde d x_{n-4} = \tilde b y_{n-2}, \ldots$$

The element x represents the trivial homology class in $\mathsf{HM}_n(\tilde\Omega/\mathsf{im}\,\tilde b)$ if and only if there are elements $z = (z_{n+1}, z_{n-1}, \ldots), t \in \mathsf{T}_{n+1}(\tilde\Omega)$ such that

$$\tilde b z_{n+1} + \tilde d z_{n-1} = x_n + \tilde b t_{n+1}, \quad \tilde b z_{n-1} + \tilde d z_{n-3} = x_{n-2} + \tilde b t_{n-1}, \ldots \tag{14}$$

Recall that $\tilde\xi = 0$ means that $\tilde B_n = \beta_n \tilde d_n$ where $\beta_n \in k$ is the constant coefficient of c_n. Hence $u = \varphi_n(x) \in \mathsf{T}_n(\tilde\Omega)$ satisfies

$$\tilde b u_n + \tilde B u_{n-2} = \tilde b v_n, \quad \tilde b u_{n-2} + \tilde B u_{n-4} = \tilde b v_{n-2}, \ldots$$

where $v = \varphi_n(y)$.

Furthermore, (14) implies

$$\tilde b w_{n+1} + \tilde B w_{n-1} = u_n + \tilde b s_{n+1}, \ldots$$

with $w = \varphi_{n+1}(z)$, $s = \varphi_{n+1}(t)$. This shows that φ_n induces a well-defined map on homology which is clearly bijective. The image of $\mathsf{HM}_n(\tilde\Omega)$ in $\mathsf{HM}_n(\tilde\Omega/\mathsf{im}\,\tilde b)$ consists of those classes that can be represented as above with $y = 0$, and then $v = 0$ means that the image in $\mathsf{HC}_n(\tilde\Omega/\mathsf{im}\,\tilde b)$ is also in the image of $\mathsf{HC}_n(\tilde\Omega)$. The other isomorphisms follow in an exactly analogous way. $\square$

Remark 4. For most of the isomorphisms required in Lemma 8, there is little restriction on the particular isomorphism we use; we could, for example, take the identity instead of φ. However, this causes (12) to fail.

Proof of Theorem 1 (1). The short exact sequences of chain complexes

$$0 \to \mathsf{T}(\mathsf{im}\,\tilde b) \to \mathsf{T}(\tilde\Omega) \to \mathsf{T}(\tilde\Omega/\mathsf{im}\,\tilde b) \to 0$$

with respect to $\tilde b + \tilde d$ and $\tilde b + \tilde B$ yield long exact sequences

$$\ldots \to \mathsf{HM}_{n+1}(\tilde\Omega/\mathsf{im}\,\tilde b) \xrightarrow{\partial^{\mathsf{M}}_{n+1}} \mathsf{HM}_n(\mathsf{im}\,\tilde b) \to \mathsf{HM}_n(\tilde\Omega) \to \mathsf{HM}_n(\tilde\Omega/\mathsf{im}\,\tilde b) \xrightarrow{\partial^{\mathsf{M}}_n} \ldots$$

and

$$\ldots \to \mathsf{HC}_{n+1}(\tilde{\Omega}/\mathsf{im}\,\tilde{b}) \xrightarrow{\partial^{\mathsf{C}}_{n+1}} \mathsf{HC}_n(\mathsf{im}\,\tilde{b}) \to \mathsf{HC}_n(\tilde{\Omega}) \to \mathsf{HC}_n(\tilde{\Omega}/\mathsf{im}\,\tilde{b}) \xrightarrow{\partial^{\mathsf{C}}_{n}} \ldots$$

which split into short exact sequences

$$0 \to \mathsf{HM}_n(\mathsf{im}\,\tilde{b})/\mathsf{im}\,\partial^{\mathsf{M}}_{n+1} \to \mathsf{HM}_n(\tilde{\Omega}) \to \ker \partial^{\mathsf{M}}_n \to 0$$

and

$$0 \to \mathsf{HC}_n(\mathsf{im}\,\tilde{b})/\mathsf{im}\,\partial^{\mathsf{C}}_{n+1} \to \mathsf{HC}_n(\tilde{\Omega}) \to \ker \partial^{\mathsf{C}}_n \to 0.$$

The theorem now follows in view of the isomorphisms (of k-modules)

$$\mathsf{HM}_n(\mathsf{im}\,\tilde{b})/\mathsf{im}\,\partial^{\mathsf{M}}_{n+1} \cong \mathsf{HC}_n(\mathsf{im}\,\tilde{b})/\mathsf{im}\,\partial^{\mathsf{C}}_{n+1}, \qquad \ker \partial^{\mathsf{M}}_n \cong \ker \partial^{\mathsf{C}}_n$$

established in Lemma 8 by inserting the explicit definition of $\mathsf{HC}_n(\mathsf{im}\,\tilde{b})$ and of ∂^{C} (which is induced by $\tilde{b} + \tilde{B}$). □

Example 7. For a basic nontrivial example of the main theorem let k be any commutative ring, $q \in k$, and R be the unital associative k-algebra generated by x, y satisfying

$$x^2 = y^2 = xy + qyx = 0,$$

so that R is a free k-module with basis $\{1, x, y, yx\}$.

We obtain a mixed complex of R-modules with

$$\Omega_n := \begin{cases} R/Ryx & n = 0, \\ R & n > 0 \end{cases}$$

and b_n given by right multiplication by x and d_n given by right multiplication by y. With $c_n = q^n$, that is, B_n given by right multiplication by $q^n y$, we obtain for $n > 0$ and $r \in R = \Omega_n$

$$(b_{n-1}B_n + B_{n-1}b_n)r = r(q^n yx + q^{n-1}xy) = 0,$$
$$(b_{n-1}d_n + d_{n-1}b_n)r = r(yx + xy) = (1-q)ryx,$$

and for $n = 0$

$$(b_1 B_0)(r + Ryx) = (b_1 d_0)(r + Ryx) = ryx + Ryx = 0.$$

In particular,

$$\mathsf{HC}_1(\mathsf{im}\,\xi) = \mathsf{im}\,\xi_1 \cong k/I,$$

where $I \lhd k$ is the annihiltor of $1-q$ in k.

We furthermore see

$$\hat{\Omega}_n = \tilde{\Omega}_n = \begin{cases} R/Ryx & n = 0, \\ R/R(1-q)yx & n > 0, \end{cases}$$

and direct computation yields

$$\mathsf{HC}_2(\Omega) \cong k$$

with basis given by the class of $(0, y + Ryx)$, while in $\mathsf{HC}_2(\hat{\Omega})$ and $\mathsf{HM}_2(\Omega)$ there is an additional generator represented by $((1-q)y, 0)$,

$$\mathsf{HC}_2(\hat{\Omega}) \cong \mathsf{HM}_2(\Omega) \cong k \oplus k/I \cong \mathsf{HC}_2(\Omega) \oplus \mathsf{HC}_1(\mathsf{im}\, \xi).$$

Note also that $\mathsf{HB}_2(\Omega) = 0$ here, so the first isomorphism above is canonical in this example.

Example 8. Our final example demonstrates that $\mathsf{HB}(\Omega)$ can be non-trivial. To see this, consider the mixed complex

$$\Omega_n := \begin{cases} \mathbb{C} & n = 0, 1, 2, \\ 0 & n > 2, \end{cases}$$

with (co)boundary maps

$$b_n := \begin{cases} \mathrm{id} & n = 1, \\ 0 & n \neq 1, \end{cases} \qquad d_n := \begin{cases} \mathrm{id} & n = 1, \\ 0 & n \neq 1. \end{cases}$$

Taking $c_n := 1$ for all n, we obtain

$$\mathsf{T}_n(\tilde{\Omega}) = \begin{cases} \mathbb{C} & n = 0 \text{ or } n = 2k + 1, \\ \mathbb{C} \oplus \mathbb{C} & n = 2k + 2. \end{cases}$$

Here $(0, 1) = \tilde{b}(1) \in \mathsf{ker}\, \tilde{B} \cap \mathsf{im}\, \tilde{b} \subset \mathsf{T}_2(\tilde{\Omega})$ generates $\mathsf{HB}_2(\Omega) \cong \mathbb{C}$, with $\mathsf{im}\, \tilde{b} \cap \mathsf{im}\, (\tilde{b} + \tilde{B}) = 0$.

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