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On Quantum Double Groups and Quantum Flag Manifolds. Two Mathematical Structures with Possible Applications in q-Deformed Field Theories.

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Abstract

In the first part of this work, l-functionals on coquasitriangular Hopf algebras are studied. First, a formula from [KS1] relating the l-functionals on $\mathbb{C}_q[SL(N+1,\mathbb{C})]$ to Lusztig's quantum root vectors in $U_q(\mathfrak{sl}(N+1,\mathbb{C}))$ is generalized to the other classical matrix Lie groups. It allows to substitute the root vectors in the Poincaré-Birkhoff-Witt basis by l-functionals. Then one has explicit formulas for the commutation relations and the coproduct of the basis elements. Afterwards, the Hopf algebra generated by the l-functionals on the quantum double $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$ is shown to be isomorphic to $\mathbb{C}_q[G]^{\mathrm{op}} \bowtie U_q(\mathfrak{g})$ for all semisimple G. This was conjectured by T. Hodges in [Ho]. As an algebra, $\mathbb{C}_q[G]^{\mathrm{op}} \bowtie U_q(\mathfrak{g})$ can be embedded into $U_q(\mathfrak{g} \oplus \mathfrak{g})$, see [Ho]. Here it is proven that there is no bialgebra structure on $U_q(\mathfrak{g} \oplus \mathfrak{g})$, for which this embedding becomes a homomorphism of bialgebras. In particular, it is not an isomorphism.
The second part deals with the theory of covariant differential calculi on quantum homogeneous spaces, and with their relation to A. Connes' non-commutative geometry. A Dirac operator on quantized irreducible flag manifolds is defined. This yields a Hilbert space realization of the covariant differential calculi constructed by I. Heckenberger and S. Kolb in [HK1]. All differentials $\mathrm{d}f = i[D, f]$ are bounded operators. In the simplest case of Podleś' standard quantum sphere one obtains the spectral triple found by L. Dąbrowski and A. Sitarz [DS1].
In a short third part we prove that in contrast to their C^*-completions, the coordinate algebras of the non-standard Podleś spheres depend on the additional deformation parameter. This was posed as an open problem in [HMS].
The results of the first two parts of this work are published in [Kr1, Kr2].

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Contents

Chapter 1

Overview

The aim of this chapter is to explain and motivate the results of the present work. Its sections correspond to the following chapters of the work. As in any of them, the last section contains some bibliographical notes.

1.1 Quantum groups

Let $\{\cdot, \cdot\}$ be a Poisson bracket on the coordinate algebra $\mathbb{C}[G]$ of an affine algebraic group G over $\mathbb{C}$. Then G is said to be a Poisson algebraic group, provided that its group multiplication is a Poisson map with respect to the canonical Poisson bracket $\{f_1 \otimes f_2, g_1 \otimes g_2\} := f_1 g_1 \otimes \{f_2, g_2\} + \{f_1, g_1\} \otimes f_2 g_2$ on $\mathbb{C}[G \times G] := \mathbb{C}[G] \otimes \mathbb{C}[G]$. Analogously one defines Poisson-Lie groups.
A quantization of a Poisson algebraic group is a ($\hbar$-adic) Hopf algebra structure on the vector space $\mathbb{C}[G][[\hbar]]$ of formal power series in an indeterminate $\hbar$ with coefficients in $\mathbb{C}[G]$, whose product $\cdot_\hbar$ and coproduct $\Delta_\hbar$ satisfy [Dr2]

$$f \cdot_\hbar g = fg + \frac{\hbar}{2}\{f, g\} + O(\hbar^2), \quad \Delta_\hbar(f) = \Delta(f) + O(\hbar^2) \quad \forall f, g \in \mathbb{C}[G].$$

Here Δ is the coproduct of the standard Hopf algebra structure on $\mathbb{C}[G]$,

$$\Delta : \mathbb{C}[G] \to \mathbb{C}[G \times G], \quad \Delta(f)(x, y) := f(xy), \quad x, y \in G.$$

Any Poisson algebraic group admits a quantization [EK]. In the main examples one even has $\Delta_\hbar = \Delta$, and the formal power series $q := e^\hbar$ can be specialized to a fixed complex number. This yields a family of Hopf algebras $\mathbb{C}_q[G]$, $q \in \mathbb{C}$. They coincide as a coalgebra with $\mathbb{C}[G]$, but their product is essentially given by $\cdot_\hbar$. In a sense, the space G is quantized without touching its group structure. Hence one speaks of $\mathbb{C}_q[G]$ as of a 'coordinate algebra of a quantum group', although the latter remains a purely imaginary object.
Many geometric concepts can be generalized to quantum groups. For example, the notion of a G-variety is extended by that of a $\mathbb{C}_q[G]$-comodule algebra. These are consequently called 'coordinate algebras of quantum spaces'.
Historically, quantum groups were discovered first as a sort of symmetry of certain anisotropic completely integrable quantum systems. Later they found an application as a source of very effective knot invariants. Many authors speculate that using quantum spaces as models of space-time could also help dealing with some fundamental problems of quantum field theory. The last point is mentioned in the subtitle of this thesis, because one of the few common features of its different parts is that the investigated structures all arose from this motivation.

1.2 On the quantum Iwasawa decomposition

1.2.1 Poisson group duality and quantum group duality

To any connected and simply connected Poisson-Lie group G_0 one can associate a connected and simply connected Poisson-Lie group G_0^* of the same dimension, whose group structure corresponds to the Poisson structure of G_0 and vice versa. This follows from the infinitesimal description of Poisson-Lie groups in terms of Lie bialgebras, see [LW]. One calls G_0^* the dual of G_0, and $(G_0^*)^* = G_0$. Both G_0 and G_0^* can be embedded into a Poisson-Lie group G, such that the multiplication in G defines a diffeomorphism $\zeta : G_0 \times G_0^* \to G$. One calls G the double of G_0 and (G, G_0, G_0^*) a Manin triple of Poisson-Lie groups.
The standard example of such a Manin triple is given by the Iwasawa decomposition of a complex semi-simple Lie group G into its compact real form G_0 and an exponential Lie group G_0^* (i.e. one whose exponential map is a diffeomorphism). If we treat G as an algebraic group, then G_0 and a finite covering of G_0^* are real algebraic subgroups. Complexifying these yields a Manin triple $(G \times G, G, G^*)$ of Poisson algebraic groups [DP]. Now G, G^* are only locally the duals of each other, but this will play no essential role in the following.
Both G and G^* lead to quantum groups as described in the previous section. Mostly, $\mathbb{C}_q[G^*]$ is denoted by $U_q(\mathfrak{g})$ and considered as a quantization of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} := \mathrm{Lie}(G)$. This is as a Lie algebra indeed isomorphic to a Poisson algebra of functions on G^* [GW]. The coproduct of $\mathbb{C}_q[G^*]$ can also be considered as a quantization of the classical coproduct of $U(\mathfrak{g})$, but one should be aware that it differs rather crucially from the latter. We adopt the symbol $U_q(\mathfrak{g})$ to be compatible with the literature.
The Hopf algebra $U_q(\mathfrak{g})$ is defined in terms of generators and relations similar to those given by J. P. Serre for $U(\mathfrak{g})$. All finite-dimensional representations of G can be deformed to representations of $U_q(\mathfrak{g})$. The Hopf algebra $\mathbb{C}_q[G]$ can be derived from $U_q(\mathfrak{g})$ in analogy with the Peter-Weyl theorem as the Hopf subalgebra of the Hopf dual $U_q(\mathfrak{g})^\circ$ spanned by the matrix coefficients of these representations. Thus there exists by definition a dual pairing between $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})$. This reflects the duality of G and G^*.
Although the dual pairing of $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})$ is non-degenerate, no one is the full Hopf dual of the other. This raises the problem of formulating a duality theory of quantum groups that extends the duality of Poisson groups.
The first solution of this problem was known in advance: In [Wo1] S. L. Woronowicz proposed to generalize the classical Pontrjagin duality of locally compact abelian groups in the language of suitably defined Hopf C^*-algebras. It turned out later that quantum groups provide non-trivial examples for this concept. Together with P. Podleś, he also showed that the Iwasawa decomposition admits a straightforward generalization to the quantum setting, see [PW1].
The topic of Chapter 3 is a different approach to quantum group duality and the quantum Iwasawa decomposition relying on the notion of (co)quasitriangularity. In this formulation, the results apply to a wider class of Hopf algebras, and one avoids the technicalities entering in the C^*-algebraic theory. On the other hand, the duality is here not an operation inside of one single category, that is, the roles of $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})$ are no longer completely symmetric. We refer to [VD] for another purely algebraic duality theory avoiding this effect.

1.2.2 L-functionals on $\mathbb{C}_q[G]$

The above mentioned link between quantum groups, statistical mechanics and knot theory is closely related to the fact that the representations of $U_q(\mathfrak{g})$ used to define $\mathbb{C}_q[G]$ form a braided tensor category $\mathcal{C}_G$. This means that for $V, W \in \mathcal{C}_G$ there exists an isomorphism $R_{VW} : V \otimes W \to W \otimes V$ satisfying certain conditions. An equivalent fact is that there exists a bilinear form $\mathbf{r}$ on $\mathbb{C}_q[G]$, such that $(\mathbb{C}_q[G], \mathbf{r})$ is a coquasitriangular Hopf algebra, see the main text for details. This bilinear form can be used to associate to $f \in \mathbb{C}_q[G]$ two linear functionals on $\mathbb{C}_q[G]$ called l-functionals:

$$l^+(f) := \mathbf{r}(\cdot, f), \quad l^-(f) := \mathbf{r}(S(f), \cdot).$$

Here S is the antipode of $\mathbb{C}_q[G]$. The maps $f \mapsto l^\pm(f)$ are coalgebra homomorphisms and algebra antihomomorphisms into the dual Hopf algebra $\mathbb{C}_q[G]^\circ$.
It turns out that the Hopf algebra generated by the set of all l-functionals can be identified with $U_q(\mathfrak{g})$. Then one has

$$l^+(c^\lambda_{-\mu,\nu}) = X^+(c^\lambda_{-\mu,\nu})K_\mu, \quad l^-(c^\lambda_{-\mu,\nu}) = X^-(c^\lambda_{-\mu,\nu})K_{-\nu}. \tag{1.1}$$

Here $c^\lambda_{-\mu,\nu}$ is a matrix coefficient of (the deformation of) the irreducible representation $V(\lambda)$ with highest weight λ, and $-\mu, \nu$ are weights labeling the dual bases in $V(\lambda), V(\lambda)^*$ with respect to which the matrix coefficients are taken. The elements $K_\lambda \in U_q(\mathfrak{g})$ are the generators of a subalgebra $U_q(\mathfrak{h})$ generalizing the Cartan subalgebra of $\mathfrak{g}$, and $X^\pm(c^\lambda_{-\mu,\nu})$ lie in subalgebras $U_q(\mathfrak{n}_\pm)$ which are analogues of the standard maximal nilpotent subalgebras of $\mathfrak{g}$. Hence the above formula describes the decomposition of $l^\pm(c^\lambda_{-\mu,\nu})$ into components in $U_q(\mathfrak{h})$ and $U_q(\mathfrak{n}_\pm)$. It is proven in [Jo] under the assumption that $\mu = \nu$ and in the general case in Proposition 13 below. In [Mö] the notion of coquasitriangularity is suppressed and the relation is part of the definition of $l^\pm(f)$.
The first main result of Chapter 3 is a formula relating certain l-functionals to Lusztig's quantum root vectors [Lu]. These are elements $E_{\beta_k}, F_{-\beta_k} \in U_q(\mathfrak{g})$, where $\{\pm\beta_k\}$ are the roots of $\mathfrak{g}$. They are mainly used to generalize the Poincaré-Birkhoff-Witt theorem to $U_q(\mathfrak{g})$: The monomials $K_\lambda F^{i_1}_{-\beta_1} \cdots F^{i_n}_{-\beta_n} E^{j_1}_{\beta_1} \cdots E^{j_n}_{\beta_n}$ form a vector space basis of $U_q(\mathfrak{g})$. Unfortunately, there is no direct access to their algebraic properties. Their commutation relations are known in general only modulo terms of lower degree with respect to some filtration [KS2], Theorem 3.2.3. For the coproduct there are only a formal expression ([KS2], Proposition 3.2.1) and for special cases some explicit calculations [Xi].
The root vectors depend on a reduced expression for the longest element of the Weyl group of $\mathfrak{g}$. We will give for all classical matrix Lie groups one expression, for which the root vectors appear (with one exception in case of the symplectic groups) in (1.1) as $X^\pm(f)$, where f are certain matrix coefficients of the defining representation of G (Theorem 1). For $G = SL(N+1, \mathbb{C})$ this was stated already in [KS1]. One can modify the Poincaré-Birkhoff-Witt basis substituting the root vectors by the corresponding l-functionals. Then one has explicit formulas for the commutation relations and the coproduct of the basis elements.
One sees in particular that the coproduct of E_{β_k} and $F_{-\beta_k}$ is essentially given by classical matrix multiplication. This underlines the difference between the coalgebras $U_q(\mathfrak{g})$ and $U(\mathfrak{g})$, since the elements of $\mathfrak{g} \subset U(\mathfrak{g})$ are primitive, that is, satisfy $\Delta(X) = 1 \otimes X + X \otimes 1$.

1.2.3 L-functionals on $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$

Let A and B be two Hopf algebras and $\langle\cdot,\cdot\rangle$ be a skew-pairing of A and B, that is, a dual pairing of the opposite Hopf algebra A^{op} with B. Let $A \bowtie B$ denote the corresponding quantum double of A and B. This is a Hopf algebra which equals $A \otimes B$ as a coalgebra, but whose product is given by

$$(f \otimes g)(f' \otimes g') := (f f'_{(2)} \otimes g_{(2)} g')\,\langle S(f'_{(1)}), g_{(1)}\rangle\langle f'_{(3)}, g_{(3)}\rangle.$$

Here $\Delta(f) = f_{(1)} \otimes f_{(2)}$ is Sweedler's notation for the coproduct in a coalgebra. For example, if $(A, \mathbf{r})$ is a coquasitriangular Hopf algebra, then $\mathbf{r}$ is a skew-pairing, and $A \bowtie A$ is again coquasitriangular with respect to the bilinear form

$$\hat{\mathbf{r}}(f \otimes g, f' \otimes g') := \mathbf{r}(S(f'_{(1)} g'_{(1)}), f)\mathbf{r}(g, f'_{(2)} g'_{(2)}).$$

Let $U(A)$ and $U(A \bowtie A)$ be the Hopf algebras generated by the l-functionals on A and $A \bowtie A$. As suggested in [Ho], we call them the FRT-duals of A and $A \bowtie A$, respectively. There it was shown (the finite-dimensional case goes back to [RS], see also [Ma1]) that there exists an injective algebra homomorphism

$$\iota : U(A \bowtie A) \to U(A) \otimes U(A)$$

and a surjective Hopf algebra homomorphism

$$\zeta : A^{\mathrm{op}} \bowtie U(A) \to U(A \bowtie A).$$

Here the skew-pairing of A^{op} and $U(A)$ used to define $A^{\mathrm{op}} \bowtie U(A)$ is the restriction of the canonical pairing of A and A°.
In Section 3.3 we continue the investigation of these maps for $A = \mathbb{C}_q[G]$. We prove that ζ is then an isomorphism (Theorem 2). We furthermore show that there exists no bialgebra structure on $U(A) \otimes U(A) = U_q(\mathfrak{g} \oplus \mathfrak{g})$, such that ι becomes a bialgebra homomorphism (Theorem 3). In particular, ι is not an isomorphism (Corollary 1). But its image contains at least $1 \otimes C, C \otimes 1$ for all quantum Casimir operators C of $U_q(\mathfrak{g})$ (Proposition 16). These are central elements defined in the FRT-dual of any coquasitriangular Hopf algebra.
For real q, $\mathbb{C}_q[G]$ and $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$ become Hopf $*$-algebras $\mathbb{C}_q[G_0]$ and $\mathbb{C}_q[G_0] \bowtie \mathbb{C}_q[G_0]$ deforming the algebras of all complex-valued polynomials on the real algebraic groups G_0 and G, respectively. The quantum double $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$ appeared first in this setting in the deformation of the action of $SL(2,\mathbb{C})$ on Minkowski space [CSSW, PW1]. The map ζ is then a dual and purely algebraic version of the quantum Iwasawa decomposition from [PW1].
For arbitrary q, $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$ defines a new quantum group deformation of $G \times G$ related to the Manin triple $(G \times G, G, G^*)$.
Several authors proposed definitions of a quantized universal enveloping algebra corresponding to $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$, in particular, of a quantized Lorentz algebra. One idea was to dualize the structure of $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$ in form of a quantum codouble $U_q(\mathfrak{g}) \blacktriangleright\!\!\blacktriangleleft U_q(\mathfrak{g})$ [Ma1]. But ι would be a Hopf algebra homomorphism into such a quantum codouble. Hence it can not be well-defined by Theorem 3. The quantum double $\mathbb{C}_q[G]^{\mathrm{op}} \bowtie U_q(\mathfrak{g})$ provides a rigorously defined alternative. For the example of the q-Lorentz algebra the two factors can be interpreted as subalgebras of infinitesimal rotations and boosts, see e.g. [Bl]. The identification with $U(\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G])$ can be considered as a second example for a general philosophy to use FRT-duals as quantized enveloping algebras of quantum groups. As in case of $U_q(\mathfrak{g})$ one can think of it of course also as a definition of a quantum group dual to that given by $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$.

1.3 Dirac operators on quantum flag manifolds

1.3.1 Flag manifolds and their quantizations

Let G be as before, but we assume it now to be simply connected and simple. Let P be a parabolic subgroup, that is, a closed subgroup containing a Borel subgroup B_+ (a maximal solvable subgroup of G). Then the homogeneous space $M := G/P$ is called a (generalized) flag manifold.
Classically, a flag in $\mathbb{C}^{N+1}$ is a sequence $V_1 \subset \ldots \subset V_k$ of linear subspaces. The action of $SL(N+1,\mathbb{C})$ on $\mathbb{C}^{N+1}$ induces one on the set of all flags of a fixed length with fixed $\dim V_i$. This set becomes in this way a flag manifold in the above sense, and all flag manifolds associated to $SL(N+1,\mathbb{C})$ are of this form. For general G, the flag manifolds exhaust the compact homogeneous Kähler manifolds [Wa2]. In particular, they are compact symplectic and spin$^{\mathbb{C}}$ manifolds. When treated algebraically, they provide the basic examples of projective varieties. They also appear naturally in Yang-Mills theory, see [At, Ma3].
One can choose a compact real form G_0 of G such that $G_0^* \subset B_+$. Hence M is by the Iwasawa decomposition as a real variety or as a smooth manifold equal to G_0/L_0 with $L_0 := G_0 \cap P$. The latter is the compact real form of the Levi factor L of P (its maximal reductive subgroup). In other words, M is as a real variety a real form of G/L which is affine with coordinate ring [HK5]

$$\mathbb{C}[G/L] = \{f \in \mathbb{C}[G] \,|\, f(xy) = f(x) \,\forall x \in G, y \in L\}. \tag{1.2}$$

Equipping this algebra with the involution induced on $\mathbb{C}[G]$ by G_0 one obtains a $*$-algebra $\mathbb{C}[M]$ describing M. There exist analogues $\mathbb{C}_q[M]$ of this $*$-algebra inside of $\mathbb{C}_q[G_0]$. Like $\mathbb{C}[M]$ they are left coideals and hence define quantum spaces which are called quantum flag manifolds. They can be understood as quantizations of M with respect to some Poisson bracket.
The simplest example of a flag manifold is $\mathbb{C}P^1 = S^2$. The corresponding quantum flag manifold is Podleś' standard quantum sphere [Po]. Hence quantum flag manifolds are generalizations of this prototype of a quantum space.
We will have to assume that the considered flag manifolds are irreducible in the sense that $\mathfrak{p} := \mathrm{Lie}(P)$ acts irreducibly on $T_{eP}M = \mathfrak{g}/\mathfrak{p}$. This will be used at several places and entered in a crucial way in [HK1] on which Chapter 4 is based. The irreducible flag manifolds are exactly the irreducible compact Hermitian symmetric spaces and cover in particular the complex Grassmannians and the complexified spheres. A complete list will be given in 4.1.1.3 below.

1.3.2 Covariant differential calculi

Covariant differential calculi are an attempt to generalize the notion of differential 1-forms to quantum spaces [Wo2]. In general, a (first order) differential calculus over an algebra B is a pair of a B-bimodule Γ and a linear map $\mathrm{d} : B \to \Gamma$ satisfying the Leibniz rule $\mathrm{d}(fg) = f(\mathrm{d}g) + (\mathrm{d}f)g,\ f, g \in B$. Additionally, one requires that Γ is spanned (over $\mathbb{C}$) by the elements $f\mathrm{d}g$ with $f, g \in B$.
If A is a Hopf algebra and B is an A-comodule algebra (for example the coordinate algebra of a quantum space), then (Γ, d) is called covariant, if Γ is an A-comodule whose coaction is compatible with the bimodule structure and the map d in the sense that $(f\mathrm{d}g)_{(1)} \otimes (f\mathrm{d}g)_{(2)} = f_{(1)}g_{(1)} \otimes f_{(2)}\mathrm{d}(g_{(2)}) \in A \otimes \Gamma$ for all $f, g \in B$. Here we extended Sweedler's notation to coactions.

1.3.3 Differential calculi on quantum flag manifolds

On a given A-comodule algebra B there exist in general many covariant differential calculi. In particular, there is still no recipe known that singles out a 'best' one over $\mathbb{C}_q[G]$ itself. On quantized irreducible flag manifolds the situation looks better: As proven by I. Heckenberger and S. Kolb, these admit exactly two non-isomorphic irreducible covariant differential calculi $(\Gamma_\pm, \mathrm{d}_\pm)$ of finite dimension $\dim \Gamma_\pm := \dim_{\mathbb{C}} \Gamma_\pm / \mathbb{C}_q[M]^+ \Gamma_\pm$ [HK1]. Here $\mathbb{C}_q[M]^+ := \mathbb{C}_q[M] \cap \ker \varepsilon$ with ε being the counit of $\mathbb{C}_q[G_0]$, and a calculus is called irreducible if it possesses no non-trivial quotient. The elements of $\Gamma_\pm$ are analogues of holomorphic and antiholomorphic 1-forms, respectively. The direct sum (Γ, d) of Γ_+ and Γ_- is a $*$-calculus, that is, there exists an involution on Γ such that $(f\mathrm{d}g)^* = \mathrm{d}(g^*)f^*$. The aim of Chapter 4 is to show that these calculi are to some extent also compatible with the ideas of A. Connes' non-commutative geometry.

1.3.4 Dirac operators on quantum flag manifolds

Non-commutative geometry is a collection of various generalizations of geometric concepts to non-commutative algebras [Co1]. Although the basic idea is similar to that of quantum group theory, the used techniques are rather distinct and in fact both theories were developed mainly independently of each other.
The main objects studied in non-commutative geometry are spectral triples. A spectral triple consists basically of a $*$-algebra B of bounded operators on a Hilbert space H together with a self-adjoint operator D, such that the commutators $[D, f]$, $f \in B$, are all bounded. Any compact spin manifold defines a spectral triple in which B is the algebra of complex-valued smooth functions on the manifold that act as multiplication operators on the Hilbert space of square-integrable spinor fields; D is the Dirac operator. A basic question of non-commutative geometry is, to what extent classical geometry can be generalized to spectral triples over possibly non-commutative algebras.
A spectral triple (B, H, D) defines in particular a differential $*$-calculus over B with $\mathrm{d}f := i[D, f]$ and $\Gamma := \mathrm{span}_{\mathbb{C}}\{f\mathrm{d}g \,|\, f, g \in B\}$. For the spectral triple associated to a compact spin manifold, the operator $\mathrm{d}f$ acts by Clifford multiplication of a spinor field with the differential of f, see e.g. [Fr], p. 69.
It is hence a manifest question, whether a given covariant differential calculus on a quantum space can be realized by a spectral triple over a Hilbert space representation of its coordinate algebra. For the standard examples of differential $*$-calculi over $\mathbb{C}_q[SU(2)]$ this turned out to be only possible if one drops the condition that $\mathrm{d}f$ is a bounded operator [Sch2]. Hence the known spectral triples over $\mathbb{C}_q[SU(2)]$ [Co4, CP1, Go] seem to be unrelated to these calculi.
This problem does not occur for quantized irreducible flag manifolds. As we will show in Chapter 4, there exists a spectral triple over $\mathbb{C}_q[M]$ such that the associated differential calculus is isomorphic to the $*$-calculus (Γ, d) (Theorem 4). The key to its construction is that spin geometry on flag manifolds can be formulated entirely in terms of representation theory. This allows to generalize the appearing structures to the quantum case.
For the standard Podleś sphere, L. Dąbrowski and A. Sitarz derived in [DS1] a spectral triple by starting with an ansatz and implementing the axioms of (real) spectral triples with help of MAPLE. A uniqueness result proven in [DS1] shows that specializing our construction one obtains exactly this triple.

1.3.5 Outlook

It should be pointed out that the motivation for inventing spectral triples was not to define differential calculi, but to generalize some strong results from geometry such as index theorems. There are additional conditions for spectral triples needed to obtain them in the non-commutative setting, and it has *not* to be expected that they are satisfied by our spectral triples on quantum flag manifolds. This was already observed for the case of the Podleś sphere in [DS1, NT]. In particular, Connes' dimension theory based on the summability properties of the eigenvalues of D, and consequently the machinery of the local index formula [CM] computing the Chern character of a spectral triple [Co1] can not be applied directly. The latter should assign to a spectral triple (B, H, D) a cocycle in the cyclic cohomology $HC^*(B)$ of B. This is a cohomology theory for algebras derived from Hochschild cohomology. It substitutes the de Rham homology of a manifold in non-commutative geometry. It is known from several examples (including $\mathbb{C}_q[G]$) that the Hochschild and cyclic cohomology of an algebra obtained by quantizing a Poisson manifold is usually the homology of some space parametrizing the leaves of the symplectic foliation of that Poisson manifold, see [FT]. In particular, their Hochschild dimension is then smaller than that of the Poisson manifold. Hence some degeneracy of the Chern character of a spectral triple on quantum spaces is quite natural. On the other hand, the example $SU_q(2)$ showed that the 'dimension spectrum' of a spectral triple can reflect the dimension of the unquantized space, see [Co4].
In [KMT] a modified version of cyclic cohomology was proposed and shown to be closely related to the theory of covariant differential calculi. It was recently proven in [SW2] that for the spectral triple on the Podleś sphere some variations in the local index formula produce a well-defined cocycle in this 'twisted' cyclic cohomology of $\mathbb{C}_q[S^2]$. But beyond this example there are no general results in this direction. Furthermore, the integrality aspect of Connes' theory seems to get lost completely. That is, even if a 'twisted' Chern character or local index formula exists, it perhaps does not compute an index of an operator.
Some elaboration of these topics could be a next step towards a better understanding of the interplay of non-commutative geometry and the theory of quantum groups, and we hope that the spectral triples derived in Chapter 4 will provide natural examples to test new developments in this area.

1.4 On the non-standard Podleś spheres

The last chapter contains a short remark concerning the generic Podleś spheres. Like the standard one that appears in Chapter 4 as the simplest example of a quantum flag manifold, these are quantum spaces described by a family of coideal subalgebras $B_{q\rho}$, $\rho \in \mathbb{C} \cup \{\infty\}$, of $\mathbb{C}_q[SL(2, \mathbb{C})]$. For $\rho \in \mathbb{R}$ the algebras are $*$-subalgebras of $\mathbb{C}_q[SU(2)]$ deforming $\mathbb{C}[S^2]$. The parameter ρ plays the role of a radius. For $\rho = \infty$ one obtains the standard sphere $\mathbb{C}_q[S^2]$.
It was shown in [HMS] that the C^*-closure of these $*$-algebras does not depend on ρ, but it was conjectured that this is not the case for $B_{q\rho}$. Here we give an elementary proof of this conjecture (Theorem 5).

1.5 Bibliographical notes

For background on algebraic groups we refer to [Hu2, Sp], for Poisson geometry and quantization to [Va1, We]. Poisson groups were introduced by V. G. Drinfeld in [Dr1]. The duality theory and the correspondence to Manin triples was mainly developed in [LW]. For the algebraic Manin triple $(G \times G, G, G^*)$ and the interpretation of $U_q(\mathfrak{g})$ as a quantum group coordinate algebra see [DP].
Some pioneering papers on quantum groups and quantum spaces were [Dr2, FRT, Ji, KR, Ma2, Po, PW1, Wo1, Wo2]. For the mentioned applications of quantum groups see [CP2, Ka].
For introductions into non-commutative geometry see [Co1, FGV] and the review articles [Co2, Co3, Va2]. For the local index formula consult [CM, Hi].
More references will be given at the ends of each chapter.

Chapter 2

Quantum groups

In this chapter we fix notations and conventions, and also recall some frequently used results from Lie theory and quantum group theory.

2.1 Lie groups

2.1.1 Lie algebras

2.1.1.1 Roots and weights. Throughout this work, $\mathfrak{g}$ denotes a complex semi-simple Lie algebra. We refer e.g. to [Hu1, Kn] for the appearing notions and results from the theory of semi-simple Lie algebras and Lie groups.
We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a set $\{\alpha_1, \dots, \alpha_N\}$ of simple roots in the root system Φ associated to $\mathfrak{g}$ and $\mathfrak{h}$. Let $\Phi = \Phi^+ \cup \Phi^-$ be the corresponding decomposition of Φ into positive and negative roots, $\mathbf{Q} := \bigoplus_i \mathbb{Z}\alpha_i$ be the root lattice, $\{\mathfrak{g}_\beta\}_{\beta\in\Phi}$ be the root spaces, and set $\mathfrak{n}_\pm := \bigoplus_{\beta\in\Phi^\pm} \mathfrak{g}_\beta$ and $\mathfrak{b}_\pm := \mathfrak{h} \oplus \mathfrak{n}_\pm$. Both the Killing form of $\mathfrak{g}$ and the bilinear form induced by it on $\mathfrak{h}^*$ will be denoted by $\langle\cdot,\cdot\rangle$. Thus the entries of the Cartan matrix of $\mathfrak{g}$ are $a_{ij} := d_i^{-1}\langle\alpha_i, \alpha_j\rangle$ with $d_i := \frac{1}{2}\langle\alpha_i, \alpha_i\rangle$. We denote by $\{\omega_1, \dots, \omega_N\}$ the fundamental weights satisfying $\langle\omega_i, \alpha_j\rangle = \delta_{ij} d_i$ and by $\mathbf{P} := \bigoplus_i \mathbb{Z}\omega_i$ and $\mathbf{P}^+ := \bigoplus_i \mathbb{N}_0\omega_i$ the sets of integral and dominant integral weights. For $\lambda, \mu \in \mathbf{P}$ we write $\lambda < \mu$ if $\mu - \lambda$ is a sum of positive roots. The representation of $\mathfrak{g}$ with highest weight $\lambda \in \mathbf{P}^+$ and its carrier space are denoted by $(\rho_\lambda, V(\lambda))$.

2.1.1.2 Weyl group. Let $r_i : \alpha_j \mapsto \alpha_j - a_{ij}\alpha_i$ be the standard generators of the Weyl group W of $\mathfrak{g}$. For a given reduced expression $r_{i_1} \cdots r_{i_n}$ of its longest element w_0 let $\beta_k := r_{i_1} \cdots r_{i_{k-1}} \alpha_{i_k}$ be the corresponding ordering of Φ^+.

2.1.1.3 Compact real form and Iwasawa decomposition. We choose a Chevalley basis $\{H_k, E_\beta, F_{-\beta}\}_{k=1,\dots,N,\beta\in\Phi^+}$ of $\mathfrak{g}$ and abbreviate $E_k := E_{\alpha_k}$, $F_k := F_{-\alpha_k}$. Then $\mathfrak{g}_0 := \mathrm{span}_\mathbb{R}\{iH_k, E_\beta - F_{-\beta}, i(E_\beta + F_{-\beta})\}$ is a compact real form of $\mathfrak{g}$. Let $\mathfrak{g}_\mathbb{R}$ be $\mathfrak{g}$ treated as a real Lie algebra and $\mathfrak{g}_\mathbb{R} = \mathfrak{g}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}_+$ be the Iwasawa decomposition ([Kn], Proposition 6.43). Recall that this is a decomposition as a real vector space into three real Lie subalgebras. The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{g}_0$ or to $\mathfrak{a} \oplus \mathfrak{n}_+$ is real-valued. Hence the non-degenerate bilinear form on $\mathfrak{g}_\mathbb{R}$ given by its imaginary part defines an isomorphism of real vector spaces $\mathfrak{g}_0^* = \mathfrak{a} \oplus \mathfrak{n}_+$. For example, if $\mathfrak{g} = \mathfrak{sl}(N+1, \mathbb{C})$, then $\mathfrak{g}_0 = \mathfrak{su}(N+1)$, $\mathfrak{a}$ consists of all diagonal elements with real entries, and $\mathfrak{n}_+$ of the upper triangular elements with vanishing diagonal entries.

2.1.1.4 Universal enveloping algebra. We consider the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ in the usual way as a Hopf algebra. There exists a Hopf $*$-structure $*: U(\mathfrak{g}) \to U(\mathfrak{g})$ such that $* \circ S$ extends the Cartan involution θ of $\mathfrak{g}$ corresponding to $\mathfrak{g}_0$. Here S is the antipode of $U(\mathfrak{g})$. We denote the Hopf $*$-algebra $(U(\mathfrak{g}), *)$ with slight abuse of notation by $U(\mathfrak{g}_0)$.

2.1.2 Lie groups and algebraic groups

2.1.2.1 G, G_0, G_0^* and Iwasawa decomposition. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let G_0, G_0^* be the (real) subgroups of G corresponding to $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$. Then the group multiplication of G defines a diffeomorphism (of real manifolds) $G_0 \times G_0^* \to G$ called the Iwasawa decomposition of G. The group G_0^* is diffeomorphic to $\mathfrak{g}_0^*$ via its exponential map. In particular, G is contractible to G_0, and in the example $G = SL(N+1, \mathbb{C})$, the group G_0^* consists of all complex upper triangular matrices with real positive diagonal entries.

2.1.2.2 Group weights. Let $\mathbf{L}$ be the character group of a maximal torus of G with Lie algebra $\mathfrak{h}$. By passing to differentials we consider $\mathbf{L}$ as a sublattice of $\mathbf{P}$. Then $|\mathbf{P}/\mathbf{L}|$ is the order of the fundamental group $\pi_1(G) = \pi_1(G_0)$. In particular, $\mathbf{L} = \mathbf{P}$ iff G is simply connected. The set $\mathbf{L}^+ := \mathbf{L} \cap \mathbf{P}^+$ contains the highest weights of those representations of $\mathfrak{g}$ which lift to representations of G.

2.1.2.3 G as an algebraic group. We denote by the same letter G the unique affine algebraic group over $\mathbb{C}$ whose associated Lie group is G. Recall that the group structure turns its coordinate algebra $\mathbb{C}[G]$ (the algebra of regular functions on G) into a Hopf algebra. As a G-bimodule it is the span of the matrix coefficients of the irreducible finite-dimensional representations of G, $\mathbb{C}[G] = \bigoplus_{\lambda \in \mathbf{L}^+} V(\lambda)^* \otimes V(\lambda)$ (Peter-Weyl theorem). The evaluation of a matrix coefficient on an element of $\mathfrak{g}$ continues to a non-degenerate dual pairing of the Hopf algebras $\mathbb{C}[G]$ and $U(\mathfrak{g})$. In particular, the Hopf $*$-structure of $U(\mathfrak{g}_0)$ induces one on $\mathbb{C}[G]$. We denote the resulting Hopf $*$-algebra by $\mathbb{C}[G_0]$. Its involution is dual to the real structure on G whose set of real points is G_0.

2.2 Quantum groups

2.2.1 The Hopf algebra $U_q(\mathfrak{g})$

2.2.1.1 Algebras, coalgebras, Hopf algebras. In this work, all algebras are assumed to be associative, unital algebras over the ground field $\mathbb{C}$. The multiplication map of an algebra is denoted by m. The coproduct, counit and antipode of a Hopf algebra are denoted by Δ, ε and S, respectively.
We use Sweedler's notation $\Delta(X) = X_{(1)} \otimes X_{(2)}$ for the coproduct in a coalgebra or more generally for the coaction of a coalgebra on a comodule. If A is a (co)algebra, then we denote by $A^{(\mathrm{c})\mathrm{op}}$ the (co)opposite (co)algebra with (co)product $f \cdot_{\mathrm{op}} g := gf$ and $\Delta_{\mathrm{cop}}(X) := X_{(2)} \otimes X_{(1)}$, respectively.
For an algebra A, we denote by A° the dual coalgebra, that is, the set of all linear functionals on A whose kernel contains an ideal of finite codimension. This is a coalgebra with coproduct given by dualizing the multiplication map of A [Ab], Section 2.3. If A is a Hopf algebra, then A° is a Hopf algebra as well.

2.2.1.2 The algebra $U_q(\mathfrak{g})$. Let $q \in \mathbb{C} \setminus \{0\}$ be not a root of unity, fix $\hbar \in \mathbb{C}$ with $q = e^{\hbar}$ and define $q^z := e^{\hbar z}$ for $z \in \mathbb{C}$. Set $q_i := q^{d_i}$ and for $n, k \in \mathbb{N}_0$

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q := \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad [n]_q! := \prod_{k=1}^{n} [k]_q, \quad [k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}.$$

Let $U_q(\mathfrak{h}) := \mathrm{span}_{\mathbb{C}}\{K_\lambda, \lambda \in \mathbf{L} \,|\, K_0 = 1, K_\lambda K_\mu = K_{\lambda+\mu}\}$ be the group algebra of $\mathbf{L}$ and $U_q(\mathfrak{g})$ be the quotient of $U_q(\mathfrak{h})[E_1, \ldots, E_N, F_1, \ldots, F_N]$ by the relations

$$\begin{aligned} &K_\lambda E_i = q^{\langle \lambda, \alpha_i \rangle} E_i K_\lambda, \quad K_\lambda F_i = q^{-\langle \lambda, \alpha_i \rangle} F_i K_\lambda, \\ &E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad K_i := K_{\alpha_i} \end{aligned}$$

and the quantum Serre relations

$$\begin{aligned} &\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array}\right]_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0, \quad i \neq j, \\ &\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array}\right]_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0, \quad i \neq j. \end{aligned}$$

2.2.1.3 Hopf algebra structure. There is a unique Hopf algebra structure on $U_q(\mathfrak{g})$, such that Δ, ε, S are given on the generators by

$$\begin{aligned} &\Delta(K_\lambda) = K_\lambda \otimes K_\lambda, \quad \varepsilon(K_\lambda) = 1, \quad S(K_\lambda) = K_{-\lambda}, \\ &\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \\ &\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i. \end{aligned}$$

We call the Hopf algebra $U_q(\mathfrak{g})$ the quantized universal enveloping algebra corresponding to $\mathfrak{g}$ and q.

2.2.1.4 Relation with [Jo, KS1]. Note that the above definition which we learned from [HLT] depends not only on $\mathfrak{g}$, but also on $\mathbf{L}$, that is, on the choice of G. In the most common definition of $U_q(\mathfrak{g})$ one considers only the Hopf subalgebra generated by $K_i^{\pm 1}, E_i, F_i$. This contains only K_λ with $\lambda \in \mathbf{Q}$, the minimal choice for $\mathbf{L}$. However, at some depth of the theory this becomes insufficient. In [Jo] and [KS1] this leads to the definition of two Hopf algebras denoted there by $\check{U}$ and $U_q^{\mathrm{ext}}(\mathfrak{g})$, respectively. Both are special cases of $U_q(\mathfrak{g})$ as defined here and are obtained by taking G to be simply connected ($\check{U}$) or to be one of the classical matrix Lie groups ($U_q^{\mathrm{ext}}(\mathfrak{g})$). In particular, for $G = SL(N+1, \mathbb{C})$ our $U_q(\mathfrak{sl}(N+1, \mathbb{C}))$ is equal to $\check{U}$ and to $U_q^{\mathrm{ext}}(\mathfrak{sl}(N+1, \mathbb{C}))$ from [Jo] and [KS1] and obtained by adding $K_{\pm \omega_N}$ to the generators $K_i^{\pm 1}, E_i, F_i$ (with $\omega_N = -\frac{1}{N+1} \sum_{k=1}^{N} k \alpha_k$).

2.2.1.5 Compact real form. For $q \in \mathbb{R}$ the Hopf $*$-structure of $U(\mathfrak{g}_0)$ can be generalized to one on $U_q(\mathfrak{g})$. Its involution $*$ is given on the generators by

$$K_\lambda^* = K_\lambda, \quad E_i^* = K_i F_i, \quad F_i^* = E_i K_i^{-1}.$$

Hence it coincides essentially with κ' from [Jo], 3.3.3 with the only difference that κ' is continued to a linear map while $*$ is conjugate linear. We denote $(U_q(\mathfrak{g}), *)$ by $U_q(\mathfrak{g}_0)$. We call $\theta := * \circ S$ the Cartan involution of $U_q(\mathfrak{g}_0)$.

2.2.1.6 Triangular decomposition. Let $U_q(\mathfrak{n}_\pm) \subset U_q(\mathfrak{g})$ be the subalgebras generated by the E_i and F_i, respectively, and set $U_q(\mathfrak{b}_\pm) := U_q(\mathfrak{h})U_q(\mathfrak{n}_\pm)$. Then $U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{n}_+) \to U_q(\mathfrak{g})$, $K \otimes F \otimes E \mapsto KFE$ defines an isomorphism of vector spaces called the triangular decomposition of $U_q(\mathfrak{g})$.

2.2.1.7 Q-grading. There is a $\mathbf{Q}$-grading on $U_q(\mathfrak{g})$ given by

$$U_q(\mathfrak{g}) = \bigoplus_{\lambda \in \mathbf{Q}} U_q^\lambda(\mathfrak{g}), \quad U_q^\lambda(\mathfrak{g}) := \{X \in U_q(\mathfrak{g}) \,|\, K_\mu X K_\mu^{-1} = q^{\langle \lambda, \mu \rangle} X \quad \forall\, \mu \in \mathbf{L}\}.$$

We will need below the following description of the coproduct of elements of the vector spaces $U_q^\lambda(\mathfrak{n}_\pm) := U_q^\lambda(\mathfrak{g}) \cap U_q(\mathfrak{n}_\pm)$ ([Ja], Lemma 4.12):

Proposition 1 *For $X^\pm \in U_q^{\pm\lambda}(\mathfrak{n}_\pm)$, $\lambda > 0$, there are $X_{1i}^\pm \in U_q^{\pm(\lambda-\mu_i)}(\mathfrak{n}_\pm)$, $X_{2i}^\pm \in U_q^{\pm\mu_i}(\mathfrak{n}_\pm)$, $0 < \mu_i < \lambda$, such that*

$$\Delta(X^+) = X^+ \otimes 1 + \sum_i X_{1i}^+ K_{\mu_i} \otimes X_{2i}^+ + K_\lambda \otimes X^+,$$
$$\Delta(X^-) = X^- \otimes K_{-\lambda} + \sum_i X_{1i}^- \otimes X_{2i}^- K_{\mu_i - \lambda} + 1 \otimes X^-.$$

2.2.1.8 Poincaré-Birkhoff-Witt theorem. There are algebra automorphisms $\mathcal{T}_1, \ldots, \mathcal{T}_N$ of $U_q(\mathfrak{g})$ that generate an action of the braid group associated to $\mathfrak{g}$ on $U_q(\mathfrak{g})$ [Lu]. This can be used to generalize the Poincaré-Birkhoff-Witt theorem to $U_q(\mathfrak{g})$: One defines the quantum root vectors

$$E_{\beta_k} := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{k-1}}(E_{i_k}), \quad F_{-\beta_k} := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{k-1}}(F_{i_k}),$$

where $r_{i_1} \cdots r_{i_n}$ is a reduced expression of w_0. These elements depend on the choice of $r_{i_1} \cdots r_{i_n}$. But independently of the choice one has:

Proposition 2 (PBW theorem) *The monomials*

$$K_\lambda F_{\mathbf{i}} E_{\mathbf{j}} := K_\lambda F_{-\beta_1}^{i_1} \cdots F_{-\beta_n}^{i_n} E_{\beta_1}^{j_1} \cdots E_{\beta_n}^{j_n}, \quad \lambda \in \mathbf{L}, \mathbf{i}, \mathbf{j} \in \mathbb{N}_0^n.$$

form a vector space basis of $U_q(\mathfrak{g})$.

2.2.1.9 Commutation relations of $E_{\beta_k}, F_{-\beta_k}$. We have

Proposition 3 *For $i < j$ there are $x_{ij\mathbf{k}}, y_{ij\mathbf{k}} \in \mathbb{C}$, such that*

$$E_{\beta_i} E_{\beta_j} - q^{\langle \beta_i, \beta_j \rangle} E_{\beta_j} E_{\beta_i} = \sum_{\mathbf{k} \in \mathbb{N}_0^{j-i-1}} x_{ij\mathbf{k}} E_{\beta_{i+1}}^{k_1} \cdots E_{\beta_{j-1}}^{k_{j-i-1}}, \tag{2.1}$$
$$F_{-\beta_i} F_{-\beta_j} - q^{-\langle \beta_i, \beta_j \rangle} F_{-\beta_j} F_{-\beta_i} = \sum_{\mathbf{k} \in \mathbb{N}_0^{j-i-1}} y_{ij\mathbf{k}} F_{-\beta_{i+1}}^{k_1} \cdots F_{-\beta_{j-1}}^{k_{j-i-1}}. \tag{2.2}$$

If $\beta_i + \beta_j \neq \sum_{l=1}^{j-1} k_l \beta_{i+l}$, then $x_{ij\mathbf{k}} = y_{ij\mathbf{k}} = 0$.

The relations (2.1), (2.2) are proven in [KS2], Theorem 3.2.3. Conjugating them with K_λ the additional remark follows from the PBW theorem.

2.2.1.10 $U_q(\mathfrak{g})$ is an integral domain. There is a filtration of $U_q(\mathfrak{g})$ for which the terms on the right hand side of (2.1), (2.2) are of lower degree than those on the left hand side. This was used in [DK1], Corollary 1.8 to prove:

Proposition 4 *The algebra $U_q(\mathfrak{g})$ is an integral domain.*

2.2.1.11 Adjoint action and Rosso form. We abbreviate by $X \bullet Y$ the (left) adjoint action $\mathrm{ad}(X)Y := X_{(1)}YS(X_{(2)})$ of $X \in U_q(\mathfrak{g})$ on $Y \in U_q(\mathfrak{g})$. We denote by $\mathcal{F}(U_q(\mathfrak{g})) := \{X \in U_q(\mathfrak{g}) \,|\, \dim U_q(\mathfrak{g}) \bullet X < \infty\}$ the locally finite part of $U_q(\mathfrak{g})$. One has ([Jo], 7.1.3):

Proposition 5 $K_\lambda \in \mathcal{F}(U_q(\mathfrak{g})) \Leftrightarrow \lambda \in -2\mathbf{L}^+$.

We denote by φ the dual pairing of $U_q(\mathfrak{b}_-)$ and $U_q(\mathfrak{b}_+)^{\mathrm{cop}}$ as defined in [Jo], 3.1.7 and by $\langle \cdot, \cdot \rangle$ the quantum Killing form (or Rosso form) on $U_q(\mathfrak{g})$ as defined in [Jo], 3.3.3. It has the following properties ([Jo], 3.3.3, 7.2.4):

Proposition 6 *For all $X, Y, Z \in U_q(\mathfrak{g})$ one has $\langle Z \bullet X, Y \rangle = \langle X, S(Z) \bullet Y \rangle$. If $X \in U_q^\lambda(\mathfrak{g}), Y \in U_q^\mu(\mathfrak{g})$ with $\lambda \neq -\mu$, then $\langle X, Y \rangle = 0$. The restriction of $\langle \cdot, \cdot \rangle$ to $U_q^\lambda(\mathfrak{g}) \times U_q^{-\lambda}(\mathfrak{g})$ is non-degenerate. If $q \in (1, \infty)$, then for all $X, Y \in U_q(\mathfrak{g}_0)$ one has $\langle X^*, Y \rangle = \overline{\langle Y^*, X \rangle}$ and $\langle X^*, X \rangle > 0$ if $X \neq 0$.*

2.2.1.12 Representation theory. The classical theory of highest weight representations carries over almost literally to $U_q(\mathfrak{g})$, see [KS1], Chapter 7 or [CP2], Section 10.1. In particular, for every $\lambda \in \mathbf{P}^+$ there exists a finite-dimensional irreducible $U_q(\mathfrak{g})$-module $V(\lambda)$ which is generated by a highest weight vector. Let $\mathcal{C}_G$ be the category of all finite direct sums of $V(\lambda)$ with $\lambda \in \mathbf{L}^+$. For $V \in \mathcal{C}_G$ we denote by $V_\lambda := \{v \in V \,|\, K_\mu v = q^{\langle \lambda, \mu \rangle} v \,\forall \mu \in \mathbf{L}\}$ the set of vectors of weight $\lambda \in \mathbf{P}$. Then the multiplicities $\dim V_\lambda$ are the same as those in the corresponding representation of $\mathfrak{g}$. The category $\mathcal{C}_G$ is closed under the formation of duals and tensor products, where for $V, W \in \mathcal{C}_G$ the module structures on V^* and $V \otimes W$ are given by

$$
\begin{aligned}
(Xv)(w) &:= v(S(X)w), \quad X \in U_q(\mathfrak{g}), v \in V^*, w \in V, \\
X(v \otimes w) &:= X_{(1)}v \otimes X_{(2)}w, \quad X \in U_q(\mathfrak{g}), v \in V, w \in W.
\end{aligned}
$$

The irreducible components of $V \otimes W$ are the same as in the classical case. There are isomorphisms $R_{VW} : V \otimes W \to W \otimes V$ which turn $\mathcal{C}_G$ into a braided tensor category. For $q \in \mathbb{R}$ there exists a Hermitian inner product $(\cdot, \cdot)_\lambda$ on $V(\lambda)$ such that $(Xv, w)_\lambda = (v, X^* w)_\lambda$ for all $X \in U_q(\mathfrak{g}_0), v, w \in V$.

2.2.2 The Hopf algebra $\mathbb{C}_q[G]$

2.2.2.1 The Hopf algebra $\mathbb{C}_q[G]$. Let $\mathbb{C}_q[G]$ be the restricted Hopf dual $U_q(\mathfrak{g})^\circ_{\mathcal{C}_G}$ of $U_q(\mathfrak{g})$, that is, the Hopf subalgebra of the full Hopf dual $U_q(\mathfrak{g})^\circ$ spanned by the matrix coefficients $c_{v,w} : U_q(\mathfrak{g}) \to \mathbb{C}$, $c_{v,w}(X) := v(Xw)$ with $v \in V^*, w \in V, V \in \mathcal{C}_G$. Its Hopf algebra structure is given by

$$
\begin{aligned}
c_{v,w}c_{v',w'} &= c_{v' \otimes v, w \otimes w'}, \quad \varepsilon(c_{v,w}) = v(w), \\
\Delta(c_{v,w}) &= \sum_i c_{v,w_i} \otimes c_{v_i,w}, \quad S(c_{v,w}) = c_{w,v},
\end{aligned}
$$

where $\{v_i\}, \{w_i\}$ are dual bases in V^*, V. Note that $S^2 \neq \mathrm{id}$, since in the identification $V^{**} = V$ the automorphism S^2 of $U_q(\mathfrak{g})$ is hidden. If $v \in V(\lambda)^*_{-\mu}, w \in V(\lambda)_\nu$, then $c_{v,w}$ is denoted by $c^\lambda_{-\mu,\nu}$ as well. We fix dual bases $\{v_i^\lambda\}, \{w_i^\lambda\}$ in $V(\lambda)^*, V(\lambda)$ with $w_i^\lambda \in V(\lambda)_{\mu_i}$ and set $c^\lambda_{ij} := c_{v_i^\lambda, w_j^\lambda}$. We call $\mathbb{C}_q[G]$ the coordinate algebra of the standard quantum group G_q associated to G and q.

2.2.2.2 Peter-Weyl decomposition. The evaluation $\langle X, c_{v,w}\rangle := c_{v,w}(X)$ is a non-degenerate dual pairing of Hopf algebras. It turns $\mathbb{C}_q[G]$ into a $U_q(\mathfrak{g})$-bimodule with left and right action given by

$$X \triangleright f := \langle X, f_{(2)}\rangle f_{(1)}, \quad f \triangleleft X := \langle X, f_{(1)}\rangle f_{(2)}, \quad X \in U_q(\mathfrak{g}), f \in \mathbb{C}_q[G].$$

The structure of this bimodule is given by the classical Peter-Weyl decomposition. That is, the c^λ_{ij} form a vector space basis of $\mathbb{C}_q[G]$.

2.2.2.3 Compact real form. If $q \in \mathbb{R}$, there is a unique Hopf $*$-structure on $\mathbb{C}_q[G]$ such that the pairing with $U_q(\mathfrak{g})$ becomes a pairing of Hopf $*$-algebras with $U_q(\mathfrak{g}_0)$. We denote this Hopf $*$-algebra by $\mathbb{C}_q[G_0]$. We can choose the matrix coefficients c^λ_{ij} in such a way that they are unitary, $(c^\lambda_{ij})^* = S(c^\lambda_{ji})$.

2.2.2.4 $\mathbb{C}_q[G]$ is an integral domain. We will see in the next chapter that $\mathbb{C}_q[G]^{\mathrm{op}}$ is as an algebra isomorphic to a subalgebra of $U_q(\mathfrak{g} \oplus \mathfrak{g})$. By Proposition 4 we therefore have:

Proposition 7 *The algebra $\mathbb{C}_q[G]$ is an integral domain.*

2.2.2.5 Haar functional. Let h be the Haar functional on $\mathbb{C}_q[G]$. It is defined by $h(c^\lambda_{ij}) := \delta_{\lambda 0}$ and can be characterized by the following properties:

Proposition 8 *The functional h is the unique linear functional on $\mathbb{C}_q[G]$ such that $h(1) = 1$ and $h(f_{(1)})f_{(2)} = h(f_{(2)})f_{(1)} = h(f)$ for all $f \in \mathbb{C}_q[G]$.*

For real q we define $\langle f, g\rangle_h := h(fg^*)$, $f, g \in \mathbb{C}_q[G_0]$. Then we have

Proposition 9 *The sesquilinear form $\langle \cdot, \cdot \rangle_h$ is a Hermitian inner product on $\mathbb{C}_q[G_0]$. For all $X \in U_q(\mathfrak{g}_0), f, g \in \mathbb{C}_q[G_0]$ we have $\langle f \triangleleft X, g\rangle_h = \langle f, g \triangleleft X^*\rangle_h$.*

That is, $\langle \cdot, \cdot \rangle_h$ is the direct sum of the $(\cdot, \cdot)_\lambda$ on the right $U_q(\mathfrak{g}_0)$-module $\mathbb{C}_q[G_0]$.

2.2.2.6 Relation with [Jo, KS1]. If G is simply connected, then $\mathbb{C}_q[G]$ coincides with $R_q[G]$ from [Jo]. Originally it was defined only for the classical matrix Lie groups in terms of generators and relations [FRT]. The generators were the matrix coefficients of the vector representation of $U_q(\mathfrak{g})$ (the one in which G is defined as a matrix Lie group) with respect to some basis. For the relations we refer to [KS1], Chapter 9. There the resulting Hopf algebras are denoted by $\mathcal{O}(G_q)$. If q is not a root of unity, then $\mathcal{O}(G_q)$ is isomorphic to $\mathbb{C}_q[G]$ as introduced above for all G except $G = SO(2N+1)$, where $\mathcal{O}(G_{q^2}) = \mathbb{C}_q[G]$. This is a consequence of the Peter-Weyl theorem for $\mathcal{O}(G_q)$ ([KS1], Theorem 11.22). The latter is stated in [KS1] under the assumption that q is transcendental, but according to Remark 3 on p. 415 of [KS1] and Corollaries 4.15 and 5.22 from [LR] the result holds also for q not a root of unity.

2.3 Bibliographical notes

Most of the material on Lie theory is described in any textbook on the subject. Besides the mentioned references, see also [He1, Wa1] for the Iwasawa decomposition. For the Hopf algebra $U(\mathfrak{g})$ and the Hopf $*$-algebra $U(\mathfrak{g}_0)$ see [KS1], Sections 1.2.6 and 1.2.7. For the general theory of Hopf algebras we refer to [Ab, Mo, Sw].

The relation between group weights and $\pi_1(G)$ is proven for example in [Kn],

Corollary 5.108. The description of G as an algebraic group and a proof of the algebraic Peter-Weyl theorem is given in [Ta1], Section 15.8. For dual pairings of Hopf algebras see [KS1], Sections 1.2.5 and 1.2.7.
Some textbooks on quantum groups are [CP2, Ja, Jo, Ka, KS1, KS2, Lu, Ma1]. To the authors knowledge the refined definition of $U_q(\mathfrak{g})$ used here appears first in [Ta2]. For the Hopf $*$-algebra $U_q(\mathfrak{g}_0)$ see [KS1], Section 6.1.7. The quantum root vectors and the PBW theorem for $U_q(\mathfrak{g})$ are due to G. Lusztig [Lu]. For braided tensor categories see [Ka] or [CP2] (where they are called quasitensor categories). The braiding of $\mathcal{C}_G$ is described in [CP2, HLT]. For the Hopf $*$-algebra $\mathbb{C}_q[G_0]$ and the Haar functional see [KS1].

Chapter 3

On the quantum Iwasawa decomposition

After recalling the definitions of universal r-forms and l-functionals in the first section, we study in the second the structure of the l-functionals on $\mathbb{C}_q[G]$ and their relation to quantum root vectors. In the third section we turn to the l-functionals on the quantum double $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$. The main result is that the Hopf algebra generated by them is isomorphic to $\mathbb{C}_q[G]^{\mathrm{op}} \bowtie U_q(\mathfrak{g})$ as was conjectured in [Ho]. We also show that the quantum codouble $U_q(\mathfrak{g}) \blacktriangleright\!\!\blacktriangleleft U_q(\mathfrak{g})$ studied e.g. in [Ma1] is not well-defined in the purely algebraic context.

3.1 L-functionals

3.1.1 Coquasitriangular Hopf algebras

3.1.1.1 Definition. A Hopf algebra A is called coquasitriangular if there exist two bilinear forms $\mathbf{r}, \bar{\mathbf{r}}$ on A such that for all $f, g, h \in A$ one has

$$\begin{aligned}
&\mathbf{r}(f_{(1)}, g_{(1)})\bar{\mathbf{r}}(f_{(2)}, g_{(2)}) = \bar{\mathbf{r}}(f_{(1)}, g_{(1)})\mathbf{r}(f_{(2)}, g_{(2)}) = \varepsilon(f)\varepsilon(g),\\
&gf = \mathbf{r}(f_{(1)}, g_{(1)})f_{(2)}g_{(2)}\bar{\mathbf{r}}(f_{(3)}, g_{(3)}),\\
&\mathbf{r}(fg, h) = \mathbf{r}(f, h_{(1)})\mathbf{r}(g, h_{(2)}), \quad \mathbf{r}(f, gh) = \mathbf{r}(f_{(1)}, h)\mathbf{r}(f_{(2)}, g).
\end{aligned}$$

In the sequel, we will frequently use the standard shorthand notation for formulas of this type, see [KS1]. For example, the above would be written as

$$\begin{aligned}
&\mathbf{r}\bar{\mathbf{r}} = \bar{\mathbf{r}}\mathbf{r} = \varepsilon \otimes \varepsilon, \quad m_{21} = \mathbf{r}m\bar{\mathbf{r}},\\
&\mathbf{r} \circ (m \otimes \mathrm{id}) = \mathbf{r}_{13}\mathbf{r}_{23}, \quad \mathbf{r} \circ (\mathrm{id} \otimes m) = \mathbf{r}_{13}\mathbf{r}_{12}.
\end{aligned}$$

One calls $\mathbf{r}$ a universal r-form of A and $\bar{\mathbf{r}}$ its inverse. If A is a Hopf $*$-algebra, then $\mathbf{r}$ is called real (inverse real), if $\mathbf{r}(f^*, g^*) = \overline{\mathbf{r}(g, f)}$ ($\mathbf{r}(f^*, g^*) = \overline{\mathbf{r}(f, g)}$).

3.1.1.2 Properties of r. Let $(A, \mathbf{r})$ be a coquasitriangular Hopf algebra. Then the universal r-form satisfies ([KS1], Proposition 10.2)

$$\mathbf{r}(1, f) = \mathbf{r}(f, 1) = \varepsilon(f), \quad \bar{\mathbf{r}}(f, g) = \mathbf{r}(S(f), g), \quad \mathbf{r}(f, g) = \mathbf{r}(S(f), S(g))$$

and the quantum Yang-Baxter equation $\mathbf{r}_{12}\mathbf{r}_{13}\mathbf{r}_{23} = \mathbf{r}_{23}\mathbf{r}_{13}\mathbf{r}_{12}$. The bilinear form $\bar{\mathbf{r}}_{21}(f, g) := \bar{\mathbf{r}}(g, f)$ is a universal r-form as well.

3.1.1.3 Inverse of the antipode. Define two linear functionals $\mathrm{f}, \bar{\mathrm{f}}$ on A by

$$\mathrm{f}(f) := \mathbf{r}(f_{(1)}, S(f_{(2)})), \quad \bar{\mathrm{f}}(f) := \mathbf{r}(S^2(f_{(1)}), f_{(2)}).$$

Then $\mathrm{f}\bar{\mathrm{f}} = \bar{\mathrm{f}}\mathrm{f} = \varepsilon$ and one has ([KS1], Proposition 10.3):

Proposition 10 *For all $k \in \mathbb{Z}$ one has $S^{k+2}(f) = \bar{\mathrm{f}}(f_{(1)}) S^k(f_{(2)}) \mathrm{f}(f_{(3)})$.*

3.1.2 L-functionals and FRT-duals

3.1.2.1 L-functionals. Let $(A, \mathbf{r})$ be a coquasitriangular Hopf algebra. Then any $f \in A$ defines two linear functionals

$$l^+(f) := \mathbf{r}(\cdot, f), \quad l^-(f) := \bar{\mathbf{r}}(f, \cdot), \quad f \in A$$

on A. These are called the l-functionals on A.

3.1.2.2 FRT-duals. The properties of $\mathbf{r}$ imply that $l^\pm : A^{\mathrm{op}} \to A^\circ$, $f \mapsto l^\pm(f)$ are Hopf algebra homomorphisms. Hence the subalgebra $U(A)$ of A° generated by all l-functionals is a Hopf algebra. It follows from the quantum Yang-Baxter equation that

$$U(A) = \mathrm{span}\{l^+(f) l^-(g) \mid f, g \in A\} = \mathrm{span}\{l^-(f) l^+(g) \mid f, g \in A\}. \tag{3.1}$$

Following [Ho] we call $U(A)$ the FRT-dual of A.

3.2 L-functionals on $\mathbb{C}_q[G]$

3.2.1 L-functionals on $\mathbb{C}_q[G]$

3.2.1.1 Coquasitriangularity of $\mathbb{C}_q[G]$. Let U be a Hopf algebra and $\mathcal{C}$ a braided tensor category of finite-dimensional U-modules which is closed under the formation of duals. Then $A := U^\circ_{\mathcal{C}}$ is coquasitriangular with a universal r-form given by $\mathbf{r}(c_{v,w}, c_{v',w'}) := (v' \otimes v) \circ R_{VW}(w \otimes w')$, where $v \in V^*, w \in V, v' \in W^*, w' \in W$ and R_{VW} is the braiding. In particular, $\mathbb{C}_q[G]$ is coquasitriangular. It follows from the explicit form of the braiding that $\mathbf{r}$ is real if $q \in \mathbb{R}$.

3.2.1.2 Identification of $U(\mathbb{C}_q[G])$ with $U_q(\mathfrak{g})$. The braiding of $\mathcal{C}_G$ is constructed in such a way that the l-functionals on $\mathbb{C}_q[G]$ can be identified with elements of $U_q(\mathfrak{g})$. That is, there is a Hopf algebra embedding of $U(\mathbb{C}_q[G])$ into $U_q(\mathfrak{g})$. Then we have

$$\mathbf{r}(f, g) = \varphi(l^-(f), l^+(g)). \tag{3.2}$$

The embedding is in fact surjective [Jo], 9.2.12. We therefore have:

Proposition 11 *The Hopf algebras $U(\mathbb{C}_q[G])$ and $U_q(\mathfrak{g})$ are isomorphic.*

The characterization (3.2) of $\mathbf{r}$ implies (see [Ho]):

Proposition 12 *One has $c_{v,w} \in \ker l^\pm \Leftrightarrow v(U_q(\mathfrak{b}_\mp) w) = 0$.*

In what follows, we will not distinguish between $U(\mathbb{C}_q[G])$ and $U_q(\mathfrak{g})$ any more.

3.2.1.3 Structure of l-functionals. We will now generalize the description of the triangular decomposition of $l^\pm(c^\lambda_{-\mu,\nu})$ given in [Jo], 9.2.11. Recently the author became aware of the review article [Mö]. There the notion of coquasitriangularity is suppressed and the statement is part of the definition of $l^\pm(c^\lambda_{-\mu,\nu})$ (see Equations 3.8 and 3.9 therein).

Proposition 13 *For $c^\lambda_{-\mu,\nu} \in \mathbb{C}_q[G]$ there are $X^\pm(c^\lambda_{-\mu,\nu}) \in U_q^{\nu-\mu}(\mathfrak{n}_\pm)$ with*

$$l^+(c^\lambda_{-\mu,\nu}) = X^+(c^\lambda_{-\mu,\nu})K_\mu, \quad l^-(c^\lambda_{-\mu,\nu}) = X^-(c^\lambda_{-\mu,\nu})K_{-\nu}.$$

Proof. We treat only l^+, the other case is analogous. Let $c^\lambda_{-\mu,\nu} = c_{v,w}$ be given. Fix dual bases $\{v_i\}, \{w_i\}$ with $w_i \in V(\lambda)_{\nu_i}$ and $w \in \{w_i\}$. Let $w' \in V(\lambda)_\lambda$. Since l^+ is a coalgebra homomorphism, we then have

$$\Delta(l^+(c_{v,w'})) = \sum_i l^+(c_{v,w_i}) \otimes l^+(c_{v_i,w'}). \tag{3.3}$$

It is known that the claim holds for $\nu = \lambda$ [Jo], 9.2.11, so

$$l^+(c_{v,w'}) = X^+(c_{v,w'})K_\mu, \quad l^+(c_{v_i,w'}) = X^+(c_{v_i,w'})K_{\nu_i}. \tag{3.4}$$

By the first equality and Proposition 1 we also have

$$\Delta(l^+(c_{v,w'})) = l^+(c_{v,w'}) \otimes K_\mu + \sum_j X_{1j}K_{\xi_j+\mu} \otimes X_{2j}K_\mu + K_\lambda \otimes l^+(c_{v,w'}) \tag{3.5}$$

with $X_{1j} \in U_q^{\xi_j}(\mathfrak{n}_+)$, $X_{2j} \in U_q^{\lambda-\mu-\xi_j}(\mathfrak{n}_+)$, $0 < \xi_j < \lambda - \mu$. If one compares the $U_q(\mathfrak{h})$-parts of the terms in (3.3) and (3.5) in the second tensor component, one gets by the second equality in (3.4) and the PBW theorem

$$\sum_k l^+(c_{v,w_{i_k}}) \otimes l^+(c_{v_{i_k},w'}) = \sum_l X_{1j_l}K_{\xi_{j_l}+\mu} \otimes X_{2j_l}K_\mu$$

where the indices i_k and j_l are those with $\nu_{i_k} = \xi_{j_l} + \mu = \nu$.
We claim that the elements $l^+(c_{v_i,w'})$ are linearly independent. Indeed, assume that there are $x_i \in \mathbb{C}$ with $\sum_i x_i l^+(c_{v_i,w'}) = l^+(c_{\sum_i x_i v_i,w'}) = 0$. Since w' is a highest weight vector, Proposition 12 implies $\sum_n x_i v_i = 0$. Hence $x_i = 0$ for all i, because $\{v_i\}$ is a basis. It follows that all $l^+(c_{v,w_{i_k}})$ and in particular $l^+(c_{v,w})$ are linear combinations of $X_{1j_l}K_\mu$. □

3.2.2 L-functionals and quantum root vectors

3.2.2.1 On the coproduct of $E_{\beta_k}, F_{-\beta_k}$. The Lusztig automorphisms $\mathcal{T}_i$ are algebra, but not coalgebra homomorphisms. So it is not possible to derive the coproduct of the quantum root vectors $E_{\beta_k}, F_{-\beta_k}$ directly from their definition. However, it is mentioned in [KS1] on p. 278 that for $G = A_N := SL(N+1,\mathbb{C})$ there is a choice of $r_{i_1}r_{i_2}\cdots r_{i_n}$, such that the quantum root vectors appear as $X^\pm(c^\lambda_{-\mu,\nu})$ in Proposition 13. This allows to compute their coproduct explicitly. Here we generalize this result to the other classical matrix Lie groups $B_N := SO(2N+1,\mathbb{C}), C_N := Sp(2N,\mathbb{C}), D_N := SO(2N,\mathbb{C})$. So we assume for the rest of this section that G is one of these.

3.2.2.2 Some notation. We denote by $l^\pm_{ij}$ the l-functionals $l^\pm(c^{\omega_1}_{ij})$ associated to the matrix coefficients of the defining vector representation of G used in [KS1] as generators of $\mathbb{C}_q[G]$. We also adopt from there the abbreviations $j' := 2N+2-j$ for $G = B_N$ and $j' := 2N+1-j$ for $G = C_N, D_N$. For $X, Y \in U_q(\mathfrak{g})$ we write $X \equiv Y$ iff X and Y do not vanish and are linearly dependent.

3.2.2.3 A recurrence relation. With these notations, we have for all k with $i < k < j$ and $k \neq i', j'$ ([KS1], Proposition 8.29):

$$l_{ij}^{+} \equiv [l_{ik}^{+}, l_{kj}^{+}] l_{kk}^{-}, \quad l_{ji}^{-} \equiv l_{kk}^{+} [l_{jk}^{-}, l_{ki}^{+}]. \tag{3.6}$$

3.2.2.4 On the conventions in [KS1]. In the sequel we will frequently use explicit calculations carried out in [KS1]. There a different convention for the coproduct of the standard generators of $U_q(\mathfrak{g})$ is used; the elements denoted there by K_i, E_i, F_i are K_i^{-1}, F_i, E_i in our notation.
It seems unreasonable to rewrite all needed formulas from [KS1] in our notation, since the results of this section are not used in the rest of the work. Hence let us agree for the remainder of this section to use the conventions from [KS1].

3.2.2.5 Choice of $r_{i_1} r_{i_2} \cdots r_{i_n}$. We will use a special ordering of the positive roots, in which most if not all terms on the right hand side of (2.1), (2.2) vanish. To define it, we first arrange the positive roots in the following way as parts of matrices:

$\beta_{ij} \in \Phi^+$	i	j	G
$\sum_{k=i}^{j-1} \alpha_k$ $\sum_{k=i}^{N} \alpha_k + \sum_{k=j'}^{N} \alpha_k$	$1, \dots, N$	$i+1 \leq j \leq N+1$ $N+1 < j \leq 2N+1-i$	B_N
$\sum_{k=i}^{j-1} \alpha_k$ $\sum_{k=i}^{N} \alpha_k + \sum_{k=j'}^{N-1} \alpha_k$	$1, \dots, N$	$i+1 \leq j \leq N+1$ $N+1 < j \leq 2N+1-i$	C_N
$\sum_{k=i}^{j-1} \alpha_k$ $\sum_{k=i}^{N-2} \alpha_k + \alpha_N$ $\sum_{k=i}^{N} \alpha_k$ $\sum_{k=i}^{N} \alpha_k + \sum_{k=j'}^{N-2} \alpha_k$	$1, \dots, N-1$	$i+1 \leq j \leq N$ $j = N+1$ $j = N+2$ $N+2 < j \leq 2N-i$	D_N

Note that $\beta_{ij} = \lambda_j - \lambda_i$ with λ_k given by

λ_k	k	G
$-\alpha_k - \cdots - \alpha_N$ 0	$k \leq N$ $k = N+1$	B_N
$-\alpha_k - \cdots - \alpha_{N-1} - 1/2\alpha_N$	$k \leq N$	C_N
$-\alpha_k - \cdots - \alpha_{N-2} - 1/2\alpha_{N-1} - 1/2\alpha_N$ $1/2\alpha_{N-1} - 1/2\alpha_N$	$k \leq N-1$ $k = N$	D_N

and $\lambda_k = -\lambda_{k'}$ otherwise.
Now we fix the expression $a_N a_{N-1} \cdots a_1$ for the longest word of W, where

a_k	$k-N$	G
$(\prod_{i=k}^{N} r_i)(\prod_{j=N-1}^{k} r_j)$		B_N, C_N
1 $(\prod_{i=k}^{N-2} r_i) r_N (\prod_{j=N-1}^{k} r_j)$ $(\prod_{i=k}^{N-1} r_i) r_N (\prod_{j=N}^{k} r_j)$	0 odd even, $\neq 0$	D_N

We denote the induced ordering of Φ^+ by $\prec$. We have:

$\beta_{ij} \prec \beta_{kl} \Leftrightarrow$	G
$k < i$ or $i = k, l < j$	B_N, D_N
$k < i$ or $i = k, j = N+1$ or $i = k, l < j, j \neq N+1, l \neq N+1$	C_N

3.2.2.6 L-functionals and quantum root vectors. Comparing (3.6) with the commutation relations (2.1),(2.2) of the quantum root vectors we prove now:

Theorem 1 *If ij appear as indices of a positive root β_{ij}, then*

$$l^+_{ij} \equiv l^+_{ii} E_{\beta_{ij}}, \quad l^-_{ji} \equiv l^-_{ii} F_{-\beta_{ij}},$$

except if $G = C_N$ and $j = i' > N+1$. In this case, there are $x_i, y_i \in \mathbb{C}$, such that

$$l^+_{ii'} \equiv l^+_{ii}(E_{\beta_{ii'}} + x_i E_{\beta_{ii'-1}} E_i), \quad l^-_{i'i} \equiv (l^-)^i_i (F_{-\beta_{ii'}} + y_i F_{-\beta_{ii'-1}} F_i).$$

Proof. We consider only l^+_{ij}; l^-_{ji} is treated analogously. The proof is by induction over $j-i$. By the formulas given for $l^\pm_{ij}$ in Section 8.5.2 of [KS1] we have

$$l^\pm_{kk} = K_{\pm\lambda_k}, \quad l^+_{j-1j} \equiv l^+_{j-1j-1} E_{f(j-1)}, \quad f(k) := \begin{cases} k & k \le N, \\ k'-1 & k > N. \end{cases} \tag{3.7}$$

Hence the claim holds for $j-i=1$. All occurring l^+_{ij} except l^+_{N-1N+1} for $G = C_N, D_N$ can be calculated from the recurrence relation (3.6). We apply it with $k = j-1$. This is admissible in all cases except $G = C_N, D_N$ and $j = N+1$. These must be treated separately afterwards.
Inserting (3.7) and the induction hypothesis into (3.6) we get

$$l^+_{ij} \equiv l^+_{ii}(E_{f(j-1)} E_{\beta_{ij-1}} - q^{-g(i,j-1)} E_{\beta_{ij-1}} E_{f(j-1)}) \tag{3.8}$$

with $g(i,j) := \langle \lambda_j, \beta_{ij} \rangle - \langle \lambda_i, \alpha_{f(j)} \rangle$. Inserting $\lambda_k, \beta_{ij}, \langle \alpha_i, \alpha_j \rangle$ one gets

$$g(i,j-1) = \begin{cases} 2 & G = C_N, j = i' \\ -\langle \alpha_{f(j-1)}, \beta_{ij-1} \rangle & \text{otherwise} \end{cases}. \tag{3.9}$$

The calculations are straightforward, but rather lengthy, so we moved them to an additional paragraph at the end of this section.
We have $\alpha_{f(j-1)} \prec \beta_{ij-1}$ for $i < j-1$ and $j \neq i'$ which holds in all cases except $G = C_N, j = i'$. Since $G = D_N, j = N+1$ was excluded we furthermore have $\alpha_{f(j-1)} + \beta_{ij-1} = \beta_{ij}$, and there is no other linear combination of roots between $\alpha_{f(j-1)}$ and β_{ij-1} equal to β_{ij}. Hence the exponent in (3.8) is in all considered cases except $G = C_N, j = i'$ the same as the one which appears on the left hand side of (2.1). Thus the claim reduces to Proposition 3.
For $G = C_N, j = i'$ we obtain

$$l^+_{ii'} \equiv l^+_{ii}(E_i E_{\beta_{ii'-1}} - q^{-2} E_{\beta_{ii'-1}} E_i) \equiv l^+_{ii}(E_{\beta_{ii'}} + x_i E_{\beta_{ii'-1}} E_i)$$

for some $x_i \in \mathbb{C}$, because $\langle \alpha_i, \beta_{ij-1} \rangle = 0$.
It remains to treat the excluded cases l^+_{iN+1} for $G = C_N, D_N$. By the explicit lists of l^+_{ij} in [KS1] we have for $G = C_N$, $i = N-1$

$$\begin{aligned} l^+_{N-1N+1} &\equiv l^+_{N-1N-1}(E_N E_{N-1} - q^{-2} E_{N-1} E_N) \\ &= l^+_{N-1N-1}(E_N E_{N-1} - q^{\langle \alpha_{N-1}, \alpha_N \rangle} E_{N-1} E_N) \\ &\equiv l^+_{N-1N-1} E_{\alpha_{N-1}+\alpha_N} \end{aligned}$$

by the same argument as above. For $G = D_N$ the lists directly contain

$$l^+_{N-1N+1} \equiv l^+_{N-1N-1} E_N,$$

so the claim holds in these cases. For $i < N-1$ we need a second induction on i starting with $i = N-1$. We again use the recurrence relation (3.6), but now with $k = i+1$ (which is possible for $i < N-1$). By induction we get

$$\begin{aligned} l^+_{iN+1} &\equiv [l^+_{ii+1}, l^+_{i+1N+1}] l^-_{i+1i+1} \\ &\equiv l^+_{ii}(E_i E_{\beta_{i+1N+1}} - q^{\langle \lambda_{i+1}, \alpha_i\rangle - \langle \lambda_i, \beta_{i+1N+1}\rangle} E_{\beta_{i+1N+1}} E_i). \end{aligned}$$

In all cases we have $\langle \lambda_{i+1}, \alpha_i \rangle = 1$ and the second term in the exponent vanishes, since in β_{i+1N+1} only α_j with $j > i$ occur. Since $\langle \alpha_i, \beta_{i+1N+1}\rangle = -1$ and $\alpha_i \succ \beta_{i+1N+1}$, the same argumentation as above yields

$$l^+_{iN+1} \equiv l^+_{ii}(E_{\beta_{i+1N+1}} E_i - q^{\langle \alpha_i, \beta_{i+1N+1}\rangle} E_i E_{\beta_{i+1N+1}}) \equiv l^+_{ii} E_{\beta_{iN+1}}.$$

$\square$

3.2.2.7 Modifying the PBW basis. Now we can substitute $E_{\beta_{ij}}$ and $F_{-\beta_{ij}}$ in the PBW monomials $K_\lambda F_{\mathbf{i}} E_{\mathbf{j}}$ by l^+_{ij} and l^-_{ji}, respectively. Theorem 1 implies that we obtain again a vector space basis of $U_q(\mathfrak{g})$. That this applies also for $G = C_N, j = i'$ can be shown for example by noticing that the additional terms are of lower degree with respect to the filtration studied in [DK1].

The coproduct of the new basis elements is directly available since $\Delta(l^\pm_{ij}) = \sum_k l^\pm_{ik} \otimes l^\pm_{kj}$. One also can compute explicitly their commutation relations.

3.2.2.8 Proof of (3.9). Inserting the explicit entries of the symmetrized Cartan matrix one computes for $k \le N$

$\langle \lambda_k, \alpha_l \rangle$	k, l	G
-1	$k = l = N,$	B_N
-2	$k = l \neq N,$	
$+2$	$k = l+1$	
0	otherwise	
-1	$k = l < N$	C_N
-2	$k = l = N$	
$+1$	$k = l+1$	
0	otherwise	
-1	$k = l$	D_N
$+1$	$k = l+1 \neq N$	
-1	$k = l+1 = N$	
-1	$k = l-1 = N-1$	
0	otherwise	

For $j-1 \le N$ we have $f(j-1) = j-1 > i$, but also for $j-1 > N$ we have $f(j-1) = (j-1)' - 1 = j' > i$ except for $j = i'$ in the case $G = C_N$ where $f(j-1) = i$. Hence (for $i \neq N$, the case $i = N$ does not appear)

$$\langle \lambda_i, \alpha_{f(j-1)} \rangle = \begin{cases} -1 & j = i', G = C_N \\ 0 & \text{otherwise} \end{cases}.$$

Now we consider $\langle \lambda_{j-1}, \beta_{ij-1} \rangle$ by inserting the explicit expressions for β_{ij-1} in all cases. Let first $j-1 \le N$. Then for $G = B_N$ we have

$$\langle \lambda_{j-1}, \beta_{ij-1} \rangle = \langle \lambda_{j-1}, \alpha_i + \dots + \alpha_{j-2} \rangle = 2 = -\langle \alpha_{j-1}, \beta_{ij-1} \rangle,$$

for $G = C_N, D_N$ we have (recall that $j = N+1$ was excluded)

$$\langle \lambda_{j-1}, \beta_{ij-1} \rangle = \langle \lambda_{j-1}, \alpha_i + \dots + \alpha_{j-2} \rangle = 1 = -\langle \alpha_{j-1}, \beta_{ij-1} \rangle.$$

If $j-1 = N+1$, then for $G = B_N$ we have

$$\langle \lambda_{N+1}, \beta_{iN+1} \rangle = \langle 0, \alpha_i + \dots + \alpha_N \rangle = 0 = -\langle \alpha_N, \beta_{iN+1} \rangle = -\langle \alpha_{f(N+1)}, \beta_{iN+1} \rangle,$$

for $G = C_N$ we have

$$\begin{aligned} & \langle \lambda_{N+1}, \beta_{iN+1} \rangle = -\langle \lambda_N, \alpha_i + \dots + \alpha_N \rangle = 1 \\ = \quad & -\langle \alpha_{N-1}, \beta_{iN+1} \rangle = -\langle \alpha_{f(N+1)}, \beta_{iN+1} \rangle \end{aligned}$$

and for $G = D_N$ we have

$$\begin{aligned} & \langle \lambda_{N+1}, \beta_{iN+1} \rangle = -\langle \lambda_N, \alpha_i + \dots + \alpha_{N-2} + \alpha_N \rangle = 1 \\ = \quad & -\langle \alpha_{N-1}, \beta_{iN+1} \rangle = -\langle \alpha_{f(N+1)}, \beta_{iN+1} \rangle. \end{aligned}$$

If $j-1 > N+1$, then for $G = B_N$ we have

$$\begin{aligned} & \langle \lambda_{j-1}, \beta_{ij-1} \rangle = -\langle \lambda_{2N+3-j}, \alpha_i + \dots + \alpha_{2N+2-j} + 2\alpha_{2N+3-j} + \dots + 2\alpha_N \rangle \\ = \quad & 2 = -\langle \alpha_{2N+2-j}, \beta_{ij-1} \rangle = -\langle \alpha_{f(j-1)}, \beta_{ij-1} \rangle, \end{aligned}$$

for $G = C_N$ we have

$$\begin{aligned} & \langle \lambda_{j-1}, \beta_{ij-1} \rangle \\ = \quad & -\langle \lambda_{2N+2-j}, \alpha_i + \dots + \alpha_{2N+1-j} + 2\alpha_{2N+2-j} + \dots + 2\alpha_{N-1} + \alpha_N \rangle \\ = \quad & 1 \\ = \quad & \begin{cases} 1 & j = i' \\ -\langle \alpha_{2N+1-j}, \beta_{ij-1} \rangle & \text{otherwise} \end{cases} \\ = \quad & \begin{cases} \langle \alpha_{f(j-1)}, \beta_{ij-1} \rangle + 1 & j = i' \\ -\langle \alpha_{f(j-1)}, \beta_{ij-1} \rangle & \text{otherwise} \end{cases} \end{aligned}$$

and for $G = D_N$ we have for $j-1 = N+2$

$$\begin{aligned} & \langle \lambda_{N+2}, \beta_{iN+2} \rangle = -\langle \lambda_{N-1}, \alpha_i + \dots + \alpha_N \rangle = 1 \\ = \quad & -\langle \alpha_{N-2} \beta_{iN+2} \rangle = -\langle \alpha_{f(N+2)}, \beta_{iN+2} \rangle \end{aligned}$$

and otherwise

$$\begin{aligned} & \langle \lambda_{j-1}, \beta_{ij-1} \rangle \\ = \quad & -\langle \lambda_{2N+2-j}, \sum_{k=i}^{2N+1-j} \alpha_k + 2 \sum_{k=2N+2-j}^{N-2} \alpha_k + \alpha_{N-1} + \alpha_N \rangle \\ = \quad & 1 \\ = \quad & -\langle \alpha_{2N+1-j}, \beta_{ij-1} \rangle \\ = \quad & -\langle \alpha_{f(j-1)}, \beta_{ij-1} \rangle \end{aligned}$$

So (3.9) is proven.

3.3 L-functionals on $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$

3.3.1 L-functionals on the quantum double $A \bowtie A$

3.3.1.1 The Hopf algebra $A \bowtie A$. Let A be a coquasitriangular Hopf algebra with universal r-form $\mathbf{r}$. Then $\mathbf{r}$ can be used to define the quantum double $A \bowtie A$. This is a Hopf algebra which is the tensor product coalgebra $A \otimes A$ endowed with the product

$$(f \otimes g)(f' \otimes g') := (f f'_{(2)} \otimes g_{(2)} g')\, \bar{\mathbf{r}}(f'_{(1)}, g_{(1)}) \mathbf{r}(f'_{(3)}, g_{(3)}).$$

The antipode of $A \bowtie A$ is given by $S(f \otimes g) := (1 \otimes S(g))(S(f) \otimes 1)$. See [Ho, Ma1, KS1] for more information about quantum doubles.

3.3.1.2 $A \bowtie A$ as realification of A. If A is a Hopf $*$-algebra and $\mathbf{r}$ is real, then $A \bowtie A$ is a Hopf $*$-algebra with involution defined by $(f \otimes g)^* := g^* \otimes f^*$ ([Ma1], Section 7.3, [KS1], Section 10.2.7). This applies in particular to the case of the coordinate algebras $\mathbb{C}_q[G_0]$. Then any element of $A \bowtie A$ can be written uniquely as $f^* g$ with $f, g \in A$ (considered as a Hopf subalgebra of $A \bowtie A$ via the embedding $f \mapsto 1 \otimes f$ of A). One says that $A \bowtie A$ is a realification of A (in [Ma1] it is called a complexification).

3.3.1.3 Coquasitriangularity of $A \bowtie A$. By [KS1], Corollary 10.23, the Hopf algebra $A \bowtie A$ is again coquasitriangular. We define its FRT-dual $U(A \bowtie A)$ with respect to the universal r-form $\hat{\mathbf{r}} := \bar{\mathbf{r}}_{41} \bar{\mathbf{r}}_{31} \mathbf{r}_{24} \mathbf{r}_{23}$, that is,

$$\begin{aligned}
& \hat{\mathbf{r}}(f \otimes g, f' \otimes g') \\
= \; & \bar{\mathbf{r}}(g'_{(1)}, f_{(1)}) \bar{\mathbf{r}}(f'_{(1)}, f_{(2)}) \mathbf{r}(g_{(1)}, g'_{(2)}) \mathbf{r}(g_{(2)}, f'_{(2)}) \\
= \; & \bar{\mathbf{r}}(f'_{(1)} g'_{(1)}, f) \mathbf{r}(g, f'_{(2)} g'_{(2)}).
\end{aligned} \tag{3.10}$$

If A is a Hopf $*$-algebra and $\mathbf{r}$ is of real type, then $\hat{\mathbf{r}}$ is inverse real [KS1], Proposition 10.29.

3.3.1.4 L-functionals on $A \bowtie A$. Consider the multiplication map m as a map from $A \bowtie A$ to A and define a second linear map

$$\theta : A \bowtie A \to U(A), \quad f \otimes g \mapsto S(l^-(g) l^+(f)).$$

Denote by $\theta^\circ : A \to (A \bowtie A)^\circ$ and $m^\circ : U(A) \to (A \bowtie A)^\circ$ the dual maps,

$$\langle \theta^\circ(f), g \otimes h \rangle := \langle \theta(g \otimes h), f \rangle, \quad \langle m^\circ(X), f \otimes g \rangle := \langle X, fg \rangle.$$

The properties of $\mathbf{r}$ and the fact that Δ is an algebra homomorphism imply

$$\begin{aligned}
\hat{\mathbf{r}}(f \otimes g, f' \otimes g') &= \bar{\mathbf{r}}(f'_{(1)} g'_{(1)}, f) \mathbf{r}(g, f'_{(2)} g'_{(2)}) \\
&= \left\langle l^+ \left(S^{-1}(f)\right), (f'g')_{(1)} \right\rangle \left\langle l^- \left(S^{-1}(g)\right), (f'g')_{(2)} \right\rangle \\
&= \langle \theta(f \otimes g), m(f' \otimes g') \rangle.
\end{aligned}$$

For the convolution inverse $\bar{\hat{\mathbf{r}}} = \bar{\mathbf{r}}_{23} \bar{\mathbf{r}}_{24} \mathbf{r}_{31} \mathbf{r}_{41}$ of $\hat{\mathbf{r}}$ one obtains similarly

$$\bar{\hat{\mathbf{r}}}(f \otimes g, f' \otimes g') = \left\langle S^{-1}\left(\theta(f \otimes g)\right), m(f' \otimes g') \right\rangle.$$

Hence the l-functionals on $U(A \bowtie A)$ are given by

$$\hat{l}^+ = \theta^\circ \circ m, \quad \hat{l}^- = m^\circ \circ S^{-1} \circ \theta.$$

In particular, the images of $\hat{l}^+$ and $\hat{l}^-$ are contained in those of θ° and m°, respectively. The map m is obviously surjective. But $S^{-1}\circ\theta : a\otimes b \mapsto l^-(b)l^+(a)$ is also surjective by (3.1). Hence one even has

$$\operatorname{im}\hat{l}^+ = \operatorname{im}\theta^\circ, \quad \operatorname{im}\hat{l}^- = \operatorname{im} m^\circ.$$

3.3.1.5 $U(A \bowtie A)$ **as a subalgebra of** $U(A)\otimes U(A)$**.** It is shown in [Ho] that

$$\iota : U(A \bowtie A) \to A^\circ \otimes A^\circ, \quad X \mapsto \langle X_{(1)}, (\cdot \otimes 1)\rangle \otimes \langle X_{(2)}, (1 \otimes \cdot)\rangle \tag{3.11}$$

is an embedding of algebras (but not of coalgebras) and that

$$\iota \circ m^\circ = \Delta, \quad \iota \circ \theta^\circ = (l^- \otimes l^+) \circ \Delta.$$

In particular, $\operatorname{im}\iota \subset U(A) \otimes U(A)$.

3.3.1.6 Morphism properties of m°, θ°**.** It follows from $m_{21} = \mathbf{r} m \bar{\mathbf{r}}$ that $m : A \bowtie A \to A$ is a homomorphism of Hopf algebras. It is straightforward to check that the restriction of $1 \otimes \varepsilon$ defines a Hopf algebra homomorphism $U(A) \otimes U(A) \supset \iota(U(A \bowtie A)) \to U(A)$. Hence $\theta = S \circ (1 \otimes \varepsilon) \circ \iota \circ \hat{l}^-$ is a Hopf algebra homomorphism from $A \bowtie A$ to $U(A)^{\mathrm{cop}}$. By dualization we obtain

Proposition 14 *The maps $m^\circ : U(A) \to U(A \bowtie A)$ and $\theta^\circ : A^{\mathrm{op}} \to U(A \bowtie A)$ are Hopf algebra homomorphisms.*

In the sequel, we will use the product, coproduct and antipode of A to express those of A^{op}. So the product of $f, g \in A^{\mathrm{op}}$ will be written as gf and the coproduct and the antipode of A^{op} are Δ and S^{-1}, respectively.

3.3.2 Casimir operators in $U(A \bowtie A)$

3.3.2.1 Casimir operators. Let c_{ij} be the matrix coefficients of a finite-dimensional corepresentation of A. Then

$$C := \sum_{ijk} S(l^-(c_{ij}))l^+(c_{jk})\bar{\mathrm{f}}(c_{ki}).$$

belongs to the center of $U(A)$ [KS1], Proposition 10.16. We call it the Casimir operator associated to the corepresentation. The following remarks will be used below:

Proposition 15 *The Casimir operators associated to the corepresentations with matrix coefficients $S^2(c_{ij})$ and $S(c_{ji})$ coincide with C and $S(C)$, respectively. In particular, $S^2(C) = C$. If the corepresentation is irreducible, then there exists a constant λ such that $\mathrm{f}_{\bar{\mathbf{r}}_{21}}(c_{ij}) = \lambda \mathrm{f}(c_{ij})$ and $S(C) = \lambda C_{\bar{\mathbf{r}}_{21}}$, where $\mathrm{f}_{\bar{\mathbf{r}}_{21}}$ and $C_{\bar{\mathbf{r}}_{21}}$ are the analogues of f, C defined with respect to $\bar{\mathbf{r}}_{21}$. If A is a Hopf $*$-algebra, $\mathbf{r}$ is real and $c_{ij}^* = S(c_{ji})$, then $C^* = C$.*

Proof. It is clear that $\mathrm{f}(S^2(f)) = \mathrm{f}(f)$ for all $f \in A$. Inserting this into the definition of C and using two times Proposition 10 to transform S^2 into the identity, one gets

$$\sum_{ijk} S(l^-(S^2(c_{ij})))l^+(S^2(c_{jk}))\bar{\mathrm{f}}(S^2(c_{ki})) = C.$$

For the second statement we calculate with Proposition 10 and the Yang-Baxter equation

$$\begin{aligned}
S(C) &= \sum_{ijk} S(l^+(c_{jk}))S^2(l^-(c_{ij}))\bar{\mathrm{f}}(c_{ki}) \\
&= \sum_{ijk} l^+(S^{-1}(c_{jk}))l^-(S^{-2}(c_{ij}))\bar{\mathrm{f}}(c_{ki}) \\
&= \sum_{ijkrstu} \bar{\mathbf{r}}(S^{-2}(c_{ir}), S^{-1}(c_{uk}))l^-(S^{-2}(c_{rs}))l^+(S^{-1}(c_{tu})) \\
&\quad \mathbf{r}(S^{-2}(c_{sj}), S^{-1}(c_{jt}))\bar{\mathrm{f}}(c_{ki}) \\
&= \sum_{ikrstu} \mathbf{r}(c_{ir}, c_{uk})l^-(S^{-2}(c_{rs}))l^+(S^{-1}(c_{tu}))\mathrm{f}(c_{st})\bar{\mathrm{f}}(c_{ki}) \\
&= \sum_{ikrstu} \mathbf{r}(S^2(c_{ki}), c_{uk})\bar{\mathrm{f}}(c_{ir})l^-(S^{-2}(c_{rs}))l^+(S^{-1}(c_{tu}))\mathrm{f}(c_{st}) \\
&= \sum_{itu} \bar{\mathrm{f}}(S^{-1}(c_{ui}))l^-(c_{it}))l^+(S^{-1}(c_{tu})) \\
&= \sum_{itu} S(l^-(S(c_{it})))l^+(S(c_{ui}))\bar{\mathrm{f}}(S^{-1}(c_{tu})) \\
&= \sum_{itu} S(l^-(S(c_{it})))l^+(S(c_{ui}))\bar{\mathrm{f}}(S(c_{tu})).
\end{aligned}$$

If the matrix corepresentation $\{c_{ij}\}$ is irreducible, then Schur's lemma implies that $(\mathrm{f}_{\bar{\mathbf{r}}_{21}}\bar{\mathrm{f}})(c_{ij}) = \lambda\delta_{ij}$ for some constant λ, because $\mathrm{f}_{\bar{\mathbf{r}}_{21}}\bar{\mathrm{f}}$ is central in the convolution algebra of linear functionals on A [Sch1]. Hence $\mathrm{f}_{\bar{\mathbf{r}}_{21}}(c_{ij}) = \lambda\mathrm{f}(c_{ij})$. Then the second equality in the above computation of $S(C)$ gives

$$S(C) = \sum_{ijk} S(l^+(c_{jk}))l^-(c_{ki})\bar{\mathrm{f}}(c_{ki}) = \lambda C_{\bar{\mathbf{r}}_{21}}.$$

If A is a Hopf $*$-algebra, $\mathbf{r}$ is real and $c_{ij}^* = S(c_{ji})$, then $l^{\pm}(c_{ij})^* = S(l^{\mp}(c_{ji}))$ [KS1], Proposition 10.14. It is immediate that $\mathrm{f}(c_{ij}) \in \mathbb{R}$. Inserting this into the definition of C^* one gets $C^* = C$. □

3.3.2.2 Casimir operators in $U(A \bowtie A)$. As we will prove below, the embedding ι is in general not surjective. But for any Casimir operator $C \in U(A)$, the elements $C \otimes 1, 1 \otimes C$ are contained in $\operatorname{im}\iota$:

Proposition 16 *Let c_{ij} be the matrix coefficients of a corepresentation of A. Let C^1, C^2 be the Casimir operators of $U(A \bowtie A)$ associated to the corepresentations with matrix coefficients $c^1_{ij} := c_{ij} \otimes 1$ and $c^2_{ij} = 1 \otimes c_{ij}$, respectively. Then*

$$\iota(C^1) = S(C) \otimes 1, \quad \iota(C^2) = 1 \otimes C,$$

where $C \in U(A)$ is the Casimir operator associated to $\{c_{ij}\}$. If A is a cosemisimple Hopf $*$-algebra and $\mathbf{r}$ is real, then

$$\iota\left((C^1)^*\right) = 1 \otimes S(C), \quad \iota\left((C^2)^*\right) = C \otimes 1.$$

Proof. Let $\mathrm{f}_{\hat{\mathbf{r}}}$ be the analogue of f for $A \bowtie A$. We have

$$\overline{\mathrm{f}_{\hat{\mathbf{r}}}}(c^2_{ij}) = \sum_k \hat{\mathbf{r}}\left(S^2(1 \otimes c_{ik}), 1 \otimes c_{kj}\right) = \sum_k \mathbf{r}\left(S^2(c_{ik}), c_{kj}\right) = \bar{\mathrm{f}}(c_{ij}).$$

Furthermore, $\iota \circ S(\hat{l}^-(c^2_{ij})) = \Delta \circ S\left(l^-(c_{ij})\right)$ and $\iota \circ \hat{l}^+(c^2_{ij}) = (l^- \otimes l^+) \circ \Delta(c_{ij})$. Hence

$$\sum_k \iota \circ S(\hat{l}^-(c^2_{ik}))\iota \circ \hat{l}^+(c^2_{kj}) = \sum_k 1 \otimes S\left(l^-(c_{ik})\right) l^+(c_{kj}).$$

Therefore $C^2 = 1 \otimes C$. The calculation for C^1 is analogous.
If A is a Hopf $*$-algebra, then for $f \otimes g$ we have

$$\langle (1 \otimes C)^*, f \otimes g\rangle = \overline{\langle 1 \otimes C, S(f \otimes g)^*\rangle}.$$

Furthermore, we have

$$\begin{aligned}
S(f \otimes g)^* &= ((1 \otimes S(g))(S(f) \otimes 1))^* \\
&= (S(f) \otimes 1)^*(1 \otimes S(g))^* \\
&= (1 \otimes S(f)^*)(S(g)^* \otimes 1) \\
&= (S(g)^*_{(2)} \otimes S(f)^*_{(2)})\bar{\mathbf{r}}(S(f)^*_{(1)}, S(g)^*_{(1)})\mathbf{r}(S(f)^*_{(3)}, S(g)^*_{(3)}) \\
&= (S(g_{(2)})^* \otimes S(f_{(2)})^*)\bar{\mathbf{r}}(S(f_{(3)})^*, S(g_{(3)})^*)\mathbf{r}(S(f_{(1)})^*, S(g_{(1)})^*) \\
&= (S(g_{(2)})^* \otimes S(f_{(2)})^*)\mathbf{r}(f^*_{(3)}, S(g_{(3)})^*)\mathbf{r}(S(f_{(1)})^*, S(g_{(1)})^*).
\end{aligned}$$

Inserting this into the first equation we get for real $\mathbf{r}$

$$\begin{aligned}
\langle (1 \otimes C)^*, f \otimes g\rangle &= \overline{\langle 1 \otimes C, S(g_{(2)})^* \otimes S(f_{(2)})^*\rangle} \\
&\quad \overline{\mathbf{r}(f^*_{(3)}, S(g_{(3)})^*)\mathbf{r}(S(f_{(1)})^*, S(g_{(1)})^*)} \\
&= \varepsilon(g_{(2)})\langle C, f_{(2)}\rangle \mathbf{r}(S(g_{(3)}), f_{(3)})\mathbf{r}(S(g_{(1)}), S(f_{(1)})).
\end{aligned}$$

By cosemisimplicity of A we may assume that f and hence all $f_{(2)}$ are matrix coefficients of an irreducible corepresentation. Then Schur's lemma implies $\langle C, f_{(2)}\rangle = \lambda\varepsilon(f_{(2)})$ for some constant λ which is independent of $f_{(2)}$. Inserting this we get

$$\begin{aligned}
\ldots &= \lambda\varepsilon(g_{(2)})\varepsilon(f_{(2)})\mathbf{r}(S(g_{(3)}), f_{(3)})\mathbf{r}(S(g_{(1)}), S(f_{(1)})) \\
&= \lambda\mathbf{r}(S(g_{(2)}), f_{(2)})\mathbf{r}(S(g_{(1)}), S(f_{(1)})) \\
&= \lambda\mathbf{r}(S(g), S(f_{(1)})f_{(2)}) \\
&= \lambda\varepsilon(g)\varepsilon(f) \\
&= \langle C \otimes 1, f \otimes g\rangle.
\end{aligned}$$

The calculation of $\iota\left((C^1)^*\right)$ is analogous. □

Note that coquasitriangular Hopf algebras need not to be cosemisimple; a counterexample is given by Sweedler's Hopf algebra.

3.3.3 The quantum Iwasawa decomposition

3.3.3.1 Quantum Iwasawa decomposition. Define a linear map

$$\zeta : A^{\mathrm{op}} \bowtie U(A) \to U(A \bowtie A), \quad \zeta(f \otimes X) := \theta^\circ(f)m^\circ(X)$$

with the quantum double $A^{\mathrm{op}} \bowtie U(A)$ constructed with respect to the canonical pairing of $U(A)$ and A. Then we have [Ho, Ma1, RS]:

Proposition 17 *The map ζ is an epimorphism of Hopf algebras.*

Proof. The surjectivity follows from the above discussion of $\hat{l}^{\pm}$ and (3.1). It is immediate that ζ is a coalgebra homomorphism. That it is an algebra homomorphism follows from the relation

$$\theta^{\circ}(f_{(1)})m^{\circ}(X_{(1)})\langle X_{(2)}, f_{(2)}\rangle = m^{\circ}(X_{(2)})\theta^{\circ}(f_{(2)})\langle X_{(1)}, f_{(1)}\rangle, \quad f \in A, X \in U(A).$$

To prove this relation one can assume by (3.1) and linearity that $X = l^-(g)l^+(h)$ with $g, h \in A$. By applying it then to $i \otimes j \in A \bowtie A$ it turns out to be equivalent to

$$\bar{\mathbf{r}}_{12}\mathbf{r}_{31}\bar{\mathbf{r}}_{42}\mathbf{r}_{25}\bar{\mathbf{r}}_{43}\mathbf{r}_{35}\bar{\mathbf{r}}_{41}\mathbf{r}_{15} = \bar{\mathbf{r}}_{41}\mathbf{r}_{15}\bar{\mathbf{r}}_{42}\mathbf{r}_{25}\bar{\mathbf{r}}_{43}\mathbf{r}_{35}\bar{\mathbf{r}}_{12}\mathbf{r}_{31}.$$

This equation in turn is proven by use of $\bar{\mathbf{r}}_{12}\mathbf{r}_{23}\bar{\mathbf{r}}_{14}\mathbf{r}_{43} = \bar{\mathbf{r}}_{24}\bar{\mathbf{r}}_{14}\mathbf{r}_{43}\bar{\mathbf{r}}_{12}\mathbf{r}_{23}\mathbf{r}_{24}$ which follows from the Yang-Baxter equation. □

3.3.3.2 The case of $*$-algebras. If A is a Hopf $*$-algebra and $\mathbf{r}$ is real, then there is an involution on $A^{\mathrm{op}} \bowtie U(A)$ defined by

$$(f \otimes X)^* := (1 \otimes X^*)\left(S^2(f)^* \otimes 1\right),$$

for which $A^{\mathrm{op}} \bowtie U(A)$ becomes a Hopf $*$-algebra and ζ a $*$-homomorphism, see [Ma1], Proposition 7.1.4 and Theorem 7.3.5.

3.3.3.3 Main result. Now we prove the main result of this chapter.

Theorem 2 *For $A = \mathbb{C}_q[G]$ the map ζ is an isomorphism.*

Proof. It remains to check the injectivity. We prove that

$$\zeta' := \iota\circ\zeta : \mathbb{C}_q[G]^{\mathrm{op}} \bowtie U_q(\mathfrak{g}) \to U_q(\mathfrak{g})\otimes U_q(\mathfrak{g}), f\otimes X \mapsto l^-(f_{(1)})X_{(1)}\otimes l^+(f_{(2)})X_{(2)}$$

with ι from (3.11) is injective.

Suppose $X \in \ker\zeta'$, $X = \sum_{\lambda\in\mathbf{L},\mathbf{i},\mathbf{j}\in\mathbb{N}_0^n} f_{\lambda\mathbf{ij}} \otimes K_\lambda F_{\mathbf{i}} E_{\mathbf{j}}$ with $f_{\lambda\mathbf{ij}} = 0$ for almost all $\lambda\mathbf{ij}$. We have to show that X vanishes.

Order $\mathbb{N}_0^n$ in such a way that the weights $\mu_{\mathbf{j}}$ of $E_{\mathbf{j}}$ form a non-decreasing (with respect to $<$) sequence. Let $\mathbf{j}_0$ be a maximal $\mathbf{j}$ for which there exists an $f_{\lambda\mathbf{ij}} \neq 0$. Recall that $\iota \circ m^{\circ} = \Delta$ and $\iota \circ \theta^{\circ} = (l^- \otimes l^+) \circ \Delta$. Note that by Proposition 13 and Proposition 1 we have

$$(l^- \otimes l^+) \circ \Delta(f_{\lambda\mathbf{ij}}) \in U_q(\mathfrak{b}_-) \otimes U_q(\mathfrak{b}_+), \quad \Delta(K_\lambda F_{\mathbf{i}}) \in U_q(\mathfrak{b}_-) \otimes U_q(\mathfrak{b}_-).$$

Hence only $\Delta(E_{\mathbf{j}})$ contribute to the $U_q(\mathfrak{n}_+)$-part in the first tensor component. Expand them according to Proposition 1. Then the PBW theorem implies that

$$\sum_{\lambda\mathbf{i}}(l^- \otimes l^+) \circ \Delta(f_{\lambda\mathbf{ij}_0}) \cdot \Delta(K_\lambda F_{\mathbf{i}}) \cdot (E_{\mathbf{j}_0} \otimes 1)$$

is linearly independent from the other terms occurring in $\zeta'(X)$ and vanishes separately. Since $U_q(\mathfrak{g})\otimes U_q(\mathfrak{g}) = U_q(\mathfrak{g}\oplus\mathfrak{g})$ is an integral domain (Proposition 4), we get

$$\sum_{\lambda\mathbf{i}}(l^- \otimes l^+) \circ \Delta(f_{\lambda\mathbf{ij}_0}) \cdot \Delta(K_\lambda F_{\mathbf{i}}) = 0.$$

The same argument applied to a maximal $\mathbf{i}_0$ and the second tensor component shows

$$\sum_{\lambda}(l^- \otimes l^+) \circ \Delta(f_{\lambda \mathbf{i}_0 \mathbf{j}_0}) \cdot \Delta(K_\lambda) = 0.$$

By Proposition 13 we can write $(l^- \otimes l^+) \circ \Delta(f_{\lambda \mathbf{i}_0 \mathbf{j}_0})$ as $\sum_{\xi \in \mathbf{L}} X_{\lambda\xi} K_\xi \otimes Y_{\lambda\xi} K_{-\xi}$ for some $X_{\lambda\xi} \otimes Y_{\lambda\xi} \in U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{n}_+)$. Then the last equation becomes

$$\sum_{\xi\lambda} X_{\lambda\xi} K_{\xi+\lambda} \otimes Y_{\lambda\xi} K_{-\xi+\lambda} = 0.$$

This implies $X_{\lambda\xi} \otimes Y_{\lambda\xi} = 0$ for all λ, ξ. Finally,

$$(l^- \otimes l^+) \circ \Delta(f_{\lambda \mathbf{i_0 j_0}}) = 0$$

implies $f_{\lambda \mathbf{i_0 j_0}} = 0$ in contradiction with the assumption, because $(l^- \otimes l^+) \circ \Delta$ is injective by the definition of $U(\mathbb{C}_q[G])$. □

3.3.4 On the quantum codouble $U_q(\mathfrak{g}) \blacktriangleright\!\blacktriangleleft U_q(\mathfrak{g})$

3.3.4.1 Finite-dimensional case. If A is finite-dimensional, then any universal r-form $\mathbf{r}$ is simultaneously a universal R-matrix R for the dual Hopf algebra A° which therefore is quasitriangular. This R-matrix can be used to form a quantum codouble $A^\circ \blacktriangleright\!\blacktriangleleft A^\circ$ of two copies of A°, see [Ma1]. Its structure is completely dual to that of $A \bowtie A$ - it is the tensor product algebra $A^\circ \otimes A^\circ$ with a twisted coproduct

$$\Delta(X \otimes Y) := X_{(1)} \otimes R(Y_{(1)} \otimes X_{(2)})R^{-1} \otimes Y_{(2)}. \tag{3.12}$$

The map ι becomes a Hopf algebra homomorphism into $A^\circ \blacktriangleright\!\blacktriangleleft A^\circ$. If A is in addition factorizable, i.e., if $f \mapsto S(l^-(f_{(1)}))l^+(f_{(2)})$ is injective, then both ι and ζ are isomorphisms [Ma1], Theorem 7.3.5. As we will see now, there is no way to define the above coproduct in a rigorous way for arbitrary coquasitriangular Hopf algebras A.

3.3.4.2 $U_q(\mathfrak{g}) \blacktriangleright\!\blacktriangleleft U_q(\mathfrak{g})$ does not exist. The universal R-matrix dual to the universal r-form $\mathbf{r}$ of $\mathbb{C}_q[G]$ does not exist as an element of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, but only in some topological completion. Nevertheless, parts of the theory of quasitriangular Hopf algebras carry over to $U_q(\mathfrak{g})$, since the l-functionals contain essentially the same information. Hence it is not a priori clear that there is no way to define the twisted coproduct (3.12) as well on $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. But we show now that this is in fact impossible.

Theorem 3 *There exists no bialgebra structure on $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ such that ι becomes a homomorphism of bialgebras.*

Proof. Suppose that the opposite holds. Then $\iota \circ \theta^\circ$ is a bialgebra homomorphism as well. Note that $\pm(\nu - \mu) \notin \sum_{i=1}^N \mathbb{N}_0 \alpha_i$ implies $l^\pm(c^\lambda_{-\mu,\nu}) = 0$ by Proposition 12. Using this and Proposition 13 one computes

$$\begin{aligned}
\Delta(K_\lambda \otimes K_{-\lambda}) &= \Delta \circ \iota \circ \theta^\circ(c^\lambda_{\lambda,-\lambda}) \\
&= (\iota \circ \theta^\circ \otimes \iota \circ \theta^\circ) \circ \Delta(c^\lambda_{\lambda,-\lambda}) \\
&= \sum_n K_\lambda \otimes X^+(c^\lambda_{\lambda,\nu_n})K_{-\lambda} \otimes X^-(c^\lambda_{-\nu_n,-\lambda})K_\lambda \otimes K_{-\lambda}.
\end{aligned}$$

This must be an invertible element of $U_q(\mathfrak{g})^{\otimes 4}$, because Δ is an algebra homomorphism and $K_\lambda \otimes K_{-\lambda}$ is invertible. Since $K_\lambda \otimes K_{-\lambda} \otimes K_\lambda \otimes K_{-\lambda}$ is invertible, $\sum_n X^+(c^\lambda_{\lambda,\nu_n}) \otimes X^-(c^\lambda_{-\nu_n,-\lambda})$ is an invertible element of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$.
An invertible element of a graded algebra must be homogeneous - the product of the homogeneous components of highest degrees n_0, m_0 of the element and its inverse must be of degree zero, so $m_0 = -n_0$, the same must hold for the components of lowest degrees n_1, m_1, so $m_1 = -n_1$ and $n_1 \leq n_0$ and $m_1 \leq m_0$ implies then $m_0 = m_1 = -n_0 = -n_1$. By Proposition 13 $\sum_n X^+(c^\lambda_{\lambda,\nu_n}) \otimes X^-(c^\lambda_{-\nu_n,-\lambda})$ is not homogeneous with respect to the $\mathbf{Q} \times \mathbf{Q}$-grading of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, so we obtain a contradiction. $\square$

Although $(\mathbb{C}_q[G], \mathbf{r})$ is factorizable [BS], we therefore have:

Corollary 1 *The map ι is not surjective.*

3.4 Bibliographical notes

Coquasitriangular Hopf algebras were introduced in [LT]. We refer to Chapter 10 of [KS1] for more information. The quantum Iwasawa decomposition was introduced in [PW1], but in the C^*-algebraic framework with the duality given by the Haar functional instead of the r-form. The purely algebraic formulation was intensively studied by S. Majid [Ma1], but the basic structures appeared already earlier [RS]. For the analogous structures for Poisson-Lie groups see [HY]. For proofs and further details on $A \bowtie A$ and the constructions used in this chapter see [Ho] and Chapter 10 of [KS1].

Chapter 4

Dirac operators on quantum flag manifolds

The first section contains preliminaries on quantum flag manifolds and quantum homogeneous vector bundles. The second summarizes the theory of covariant differential calculi on quantum homogeneous spaces developed in [HK3]. At its end the notion of a Hilbert space representation (commutator representation) of a differential calculus is explained. The main section is the third, where such a representation is constructed for the irreducible finite-dimensional covariant differential calculi on quantized irreducible flag manifolds found in [HK1].
In this chapter we assume $q \in (1, \infty)$ and that G is simple and simply connected.

4.1 Quantum flag manifolds

4.1.1 Flag manifolds

4.1.1.1 Definition. Let $P \subset G$ be a parabolic subgroup, that is, a closed subgroup whose Lie algebra $\mathfrak{p}$ contains a maximal solvable Lie subalgebra of $\mathfrak{g}$. Then the homogeneous space $M := G/P$ is called a (generalized) flag manifold. Given a maximal solvable subalgebra of $\mathfrak{g}$, there exists an inner automorphism of $\mathfrak{g}$ mapping this subalgebra to $\mathfrak{b}_+$. Hence we assume that $\mathfrak{b}_+ \subset \mathfrak{p}$ from now on. We denote by $\mathfrak{l}$ the Levi factor of $\mathfrak{p}$ (its maximal reductive subalgebra). Then $\mathfrak{l} =: \mathfrak{h} \oplus \bigoplus_{\Phi_\mathfrak{l} \subset \Phi} \mathfrak{g}_\beta$, and $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_+$, where $\mathfrak{u}_+$ is the nilpotent ideal $\bigoplus_{\Phi^+ \setminus \Phi_\mathfrak{l}^+} \mathfrak{g}_\beta$, $\Phi_\mathfrak{l}^\pm := \Phi^\pm \cap \Phi_\mathfrak{l}$ [Kn], (5.93). Note that $\mathfrak{l}$ determines $\mathfrak{p}$.
As a complex variety, M is projective. Thus it can not be described globally by a single ring. But as explained in Chapter 1, it is as a real variety isomorphic to the real form G_0/L_0 of the complex affine variety G/L. Here L_0 is the compact real form of the subgroup L of G corresponding to $\mathfrak{l}$. This allows to describe M as a real variety by the $*$-algebra $\mathbb{C}[M] \subset \mathbb{C}[G_0]$ of L_0-invariant elements.

4.1.1.2 The tangent space. The Lie algebra $\mathfrak{p}$ acts on the complex tangent space $T_{eP}M = \mathfrak{g}/\mathfrak{p}$ with action induced by the adjoint action of $\mathfrak{g}$. If $\mathfrak{u}$ denotes the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{g}$ with respect to the Killing form, then $\mathfrak{g}/\mathfrak{p}$ is as a representation only of $\mathfrak{l}$ isomorphic to $\mathfrak{u}_- := \mathfrak{u} \cap \mathfrak{n}_-$ (with action being the adjoint action). The Killing form defines a non-degenerate pairing between $\mathfrak{u}_+$ and $\mathfrak{u}_-$, so $\mathfrak{u}_+$ gets identified with the dual representation $\mathfrak{u}_-^*$ of $\mathfrak{l}$. The action of $\mathfrak{l}$ on $\mathfrak{u}$ defines an embedding of $\mathfrak{l}$ into $\mathfrak{so}(2m, \mathbb{C})$, where $m = \dim_\mathbb{C} M = \dim_\mathbb{C} \mathfrak{u}_\pm$.

4.1.1.3 Irreducible flag manifolds. We will have to assume that all considered flag manifolds are irreducible. By this we mean that $\mathfrak{g}/\mathfrak{p}$ is an irreducible representation of $\mathfrak{p}$. Engel's theorem implies that $\mathfrak{u}_+$ acts then trivially on $\mathfrak{g}/\mathfrak{p}$. In this case there exists $r \in \{1,\dots,N\}$ such that $\Phi_\mathfrak{l} = \Phi \cap \bigoplus_{i\neq r} \mathbb{Z}\alpha_i$, that is, $\mathfrak{l}$ is the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and all E_k, F_k except E_r, F_r [BE], Example 3.1.10. The representations $\mathfrak{u}_\pm$ of $\mathfrak{l}$ are irreducible with highest (lowest) weight $\mp\alpha_r$. The following is a complete list of all possible values for r:

$\mathfrak{g}$	A_N	B_N	C_N	D_N	E_6	E_7
r	$1,\dots,N$	1	N	$1, N-1, N$	1	N

The cases A_N and $B_N, D_N, r = 1$ give the complex Grassmannians $\mathrm{Gr}(r, N+1)$ and the complexified spheres $\mathbb{C}S^m$. Note also that the irreducible flag manifolds coincide with the irreducible compact Hermitian symmetric spaces [He1].

4.1.2 Quantum flag manifolds

4.1.2.1 Quantum homogeneous spaces. Let A be a Hopf algebra and $B \subset A$ be a left coideal subalgebra of A. Then the coproduct of A defines a left coaction of A on B such that B becomes an A-comodule algebra. We call B an algebra of functions on a quantum homogeneous space, if A is faithfully flat as a B-module [MS]. This condition is natural from several points of view. In particular, it enters in a crucial way the theory of covariant differential calculi on B as was realized by U. Hermisson [He2], see the sections below. In the commutative case it is necessarily fulfilled if the prime spectrum of B can be identified with the quotient of the affine group scheme associated to A by the closed affine subgroup scheme corresponding to the quotient Hopf algebra A/AB^+, cf. [DG], Proposition III.3.2.5. Here as in what follows we define $B^+ := B \cap \ker \varepsilon$ for a vector subspace B of a coalgebra.

4.1.2.2 Quantum homogeneous spaces as infinitesimal invariants. The quantum homogeneous spaces discussed in this chapter are all of the following type: Let U be a Hopf algebra with bijective antipode, $K \subset U$ be a right coideal subalgebra, $\mathcal{C}$ be a category of finite-dimensional U-modules which is closed under formation of tensor products and duals. Let A be the restricted Hopf dual $U^\circ_\mathcal{C}$ of U with respect to $\mathcal{C}$. We assume that the pairing $\langle\cdot,\cdot\rangle$ between U and A is non-degenerate and that the antipode of A is bijective as well. Set

$$B := \{f \in A \,|\, X \triangleright f = \varepsilon(X)f \;\forall X \in K\}, \quad X \triangleright f := \langle X, f_{(2)}\rangle f_{(1)}.$$

Then B is a left coideal subalgebra of A and A is faithfully flat over B if the modules of $\mathcal{C}$ are semi-simple over K [MS], Theorem 2.2. The latter condition is always fulfilled if U is a pointed Hopf $*$-algebra (i.e. one whose simple subcoalgebras are all one-dimensional), if any $V \in \mathcal{C}$ admits a Hermitian inner product $(\cdot,\cdot)$ such that $(Xv, w) = (v, X^*w)$ for all $X \in U, v, w \in V$, and if there exists a coideal $I \subset U$ with $S(I)^* \subset I$ such that

$$K = \{X \in U \,|\, \pi(X_{(1)}) \otimes X_{(2)} = \pi(1) \otimes X\},$$

where $\pi : U \to \bar{U} := U/UI$ is the canonical projection.
Since this is more convenient in our applications, we will consider B usually not as a left A-comodule algebra, but as a right U-module algebra with respect to the action $f \triangleleft X = \langle X, f_{(1)}\rangle f_{(2)}$, $X \in U, f \in B$.

4.1.2.3 Quantum flag manifolds. Let $M = G/P$ be an irreducible flag manifold and r be as in 4.1.1.3. Choose

$$U := U_q(\mathfrak{g}_0), \quad \mathcal{C} := \mathcal{C}_G, \quad I := \operatorname{span}\{K_\lambda - 1, E_i, F_i \,|\, \lambda \in \mathbf{P}, i \neq r\}.$$

Then U is pointed ([Mo], Lemma 5.5.5) and A is equal to $\mathbb{C}_q[G_0]$. It follows from [MS], Theorem 1.2 and the fact that the coalgebra $\bar{U} = U/UI$ is equal to U/UK^+, that K is the Hopf $*$-subalgebra $U_q(\mathfrak{l}_0) \subset U_q(\mathfrak{g}_0)$ generated by 1 and I. The algebra $B =: \mathbb{C}_q[M]$ is clearly a quantum analogue of $\mathbb{C}[M]$. We call it the coordinate algebra of the quantum flag manifold M_q.

4.1.2.4 Quantum flag manifolds and quantum double groups. There is a generalization of the isomorphism $G/P = G_0/L_0$ in the quantum case. Let A, U, B, K be as in 4.1.2.2 with A coquasitriangular and $U \subset U(A)$. Define

$$B' := \{f \in A \bowtie A \,|\, X \triangleright f = \varepsilon(X) f \,\forall X \in A^{\mathrm{op}} \bowtie K\}.$$

This is an $A \bowtie A$-comodule algebra. The composition of the coaction with the map $m : A \bowtie A \to A, f \otimes g \mapsto fg$ turns B' into an A-comodule algebra.

Proposition 18 *If the pairing of $U(A \bowtie A)$ and $A \bowtie A$ is non-degenerate, then m defines an isomorphism of A-comodule algebras between B' and B.*

Proof. The map m is an epimorphism of A-comodule algebras. Suppose that $m(f) = 0$ for some $f \in B'$. Under our assumptions the map $\zeta^\circ := (m \otimes \theta) \circ \Delta$ dual to ζ from 3.3.3.1 is injective [Ho], Corollary 2.5. We can assume that the summands of $\zeta^\circ(f) = \sum m(f_{(1)}) \otimes \theta(f_{(2)})$ have linearly independent second tensor components. Hence for any of them there exists $g \in A$ vanishing on all of them but the considered one where it gives 1 (if the pairing of $U(A \bowtie A)$ with $A \bowtie A$ is non-degenerate, then that of $U(A)$ and A also is). Applying $\mathrm{id} \otimes g$ to $\zeta^\circ(f)$ we hence obtain the corresponding term of the first tensor component. On the other hand we have

$$(\mathrm{id} \otimes g)(\zeta^\circ(f)) = m(f_{(1)}) \otimes \langle \theta(f_{(2)}), g \rangle = m(\theta^\circ(g) \triangleright f) = \varepsilon(g) m(f) = 0,$$

because $f \in B'$. Hence $\zeta^\circ(f) = 0$ and the claim follows. □

Hence the homogeneous space of the quantum Lorentz group used to define induced representations in [PW2] is just Podleś' standard quantum sphere (compare Remark 4.5 there), and in general quantum flag manifolds could be used to construct unitary corepresentations of $\mathbb{C}_q[G_0] \bowtie \mathbb{C}_q[G_0]$ for arbitrary G.

4.1.3 Quantum homogeneous vector bundles

4.1.3.1 Homogeneous vector bundles. Let $M = G/P = G_0/L_0$ be a flag manifold. Then any representation ρ of L_0 on a vector space V associates to the L_0-principal fiber bundle $G_0 \to G_0/L_0$ a vector bundle over M. Its total space $G_0 \times_{L_0} V$ is the quotient of $G_0 \times V$ by the right L_0-action $(x, v) \triangleleft y := (xy, \rho(y^{-1})v)$, $x \in G_0, y \in L_0, v \in V$. The projection of the vector bundle is defined by the projection of $G_0 \times V$ onto the first factor. Its sections are represented by functions $\psi : G_0 \to V$ satisfying $\psi(xy) = \rho(y^{-1})\psi(x)$ for all $x \in G_0, y \in L_0$. The vector space of all of them forms by definition the representation $\mathrm{Ind}_{L_0}^{G_0}\rho$ of G_0 induced by ρ. One calls such vector bundles over M homogeneous vector bundles [BE, Wa1].

4.1.3.2 Quantum homogeneous vector bundles. Any representation of L_0 admits a deformation to a $*$-representation of $U_q(\mathfrak{l}_0)$. Thus one can introduce the following direct analogue of the vector space of sections of $G_0 \times_{L_0} V$ [GZ]:

$$\begin{aligned}\Gamma(M_q, V) &:= \{\psi \in \mathbb{C}_q[G_0] \otimes V \mid X \triangleright \psi = \rho(S(X))\psi \; \forall X \in U_q(\mathfrak{l}_0)\} \\ &= \bigoplus_{\lambda \in \mathbf{P}^+} V(\lambda) \otimes \mathrm{Hom}_{U_q(\mathfrak{l}_0)}(V(\lambda), V). \qquad (4.1)\end{aligned}$$

The last identification is induced by the Peter-Weyl decomposition of $\mathbb{C}_q[G_0]$. We recall several facts on quantum homogeneous vector bundles needed in the sequel. For proofs we refer to [GZ].

4.1.3.3 A basis for $\Gamma(M_q, V)$. For any λ, let $\{A_i^\lambda\}$ be a vector space basis of $\mathrm{Hom}_{U_q(\mathfrak{l}_0)}(V(\lambda), V)$. Then the elements $\psi_{ij}^\lambda := \sum_k S(c_{kj}^\lambda) \otimes A_i^\lambda(w_k^\lambda)$ form a basis of $\Gamma(M_q, V)$. Here $\{w_i^\lambda\}$ is the basis of $V(\lambda)$ with respect to which the matrix coefficients c_{ij}^λ are defined.

4.1.3.4 Unitarity. There is a Hermitian inner product $\langle \cdot, \cdot \rangle_V$ on $\Gamma(M_q, V)$ defined by applying $\langle \cdot, \cdot \rangle_h$ to $\mathbb{C}_q[G_0]$ and the invariant Hermitian inner product $(\cdot, \cdot)_V$ to V. We can choose the basis ψ_{ij}^λ to be orthonormal. We complete $\Gamma(M_q, V)$ to a Hilbert space $H(M_q, V)$ which we call the space of square-integrable sections of the quantum homogeneous vector bundle on M_q.

4.1.3.5 Module structure. The right action on $\mathbb{C}_q[G_0]$ turns $\Gamma(M_q, V)$ into a right $U_q(\mathfrak{g}_0)$-module. The multiplication in $\mathbb{C}_q[G_0]$ defines a $\mathbb{C}_q[M]$-bimodule structure on $\Gamma(M_q, V)$ and when restricting to a one-sided action one obtains a projective module (this is proven quite explicitly in [GZ], but follows also from more abstract arguments, see e.g. [Du], Appendix B or [HK2], Remark 2.4.(iii)).

4.2 Covariant differential calculi

4.2.1 Covariant differential calculi

4.2.1.1 Differential calculi. A (first order) differential calculus over an algebra B is a bimodule Γ together with a $\mathbb{C}$-linear map $\mathrm{d} : B \to \Gamma$ such that

$$\Gamma = \mathrm{span}_{\mathbb{C}}\{f\mathrm{d}g \mid f, g \in B\}, \quad \mathrm{d}(fg) = (\mathrm{d}f)g + f(\mathrm{d}g) \quad \forall\, f, g \in B.$$

A differential calculus over a $*$-algebra is called a $*$-calculus, if it admits a conjugate linear involution $*$ such that $(f\mathrm{d}g)^* = \mathrm{d}(g^*)f^*$. A morphism of differential calculi is a morphism of bimodules which intertwines the two differentials. Note that a morphism of differential calculi is automatically surjective.

4.2.1.2 The universal calculus. Let B be any algebra and denote by $\bar{f}$ the class of $f \in B$ in $\bar{B} := B/\mathbb{C}\cdot 1$. Then $\Gamma_{\mathrm{u}} := B \otimes \bar{B}$ becomes a differential calculus over B with differential $\mathrm{d}_{\mathrm{u}}(f) := 1 \otimes \bar{f}$, see [KS1]. The left module structure is given by multiplication in B. The right one is determined by the Leibniz rule. If (Γ, d) is any differential calculus over B, then $f\mathrm{d}_{\mathrm{u}}g \mapsto f\mathrm{d}g$ extends to a morphism $\Gamma_{\mathrm{u}} \to \Gamma$. Hence Γ is isomorphic to the quotient of Γ_{u} by the kernel of this morphism. In particular, this implies:

Proposition 19 *Two covariant differential calculi (Γ, d) and (Γ', d') over an algebra B are isomorphic iff $\sum_i f_i \mathrm{d}g_i = 0 \Leftrightarrow \sum_i f_i \mathrm{d}' g_i = 0$.*

4.2.1.3 Covariant differential calculi. Let A be a Hopf algebra and B be a left A-comodule algebra. Then a differential calculus (Γ, d) over B is called covariant, if Γ is a left A-comodule and the coaction is compatible with the B-action and d in the sense that

$$(f\mathrm{d}g)_{(1)} \otimes (f\mathrm{d}g)_{(2)} = f_{(1)}g_{(1)} \otimes f_{(2)}\mathrm{d}(g_{(2)}) \in A \otimes \Gamma$$

for all $f, g \in B$. If U is a Hopf subalgebra of A°, then Γ becomes a right U-module and $X(f\mathrm{d}g) = (f \triangleleft X_{(1)})\mathrm{d}(g \triangleleft X_{(2)})$ for all $X \in U$. In particular, d is U-linear.

4.2.2 Differential calculi on quantum flag manifolds

4.2.2.1 Differential calculi on quantum homogeneous spaces. Assume now that B is a quantum homogeneous space of the type introduced in 4.1.2.2. Define the dimension of a covariant differential calculus (Γ, d) over B to be $\dim \Gamma := \dim_{\mathbb{C}} \Gamma / B^+\Gamma$ [He2]. Then finite-dimensional covariant differential calculi over B can be characterized as follows [HK3], Corollary 5:

Proposition 20 *There is a canonical one-to-one correspondence between m-dimensional covariant differential calculi over B and $m+1$-dimensional subspaces T of the dual coalgebra B° such that*

$$\varepsilon \in T, \quad \Delta(T) \subset T \otimes B^\circ, \quad KT \subset T.$$

Here KT is the orbit of T under the canonical action of K on B° given by $\langle XY, f\rangle := \langle X, f_{(1)}\rangle\langle Y, f_{(2)}\rangle$, $X \in K, Y \in B^\circ, f \in B$.
The vector space T corresponding to a given calculus (Γ, d) is

$$T = \Big\{ X \in B^\circ \,\Big|\, \sum_i f_i \mathrm{d}g_i = 0 \Rightarrow \sum_i \varepsilon(f_i) X(g_i - \varepsilon(g_i)) = 0 \Big\}.$$

The vector space $T^+ = T \cap \ker \varepsilon$ is called the quantum tangent space of the corresponding differential calculus.

4.2.2.2 Calculi obtained by restriction. Let $B \subset A$ be as above and (Γ, d) be a covariant differential calculus over A. Then $\Gamma|_B := \mathrm{span}\{f\mathrm{d}g \,|\, f, g \in B\}$ is a covariant differential calculus over B. We will see below that all finite-dimensional covariant differential calculi on quantum flag manifolds can be obtained in this way from calculi over $\mathbb{C}_q[G_0]$. To show this we need the following result, in which $\pi : A^\circ \to B^\circ$ denotes the restriction [HK3], Corollary 9:

Proposition 21 *If Γ is finite-dimensional with quantum tangent space T^+, then $\Gamma|_B$ is finite-dimensional iff $K\pi(T^+)$ is finite-dimensional, and in this case this is the quantum tangent space of $\Gamma|_B$.*

4.2.2.3 Differential calculi on quantum flag manifolds. By the above, the main step to classify all finite-dimensional covariant differential calculi over B is to determine the vector space $\mathcal{F}(B^\circ, K) := \{X \in B^\circ \,|\, \dim KX < \infty\}$ which was called in [HK3] the locally finite part of B°. For quantized irreducible flag manifolds one has $\mathcal{F}(B^\circ, K) = \bar{U}$ [HK1], Theorem 6.5. Hence in this case the quantum tangent spaces of finite-dimensional covariant differential calculi are contained in $\pi(U)$, where $\pi : A^\circ \to B^\circ$ is again the restriction map. Then $KT \subset T$ means that $\pi(XY) \in T$ for all $X \in K$ and $Y \in U$ with $\pi(Y) \in T$. Using this the following classification was obtained [HK1], Theorem 7.2:

Proposition 22 *There exist exactly two non-isomorphic finite-dimensional irreducible covariant differential calculi* $(\Gamma_\pm, \mathrm{d}_\pm)$ *over* $\mathbb{C}_q[M]$*. Both have dimension* m *and their direct sum* (Γ, d) *is a* $*$*-calculus.*

Here a covariant differential calculus $\Gamma \neq \{0\}$ is called irreducible, if it possesses no non-trivial quotient by a covariant B-subbimodule. The quantum tangent spaces of $\Gamma_\pm$ are (cf. [HK1], Theorem 7.2 and Proposition 5.5)

$$T_+^+ = \mathrm{span}\{\pi(E_\beta) \,|\, \beta \in \Phi^+ \setminus \Phi_{\mathfrak{l}}^+\}, \quad T_-^+ = \mathrm{span}\{\pi(F_{-\beta}) \,|\, \beta \in \Phi^+ \setminus \Phi_{\mathfrak{l}}^+\}. \quad (4.2)$$

4.2.3 Hilbert space representations

4.2.3.1 Spectral triples. Let B be a $*$-algebra and $\rho : B \to B(H)$ be a $*$-homomorphism into the algebra of bounded operators on a Hilbert space H. Assume that D is a symmetric operator defined on a dense subspace of H which is invariant under D and $\rho(B)$. Then we will call (B, H, D) a spectral triple, provided that all commutators $\mathrm{d}f := i[D, \rho(f)]$, $f \in B$, are bounded on the domain of definition of D. Note that this drastically simplifies the usual terminology. For a more general setting with possibly unbounded $\mathrm{d}f$ see [Sch2].

4.2.3.2 Representations of differential calculi. If (B, H, D) is a spectral triple, then $\Gamma := \mathrm{span}\{\rho(f)\mathrm{d}g \,|\, f, g \in B\} \subset B(H)$ is a differential $*$-calculus over B. The spectral triple is called a (faithful) representation of a given differential $*$-calculus (Γ', d'), if the map

$$\psi : \Gamma' \to \Gamma, \quad \sum_k f_k \mathrm{d}' g_k \mapsto \sum_k f_k \mathrm{d} g_k$$

is an (injective) morphism of differential $*$-calculi.

4.3 Dirac operators on quantum flag manifolds

4.3.1 Lifting the quantum tangent spaces to $U_q(\mathfrak{g}_0)$

4.3.1.1 Definition of $\mathfrak{u}_-$. Recall the notations from Section 4.1. Define $\lambda := -2n \cdot \omega_r$, where r is as in 4.1.1.3. The number $n \in \mathbb{N} \setminus \{0\}$ is arbitrary but fixed and will play no role in the sequel. Set $X_0 := K_\lambda - 1$ and

$$X_1 := F_r \bullet X_0 = F_r \bullet K_\lambda = F_r K_\lambda K_r - K_\lambda F_r K_r = (1 - q^{2nd_r}) F_r K_r K_\lambda.$$

Proposition 23 *The adjoint action turns* $\mathfrak{u}_- := U_q(\mathfrak{l}_0) \bullet X_1$ *into the irreducible finite-dimensional representation of* $U_q(\mathfrak{l}_0)$ *with highest weight* $-\alpha_r$*.*

Proof. Since $\Delta(K_\mu) = K_\mu \otimes K_\mu$ and $S(K_\mu) = K_\mu^{-1}$ we have

$$K_\mu \bullet X_1 = K_\mu X_1 K_\mu^{-1} = q^{-\langle \mu, \alpha_r \rangle} X_1 \quad \forall \mu \in \mathbf{P}.$$

Furthermore, K_λ commutes with all E_i, F_i for $i \neq r$. Therefore we have

$$\begin{aligned} & E_i \bullet K_\lambda = E_i K_\lambda - K_i K_\lambda K_i^{-1} E_i = E_i K_\lambda - E_i K_\lambda = 0 \\ \Rightarrow \quad & E_i \bullet X_1 = E_i F_r \bullet K_\lambda = F_r E_i \bullet K_\lambda = 0. \end{aligned}$$

By Proposition 5, $\mathfrak{u}_-$ is finite-dimensional. Hence the claim follows. $\square$

4.3.1.2 Definition of $\mathfrak{u}_+$ and $\mathfrak{u}$. Fix a basis X_i of $\mathfrak{u}_-$ consisting of weight vectors and define $X^i := X_i^*$. Then the X^i form a basis of a vector space which we denote by $\mathfrak{u}_+$. The adjoint action and the involution commute by

$$(X \bullet Y)^* = S(X_{(2)})^* Y^* X_{(1)}^* = S(X)_{(1)}^* Y^* S(S(X)_{(2)}^*) = \theta(X) \bullet Y^*, \qquad (4.3)$$

since $S \circ * \circ S \circ * = \mathrm{id}$ in any Hopf $*$-algebra. Furthermore, $U_q(\mathfrak{l}_0)$ is a Hopf $*$-subalgebra of $U_q(\mathfrak{g}_0)$. Hence $\mathfrak{u}_+$ is $U_q(\mathfrak{l}_0)$-invariant as well. Set $\mathfrak{u} := \mathfrak{u}_+ \oplus \mathfrak{u}_-$.

4.3.1.3 Properties. The weight structure of the $U_q(\mathfrak{l}_0)$-modules $\mathfrak{u}_\pm$ is the classical one. Hence their complex dimension equals m, and the weights of $\mathfrak{u}_\pm$ are the roots $\beta \in \Phi^\pm \setminus \Phi_\mathfrak{l}^\pm$, all with multiplicity 1. Therefore Proposition 6 implies that after an appropriate normalization we have $\langle X_i, X^j \rangle = \delta_{ij}$.

4.3.2 Quantum γ-matrices

4.3.2.1 The classical Clifford algebra. The Clifford algebra $\mathrm{Cl}(2m, \mathbb{C})$ is the universal algebra with a vector space embedding $\gamma : \mathbb{C}^{2m} \to \mathrm{Cl}(2m, \mathbb{C})$ such that $\gamma(v)^2 = -\sum_i v_i^2$ for all $v \in \mathbb{C}^{2m}$. The spin representation σ on the space $\Sigma_{2m} := \mathbb{C}^{2^m}$ of $2m$-spinors yields an isomorphism $\mathrm{Cl}(2m, \mathbb{C}) = \mathrm{End}(\Sigma_{2m})$ [Fr].

4.3.2.2 $\mathrm{Cl}(2m, \mathbb{C})$ as a representation of $\mathfrak{so}(2m, \mathbb{C})$. Consider $\mathbb{C}^{2m}$ and $\mathrm{End}(\Sigma_{2m}) = \Sigma_{2m} \otimes \Sigma_{2m}^*$ as the carrier spaces of the vector representation ρ of $\mathfrak{so}(2m, \mathbb{C})$ and of the tensor product representation $\sigma \otimes \sigma^*$, respectively. Then γ is even an embedding of $\mathfrak{so}(2m, \mathbb{C})$-representations. In fact, the standard vector space isomorphism $\mathrm{Cl}(2m, \mathbb{C}) = \Lambda^* \mathbb{C}^{2m}$ is an isomorphism of $\mathfrak{so}(2m, \mathbb{C})$-representations, and γ is the restriction to $\mathbb{C}^{2m} = \Lambda^1 \mathbb{C}^{2m}$. Since σ is itself the direct sum of the two half-spin representations $\sigma_\pm$ which are dual to each other (see e.g. [FH]), σ is self-dual, $\sigma = \sigma^*$, but this is irrelevant for our purposes.

4.3.2.3 Definition of quantum γ-matrices. Not all flag manifolds are spin [CG], but as Kähler manifolds they all admit spin$^\mathbb{C}$ structures [Fr], Section 3.4. In any case the embedding $\mathfrak{l} \subset \mathfrak{so}(2m, \mathbb{C})$ obtained in 4.1.1.2 from the action of $\mathfrak{l}$ on $(\mathfrak{u}_+ \oplus \mathfrak{u}_-, \langle \cdot, \cdot \rangle)$ defines the representations ρ and σ of $\mathfrak{l}$, and ρ appears as a subrepresentation in $\sigma \otimes \sigma^*$. These representations can be deformed to representations of $U_q(\mathfrak{l}_0)$ which we denote by the same symbols. The representation ρ was realized in the previous section on the vector space $\mathfrak{u}_+ \oplus \mathfrak{u}_- \subset U_q(\mathfrak{g}_0)$ defined there. Since the decomposition of $\sigma \otimes \sigma^*$ into irreducible components remains the same, we therefore have:

Proposition 24 *There is a $U_q(\mathfrak{l}_0)$-equivariant embedding*

$$\gamma : \mathfrak{u}_+ \oplus \mathfrak{u}_- \to \mathrm{End}(\Sigma_{2m}).$$

Without loss of generality we can assume that

$$\gamma(X^i) = \overline{\gamma(X_i)}^T =: \gamma(X_i)^*,$$

because we can embed first only $\mathfrak{u}_-$ and define $\gamma(X^i)$ then by the above formula. The involution on $\mathrm{End}(\Sigma_{2m})$ is defined here with respect to the $U_q(\mathfrak{l}_0)$-invariant Hermitian inner product $(\cdot, \cdot)_\sigma$ on Σ_{2m}.

4.3.2.4 γ is not unique. The map γ is not uniquely determined by these requirements. First of all, $\Lambda^1\mathbb{C}^{2m}$ is isomorphic to $\Lambda^{2m-1}\mathbb{C}^{2m}$ as a $\mathfrak{so}(2m,\mathbb{C})$-representation, so there are for $m > 1$ two embeddings $\gamma : \rho \to \sigma \otimes \sigma^*$ of $\mathfrak{so}(2m,\mathbb{C})$-representations. But the representations $\mathfrak{u}_\pm$ of $\mathfrak{l}$ appear in general even more often in $\sigma \otimes \sigma^*$. For example, if X_i, X^i are dual bases in $\mathfrak{u}_-$ and $\mathfrak{u}_+$, and if we identify $\mathfrak{u}_+ \oplus \mathfrak{u}_-$ with $\Lambda^1\mathbb{C}^{2m}$, then $Y := \sum_i X_i \wedge X^i$ is a non-vanishing $\mathfrak{l}$-invariant element in $\Lambda^2\mathbb{C}^{2m}$. For $m > 1$ one has $X_1 \wedge Y \neq 0$, and hence this element is a highest weight vector generating a copy of $\mathfrak{u}_-$ in $\Lambda^3\mathbb{C}^{2m}$.
The results derived below do not depend on the choice of γ. But it is not evident that the same is true for example for the algebra generated by the matrices $\gamma(X_i), \gamma(X^i)$. This could be an interesting problem for further studies.

4.3.3 The Dirac operator

4.3.3.1 The spinor bundle. We now can define the Dirac operator using the classical formulas given for example in [CFG]. The spinor fields on which the Dirac operator acts are introduced as the sections of the quantum homogeneous vector bundle associated to the spinor representation σ of $U_q(\mathfrak{l}_0)$.
We retain all notations introduced in Section 4.1.3 with $V = \Sigma_{2m}$. We call $H := H(M_q, \Sigma_{2m})$ the Hilbert space of square-integrable spinor fields on M_q.
4.3.3.2 Definition of D. We first define a map D_- on $\mathrm{Hom}_{U_q(\mathfrak{l}_0)}(V(\lambda), \Sigma_{2m})$.

Proposition 25 *Suppose that $A \in \mathrm{Hom}_{U_q(\mathfrak{l}_0)}(V(\lambda), \Sigma_{2m})$. Then*

$$D_-(A) := -\sum_i \gamma(X^i) \circ A \circ \rho_\lambda(X_i)$$

is again an element of $\mathrm{Hom}_{U_q(\mathfrak{l}_0)}(V(\lambda), \Sigma_{2m})$.

Proof. For $Y \in U_q(\mathfrak{l}_0)$ we have

$$\begin{aligned}
& \sum_i \gamma(X^i) \circ A \circ \rho_\lambda(X_i)\rho_\lambda(S(Y)) \\
= & \sum_i \gamma(X^i) \circ A \circ \rho_\lambda(S(Y_{(1)})Y_{(2)}X_i S(Y_{(3)})) \\
= & \sum_i \gamma(X^i)\sigma(S(Y_{(1)})) \circ A \circ \rho_\lambda(Y_{(2)} \bullet X_i) \\
= & \sum_{ij} \gamma(X^i)\sigma(S(Y_{(1)})) \circ A \circ \rho_\lambda(\langle Y_{(2)} \bullet X_i, X^j\rangle X_j) \\
= & \sum_{ij} \gamma(\langle X_i, S(Y_{(2)}) \bullet X^j\rangle X^i)\sigma(S(Y_{(1)})) \circ A \circ \rho_\lambda(X_j) \\
= & \sum_j \gamma(S(Y_{(2)}) \bullet X^j)\sigma(S(Y_{(1)})) \circ A \circ \rho_\lambda(X_j) \\
= & \sum_j \sigma(S(Y_{(3)}))\gamma(X^j)\sigma(S^2(Y_{(2)}))\sigma(S(Y_{(1)})) \circ A \circ \rho_\lambda(X_j) \\
= & \sigma(S(Y)) \sum_j \gamma(X^j) \circ A \circ \rho_\lambda(X_j),
\end{aligned}$$

where we used the Hopf algebra axioms and the equivariance of γ. □

The resulting operator on $\Gamma(M_q, \Sigma_{2m})$ which acts trivially on $V(\lambda)$ in (4.1) will be denoted by the same symbol. It can be extended to a linear operator

$$D_- : \mathbb{C}_q[G_0] \otimes \Sigma_{2m} \to \mathbb{C}_q[G_0] \otimes \Sigma_{2m}, f \otimes v \mapsto -\sum_j (S^{-1}(X_j) \triangleright f) \otimes \gamma(X^j)v.$$

We consider D_- as densely defined operator on H. Analogously there is an operator D_+ acting on $\mathbb{C}_q[G_0] \otimes \Sigma_{2m}$ by

$$D_+ : f \otimes v \mapsto -\sum_j (S^{-1}(X^j) \triangleright f) \otimes \gamma(X_j)v.$$

Finally we define the Dirac operator $D := D_+ + D_-$.

4.3.3.3 Self-adjointness of D. Notice that for $X \in U_q(\mathfrak{g}_0)$ and $f, g \in \mathbb{C}_q[G_0]$ the $U_q(\mathfrak{g}_0)$-invariance of h and (4.3) imply

$$\begin{aligned} h((X \triangleright f)g^*) &= h((X_{(1)} \triangleright f)(X_{(2)}S(X_{(3)}) \triangleright g^*)) \\ &= h(X_{(1)} \triangleright (f(S(X_{(2)}) \triangleright g^*))) \\ &= h(f(S^2(X)^* \triangleright g)^*). \end{aligned}$$

Hence

$$h((S^{-1}(X_i) \triangleright f)g^*) = h(f(S(X_i)^* \triangleright g)^*) = h(f(S^{-1}(X^i) \triangleright g)^*).$$

Together with $\gamma(X^i)^* = \gamma(X_i)$ this shows that $D^*_\pm|_{\Gamma(M_q,\Sigma_{2m})} = D_\mp$, so D is symmetric on the domain $\Gamma(M_q, \Sigma_{2m})$. This is the orthogonal sum of the finite-dimensional D-invariant spaces $V(\lambda) \otimes \mathrm{Hom}_{U_q(\mathfrak{l}_0)}(V(\lambda), \Sigma_{2m})$. Hence D becomes diagonal in a suitable orthonormal basis and extends to a self-adjoint operator on H which we denote by D as well.

4.3.3.4 On the spectrum of D. Classically, the action of D^2 on $A \in \mathrm{Hom}_{U(\mathfrak{l}_0)}(V(\lambda), \Sigma_{2m})$ can be expressed as

$$D^2(A) = A \circ \rho_\lambda(C_{\mathfrak{g}_0} + C_{\mathfrak{l}_0}), \tag{4.4}$$

where $C_{\mathfrak{g}_0}$ and $C_{\mathfrak{l}_0}$ are quadratic Casimir elements in $U(\mathfrak{g}_0)$ and $U(\mathfrak{l}_0)$, respectively. This formula goes back to [Pa], in the above form it can be found e.g. in [SS]. It can be used in particular to calculate the spectrum of D^2 from which one deduces that of D. See [CFG, SS] for the explicit calculation for the example of odd-dimensional projective spaces.

It seems a non-trivial task to generalize such calculations to the quantum case. It might of course be possible that there exists an analogue of (4.4) relating D^2 to the Casimir elements of $U_q(\mathfrak{g}_0)$ and $U_q(\mathfrak{l}_0)$ introduced in 3.3.2.1 (in the case of the Podleś sphere it can indeed be checked by hand that D^2 acts by the Casimir of $U_q(\mathfrak{su}(2))$ plus a constant [NT, SW2]). But to extend the classical proof one would need to know the commutation relations of the quantum γ-matrices. And even with such a result at hand it would remain to compute the values of the Casimir elements in a given irreducible representation as a function of its highest weight. This is to the author's knowledge also an unsolved problem.

Since only finite matrices are involved which are smooth deformations of those describing the classical Dirac operator, the spectrum should as well be a smooth deformation of the classical spectrum. Hence it can at least be conjectured that D has compact resolvent which is one of Connes' axioms for the Dirac operator of a spectral triple.

4.3.4 The case of Podleś' quantum sphere

4.3.4.1 The standard quantum sphere. The simplest example of a generalized flag manifold is the projective line $\mathbb{C}P^1 = S^2$. The corresponding quantum flag manifold is the standard quantum sphere introduced by P. Podleś [Po]. In fact, Podleś introduced a whole family of quantum spheres which are parametrized by a parameter $\rho \in \mathbb{C} \cup \{\infty\}$, see the next chapter for their definition. The one which we call 'standard' is that corresponding to $\rho = \infty$.

4.3.4.2 The Dirac operator of Dąbrowski-Sitarz. The Dirac operator on $\mathbb{C}_q[S^2]$ was constructed before by L. Dąbrowski and A. Sitarz [DS1]. They in fact classified all spectral triples sharing a list of properties with the classical spectral triple on S^2 and showed that under the assumptions made D is unique up to a rescaling of D_+ by a non-zero complex constant. Their starting point was the ansatz that the Hilbert space representation H of $\mathbb{C}_q[S^2]$ of the triple can be extended to a representation of the crossed product with $U_q(\mathfrak{su}(2))$, and that this is as in the classical case the direct sum of two non-equivalent subrepresentations $H_\pm$ which are as representations of $U_q(\mathfrak{su}(2))$ both equal to $\bigoplus_{n\in\mathbb{N}_0} V((2n+1)\omega_1)$. The two representations $H_\pm$ are classically the sections of the homogeneous vector bundles associated to the half-spin representations $\sigma_\pm$ (the irreducible components of the spin representation σ). It was shown with help of MAPLE that this fixes the Hilbert space representation uniquely, see Lemma 5 in [DS1] and also [SW1]. Since (the completion of) $\Gamma(S^2_q, \Sigma_2)$ fulfills these conditions, the uniqueness result Theorem 8 of [DS1] implies that our spectral triple reduces to the one from [DS1] for $M_q = S^2_q$.

4.3.5 The associated differential calculus

4.3.5.1 Main result. The main result of this chapter is the following:

Theorem 4 *The Hilbert space representation of $\mathbb{C}_q[M]$ derived in the previous section defines together with the operators D and $D_\pm$ faithful representations of the differential calculi (Γ, d) and $(\Gamma_\pm, \mathrm{d}_\pm)$, respectively.*

The rest of this chapter is devoted to the proof. We will treat only Γ_-, the analogous results for Γ_+ and Γ are immediate.

4.3.5.2 Lifting Γ_- to $\mathbb{C}_q[G_0]$. We first show that Γ_- can be obtained by restricting the differential calculus over $\mathbb{C}_q[G_0]$ with quantum tangent space

$$T_-^{G+} := \mathbb{C}S^{-1}(X_0) \oplus S^{-1}(\mathfrak{u}_-) \subset U_q(\mathfrak{g}_0).$$

Using that S^{-1} is a coalgebra antihomomorphism and that K_λ commutes with all elements of $U_q(\mathfrak{l}_0)$ one calculates that

$$\begin{aligned} & \Delta(S^{-1}(Y \bullet X_1)) \\ = \quad & K_\lambda^{-1} \otimes S^{-1}(Y \bullet X_1) + S^{-1}(Y_{(2)} \bullet X_1) \otimes S^{-1}(Y_{(1)} K_r K_\lambda S(Y_{(3)})) \\ \in \quad & T_-^G \otimes U_q(\mathfrak{g}_0) \end{aligned}$$

for all $Y \in U_q(\mathfrak{l}_0)$, where $T_-^G := \mathbb{C} \cdot 1 \oplus T_-^{G+}$. Thus Proposition 20 implies that there exists a differential calculus $(\Gamma_-^G, \mathrm{d}_-^G)$ over $\mathbb{C}_q[G_0]$ with quantum tangent space T_-^{G+} (the last condition in Proposition 20 becomes trivial).
Note now that the quantum tangent space of Γ_- can be described as follows, where $\pi : \mathbb{C}_q[G_0]^\circ \to \mathbb{C}_q[M]^\circ$ is as above the restriction map:

Proposition 26 *The vector space spanned by $\pi \circ S^{-1}(X_i)$, $i = 1, \dots, m$ coincides with the quantum tangent space T_-^+ of Γ_-.*

Proof. For $f \in \mathbb{C}_q[M]$ we have

$$\langle F_r K_r K_\lambda, f\rangle = \langle F_r, (K_r K_\lambda) \triangleright f\rangle = \langle F_r, f\rangle$$

and similarly

$$\langle Y \bullet X_1, f\rangle = \langle Y X_1, f\rangle \quad \forall Y \in U_q(\mathfrak{l}_0).$$

Hence the claim reduces to the fact that T_-^+ is $U_q(\mathfrak{l}_0)\pi(F_r)$. This follows from Proposition 20 since $\pi(F_r) \in T_-^+$, cf. (4.2). □

Therefore $\pi(T_-^{G+}) = T_-^+$, and Proposition 21 gives:

Corollary 2 *We have $\Gamma_-^G|_{\mathbb{C}_q[M]} = \Gamma_-$.*

4.3.5.3 The calculus defined by D_-. Recall that $\Gamma(M_q, \Sigma_{2m})$ is a $\mathbb{C}_q[M]$-bimodule. We treat the elements of $\mathbb{C}_q[M]$ from now on as linear operators on $\Gamma(M_q, \Sigma_{2m})$ by considering the *right* action. Usually, one considers spectral triples with left actions of the algebra on the Hilbert space. That we consider a right action is only a matter of convention; if one rewrites this chapter starting with the left coset space $P \setminus G$, then one ends up with a left action.
We denote by $(\Gamma'_-, \mathrm{d}'_-)$ the differential calculus over $\mathbb{C}_q[M]$ defined by D_-:

$$\Gamma'_- := \mathrm{span}_{\mathbb{C}}\{f\mathrm{d}'_- g \mid f, g \in \mathbb{C}_q[M]\} \subset \mathrm{End}(\Gamma(M_q, \Sigma_{2m})), \quad \mathrm{d}'_- f := i[D_-, f].$$

4.3.5.4 A formula for d'_-. We derive now a formula for the differential d'_- of Γ'_-. It is an analogue of the classical $\mathrm{d}f = \sum_i \frac{\partial f}{\partial x^i}\mathrm{d}x^i$, but the counterparts of the $\mathrm{d}x^i$ are not elements of the calculus.

Proposition 27 *For all $f \in \mathbb{C}_q[M]$ we have*

$$\mathrm{d}'_- f = -i \sum_{i=1}^m S^{-1}(X_i) \triangleright f \otimes \sigma(K_\lambda)\gamma(X^i).$$

Proof. The coproduct of $S^{-1}(X_1) = -(1 - q^{2nd_r})K_\lambda^{-1}F_r$ is given by

$$S^{-1}(X_1) \otimes K_r^{-1}K_\lambda^{-1} + K_\lambda^{-1} \otimes S^{-1}(X_1).$$

Since $X_j = Y \bullet X_1$ for some $Y \in U_q(\mathfrak{l}_0)$ one obtains for $\sum_i g_i \otimes v_i \in \Gamma(M_q, \Sigma_{2m})$ and $f \in \mathbb{C}_q[M]$ the relation

$$\begin{aligned}
& \sum_i S^{-1}(X_j) \triangleright (g_i f) \otimes v_i \\
= & \sum_i (Y_{(3)}S^{-1}(X_1)S^{-1}(Y_{(2)}) \triangleright g_i)(Y_{(4)}K_r^{-1}K_\lambda^{-1}S^{-1}(Y_{(1)}) \triangleright f) \otimes v_i \\
& + (Y_{(3)}K_\lambda^{-1}S^{-1}(Y_{(2)}) \triangleright g_i)(Y_{(4)}S^{-1}(X_1)S^{-1}(Y_{(1)}) \triangleright f) \otimes v_i \\
= & \sum_i (S^{-1}(X_j) \triangleright g_i) f \otimes v_i + g_i(S^{-1}(X_j) \triangleright f) \otimes K_\lambda \triangleright v_i,
\end{aligned}$$

where we used the defining properties of $\mathbb{C}_q[M], \Gamma(M_q, \Sigma_{2m})$ and the fact that K_λ commutes with elements of $U_q(\mathfrak{l}_0)$. □

4.3.5.5 $\mathrm{d}'_- f$ **is bounded.** Since the multiplication operators $R_g : f \mapsto fg$, $f, g \in \mathbb{C}_q[G_0]$ extend to bounded operators on the Hilbert space obtained by completing $\mathbb{C}_q[G_0]$ with respect to the Haar measure, Proposition 27 implies:

Corollary 3 *The elements of* Γ'_- *extend to bounded operators on* H.

4.3.5.6 Completing the proof. The general theory of covariant differential calculi over Hopf algebras with invertible antipode (see [KS1], Section 14.1) implies that in Γ^G_- the differential can be written as

$$\mathrm{d}^G_- f = \sum_{i=0}^{m} (S^{-1}(X_i) \triangleright f) \cdot \omega^i \quad \forall f \in \mathbb{C}_q[G_0], \tag{4.5}$$

where $\{\omega^i\}$ is a basis of Γ^G_- consisting of invariant 1-forms. Proposition 27 generalizes the above formula to differential calculi over quantum flag manifolds. The relation (4.5) implies in particular that

$$\sum_i f_i \mathrm{d}^G_- g_i = 0 \quad \Leftrightarrow \quad \sum_i f_i (S^{-1}(X_j) \triangleright g_i) = 0 \quad \forall j. \tag{4.6}$$

The matrices $\sigma(K_\lambda)\gamma(X^i)$ are linearly independent, because $\sigma(K_\lambda)$ is invertible, γ is injective and the X^i are linearly independent. Furthermore, $\mathbb{C}_q[G]$ is free of zero divisors (Proposition 7). Hence Proposition 27 implies

$$\sum_i f_i \mathrm{d}'_- g_i = 0 \quad \Leftrightarrow \quad \sum_i f_i (S^{-1}(X_j) \triangleright g_i) = 0. \tag{4.7}$$

Theorem 4 follows in view of (4.6) and (4.7) from Proposition 19.

4.4 Bibliographical notes

For the theory of generalized flag manifolds we refer to [Ak, BE, FH, Wa1]. That G/L is affine with coordinate ring given by (1.2) can be found in [HK5], Theorem 5.1. In general, a quotient of a reductive group is affine iff the stabilizer subgroup is reductive (Matsushima-Onishchik's criterion), see [Ri].
For further information on quantum flag manifolds see [HK1, HK2, DS2, St].
The notion of covariant differential calculus on quantum spaces was introduced by S. L. Woronowicz. See [KS1], Chapters 12-14 for more information on covariant differential calculi. For classical spin geometry we refer to [Fr].

Chapter 5

On the non-standard Podleś spheres

Here the coordinate algebras $B_{q\rho}$ of the non-standard Podleś spheres are considered. It is proven that $B_{q\rho}, B_{q\rho'}$ are isomorphic iff $\rho' = \pm\rho$. This proves a conjecture from [HMS].

5.1 On the non-standard Podleś spheres

5.1.1 The algebras $B_{q\rho}$

5.1.1.1 Definition. Let $q \in \mathbb{C} \setminus \{0\}$ be not a root of unity and $\rho \in \mathbb{C}$. Define $B_{q\rho}$ to be the algebra with generators x_{-1}, x_0, x_1 and relations

$$x_0 x_{\pm 1} = q^{\pm 2} x_{\pm 1} x_0, \quad x_{\mp 1} x_{\pm 1} = q^{\pm 2} x_0^2 + (1 + q^{\pm 2})\rho x_0 - 1. \tag{5.1}$$

Analogously one defines $B_{q\infty}$ by the relations

$$x_0 x_{\pm 1} = q^{\pm 2} x_{\pm 1} x_0, \quad x_{\mp 1} x_{\pm 1} = q^{\pm 2} x_0^2 + (1 + q^{\pm 2}) x_0. \tag{5.2}$$

5.1.1.2 Embedding into $\mathbb{C}_q[SL(2,\mathbb{C})]$. Define the following set of generators

$$K^{\pm 1} := K_{\pm\omega_1}, \quad E := E_1 K_{-\omega_1}, \quad F := K_{\omega_1} F_1$$

of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ and set

$$X_\rho := \begin{cases} \rho q^{1/2} \frac{q+q^{-1}}{q-q^{-1}}(K - K^{-1}) + qE + F & \rho \in \mathbb{C}, \\ K - K^{-1} & \rho = \infty. \end{cases}$$

Then $\Delta(X_\rho) = X_\rho \otimes K^{-1} + K \otimes X_\rho$, and hence

$$\{f \in \mathbb{C}_q[SL(2,\mathbb{C})] \,|\, X_\rho \triangleright f = 0\}$$

is a left coideal subalgebra of $\mathbb{C}_q[SL(2,\mathbb{C})]$. This is isomorphic to $B_{q\rho}$, see [KS1], Proposition 4.31. Note that in [KS1] right invariants are considered, we switched to left invariants using an isomorphism $U_q(\mathfrak{sl}(2,\mathbb{C})) = U_q(\mathfrak{sl}(2,\mathbb{C}))^{\mathrm{cop}}$ hidden in the above definition of K, E, F whose coproduct is the coopposite of that of the elements denoted by the same symbols in [KS1] (cf. Section 3.1.2 therein). Note also that our x_0 is denoted by $x_0 - \rho$ ($\rho \neq \infty$) and $x_0 - 1$ ($\rho = \infty$) in [KS1].
In particular, $B_{q\infty}$ is the coordinate algebra of the standard Podleś sphere discussed in the previous chapter. It was shown in [MS] for all but an exceptional sequence $\{\rho_n\}$ of values for ρ that $\mathbb{C}_q[SL(2,\mathbb{C})]$ is faithfully flat over $B_{q\rho}$.

5.1.2 Some properties

5.1.2.1 A vector space basis. For $i \in \mathbb{N}_0, j \in \mathbb{Z}$ define

$$e_{ij} := \begin{cases} x_0^i x_1^j & j \geq 0 \\ x_0^i x_{-1}^{-j} & j < 0. \end{cases}$$

Proposition 28 *The elements $\{e_{ij}\}$ form a vector space basis of $B_{q\rho}$.*

See [KS1], p. 125, for a proof.

5.1.2.2 $\mathbb{Z}$-grading. It follows from the defining relations that $B_{q\rho}$ is $\mathbb{Z}$-graded, $B_{q\rho} = \bigoplus_{j \in \mathbb{Z}} B^j$, $B^j B^k \subset B^{j+k}$, with

$$B^j := \operatorname{span}\{e_{ij} \mid i \in \mathbb{N}_0\} = \{f \in B_{q\rho} \mid x_0 f = q^{2j} f x_0\}.$$

5.1.2.3 The ideal I. Let I be the ideal generated by x_0. Using the basis $\{e_{ij}\}$ one sees that $I = x_0 B_{q\rho} = B_{q\rho} x_0$. We denote by $\pi : B_{q\rho} \to B_{q\rho}/I$ the canonical projection.

5.1.2.4 Characters. We recall the (well-known) character theory of $B_{q\rho}$:

Proposition 29 *The following is a complete list of the characters of $B_{q\rho}$:*

$$\begin{aligned} \rho \neq \infty, \pm i : \quad & \chi_\lambda(x_0) = 0, \chi_\lambda(x_{\pm 1}) = \lambda^{\pm 1}, \quad \lambda \in \mathbb{C} \setminus \{0\}, \\ \rho = \pm i : \quad & \chi_\lambda(x_0) = 0, \chi_\lambda(x_{\pm 1}) = \lambda^{\pm 1}, \quad \lambda \in \mathbb{C} \setminus \{0\}, \\ & \chi'(x_{\pm 1}) = 0, \chi'(x_0) = \mp i, \\ \rho = \infty : \quad & \chi_\lambda^{\pm}(x_{\pm 1}) = \chi_\lambda^{\pm}(x_0) = 0, \chi_\lambda^{\pm}(x_{\mp 1}) = \lambda, \quad \lambda \in \mathbb{C}. \end{aligned}$$

Proof. It is straightforward to check that the above equations define characters of $B_{q\rho}$. Let conversely χ be any character. We write $\chi(x)$ as x for simplicity. The first relations in (5.1), (5.2) show $x_0 = 0$ or $x_1 = x_{-1} = 0$. Suppose $x_0 = 0$. Then the other two relations both become

$$x_{-1} x_1 = \begin{cases} -1 & \rho \neq \infty \\ 0 & \rho = \infty. \end{cases}.$$

Hence $\chi = \chi_\lambda$ $(\rho \neq \infty)$ or $\chi = \chi_\lambda^{\pm}$ $(\rho = \infty)$ for some λ.
If $x_0 \neq 0$ and $x_1 = x_{-1} = 0$, then the second relations in (5.1) yield for $\rho \neq \infty$

$$x_0^2 + (1 + q^{\mp 2}) \rho x_0 - q^{\mp 2} = 0.$$

These equations are equivalent to $x_0 = \pm i, \rho = \mp i$, so $\chi = \chi'$. For $x_0 \neq 0, x_1 = x_{-1} = 0$ and $\rho = \infty$ one gets by the second relations in (5.2)

$$x_0(q^{\pm 2} x_0 + (1 + q^{\pm 2})) = 0$$

which can not be solved under the assumption $x_0 \neq 0$. $\square$

5.1.2.5 The ideal J. We denote by $J \subset B_{q\rho}$ the intersection of the kernels of all characters. For $\rho \neq \infty, \pm i$ an element $x = \sum_{ij} \xi_{ij} e_{ij} \in B_{q\rho}$, $\xi_{ij} \in \mathbb{C}$, is mapped by χ_λ to $f(\lambda)$, where f is the Laurent polynomial $f(z) = \sum_{j \in \mathbb{Z}} \xi_{0j} z^j$. Thus $\chi_\lambda(x) = 0$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ iff $f = 0$. Hence $J = I$. The same is true for $\rho = \infty$ as one checks similarly. For $\rho = \pm i$ one obtains the smaller ideal $I \cap \ker \chi'$.

5.1.3 $B_{q\rho}$ depends on ρ

It is shown in [KS1], Proposition 4.27 that $B_{q\rho}, B_{q\rho'}$ are isomorphic as comodule algebras iff $\rho' = \pm\rho$. Here we show that the same holds already if only the algebra structure is considered.

Theorem 5 *The algebras $B_{q\rho}, B_{q\rho'}$ are isomorphic iff $\rho' = \pm\rho$ ($-\infty = \infty$).*

Proof. We first note that $B_{q\infty}$ can not be isomorphic to $B_{q\rho}$ with $\rho \neq \infty$: Otherwise $B_{q\infty}/J$ would be isomorphic to $B_{q\rho}/J$. The first algebra is isomorphic to $\mathbb{C}[z] \oplus \mathbb{C}[z]$ with $\pi(x_{\pm 1})$ as generators. This follows from adding $x_0 = 0$ to (5.2). For $\rho \neq \infty, \pm i$ the algebra $B_{q\rho}/J$ is instead isomorphic to $\mathbb{C}[z, z^{-1}]$ with $z^{\pm 1}$ corresponding to $\pm\pi(x_{\pm 1})$. For $\rho = \pm i$ we have $J = I \cap \ker \chi' \subset I$, and $B_{q\pm i}/I$ is as above isomorphic to $\mathbb{C}[z, z^{-1}]$. That is, this is a quotient algebra of $B_{q\pm i}/J$, hence the latter can also not be isomorphic to $B_{q\infty}/J = \mathbb{C}[z] \oplus \mathbb{C}[z]$.
Suppose now that $\psi : B_{q\rho'} \to B_{q\rho}$ is an isomorphism with $\rho, \rho' \neq \infty$. We denote by $X_i \in B_{q\rho}$ the images of the generators of $B_{q\rho'}$ under ψ.
Since X_i generate $B_{q\rho}$, $\pi(X_i)$ generate $\pi(B_{q\rho}) = \mathbb{C}[z, z^{-1}]$. This algebra is a commutative integral domain, so $\pi(X_0)\pi(X_{\pm 1}) = q^{\pm 2}\pi(X_{\pm 1})\pi(X_0)$ implies that either $\pi(X_0)$ or both $\pi(X_{\pm 1})$ vanish. But $\mathbb{C}[z, z^{-1}]$ can not be generated by a single element, so $\pi(X_0) = 0$. Hence $X_0 = \lambda_0 x_0$ for some $\lambda_0 \in B_{q\rho}$. Repeating the whole argumentation with the roles of x_i and X_i interchanged one gets $x_0 = \mu_0 X_0$, that is, $X_0 = \mu_0\lambda_0 X_0$ for some $\mu_0 \in B_{q\rho}$. Now $\mathbb{C}_q[G]$ is an integral domain (Proposition 7), and the only invertible elements in $\mathbb{C}_q[G]$ are the non-zero multiples of 1 [Jo], 9.1.14). Thus $\lambda_0 = \mu_0^{-1} \in \mathbb{C} \setminus \{0\}$.
Therefore $x_0 X_{\pm 1} = q^{\pm 2} X_{\pm 1} x_0$. Hence $X_{\pm 1} \in B^{\pm 1}$, so $X_{\pm 1} = P_\pm(x_0) x_{\pm 1}$ for some polynomials $P_\pm \in \mathbb{C}[z]$. Inserting this into (5.1) one sees that both $P_\pm$ must be of degree zero. So $X_i = \lambda_i x_i$ for three non-zero constants λ_i. Inserting this again into the relations (5.1) we get

$$q^{\pm 2}\lambda_0^2 x_0^2 + (1 + q^{\pm 2})\rho'\lambda_0 x_0 - 1 = \lambda_1\lambda_{-1}(q^{\pm 2}x_0^2 + (1 + q^{\pm 2})\rho x_0 - 1),$$

which is equivalent to

$$\lambda_0 = \pm 1, \quad \rho' = \pm\rho, \quad \lambda_1\lambda_{-1} = 1.$$

If conversely $\rho' = -\rho$, then it is immediate that the assignment $x_{-1}, x_0, x_1 \mapsto x_{-1}, -x_0, x_1$ extends to an isomorphism $B_{q\rho'} \to B_{q\rho}$. □

5.2 Bibliographical notes

The theory of the Podleś spheres as used here was developed in [Po] and [DK2]. See also [HK4, MS, NM] for more information. For similar quantizations of general symmetric spaces see [Le]. For the semi-classical picture of the quantum 2-spheres see [Ci, Ro].

List of notations

$\equiv$	$X \equiv Y \Leftrightarrow \exists\, x \in \mathbb{C} \setminus \{0\} : X = xY$	3.2.2.2
$\bullet$	adjoint action of $U_q(\mathfrak{g})$	2.2.1.11
$\triangleright, \triangleleft$	left, right actions	2.2.2.2
$\mathbb{C}_{(q)}[G]$	(quantized) coordinate algebra of G	2.1.2.3, 2.2.2.1
$\mathbb{C}_{(q)}[G_0]$	compact real form of $\mathbb{C}_{(q)}[G]$	2.1.2.3, 2.2.2.3
$\mathbb{C}_{(q)}[M]$	a (quantized) algebra of functions on M	4.1.1.1, 4.1
$*$	involution of a $*$-algebra	2.2.1.5
$\langle \cdot, \cdot \rangle$	Killing form of $\mathfrak{g}$ (on $\mathfrak{h}$ and $\mathfrak{h}^*$);	2.1.1.1
	dual pairing between two Hopf algebras;	
	quantum Killing form of $U_q(\mathfrak{g})$	2.2.1.11
$\langle \cdot, \cdot \rangle_h$	Hermitian inner product on $\mathbb{C}_q[G_0]$	2.2.2.5
$(\cdot, \cdot)_\lambda$	Hermitian inner product on $V(\lambda)$	2.2.1.12
$A \bowtie B$	a quantum double	3.3.1.1
$A \blacktriangleright\!\!\blacktriangleleft B$	a quantum codouble	3.3.4.1
B^+	$B \cap \ker \varepsilon$	4.1.2.1
$\mathfrak{a} \oplus \mathfrak{n}_+$	2nd factor in Iwasawa decomposition	2.1.1.3
$\mathfrak{b}_\pm$	Borel subalgebras of $\mathfrak{g}$	2.1.1.1
$\mathfrak{g}$	complex semi-simple Lie algebra	2.1.1.1
$\mathfrak{g}_\mathbb{R}$	$\mathfrak{g}$ as a real Lie algebra	2.1.1.3
$\mathfrak{g}_0$	compact real form of $\mathfrak{g}$	2.1.1.3
$\mathfrak{g}_0^*$	dual space of $\mathfrak{g}_0$, identified with $\mathfrak{a} \oplus \mathfrak{n}_+$	2.1.1.3
$\mathfrak{g}_\beta$	root space ($\beta \in \Phi$)	2.1.1.1
$\mathfrak{h}$	Cartan subalgebra of $\mathfrak{g}$	2.1.1.1
$\mathfrak{l}$	Levi factor of $\mathfrak{p}$	4.1.1.1
$\mathfrak{l}_0$	$\mathfrak{l} \cap \mathfrak{g}_0$	4.1.1.1
$\mathfrak{n}_\pm$	maximal nilpotent subalgebras of $\mathfrak{g}$	2.1.1.1
$\mathfrak{p}$	parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}_+$	4.1.1.1
$\mathfrak{u}$	orthogonal complement of $\mathfrak{l}$ in $\mathfrak{g}$ or its	4.1.1.2, 4.3.1.2
	quantum analogue	
$\mathfrak{u}_\pm$	$\mathfrak{u} \cap \mathfrak{n}_\pm$ or quantum analogues	4.1.1.1, 4.1.1.2,
		4.3.1.1, 4.3.1.2
$\alpha_1, \ldots, \alpha_N$	simple roots	2.1.1.1
β_k	k-th root in ordering of Φ^+	2.1.1.2
Γ	a differential calculus	4.2.1.1
$\Gamma_\pm$	differential calculi over $\mathbb{C}_q[M]$	4.2.2.3
$\Gamma(M_q, V)$	sections of quantum homogeneous	4.1.3.2
	vector bundle	
Δ	coproduct of a Hopf algebra	

ε	counit of a Hopf algebra	
λ, μ, ν, ξ	weights of $\mathfrak{g}$	
φ	dual pairing of $U_q(\mathfrak{b}_-)$ and $U_q(\mathfrak{b}_+)^{\mathrm{cop}}$	2.2.1.11
$\Phi, \Phi^{\pm}$	(positive and negative) roots of $\mathfrak{g}$	2.1.1.1
$\Phi_{\mathfrak{l}}, \Phi_{\mathfrak{l}}^{\pm}$	(positive, negative) roots of $\mathfrak{l}$	4.1.1.1
θ	Cartan involution of $\mathfrak{g}$ or $U_q(\mathfrak{g})$;	2.1.1.4, 2.2.1.5
	a morphism from A^{op} to $U(A \bowtie A)$	3.3.1.4
$\omega_1, \dots, \omega_N$	fundamental weights	2.1.1.1
a_{ij}	Cartan matrix of $\mathfrak{g}$	2.1.1.1
$c_{v,w}, c^{\lambda}_{-\mu,\nu}, c^{\lambda}_{ij}$	matrix coefficients of U-modules	2.2.2.1
d	differential of a differential calculus	4.2.1.1
$\mathrm{f}, \bar{\mathrm{f}}$	$S^{-1} = \mathrm{f} S \bar{\mathrm{f}}$	3.1.1.3
d_i	$\frac{1}{2}\langle \alpha_i, \alpha_i \rangle$	2.1.1.1
e	unit element of G	
f, g, h	elements of $A, B, \mathbb{C}_q[G]$ or $\mathbb{C}_q[M]$	
h	the Haar functional on $\mathbb{C}_q[G]$	2.2.2.5
m	$\dim_{\mathbb{C}} M$;	
	multiplication map of an algebra	2.2.1.1
q_i	q^{d_i}	2.2.1.2
$\mathbf{r}, \bar{\mathbf{r}}$	universal r-form and its inverse	3.1.1.1
r_i	generator of W	2.1.1.2
v, w	vectors in U- or $U_q(\mathfrak{g})$-modules	
w_0	longest element of W	2.1.1.2
A	a Hopf algebra (think of $\mathbb{C}_q[G]$)	3.1.1.1, 4.1.2.2
A°	the dual Hopf algebra	
B	an algebra (think of $\mathbb{C}_q[M]$)	4.1.2.2
B°	the dual coalgebra	4.2.2.1
$B_{q\rho}$	coordinate algebras of non-standard Podleś spheres	5.1.1.1
$\mathcal{C}$	a tensor category of U-modules	4.1.2.2
$\mathcal{C}_G$	the $U_q(\mathfrak{g})$-modules with highest weights in $\mathbf{L}^+$	2.2.1.12
E_β	(quantum) root vector ($\beta \in \Phi^+$)	2.1.1.3, 2.2.1.8
E_k	E_{α_k}	2.1.1.3
$F_{-\beta}$	(quantum) root vector ($\beta \in \Phi^+$)	2.1.1.3, 2.2.1.8
F_k	$F_{-\alpha_k}$	2.1.1.3
$\mathcal{F}(U_q(\mathfrak{g}))$	locally finite part of $U_q(\mathfrak{g})$	2.2.1.11
$\mathcal{F}(B^{\circ}, K)$	locally finite part of B°	4.2.2.3
G	connected Lie group with Lie algebra $\mathfrak{g}$	2.1.2.1
G_0, G_0^*	subgroups of G corresponding to $\mathfrak{g}_0, \mathfrak{g}_0^*$	2.1.2.1
H	a Hilbert space	4.2.3.1
H_k	basis of $\mathfrak{h}$	2.1.1.3
I	a coideal of U	4.1.2.2
K	a right coideal subalgebra of U (think of $U_q(\mathfrak{l}_0)$)	4.1.2.2
K_i	$K_i := K_{\alpha_i}$	2.2.1.2
L	subgroup of G corresponding to $\mathfrak{l}$	4.1.1.1
L_0	$L \cap G_0$	4.1.1.1

$\mathbf{L}$	weights of G	2.1.1.1
M	$G/P = G_0/L_0$	4.1.1.1
P	subgroup of G corresponding to $\mathfrak{p}$	4.1.1.1
$\mathbf{P}, \mathbf{P}^+$	(dominant) integral weights	2.1.1.1
$\mathbf{Q}$	root lattice of $\mathfrak{g}$	2.1.1.1
S	antipode of a Hopf algebra	
T^+	tangent space of a differential calculus	4.2.2.1
$\mathcal{T}_i$	Lusztig automorphism of $U_q(\mathfrak{g})$	2.2.1.8
U	a Hopf algebra (think of $U_q(\mathfrak{g})$)	4.1.2.2
$U(\mathfrak{g})$	universal enveloping algebra of $\mathfrak{g}$	2.1.1.4
$U_q(\mathfrak{g})$	quantized universal enveloping algebra	2.2.1.2
$U_{(q)}(\mathfrak{g}_0)$	compact real form of $U_{(q)}(\mathfrak{g})$	2.1.1.4, 2.2.1.5
$V(\lambda)$	irred. highest weight module	2.1.1.1, 2.2.1.12
W	Weyl group of $\mathfrak{g}$	2.1.1.2
X, Y, Z	elements of U or $U_q(\mathfrak{g})$	

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Selbständigkeitserklärung

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