ON THE HOCHSCHILD (CO)HOMOLOGY OF QUANTUM HOMOGENEOUS SPACES

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Abstract. The recent result of Brown and Zhang establishing Poincaré duality in the Hochschild (co)homology of a large class of Hopf algebras is extended to right coideal subalgebras over which the Hopf algebra is faithfully flat, and applied to the standard Podleś quantum 2-sphere.

1. Introduction

1.1. **Theory.** As work in particular by Takeuchi [41], Masuoka and Wigner [31], and Müller and Schneider [34] has shown, the following definition provides a reasonable generalisation of affine homogeneous spaces of algebraic groups (see Section 1.3 below for some discussion of the commutative case):

Definition 1. *A* quantum homogeneous space *is a right faithfully flat ring extension $B \subset A$ where $A = (A, \mu, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra with bijective antipode S over a field k and B is a right coideal subalgebra, $\Delta(B) \subset B \otimes A$.*

Our aim here is to generalise a theorem by Brown and Zhang [5] from Hopf algebras to such subalgebras. For its statement we adopt the following terminology (see Section 1.3 for background information and motivation):

Definition 2. *Let k be field and B be a (unital, associative) k-algebra.*

(1) *The* dimension $\dim(B)$ *of B is its projective dimension in the category of finitely generated B-bimodules. B is* smooth *if* $\dim(B) < \infty$.
(2) *A character $\varepsilon : B \to k$ is* Cohen-Macaulay *if for the induced left B-module structure on k and some $d \geq 0$ one has $\mathrm{Ext}^n_B(k, B) = 0$ for $n \neq d$, and* Gorenstein *if in addition $\mathrm{Ext}^d_B(k, B) \simeq k$ as k-modules.*

Under these conditions we can deduce a Poincaré-type duality in the Hochschild (co)homology of B as studied by Van den Bergh in [42]:

Theorem 1. *If $B \subset A$ is a smooth quantum homogeneous space and the restriction of ε to B is Cohen-Macaulay, then there are isomorphisms*

$$\mathrm{Ext}^n_{B^e}(B, \cdot\,) \simeq \mathrm{Tor}^{B^e}_{\dim(B)-n}(\omega \otimes_B \cdot\,, B), \quad \omega := \mathrm{Ext}^{\dim(B)}_{B^e}(B, B^e) \tag{1}$$

of functors on the category of B-bimodules that are right flat. Here $\otimes := \otimes_k$, $B^e := B \otimes B^{\mathrm{op}}$, we identify left and right B^e-modules and B-bimodules, and the B-bimodule structure on ω is induced by right multiplication in B^e.

If $B = A$ and ε is Gorenstein, then Brown and Zhang's result also says that $\omega \simeq A_\sigma$ for some $\sigma \in \mathrm{Aut}(A)$ [5], by which we mean it is isomorphic to A as left module but the right action is given by $a \blacktriangleleft b := a\sigma(b)$. In particular, ω is an invertible bimodule with inverse $\omega^{-1} \simeq A_{\sigma^{-1}}$, so the duality (1) can be reversed to

$$(2)\quad \mathrm{Tor}_n^{B^e}(\,\cdot\,,B) \simeq \mathrm{Ext}_{B^e}^{\dim(B)-n}(B,\omega^{-1}\otimes_B\cdot), \quad \omega\otimes_B\omega^{-1} \simeq \omega^{-1}\otimes_B\omega \simeq B,$$

and this holds in fact on the category of all B-bimodules (the flatness assumption becomes obsolete), see [42]. Then the duality is not only of theoretical interest but a valuable tool when explicitly computing the Hochschild cohomology of B, see [28] for a concrete demonstration.

Algebraic geometry suggests that the Gorenstein condition implies the invertibility of ω in greater generality: we will show in Theorem 7 that ω carries in the Gorenstein case the structure of a (B,A)-Hopf bimodule. These are noncommutative generalisations of the modules of sections of homogeneous vector bundles, and $\mathrm{Ext}_B^{\dim(B)}(k,B)$ reduces for commutative rings to the typical fibre. So the Gorenstein condition means here that we are dealing with a line bundle whose module of sections is invertible.

We will recall that any quantum homogeneous space can be written as

$$(3)\qquad B = \{a \in A \,|\, (\pi \otimes \mathrm{id}_A) \circ \Delta(a) = \pi(1) \otimes a\},$$

where Δ is the coproduct in A and π is the canonical projection onto A/B^+A, $B^+ := B \cap \ker\varepsilon$, see Section 2.3. This is a Hopf algebra map if and only if $AB^+ = B^+A$ (since $B^+A = S(AB^+)$ as observed by Koppinen, see [34], Lemma 1.4). Our second main result applies to this case:

Theorem 2. *If $B \subset A$ is a smooth and Gorenstein quantum homogeneous space with $AB^+ = B^+A$, then $\mathrm{Ext}_{B^e}^{\dim(B)}(B,B^e)$ is an invertible B-bimodule.*

The condition $AB^+ = B^+A$ holds trivially if A is the commutative coordinate ring of an algebraic group G. Then B is the coordinate ring $k[X]$ of an affine homogeneous space of G, and A/B^+A is the coordinate ring $k[H]$ of the isotropy group $H \subset G$ of $X \simeq H\backslash G$, see Section 1.3. Important noncommutative examples with $AB^+ = B^+A$ are quantisations of quotients $H\backslash G$ of a Poisson group by a Poisson subgroup, such as the standard quantisations of the generalised flag manifolds studied e.g. in [7, 17, 18, 19, 27, 39].

There are, however, plenty examples of quantum homogeneous spaces with $AB^+ \neq B^+A$ such as the nonstandard Podleś spheres [35, 34] and more generally quantisations of quotients of Poisson groups by coisotropic subgroups. We use the antipode of A/B^+A explicitly when constructing ω^{-1} but we are not aware of a counterexample to Theorem 2 with the assumption $AB^+ = B^+A$ removed, and we expect its conclusion holds for the nonstandard Podleś spheres. Hence we ask:

Question 1. *Is ω invertible for all smooth quantum homogeneous spaces when ε is Gorenstein?*

1.2. **Application.** Our main motivation is to apply our results to the paradigmatic example of a quantum homogeneous space which is Podleś' standard quantum sphere [35]. Here A is the quantised coordinate ring $\mathbb{C}_q[SL(2)]$, and $A/B^+A \simeq \mathbb{C}[z,z^{-1}]$. The quotient π deforms the map dual to the embedding of a maximal torus $T \simeq \mathbb{C}^*$ into $SL(2,\mathbb{C})$, so B deforms the coordinate ring of the coset space $T\backslash SL(2,\mathbb{C})$ which is isomorphic to the complexified 2-sphere given in $\mathbb{C}^3$ by $x^2+y^2+z^2=1$. We will prove that B satisfies all the homological assumptions of Theorem 2 and compute ω:

Theorem 3. *Let $q \in \mathbb{C}^*$ be not a root of unity and A be the quantised coordinate ring of $SL(2,\mathbb{C})$. Then the standard Podleś quantum 2-sphere $B \subset A$ is smooth with $\dim(B)=2$, $\varepsilon|_B$ is Gorenstein, and we have $\omega \simeq B_\sigma$, where σ is the restriction of the square S^2 of the antipode of A to B.*

This form of ω had to be expected from Dolgushev's results [10] in the setting of formal deformation quantisations, and Hadfield's computations [16], since $S^2|_B$ quantises the flow of the modular vector field of the quantised Poisson structure on the 2-sphere, and also coincides with the modular automorphism of the Haar functional of A, see Section 3.1 for further details.

As we mentioned above, the standard quantum 2-sphere can be further deformed to quantum homogeneous spaces of $\mathbb{C}_q[SL(2)]$ where $AB^+ \neq B^+A$ [35, 34]. The Gorenstein condition is checked for these in the same way as for the standard sphere. It was shown in [1] that their global dimension is 2, but the methods used there seem not to allow us to answer

Question 2. *Are the nonstandard Podleś spheres smooth?*

1.3. **The case of coordinate rings.** For the reader's convenience we briefly recall here the geometric background of the theory in the case that $B \subset A$ are coordinate rings of affine varieties over an algebraically closed field.

A Hopf algebra structure on the coordinate ring $A=k[G]$ of an affine variety G corresponds directly to an algebraic group structure on G. Furthermore, a faithfully flat embedding $B=k[X] \subset A$ corresponds to a surjection $G \to X$ ([32], Theorem 7.3 on p. 48). Since $\Delta(B) \subset B \otimes A \simeq k[X \times G]$, Δ defines an algebraic action $X \times G \to X$ of G on X for which the quotient map $G \to X$ is equivariant. Hence X is indeed a homogeneous space of G, that is, the action is transitive and $X \simeq H\backslash G$ for a closed subgroup $H \subset G$.

Recall next that a variety X is smooth in a point if and only if its local ring in the point has finite global dimension which is then equal to $\dim(X)$ ([32], Theorem 19.2 on p. 156). Since Ext is compatible with localisations in the sense that for all maximal ideals $\mathfrak{m}$ in a commutative Noetherian ring B and all finitely generated modules M,N over B one has ([44], Proposition 3.3.10)

$$(\mathrm{Ext}^\bullet_B(M,N))_\mathfrak{m} \simeq \mathrm{Ext}^\bullet_B(M,N) \otimes_B B_\mathfrak{m} \simeq \mathrm{Ext}^\bullet_{B_\mathfrak{m}}(M_\mathfrak{m},N_\mathfrak{m}), \tag{4}$$

X is smooth in all points if and only if $\mathrm{gl.dim}(k[X]) < \infty$.

One has in general $\mathrm{gl.dim}(B) \leq \dim(B)$ (see Lemma 3 in Section 2.5), so the smoothness from Definition 2 implies for $B=k[X]$ that X is smooth in all points. It can happen that $\dim(B)=\infty$ even when $\mathrm{gl.dim}(B)=0$ (consider e.g. $B=\mathbb{C}$ over $k=\mathbb{Q}$), but for $k=\bar{k}$, $k[X]$-bimodules are the same as modules over $k[X] \otimes k[X] \simeq k[X \times X]$, and this has finite global dimension if $k[X]$ has ([20], Theorem 2.1) and is Noetherian. Hence the finitely generated $k[X] \otimes k[X]$-module $k[X]$ admits a finitely generated projective resolution of finite length and $\dim(k[X]) < \infty$. Thus smoothness as in Definition 2 is really equivalent to geometric smoothness of X.

For a classical homogeneous space the smoothness condition in Theorem 1 becomes in fact void in characteristic zero: Corollary 5 below tells that an affine homogeneous space $X \simeq H\backslash G$ is smooth if G is so, and affine algebraic groups are smooth in characteristic zero, see [43] Sections 11.6 and 11.7.

Similarly, a smooth character of a coordinate ring is Gorenstein since this is for these equivalent to the finiteness of the injective dimension of the corresponding local ring as a module over itself ([32], Theorem 18.1 on p. 141 in combination with (4)). In the noncommutative case this equivalence breaks down which results in various nonequivalent generalisations of the Gorenstein and similarly the Cohen-Macaulay condition. The ones from Definition 2 are closest in spirit to the notions of AS Gorenstein and AS Cohen-Macaulay rings [22] but still more naive and just meant as a working terminology to be used within this paper.

Lastly we remark that the coordinate ring of any smooth affine variety satisfies the duality from Theorem 1 with ω being the inverse of the module of top degree Kähler differentials (algebraic differential forms), see e.g. [26].

1.4. **Structure of the paper.** Theorems 1 and 2 are proved in Section 2. Sections 2.1-2.3 recall background material on Hochschild (co)homology and quantum homogeneous spaces, mainly from [25, 42] and [31, 34, 41]. Section 2.4 extends the description of the Hochschild cohomology of a Hopf algebra A as a derived functor over A rather than A^e to quantum homogeneous spaces. Using this we prove Theorem 1 in Section 2.5.

In Section 2.6 we give $\omega = \mathrm{Ext}^{\dim(B)}_{B^e}(B, B^e)$ for smooth and Gorenstein quantum homogeneous spaces $B \subset A$ the structure of a (B, A)-Hopf bimodule and deduce that it is as a left B-module isomorphic to

$$\{a \in A \,|\, (\pi \otimes \mathrm{id}_A) \circ \Delta(a) = g \otimes a\}$$

for some group-like element $g \in C = A/B^+A$. Using this we construct in Section 2.7 under the assumption $AB^+ = B^+A$ a B-bimodule $\bar{\omega}$ with $\bar{\omega} \otimes_B \omega \simeq B_\sigma$ for some algebra endomorphism σ of B. Section 2.8 discusses a generalisation of the transitive action of a group G on $X = H\backslash G$ to characters on quantum homogeneous spaces. This is used to show in Section 2.9 that σ is an automorphism which implies Theorem 2.

A short Section 2.10 contains a criterion to prove the smoothness of some quantum homogeneous spaces which is applied later in the proof of Theorem 3, and Section 2.11 gives three examples of quantum homogeneous spaces that illustrate certain aspects of the general theory developed so far.

Section 3 is devoted to the Podleś sphere and the proof of Theorem 3.

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2. Theory

2.1. **Hochschild (co)homology.** Let k be a field. For a k-algebra B, we denote by B^{op} the opposite algebra (same vector space, opposite multiplication) and by $B^e := B \otimes B^{\mathrm{op}}$ the enveloping algebra of B (here and in the rest of the paper, an unadorned $\otimes$ denotes the tensor product over k). The tensor flip $\tau(a \otimes b) := b \otimes a$ defines a canonical isomorphism $(B^e)^{\mathrm{op}} \simeq B^e$ and hence identifies left and right B^e-modules, and these are also the same as B-bimodules (with symmetric action of k). For any such bimodule M, the Hochschild (co)homology of B with coefficients in M is

$$H_\bullet(B, M) := \mathrm{Tor}_\bullet^{B^e}(M, B), \quad H^\bullet(B, M) := \mathrm{Ext}_{B^e}^\bullet(B, M),$$

where B is considered as a B-bimodule using multiplication in B.

The bar resolution of B yields canonical (co)chain complexes computing $H_\bullet(B, M)$ and $H^\bullet(B, M)$. For cohomology, this cochain complex is

$$C^\bullet(B, M) := \mathrm{Hom}_k(B^{\otimes \bullet}, M)$$

with the coboundary operator $\mathsf{b} : C^n(B, M) \to C^{n+1}(B, M)$ given by

$$\begin{aligned} \mathsf{b}\varphi(b^1, \ldots, b^{n+1}) &= b^1\varphi(b^2, \ldots, b^{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i \varphi(b^1, \ldots, b^i b^{i+1}, \ldots, b^{n+1}) \\ &\quad + (-1)^{n+1}\varphi(b^1, \ldots, b^n) b^{n+1}. \end{aligned} \tag{5}$$

For further information see e.g. [6, 29, 44].

2.2. **Van den Bergh's theorem.** The following theorem was proven by Van den Bergh in [42]. To be precise, Van den Bergh considered the case in which the bimodule ω is invertible. For the sake of clarity we include the sketch of a proof not using this assumption, see [25] for details.

Theorem 4. *Let B be a smooth algebra and assume there exists $d \geq 0$ such that $H^n(B, B^e) = 0$ for $n \neq d$. Then $d = \dim(B)$ and there is for all $n \geq 0$ and for every right B-flat B-bimodule M a canonical isomorphism*

$$H_n(B, \omega \otimes_B M) \simeq H^{d-n}(B, M), \quad \omega := H^d(B, B^e), \tag{6}$$

where the bimodule structure of ω is induced by right multiplication in B^e.

Proof. The assumption that B is smooth means that the B^e-module B admits a resolution $P_\bullet$ of finite length consisting of finitely generated projective B^e-modules. Using $H^n(B, B^e) = 0$ for $n \neq d$ and Schanuel's lemma one can assume without loss of generality (see the proof of Theorem 23 in [25] for the detailed argument) that this resolution has length d, and then $P^*_{d-\bullet} := \mathrm{Hom}_{B^e}(P_{d-\bullet}, B^e)$ is a finitely generated projective resolution of ω. Therefore we have canonical isomorphisms

$$\mathrm{Hom}_{B^e}(P_\bullet, M) \simeq P^*_\bullet \otimes_{B^e} M \simeq (P^*_\bullet \otimes_B M) \otimes_{B^e} B. \tag{7}$$

As a right B^e-projective module, $P^*_\bullet$ is right B-flat, so $P^*_\bullet \otimes_B M$ is a resolution of $\omega \otimes_B M$. Furthermore, one easily convinces oneself that $P^*_\bullet \otimes_B M$ is B^e-flat if M is right B-flat (taking into account that $P^*_\bullet$ is finitely generated projective over B^e). Hence taking homology in the above equation gives

$$\mathrm{Ext}^n_{B^e}(B, M) \simeq \mathrm{Tor}^{B^e}_{d-n}(\omega \otimes_B M, B)$$

as claimed. ∎

The point of our main result Theorem 1 is that the condition about $H^\bullet(B, B^e)$ can be replaced for quantum homogeneous spaces by the Cohen-Macaulay condition which is easier to check for concrete examples as we shall see below (it boils down to constructing resolutions of the B-module k rather than of the B-bimodule B). In the commutative case, Van den Bergh's condition is a global one concerned with the behaviour of the embedding of the corresponding space X as the diagonal into $X \times X$, while the Cohen-Macaulay condition in Theorem 1 is local in nature, dealing only with the local properties of X around the point corresponding to ε.

2.3. **Quantum homogeneous spaces.** We will freely use standard conventions and notations from Hopf algebra theory. In particular, we denote by Δ, ε, S the coproduct, counit and antipode of a co- or Hopf algebra and use Sweedler's notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for coproducts and $m \mapsto m_{(-1)} \otimes m_{(0)}$ and $n \mapsto n_{(0)} \otimes n_{(1)}$ for left and right coactions, see e.g. [23, 40].

We recall in this section from [31, 34, 41] various characterisations of the right faithful flatness of a Hopf algebra A over a right coideal subalgebra B that we use later. Some of them are given in terms of the left coaction

$$A \to C \otimes A, \quad a \mapsto a_{(-1)} \otimes a_{(0)} := \pi(a_{(1)}) \otimes a_{(2)},$$

where we write as in the introduction

$$\pi : A \to C := A/B^+A, \quad a \mapsto \pi(a) := a \bmod B^+A, \quad B^+ := B \cap \ker \varepsilon.$$

Yet others involve the categories $\mathcal{M}^C$ and ${}_B\mathcal{M}^A$ of right C-comodules and of (B, A)-Hopf modules, meaning left B-modules and right A-comodules M for which the coaction $M \to M \otimes A$ is B-linear if B acts on $M \otimes A$ via

$$b(m \otimes a) := b_{(1)}m \otimes b_{(2)}a \quad b \in B, m \in M, a \in A.$$

There are two functors relating these two categories. The first one is

$$ {}_B\mathcal{M}^A \to \mathcal{M}^C, \quad M \mapsto M/B^+M, \tag{8}$$

where the C-coaction on M/B^+M is induced by the A-coaction on M, and the second one is the cotensor product

$$\mathcal{M}^C \to {}_B\mathcal{M}^A, \quad N \mapsto N\Box_C A \tag{9}$$

given for $N \in \mathcal{M}^C$ with coaction $N \to N \otimes C$, $n \mapsto n_{(0)} \otimes n_{(1)}$ by

$$N\Box_C A := \{\sum_i n^i \otimes a^i \in N \otimes A \,|\, \sum_i n^i_{(0)} \otimes n^i_{(1)} \otimes a^i = \sum_i n^i \otimes a^i_{(-1)} \otimes a^i_{(0)}\}$$

on which the B-action and A-coaction are given by (co)multiplication in A.

The following is [31], Theorem 2.1 and [34], Theorem 1.2 and Remark 1.3:

Theorem 5. *Let A be a Hopf algebra with bijective antipode and $B \subset A$ be a right coideal subalgebra. Then the following are equivalent:*

1. *A is faithfully flat as a right module over B.*
2. *A is projective as a right B-module and there exists $B^\perp \subset A$ such that $A = B \oplus B^\perp$ as right B-module.*
3. *The functors (8) and (9) are (quasi)inverse equivalences.*
4. *A is left $C := A/B^+A$-coflat and we have*

$$B = k\Box_C A = \{b \in A \,|\, \pi(b_{(1)}) \otimes b_{(2)} = \pi(1) \otimes b\}.$$

If $AB^+ = B^+A$, then Remark 1.3 in [34] also tells that A is faithfully flat as a left module if it is faithfully flat as a right module.

Question 3. *Is this true in general?*

By Theorem 5 (4) a quantum homogeneous space can be recovered from $\pi : A \to C$ as $k\Box_C A$. Many examples are in fact defined in this way starting with π. This works in particular when C is cosemisimple (equals the direct sum of its simple subcoalgebras), see e.g. [34], Corollary 1.5:

Corollary 1. *Let A be a Hopf algebra with bijective antipode, $\pi : A \to C$ be a coalgebra and right A-module quotient, and assume that C is cosemisimple. Then $B := k\Box_C A \subset A$ is a quantum homogeneous space and $C \simeq A/B^+A$.*

For $C = k[H]$ cosemisimplicity means that H is reductive, so a quotient $H \setminus G$ of an algebraic group G by a reductive subgroup H is affine with coordinate ring B isomorphic to the ring of H-invariant regular functions on G (this is essentially the classical Matsushima-Onishchik theorem). A (B, A)-Hopf module $M \in {}_B\mathcal{M}^A$ is here isomorphic to the module of sections of the G-homogeneous vector bundle with typical fibre M/B^+M.

Later we will also use categories that we denote by $\mathcal{M}^C_{B,\tau}$ and by ${}_B\mathcal{M}^A_{B,\tau}$, where $\tau : B \to A$ is an algebra map. By the first we shall mean the category of right C-comodules and right B-modules N that satisfy

$$(nb)_{(0)} \otimes (nb)_{(1)} = n_{(0)}b_{(1)} \otimes n_{(1)}\tau(b_{(2)}), \tag{10}$$

where we use in the second tensor component on the right hand side the right A-action on A/B^+A. Similarly, objects in ${}_B\mathcal{M}^A_{B,\tau}$ are objects in ${}_B\mathcal{M}^A$ with an additional right B-action that commutes with the left one and satisfies (10), now being an equation in $M \otimes A$. Clearly, the equivalence ${}_B\mathcal{M}^A \simeq \mathcal{M}^C$ that holds in the faithfully flat case also induces ${}_B\mathcal{M}^A_{B,\tau} \simeq \mathcal{M}^C_{B,\tau}$.

2.4. $H^\bullet(B, M)$ **and** $\mathrm{Ext}^\bullet_B(k, \mathrm{ad}(M))$**.** Here we remark that the description of the Hochschild (co)homology of a Hopf algebra A used in [5] works almost as well for quantum homogeneous spaces $B \subset A$. The proof is the same as for $B = A$ [12, 15], we recall it only for the convenience of the reader:

Lemma 1. *Let A be a Hopf algebra, $B \subset A$ be a right coideal subalgebra, and M be a B-A-bimodule. Consider k as left B-module with action given by the counit ε of A, and let $\mathrm{ad}(M)$ be the left B-module which is M as vector space with left action given by the adjoint action $\mathrm{ad}(b)m := b_{(1)}mS(b_{(2)})$. Then there is a vector space isomorphism $H^\bullet(B, M) \simeq \mathrm{Ext}^\bullet_B(k, \mathrm{ad}(M))$.*

Proof. Compute $\mathrm{Ext}^\bullet_B(k, \mathrm{ad}(M))$ using the free resolution

$$\ldots \to B^{\otimes 3} \to B^{\otimes 2} \to B$$

of the B-module k whose boundary map is given by

$$b^0 \otimes \cdots \otimes b^n \mapsto \sum_{i=0}^{n-1} (-1)^i b^0 \otimes \cdots \otimes b^i b^{i+1} \otimes \cdots \otimes b^n + (-1)^n b^0 \otimes \cdots \otimes b^{n-1}\varepsilon(b^n).$$

After identifying B-linear maps $B^{\otimes n+1} \to M$ with k-linear maps $B^{\otimes n} \to M$ (fill the zeroth tensor component with $1 \in B$), this realises $\mathrm{Ext}^\bullet_B(k, \mathrm{ad}(M))$

as the cohomology of the cochain complex which as a vector space is

$$C^\bullet(B,M)=\mathrm{Hom}_k(B^{\otimes\bullet},M),$$

the standard Hochschild cochain complex, but whose coboundary map is

$$\begin{aligned}&\mathsf{d}\varphi(b^1,\ldots,b^{n+1})\\ =\;&\mathrm{ad}(b^1)\varphi(b^2,\ldots,b^{n+1})+\sum_{i=1}^{n}(-1)^i\varphi(b^1,\ldots,b^ib^{i+1},\ldots,b^{n+1})\\ &+(-1)^{n+1}\varphi(b^1,\ldots,b^n)\varepsilon(b^{n+1}).\end{aligned}$$

Now consider the k-linear isomorphism

$$\xi: C^\bullet(B,M)\to C^\bullet(B,M),\quad (\xi(\varphi))(b^1,\ldots,b^n):=\varphi(b^1_{(1)},\ldots,b^n_{(1)})b^1_{(2)}\cdots b^n_{(2)}$$

whose inverse is given by

$$(\xi^{-1}(\varphi))(b^1,\ldots,b^n):=\varphi(b^1_{(1)},\ldots,b^n_{(1)})S(b^1_{(2)}\cdots b^n_{(2)}).$$

Then $\mathsf{b}\circ\xi=\xi\circ\mathsf{d}$, where b is the standard Hochschild coboundary operator (5), so $(C^\bullet(B,M),\mathsf{d})\simeq(C^\bullet(B,M),\mathsf{b})$ as cochain complexes. □

One can apply Theorem VIII.3.1 from [6] to the map $B\to B\otimes A^{\mathrm{op}}$, $b\mapsto b_{(1)}\otimes S(b_{(2)})$ to show $\mathrm{Ext}^\bullet_{B\otimes A^{\mathrm{op}}}(A,M)\simeq\mathrm{Ext}^\bullet_B(k,\mathrm{ad}(M))$. When A is flat over B, then the same theorem applied to the obvious embedding of $B\otimes B^{\mathrm{op}}$ into $B\otimes A^{\mathrm{op}}$ also implies $\mathrm{Ext}^\bullet_{B\otimes A^{\mathrm{op}}}(A,M)\simeq H^\bullet(B,M)$ and hence the above lemma. We included the above proof since it does not require flatness. On the other hand, this seems to be a rather weak condition. It is always satisfied in the commutative case [31], note also the recent results of Skryabin [38]. For a counterexample see [37], Corollary 2.8 and Remark 2.9.

2.5. **The proof of Theorem 1.** To get Theorem 1 we only have to consider the special case $M=B\otimes A$ of Lemma 1 in more detail. We first recall:

Lemma 2. *Let R,S be rings, L be an R-module, M be an R-S-bimodule and N be an S-module. Then the canonical map*

$$\mathrm{Ext}^n_R(L,M)\otimes_S N\to\mathrm{Ext}^n_R(L,M\otimes_S N)$$

is bijective if N is flat and L admits a finitely generated projective resolution.

Proof. Fix a finitely generated projective resolution $P_\bullet\to L$. Then one has $\mathrm{Hom}_R(P_\bullet,M)\otimes_S N\simeq\mathrm{Hom}_R(P_\bullet,M\otimes_S N)$, see e.g. [3], Proposition 8.b) on p. 16. Now pass to cohomology taking into account that N is flat (see e.g. [ibid.], Corollary 2 on p. 74). □

This will be used with $R=M=B,S=L=k$ and $N=A$. For the assumption on $L=k$ we recall from [6]:

Lemma 3. *If B is an algebra over a field k and $P_\bullet\to B$ is a (finitely generated) projective resolution of B^e-modules, then $P_\bullet\otimes_B L$ is for any left B-module a (finitely generated) projective resolution of B-modules. In particular, one has for any algebra* $\mathrm{gl.dim}(B)\le\dim(B)$.

Proof. The complex $\ldots\to P_d\to\ldots\to P_0\to B\to 0$ is a flat resolution of the right B-module 0. Therefore, $H_\bullet(P\otimes_B L)\simeq\mathrm{Tor}^B_\bullet(0,L)=0$, so $P_\bullet\otimes_B L$ is and it consists of (finitely generated) projective left B-modules. □

Secondly, we need the following direct generalisation of the case $B = A$:

Lemma 4. *Let A be a Hopf algebra and $B \subset A$ be a right coideal subalgebra. Then the B-$B \otimes A^{\mathrm{op}}$-bimodule $B \otimes A$ with actions*

$$\mathrm{ad}(x)(b \otimes a)(y \otimes z) := x_{(1)}by \otimes zaS(x_{(2)})$$

is isomorphic to the B-$B \otimes A^{\mathrm{op}}$-bimodule $B \otimes A$ with actions

$$x(b \otimes a) \triangleleft (y \otimes z) := xby_{(1)} \otimes zaS^2(y_{(2)}).$$

Proof. The isomorphism is given explicitly by

$$\rho : B \otimes A \to B \otimes A, \quad b \otimes a \mapsto b_{(1)} \otimes aS^2(b_{(2)}).$$

Its inverse is given by

$$\rho^{-1} : b \otimes a \mapsto b_{(1)} \otimes aS(b_{(2)}),$$

and it follows straightforwardly from the Hopf algebra axioms that

$$\rho(x_{(1)}by \otimes zaS(x_{(2)})) = x\rho(b \otimes a) \triangleleft (y \otimes z). \qquad \square$$

Combining the lemmata gives:

Theorem 6. *Let $B \subset A$ be a right coideal subalgebra and consider $B \otimes A$ as a $B \otimes A^{\mathrm{op}}$-bimodule via multiplication in $B \otimes A^{\mathrm{op}}$. If the left B-module k admits a finitely generated projective resolution, then there is an isomorphism*

$$H^\bullet(B, B \otimes A) \simeq \mathrm{Ext}^\bullet_B(k, B) \otimes A$$

of right $B \otimes A^{\mathrm{op}}$-modules, where $\mathrm{Ext}^\bullet_B(k, B) \otimes A$ is a $B \otimes A^{\mathrm{op}}$-module via

$$([\varphi] \otimes a)(x \otimes y) := [\varphi]x_{(1)} \otimes yaS^2(x_{(2)}), \quad x \in B, [\varphi] \in \mathrm{Ext}^\bullet_B(k, B), a, y \in A$$

with the right B-action on $\mathrm{Ext}^\bullet_B(k, B)$ induced by right multiplication in B.

Proof. Apply Lemma 1 with $M = B \otimes A$. The cochain complexes and the isomorphisms ξ, ξ^{-1} defined in its proof are clearly right $B \otimes A^{\mathrm{op}}$-linear in this case, so the lemma gives a right $B \otimes A^{\mathrm{op}}$-module isomorphism

$$H^\bullet(B, B \otimes A) \simeq \mathrm{Ext}^\bullet_B(k, \mathrm{ad}(B \otimes A)), \tag{11}$$

where the right $B \otimes A^{\mathrm{op}}$-action on $\mathrm{Ext}^\bullet_B(k, \mathrm{ad}(B \otimes A)$ is induced by right multiplication in $B \otimes A^{\mathrm{op}}$ (which commutes with the left B-action given by ad). Now apply the Lemmata 4, 3 and 2 to get the isomorphisms

$$\mathrm{Ext}^\bullet_B(k, \mathrm{ad}(B \otimes A)) \simeq \mathrm{Ext}^\bullet_B(k, B \otimes A) \simeq \mathrm{Ext}^\bullet_B(k, B) \otimes A. \tag{12}$$

Composing these isomorphisms with (11) yields the claim. $\square$

Theorem 1 is an easy consequence:

Proof of Theorem 1. Theorem 5 gives a B-trimodule decomposition

$$B \otimes A \simeq B \otimes (B \oplus B^\perp) \simeq B^e \oplus (B \otimes B^\perp),$$

so we also have $H^n(B, B \otimes A) \simeq H^n(B, B^e) \oplus H^n(B, B \otimes B^\perp)$ as right B-modules. Theorem 6 and the Cohen-Macaulay assumption imply that $H^n(B, B^e) = 0$ for $n \neq \dim(B)$, so Theorem 1 follows from Theorem 4. $\square$

2.6. **ω as a Hopf bimodule.** The key step towards Theorem 2 is to turn ω into an object in ${}_B\mathcal{M}^A_{B,S^2}$. Recall that any right A-comodule N is via

$$X.n := n_{(0)}X(n_{(1)}), \quad X \in A^\circ, n \in N \tag{13}$$

a left module over the Hopf algebra A° of linear functionals on A that vanish on an ideal of finite codimension, see e.g. [40] for background. The A°-modules of this form are traditionally called rational. We define now an A°-action on $C^\bullet(B, B \otimes A)$ that restricts to $C^\bullet(B, B^e)$ and commutes with the coboundary operator b and therefore induces an A°-action on ω. While $C^\bullet(B, B^e)$ will not be rational in general we will prove afterwards that ω is.

In the definition of the searched for A°-action on $\varphi \in C^n(B, B \otimes A)$ we denote the canonical $A^\circ \otimes A^\circ$-action on $B \otimes A$ by

$$(X \otimes Y) \triangleright (x \otimes y) = X.x \otimes Y.y, \quad X, Y \in A^\circ, x \in B, y \in A,$$

where the actions of X, Y result as in (13) from the A-coactions given by the coproduct. This gets mixed with an action on the arguments of φ:

$$\begin{aligned}(X\varphi)(b^1, \ldots, b^n) := {} & (S^2(X_{(n+2)}) \otimes X_{(1)}) \triangleright \\ & \varphi(S(X_{(n+1)}).b^1, \ldots, S(X_{(2)}).b^n)).\end{aligned} \tag{14}$$

It follows from the Hopf algebra axioms that this defines a left A°-action, and in this way $C^\bullet(B, B \otimes A)$ becomes a cochain complex of A°-modules:

Lemma 5. *One has $\mathsf{b}(X\varphi) = X(\mathsf{b}\varphi)$ for all $X \in A^\circ, \varphi \in C^\bullet(B, B \otimes A)$.*

Proof. This is checked using that we have for $m \in B \otimes A, b, c \in B, X \in A^\circ$

$$\begin{aligned} &X.(bc) = (X_{(1)}.b)(X_{(2)}.c), \\ &(X \otimes 1) \triangleright (bm) = (X_{(1)}.b)((X_{(2)} \otimes 1) \triangleright m), \\ &(1 \otimes X) \triangleright (mb) = ((1 \otimes X_{(1)}) \triangleright m)(X_{(2)}.b). \end{aligned}$$

We demonstrate the claim in degree $n = 1$, the general case is analogous:

$$\begin{aligned}
&(X(\mathsf{b}\varphi))(b, c) \\
&= (S^2(X_{(4)}) \otimes X_{(1)}) \triangleright (\mathsf{b}\varphi(S(X_{(3)}).b, S(X_{(2)}).c)) \\
&= (S^2(X_{(4)}) \otimes X_{(1)}) \triangleright ((S(X_{(3)}).b)\varphi(S(X_{(2)}).c) \\
&\quad - (\varphi(S(X_{(3)}).b)(S(X_{(2)}).c)) + \varphi(S(X_{(3)}).b)(S(X_{(2)}).c)) \\
&= (S^2(X_{(4)}) \otimes X_{(1)}) \triangleright ((S(X_{(3)}).b)\varphi(S(X_{(2)}).c)) \\
&\quad - (S^2(X_{(4)}) \otimes X_{(1)}) \triangleright (\varphi((S(X_{(3)}).b)(S(X_{(2)}).c)))+ \\
&\quad (S^2(X_{(4)}) \otimes X_{(1)}) \triangleright (\varphi(S(X_{(3)}).b)(S(X_{(2)}).c)) \\
&= (S^2(X_{(4)})S(X_{(3)}).b)((S^2(X_{(5)}) \otimes X_{(1)}) \triangleright (\varphi(S(X_{(2)}).c))) \\
&\quad - (S^2(X_{(3)}) \otimes X_{(1)}) \triangleright (\varphi((S(X_{(2)})_{(1)}.b)(S(X_{(2)})_{(2)}.c)))+ \\
&\quad ((S^2(X_{(5)}) \otimes X_{(1)}) \triangleright (\varphi(S(X_{(4)}).b)))(X_{(2)}S(X_{(3)}).c) \\
&= b((S^2(X_{(3)}) \otimes X_{(1)}) \triangleright \varphi(S(X_{(2)}).c)) \\
&\quad - (S^2(X_{(3)}) \otimes X_{(1)}) \triangleright (\varphi(S(X_{(2)}).(bc)))+ \\
&\quad ((S^2(X_{(3)}) \otimes X_{(1)}) \triangleright (\varphi(S(X_{(2)}).b)))c \\
&= \mathsf{b}(X\varphi)(b, c).
\end{aligned}$$

∎

Furthermore, we obviously have:

Lemma 6. *For any right coideal subalgebra $B \subset A$, the canonical map $C^\bullet(B, B^e) \subset C^\bullet(B, B \otimes A)$ is an embedding of complexes of A°-modules.*

Thus we obtain an A°-action on $H^\bullet(B, B^e)$ and the canonical map to $H^\bullet(B, B \otimes A)$ is A°-linear. Our final aim is to prove that these two A°-modules are for a smooth and Gorenstein quantum homogeneous space rational, and that we indeed have $\omega \in {}_B\mathcal{M}^A_{B,S^2}$.

Lemma 7. *Let $B \subset A$ be a smooth right coideal subalgebra and assume $\varepsilon|_B$ is Gorenstein. Let $\chi : B \to k$ be the character defined by the right B-action on $\mathrm{Ext}^{\dim(B)}_B(k, B) \simeq k$ and define the k-algebra homomorphism*

$$\sigma : B \to A, \quad \sigma(x) := S^2(\chi(x_{(1)})x_{(2)}). \tag{15}$$

Then there are isomorphisms of A-B-bimodules and A°-modules

$$H^n(B, B \otimes A) \simeq \begin{cases} 0 & n \neq \dim(B), \\ A_\sigma & n = \dim(B), \end{cases}$$

where A° acts via the canonical action $X.a := a_{(1)}X(a_{(2)})$ on A_σ.

Proof. The claim about A-B-bimodules is a straightforward application of Theorem 6. One then has to transport the A°-action on $C^\bullet(B, B \otimes A)$ though the used isomorphisms: conjugating it by ξ from Lemma 1 gives an A°-action on the cochain complex $(C^\bullet(B, B \otimes A), \mathsf{d})$ that is given by

$$(X \blacktriangleright \varphi)(b^1, \ldots, b^n) := (S^2(X_{(2)}) \otimes X_{(1)}) \triangleright \varphi(b^1, \ldots, b^n),$$

so this action is entirely induced from an action on the coefficient bimodule. Conjugating this action with ρ from Lemma 4 gives the action

$$(X.\varphi)(b^1, \ldots, b^n) := (1 \otimes X) \triangleright \varphi(b^1, \ldots, b^n),$$

that is, in the identifications (11) and (12) in the proof of Theorem 6 the original A°-action on $H^{\dim(B)}(B, B \otimes A)$ induced by (14) is transformed into the one on $\mathrm{Ext}^{\dim(B)}_B(k, B) \otimes A$ where A° acts simply on the second tensor component A in the canonical way. $\square$

In particular, $H^{\dim(B)}(B, B \otimes A)$ is a rational A°-module, and hence so is any A°-submodule ([40], Theorem 2.1.3.a). Furthermore, A_σ and hence any B-subbimodule and A-subcomodule is an object in ${}_B\mathcal{M}^A_{B,S^2}$. This gives:

Corollary 2. *If $B \subset A$ is a smooth quantum homogeneous space and $\varepsilon|_B$ is Gorenstein, then $\omega = H^{\dim(B)}(B, B^e)$ becomes through the embedding*

$$\omega = H^{\dim(B)}(B, B^e) \to H^{\dim(B)}(B, B \otimes A) \simeq A_\sigma$$

an object in ${}_B\mathcal{M}^A_{B,S^2}$.

This allows us to describe ω finally as follows using the canonical projection $\pi : A \to C = A/B^+A$:

Theorem 7. *Let $B \subset A$ be a smooth quantum homogeneous for which $\varepsilon|_B$ is Gorenstein, and let χ be the character on B defined by its action on $\mathrm{Ext}^{\dim(B)}_B(k, B) \simeq k$. Then there exists a group-like $g \in C = A/B^+A$ with*

$$\omega \simeq \{a \in A_\sigma \mid \pi(a_{(1)}) \otimes a_{(2)} = g \otimes a\}, \quad \sigma(b) = \chi(b_{(1)})S^2(b_{(2)})$$

as an object of ${}_B\mathcal{M}^A_{B,S^2}$*, and we have for all* $b \in B$

$$g\sigma(b) = \chi(b)g, \tag{16}$$

where $g\sigma(b)$ *is defined using the right* A*-action on* $C = A/B^+A$*.*

Proof. Theorem 5 and the discussion at the end of Section 2.3 tell that $\omega \in {}_B\mathcal{M}^A_{B,S^2}$ is of the form $N\square_C A$ for some $N \in \mathcal{M}^C_{B,S^2}$. It follows that as a special case of (the proof of) Theorem 5.8 in [4] there are isomorphisms of A-B-bimodules

$$A \otimes_B \omega \simeq A \otimes_B (N\square_C A) \simeq N\square_C(A \otimes_B A) \simeq N\square_C(C \otimes A) \simeq N \otimes A,$$

where the left A-action on $N \otimes A$ is given by multiplication in A and the right B action on $N \otimes A$ is $(n \otimes a)b := nb_{(1)} \otimes S^2(b_{(2)})$. The second isomorphism (the mixed associativity of $\square_C$ and $\otimes_B$) uses the right flatness of A and the third is the Galois isomorphism for the algebra extension $B \subset A$ which is explicitly given by

$$A \otimes_B A \to C \otimes A, \quad x \otimes_B y \mapsto \pi(y_{(1)}) \otimes xy_{(2)}.$$

It follows that there is a right B-linear isomorphism

$$N \simeq (A \otimes_B \omega)/A^+(A \otimes_B \omega), \quad A^+ = \ker \varepsilon.$$

But we also have A-B-bimodule isomorphisms

$$A \otimes_B \omega = A \otimes_B \mathrm{Ext}^{\dim(B)}_{B^e}(B, B \otimes B) \simeq \mathrm{Ext}^{\dim(B)}_{B^e}(B, B \otimes A) \simeq A_\sigma$$

by Lemmata 2 and 7. Together this shows that as B-modules we have

$$N \simeq \mathrm{Ext}^{\dim(B)}_B(k, B),$$

and a coaction on the ground field is given by a group-like element $g \in C$ as

$$k \ni \lambda \mapsto \lambda \otimes g \in k \otimes C$$

that has to obey (16) in order to define an object in $\mathcal{M}^C_{B,S^2}$. The result follows now by the definition of $N\square_C A$. $\square$

2.7. **The Hopf-Galois case.** As we have recalled in the introduction, the assumption $B^+A = AB^+$ means that $\pi : A \to C = A/B^+A$ is a Hopf algebra quotient. Hence $\mathcal{M}^C$ is a monoidal category, where $M \otimes N$ is for $N, M \in \mathcal{M}^C$ the tensor product over k equipped with the coaction

$$m \otimes n \mapsto m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)}.$$

Furthermore, any $M \in \mathcal{M}^C$ is canonically an object in $\mathcal{M}^C_{B,\mathrm{id}}$ if B acts trivially (through ε) from the right. Hence $M\square_C A$ is canonically an object in ${}_B\mathcal{M}^A_B := {}_B\mathcal{M}^A_{B,\mathrm{id}}$ with B-bimodule structure

$$x(m \otimes a)y := m \otimes xay, \quad m \in M, a \in A, x, y \in B, \tag{17}$$

and with respect to this bimodule structure we have (this generalises to any faithfully flat Galois extension of an algebra B by a Hopf algebra C)

$$(M\square_C A) \otimes_B (N\square_C A) \simeq (M \otimes N)\square_C A \tag{18}$$

as B-bimodules. Any group-like element g of C is now invertible with inverse $g^{-1} = S(g)$, and Theorem 7 immediately gives:

Corollary 3. *Retain all assumptions and notation from Theorem 7 and assume in addition $AB^+ = B^+A$. Then $\sigma(B) \subset B$, and if we consider*

$$\bar{\omega} := \{a \in A \,|\, \pi(a_{(1)}) \otimes a_{(2)} = g^{-1} \otimes a\}$$

as a B-bimodule via (17), then we have a B-bimodule isomorphism

$$\bar{\omega} \otimes_B \omega \simeq B_\sigma.$$

Proof. Take in (18) for M the ground field k with the C-coaction given by $\lambda \mapsto \lambda \otimes g^{-1}$ and for N the same but with g instead of g^{-1}. Then we get $\bar{\omega} \otimes_B (N \Box_C A) \simeq B$ as B-bimodules. The B-bimodule ω is obtained from $N \Box_C A$ by twisting the right B-action by σ, so the claim follows. $\square$

The fact that $\sigma(B) \subset B$ is probably the most unexpected observation here. It illustrates how restrictive (16) is especially for $AB^+ = B^+A$ since it can in this case be multiplied from the left by g^{-1} to give

$$\pi(\sigma(b)) = \chi(b)\pi(1)$$

for all $b \in B$, and from this we indeed also compute directly that

$$\begin{aligned}
\pi(\sigma(b)_{(1)}) \otimes \sigma(b_{(2)}) &= \pi(\chi(b_{(1)})S^2(b_{(2)})) \otimes S^2(b_{(3)}) \\
&= \pi(\sigma(b_{(1)})) \otimes S^2(b_{(2)}) \\
&= \pi(1) \otimes \chi(b_{(1)})S^2(b_{(2)}) \\
&= \pi(1) \otimes \sigma(b),
\end{aligned}$$

hence $\sigma(b) \in B$ by Theorem 5.

We now want to show that in fact $\sigma(B) = B$. For this we need a small digression about characters and the following basic remark:

Lemma 8. *If $B \subset A$ is a quantum homogeneous space and $AB^+ = B^+A$, then we have $S^2(B) = B$.*

Proof. Koppinen's $S(AB^+) = B^+A$ ([34], Lemma 1.4) gives

$$S^2(B^+A) = S^2(AB^+) = S(B^+A) = S(AB^+) = B^+A$$

and hence for all $b \in B$

$$\pi(S^{\pm 2}(b)_{(1)}) \otimes S^{\pm 2}(b)_{(2)} = \pi(S^{\pm 2}(b_{(1)})) \otimes S^{\pm 2}(b_{(2)}) = \pi(1) \otimes S^{\pm 2}(b),$$

so $S^{\pm 2}(B) \subset B$ which implies $S^2(B) = B$. $\square$

2.8. **Remarks on characters.** For a Hopf algebra A the set $G := \mathrm{Char}(A)$ of characters (algebra homomorphisms $\gamma : A \to k$) is canonically an affine group scheme represented by the commutative Hopf algebra $A/J(A)$, where

$$J(A) := \{a \in A \,|\, \gamma(a) = 0 \,\forall\, \gamma \in \mathrm{Char}(A)\},$$

and for a right coideal subalgebra $B \subset A$ the set $X := \mathrm{Char}(B)$ becomes an affine G-scheme represented by $B/J(B)$. The (right) G-action on X is given like the group structure in G by the canonical product on $\mathrm{Hom}_k(A, k)$

$$(\varphi\psi)(a) := \varphi(a_{(1)})\psi(a_{(2)}), \quad \varphi, \psi \in \mathrm{Hom}_k(A, k), a \in A \tag{19}$$

for which ε is the unit element.

The inclusion $B \to A$ induces a homomorphism $B/J(B) \to A/J(A)$, and the restriction of a character from A to B is the dual morphism $G \to X$.

However, even for some well-behaved examples of quantum homogeneous spaces (such as the Podleś sphere that we will define in Section 3.1) the map $G \to X$ is not surjective, $B/J(B) \to A/J(A)$ is not faithfully flat and not even injective, and the G-action on X is not transitive.

But at least we can say the following:

Theorem 8. *If χ is a character on a quantum homogeneous space $B \subset A$, $\beta : A \to B$ is a right B-linear projection as in Theorem 5 (2) and we define*

$$\gamma : A \to k, \quad a \mapsto \chi(\beta(S^{-1}(a))),$$

then we have

$$\chi\gamma = \varepsilon$$

as functionals on B.

Proof. This follows by straightforward computation:

$$\begin{aligned} (\chi\gamma)(b) &= \chi(b_{(1)})\chi(\beta(S^{-1}(b_{(2)}))) = \chi(\beta(S^{-1}(b_{(2)})))\chi(b_{(1)}) \\ &= \chi(\beta(S^{-1}(b_{(2)}))b_{(1)}) = \chi(\beta(S^{-1}(b_{(2)})b_{(1)})) \\ &= \chi(\beta(\varepsilon(b))) = \varepsilon(b), \end{aligned}$$

where we used the properties of χ and β and the fact that in every Hopf algebra with bijective antipode we have

$$S^{-1}(a_{(2)})a_{(1)} = S^{-1}(S(S^{-1}(a_{(2)})a_{(1)})) = S^{-1}(a_{(1)}S(a_{(2)})) = \varepsilon(a)$$

for all $a \in A$ since S is always an algebra antihomomorphism. $\square$

Note that γ is in general not a character on A, though.

2.9. **The proof of Theorem 2.** Theorem 8 implies:

Corollary 4. *If χ is a character on a quantum homogeneous space $B \subset A$, then the algebra homomorphism $\sigma : B \to A$ given by*

$$\sigma(b) := \chi(b_{(1)})S^2(b_{(2)})$$

is injective. If $AB^+ = B^+A$ and $\sigma(B) \subset B$, then $\sigma(B) = B$.

Proof. An explicit left inverse of σ is given by

$$\sigma^{-1} : A \to A, \quad \sigma^{-1}(a) := \gamma(S^{-2}(a_{(1)}))S^{-2}(a_{(2)}),$$

where γ is as in Theorem 8. Under the additional assumption $AB^+ = B^+A$ we have $S^2(B) = B$ (Lemma 8), so $\sigma(B) \subset B$ implies

$$\hat{\sigma}(b) := \chi(b_{(1)})b_{(2)} = S^{-2}(\sigma(b)) \in B$$

for $b \in B$. Now abbreviate for a given $b \in B$

$$M := \{\varphi(b_{(1)})b_{(2)} \,|\, \varphi \in \mathrm{Hom}_k(B, k)\} \cap B.$$

We have

$$\hat{\sigma}(\varphi(b_{(1)})b_{(2)}) = \varphi(b_{(1)})\chi(b_{(2)})b_{(3)} = (\varphi\chi)(b_{(1)})b_{(2)} \in M,$$

that is, $\hat{\sigma}(M) \subset M$. Since σ and hence $\hat{\sigma}$ has been shown already to be injective, $\hat{\sigma}|_M$ is bijective since $\dim_k(M) < \infty$ (if $\Delta(b) = \sum_{-=0}^n x_i \otimes y_i$, then M is spanned by the y_i). Furthermore, $b = \varepsilon(b_{(1)})b_{(2)} \in M$, so $b \in \mathrm{im}\,\hat{\sigma}$. Thus $b \in \mathrm{im}\,\hat{\sigma}$ for arbitrary $b \in B$ and hence also $\sigma = S^2 \circ \hat{\sigma}$ is surjective. $\square$

Proof of Theorem 2. We have constructed in Corollary 3 a B-bimodule $\bar{\omega}$ with $\bar{\omega} \otimes_B \omega \simeq B_\sigma$, where $\sigma(b) = \chi(b_{(1)})S^2(b_{(2)})$. Corollary 4 shows that σ is under the assumptions of Theorem 2 an automorphism of B. This implies

$$B_\sigma \simeq {}_{\sigma^{-1}}B$$

as bimodules, where ${}_{\sigma^{-1}}B$ is B as right module but the left action is twisted by σ^{-1} (the isomorphism is given by σ^{-1}). Hence we get

$${}_\sigma\bar{\omega} \otimes_B \omega \simeq B.$$

To see that we also have

$$\omega \otimes_B {}_\sigma\bar{\omega} \simeq B$$

note that we know

$$\bar{\omega} \otimes_B \omega_{\sigma^{-1}} \simeq B,$$

and $\omega_{\sigma^{-1}} \in {}_B\mathcal{M}^A_B$. Applying the monoidal functor $M \mapsto M/B^+M$ gives the corresponding $g^{-1}g = 1$ in C, but here we also have $gg^{-1} = 1$. Retranslating this into Hopf bimodules yields

$$\omega \otimes_B {}_\sigma\bar{\omega} \simeq \omega_{\sigma^{-1}} \otimes \bar{\omega} \simeq B. \qquad \square$$

2.10. **A smoothness criterion.** We mention here a useful tool for proving the smoothness of $B \subset A$. The key remark is [33], Theorem 7.2.8:

Theorem 9. *Let $B \subset A$ be a ring extension such that B is a direct summand in A as a B-bimodule. Then* $\mathrm{gl.dim}(B) \leq \mathrm{gl.dim}(A) + \mathrm{proj.dim}_B(A)$.

Together with Theorem 5 this implies for example:

Corollary 5. *If $B \subset A$ is a quantum homogeneous space and A is commutative, then* $\mathrm{gl.dim}(B) \leq \mathrm{gl.dim}(A)$.

But also in many noncommutative examples it will happen that the decomposition in Theorem 5,(2) is actually a decomposition of bimodules:

Lemma 9. *Let $B \subset A$ be a quantum homogeneous space and assume that A/B^+A is cosemisimple with $AB^+ = B^+A$. Then* $\mathrm{gl.dim}(B) \leq \mathrm{gl.dim}(A)$.

Proof. As remarked above, the condition $AB^+ = B^+A$ means that $C = A/B^+A$ is a Hopf algebra quotient of A. The cosemisimplicity can be characterised as the existence of a (unique) functional $h : C \to k$ satisfying

$$h(1) = 1, \quad h(c_{(1)})c_{(2)} = c_{(1)}h(c_{(2)}) = h(c), \quad c \in C,$$

see e.g. [23], Theorem 13 in Section 11.2.1, and it is easily verified that

$$\beta : A \to A, \quad a \mapsto h(\pi(a_{(1)}))a_{(2)}$$

is then a B-bilinear projection from A onto $B \subset A$. $\square$

Note that the assumption of cosemisimplicity of C can be weakened, it suffices that there is a total integral $h : C \to A$ in the sense of [9] whose image commutes with $B \subset A$ as in [ibid.], Proposition (1.7)(b).

Corollary 6. *If $B \subset A$ is as in Lemma 9 and A^e is left Noetherian with* $\mathrm{gl.dim}(A^e) < \infty$, *then B is smooth.*

Proof. If $B \subset A$ is as in the lemma, then so is $B^e \subset A^e$, hence the lemma gives $\mathrm{gl.dim}(B^e) \leq \mathrm{gl.dim}(A^e)$. Therefore, $\mathrm{gl.dim}(A^e) < \infty$ implies that the left B^e-module B has finite projective dimension. Finally, the left Noetherianity of A^e implies that of B^e (apply $A^e \otimes_{B^e} \cdot$ to an ascending chain of left ideals in B^e and use faithful flatness). Therefore, the projective dimension of the finitely generated B^e-module B will coincide with its projective dimension in the category of finitely generated B^e-modules. □

2.11. **Some (counter)examples.** Before entering the discussion of the Podleś sphere let us mention here three simpler but instructive examples.

First of all, every Hopf subalgebra $B \subset A$ is in particular a right coideal subalgebra. If $A = U(\mathfrak{g})$ and $B = U(\mathfrak{h})$ are universal enveloping algebras of finite-dimensional Lie algebras $\mathfrak{h} \subset \mathfrak{g}$, then the Poincaré-Birkhoff-Witt theorem says that A is free over B and hence faithfully flat. However, even the basic example of the Borel subalgebra $\mathfrak{h} := \mathfrak{b}_+$ in $\mathfrak{g} := \mathfrak{sl}(2,k)$ behaves rather badly: the characters of $\mathfrak{b}_+$ are in bijection with k but only one of them (the counit) extends to A. The dualising bimodule of A is A without any twist σ, but that of B is of the form B_σ for a nontrivial automorphism (see [5], this example was suggested by Ken Brown to me). Note also that $AB^+ = B^+A$, but B is not as a B-bimodule a direct summand in A.

Secondly, consider $B = k[y]$ and for A the Hopf algebra obtained by adding a generator x satisfying

$$x^2 = 1, \quad xy = -yx,$$

so A is the smash (aka crossed or semidirect) product $B \rtimes \mathbb{Z}_2$ of B by the automorphism that sends y to $-y$. The Hopf algebra structure is given by

$$\Delta(x) = x \otimes x, \quad \Delta(y) = 1 \otimes y + y \otimes x,$$

$$\varepsilon(x) = 1, \quad \varepsilon(y) = 0, \quad S(x) = x, \quad S(y) = -yx.$$

The monomials $\{y^i, xy^i \mid i \geq 0\}$ form a k-vector space basis of A, so A is free over B with basis $\{1, x\}$. In particular, $B \subset A$ is faithfully flat. In this example one can verify directly that $H^i(B,M) \simeq H_{1-i}(B,M)$ for all B-bimodules M, and that B is Gorenstein with $\chi = \varepsilon$. However, $\sigma(b) = \chi(b_{(1)})S^2(b_{(2)}) = S^2(b)$ is not the identity automorphism since

$$S^2(y) = -S(yx) = -S(x)S(y) = xyx = -y.$$

These examples show that even if B satisfies Poincaré duality it can be difficult to read off ω from the Hopf-algebraic data given. In particular it can happen that the dualising bimodules of both A and B are of the form A_σ and B_τ, but one can have $\tau = \mathrm{id}_B, \sigma \neq \mathrm{id}_A$ or conversely $\sigma \neq \mathrm{id}_A, \tau = \mathrm{id}_B$.

Finally, we would like to mention that the cusp $X \subset k^2$ given by the equation $x^2 = y^3$ is also a quantum homogeneous space although it is surely not a homogeneous space of an algebraic group since it is not smooth. The ambient Hopf algebra is again a skew-polynomial ring $A = B \rtimes \mathbb{Z}$, $B = k[X]$, that is denoted by $B(1,1,2,3,q)$ in [14], Construction 1.2. Therein the notation is exactly the opposite of ours, their A is our B and vice versa.

3. Application

3.1. **The standard Podleś sphere.** For the rest of the paper we fix $k = \mathbb{C}$, $q \in k^*$ is not a root of unity, and A is the standard quantised coordinate ring of $SL(2, k)$ (see e.g. [23] for background information). This is the Hopf algebra with algebra generators a, b, c, d, defining relations

$$ab = qba, \quad ac = qca, \quad bc = cb, \quad bd = qdb, \quad cd = qdc,$$

$$ad - qbc = 1, \quad da - q^{-1}bc = 1$$

and the coproduct, counit, and antipode determined by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$

$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d,$$

$$\varepsilon(a) = \varepsilon(d) = 1, \quad \varepsilon(b) = \varepsilon(c) = 0,$$

$$S(a) = d, \quad S(b) = -q^{-1}b, \quad S(c) = -qc, \quad S(d) = a.$$

It follows from these relations that there is a unique Hopf algebra quotient

$$\pi : A \to C := k[z, z^{-1}], \quad \pi(a) = z, \quad \pi(d) = z^{-1}, \quad \pi(b) = \pi(c) = 0,$$

where the Hopf algebra structure of $k[z, z^{-1}]$ is determined by $\Delta(z) = z \otimes z$, that is, C is the coordinate ring of $T = k^*$, and the map π would correspond for $q = 1$ to the embedding of T as a maximal torus into $SL(2, k)$.

By Corollary 1, π gives rise to a quantum homogeneous space B as in (3). This subalgebra deforms the coordinate ring of $T \setminus SL(2, k)$ and was discovered by Podleś [35] and hence is referred to by most authors as the (standard) Podleś quantum sphere. The elements

$$y_{-1} := ca, \quad y_0 := bc, \quad y_1 := bd$$

generate B as an algebra, and B can be characterised abstractly as the algebra with three generators y_{-1}, y_0, y_1 and defining relations

$$y_0 y_{\pm 1} = q^{\pm 2} y_{\pm 1} y_0, \quad y_{\pm 1} y_{\mp 1} = q^{\mp 2} y_0^2 + q^{\mp 1} y_0, \tag{20}$$

see [8, 30, 35].

3.2. **The Koszul resolution of the B-module k.** We will construct a free resolution of the B-module k (with action given by ε) by using the probably simplest case of Priddy's noncommutative Koszul resolutions [36]:

Lemma 10. *Let B be a k-algebra and assume $z_{\pm 1} \in B$ are such that*

(1) $z_{-1} z_1 = \lambda z_1 z_{-1}$ *for some* $\lambda \in k$,
(2) $a z_{-1} = 0$ *implies* $a = 0$ *for all* $a \in B$ *and*
(3) $\nu(a z_1) = 0$ *implies* $\nu(a) = 0$ *for all* $\nu(a) := a \bmod B z_{-1} \in B / B z_{-1}$.

Then the chain complex $K_\bullet := K_\bullet(z_1, z_{-1})$ given by

$$0 \to B \to B \oplus B \to B \to 0$$

with nontrivial boundary maps

$$a \mapsto (a z_{-1}, -\lambda a z_1), \quad (b, c) \mapsto b z_1 + c z_{-1}$$

is a free resolution of B/I, $I := B z_1 + B z_{-1}$.

Proof. We clearly have $H_0(K) = B/I$ by very definition and $H_2(K) = 0$ by assumption (2). Now consider the subcomplex

$$\tilde{K} := 0 \to B \to Bz_{-1} \oplus B \to Bz_{-1} \to 0$$

of K and the quotient complex $K/\tilde{K}$ which is of the form

$$0 \to 0 \to B/Bz_{-1} \to B/Bz_{-1} \to 0.$$

Its one nontrivial boundary map map is

$$B/Bz_{-1} \to B/Bz_{-1}, \quad \nu(a) \mapsto \nu(az_1),$$

so assumption (3) means $H_1(K/\tilde{K}) = 0$. Furthermore, we have $H_1(\tilde{K}) = 0$: a cycle is an element $(bz_{-1}, c) \in Bz_{-1} \oplus B$ with

$$0 = bz_{-1}z_1 + cz_{-1} = (\lambda bz_1 + c)z_{-1},$$

so assumption (2) gives $c = -\lambda bz_1$, hence $(bz_{-1}, c) = (bz_{-1}, -\lambda bz_1)$ is a boundary. Considering the long exact homology sequence derived from the short exact sequence $0 \to \tilde{K} \to K \to K/\tilde{K} \to 0$ now yields $H_1(K) = 0$. □

From now on let B be again the Podleś sphere. Then the above gives:

Theorem 10. *The left B-module k admits a free resolution of the form $K_\bullet(z_1, z_{-1})$ with $z_{\pm 1} := y_{\pm 1} + y_0$, $\lambda := q^2$.*

Proof. It is easily seen that $B^+ := B \cap \ker \varepsilon$ is generated as a left ideal by the elements y_n. But since one has $q^{-1}y_{-1}(y_1 + y_0) - qy_0(y_{-1} + y_0) = y_0$, B^+ is in fact generated as a left ideal by the two elements $z_{\pm 1}$.

One verifies directly that $z_{-1}z_1 = q^2 z_1 z_{-1}$ which is assumption (1) in Lemma 10. Secondly, B is a domain (see e.g. [1]), so assumption (2) holds as well. For (3) we turn B into a $\mathbb{Z}$-graded algebra by assigning to y_i the degree i which is compatible with the defining relations (20). Then we have

$$B = \bigoplus_{j \in \mathbb{Z}} B_j, \quad B_i B_j \subset B_{i+j}, \quad B_j := \mathrm{span}_k\{e_{ij} \mid i \geq 0\},$$

where

$$e_{ij} := \begin{cases} y_0^i y_1^j & j \geq 0, \\ y_0^i y_{-1}^{-j} & j < 0, \end{cases} \quad i \in \mathbb{N}_0, j \in \mathbb{Z},$$

and these form a vector space basis of B. Under $\nu : B \to B/Bz_{-1}$ we have

$$\begin{aligned}
\nu(y_0^i y_{-1}^j) &= y_0^i y_{-1}^{j-1} \nu(y_{-1}) = -y_0^i y_{-1}^{j-1} \nu(y_0) \\
&= -y_0^i \nu(y_{-1}^{j-1} y_0) = -q^{2(j-1)} y_0^i \nu(y_0 y_{-1}^{j-1}) \\
&= -q^{2(j-1)} y_0^{i+1} y_{-1}^{j-2} \nu(y_{-1}) \\
&= q^{2(j-1)} y_0^{i+1} y_{-1}^{j-2} \nu(y_0) \\
&= q^{2(2j-1-2)} y_0^{i+2} y_{-1}^{j-3} \nu(y_{-1}) \\
&= -q^{2(2j-1-2)} y_0^{i+2} y_{-1}^{j-3} \nu(y_0) \\
&= \ldots \\
&= (-1)^j q^{2((j-1)j-1-2-\ldots-(j-1))} y_0^{i+j-1} y_{-1}^{j-j} \nu(y_0) \\
&= (-1)^j q^{(j-1)j} \nu(y_0^{i+j}).
\end{aligned}$$

Similarly we have for $i > 0, j > 0$

$$\begin{aligned}
\nu(y_0^i y_1^j) &= q^{2j} y_0^{i-1} \nu(y_1^j y_0) = q^{2j} y_0^{i-1} y_1^j \nu(y_{-1}) \\
&= q^{2j} y_0^{i-1} y_1^{j-1} \nu(q^{-2} y_0^2 + q^{-1} y_0) \\
&= q^{2j-2} y_0^{i-1} y_1^{j-1} \nu(y_0^2) + q^{2j-1} y_0^{i-1} y_1^{j-1} \nu(y_0) \\
&= q^{-2j+2} \nu(y_0^{i+1} y_1^{j-1}) + q\nu(y_0^i y_1^{j-1}) \\
&= q^{-2j+2}(q^{-2j+4}\nu(y_0^{i+2} y_1^{j-2}) + q\nu(y_0^{i+1} y_1^{j-2})) \\
&\quad + q(q^{-2j+4}\nu(y_0^{i+1} y_1^{j-2}) + q\nu(y_0^i y_1^{j-2})) \\
&= q^{-4j+6}\nu(y_0^{i+2} y_1^{j-2}) + q^{-2j+3}(1+q^2)\nu(y_0^{i+1} y_1^{j-2}) + q^2\nu(y_0^i y_1^{j-2}) \\
&= q^{-4j+6}(q^{-2j+6}\nu(y_0^{i+3} y_1^{j-3}) + q\nu(y_0^{i+2} y_1^{j-3})) \\
&\quad + q^{-2j+3}(1+q^2)(q^{-2j+6}\nu(y_0^{i+2} y_1^{j-3}) + q\nu(y_0^{i+1} y_1^{j-3})) \\
&\quad + q^2(q^{-2j+6}\nu(y_0^{i+1} y_1^{j-3}) + q\nu(y_0^i y_1^{j-3})) \\
&= q^{-6j+12}\nu(y_0^{i+3} y_1^{j-3}) + q^{-4j+7}(1+q^2+q^4)\nu(y_0^{i+2} y_1^{j-3}) \\
&\quad + q^{-2j+4}(1+q^2+q^4)\nu(y_0^{i+1} y_1^{j-3}) + q^3\nu(y_0^i y_1^{j-3}) \\
&= \dots \\
&= \sum_{r=0}^{j} q^{(-2r+1)j+r^2} \begin{pmatrix} j \\ r \end{pmatrix}_q \nu(y_0^{i+r})
\end{aligned}$$

where we abbreviated

$$\begin{pmatrix} j \\ r \end{pmatrix}_q := 1 + q^2 + q^4 + \dots + q^{2\begin{pmatrix} j \\ r \end{pmatrix} - 2}.$$

Thus we have

$$B/Bz_{-1} = \mathrm{span}_k\{\nu(y_0^{i+1}), \nu(y_1^i) \,|\, i \geq 0\}.$$

These residue classes are also linearly independent: assume that

$$\sum_{i\geq 0} \lambda_i \nu(y_0^i) + \sum_{j\geq 0} \mu_j \nu(y_1^{j+1}) = 0 \tag{21}$$

in B/Bz_{-1}. One easily checks that

$$B/(Bz_{-1} + By_0) = B/(By_{-1} + By_0)$$

is an algebra quotient of B (i.e. that $By_0 + By_{-1}$ is a two-sided ideal in B) and that it is as such isomorphic to the polynomial ring generated by the residue class of y_1. Hence the residue classes of y_1^j are linearly independent in this quotient of B/Bz_{-1}. Considering the image of (21) therein thus gives

$$\mu_j = 0 \quad \forall\, j \geq 0.$$

We are left with

$$\sum_{i\geq 0} \lambda_i \nu(y_0^i) = 0 \quad \Leftrightarrow \quad \sum_{i\geq 0} \lambda_i y_0^i = az_{-1} \tag{22}$$

for some $a \in B$. But $\sum_{i\geq 0} \lambda_i y_0^i$ is homogeneous of degree 0, B is a domain, and z_{-1} is not homogeneous, so the right hand side can not be homogeneous unless $a = 0$: if

$$a = a_{j_0} + \ldots + a_{j_n}, \quad a_{j_i} \in B_{j_i} \setminus \{0\}, \quad j_0 < \ldots < j_n$$

is the decomposition of a into homogeneous components, then az_{-1} has a nonzero component $a_{j_0}y_{-1}$ in degree $j_0 - 1$ and a nonzero component $a_{j_n}y_0$ in degree j_n. Thus $a = 0$ and since the y_0^i are linearly independent in B it follows that also

$$\lambda_i = 0 \quad \forall\, i \geq 0.$$

Now we compute the action of he map

$$\zeta : B/Bz_{-1} \to B/Bz_{-1}, \quad \nu(a) \mapsto \nu(az_1) \tag{23}$$

on the basis vectors. We get for $i > 0$

$$\begin{aligned} \nu(y_0^i z_1) &= \nu(y_0^i y_1) + \nu(y_0^{i+1}) \\ &= q^2 y_0^{i-1} \nu(q^{-2} y_0^2 + q^{-1} y_0) + \nu(y_0^{i+1}) \\ &= 2\nu(y_0^{i+1}) + q\nu(y_0^i) \end{aligned}$$

and for $j \geq 0$

$$\begin{aligned} \nu(y_1^j z_1) &= \nu(y_1^{j+1}) + \nu(y_1^j y_0) \\ &= \nu(y_1^{j+1}) + q^{-2j} \nu(y_0 y_1^j) \\ &= \nu(y_1^{j+1}) + q^{-2j} \sum_{r=0}^{j} q^{(-2r+1)j + r^2} \left(\begin{array}{c} j \\ r \end{array} \right)_q \nu(y_0^{1+r}). \end{aligned}$$

So if we abbreviate

$$V_j := \mathrm{span}_k \{\nu(y_0), \ldots, \nu(y_0^{j+1}), \nu(1), \nu(y_1), \ldots, \nu(y_1^j)\},$$

then we have

$$B = \bigcup_{j \geq 0} V_j, \quad \zeta(V_j) \subset V_{j+1}$$

and $\zeta|_{V_j}$ is represented with respect to our basis by a matrix of the form

$$\left(\begin{array}{cccccc} q & & & * & \ldots & * \\ 2 & \ddots & & & \ddots & \vdots \\ & \ddots & q & & & * \\ & & 2 & 0 & \ldots & 0 \\ & & & 0 & & \\ & & & 1 & \ddots & \\ & & & & \ddots & 0 \\ & & & & & 1 \end{array} \right),$$

where the $*$ denote nonzero entries and all other entries vanish. Hence ζ is evidently injective (composing $\zeta|_{V_j}$ with the canonical projection onto $V_{j+1}/\mathrm{span}_k\{\nu(y_0), \nu(1)\}$ yields an isomorphism of determinant 2^j) which is assumption (3) of Lemma 10. $\square$

3.3. **The Gorenstein condition.** From the minimal resolution of k provided by the Koszul complex one can read off $\mathrm{Ext}^n_B(k,B)$:

Lemma 11. *One has* $\mathrm{Ext}^n_B(k,B)=0$ *for* $n\neq 2$ *and* $\mathrm{Ext}^2_B(k,B)\simeq k$. *The resulting character* χ *of* B *is equal to* ε.

Proof. Apply $\mathrm{Hom}_B(\cdot,B)$ to the Koszul complex and identify $\mathrm{Hom}_B(B,B)\simeq B$. This gives the cochain complex

$$0\leftarrow B\leftarrow B\oplus B\leftarrow B\leftarrow 0$$

of right B-modules whose two nontrivial coboundary maps are given by

$$f\mapsto (z_1f,z_{-1}f),\quad (f,g)\mapsto q^{-1}z_{-1}f-qz_1g.$$

The exactness of this complex in degree 0 and 1 can be shown as the exactness of the Koszul complex using Lemma 10 (with B replaced by B^{op}). In degree 2, the cohomology is B divided by the right ideal generated by $z_{\pm 1}$. The result follows since with $qz_1y_{-1}-q^{-1}z_{-1}y_0=y_0$ one easily deduces that this ideal is again $\ker\varepsilon$. ☐

Thus the relevant twisting automorphism is

$$\sigma(f)=\chi(f_{(1)})S^2(f_{(2)})=S^2(f)$$

which is explicitly given by

$$\sigma(y_{-1})=q^2y_{-1},\quad \sigma(y_0)=y_0,\quad \sigma(y_1)=q^{-2}y_1.$$

Note this is also the restriction of Woronowicz's modular automorphism (see e.g. [23, 16] for more information) to B.

3.4. **The smoothness condition.** The smoothness of B follows from Corollary 6 since for this example $A^e\simeq k_q[SL(2)\times SL(2)]$ is left Noetherian with global dimension 4, see [13] and the references therein.

3.5. **Determining** ω**.** We now know that Theorem 2 applies to B with $\dim(B)=2$, and that B acts trivially (via ε) on $\mathrm{Ext}^2_B(k,B)\simeq k$ so that the automorphism σ from Theorem 7 equals S^2. Equation (16) becomes trivial, so g therein could be any of the group-like elements in $C=k[z,z^{-1}]$, that is, an arbitrary monomial z^n for some $n\in\mathbb{Z}$. So according to Theorem 7, ω is isomorphic as an object of ${}_B\mathcal{M}^A_{B,S^2}$ to $\omega_{n,1}$, where we define for $m,n\in\mathbb{Z}$

$$\omega_{n,m}:=\{a\in A_{S^{2m}}\mid \pi(a_{(1)})\otimes a_{(2)}=z^n\otimes a\}\in {}_B\mathcal{M}^A_{B,S^{2m}}.$$

As B-bimodules, we have isomorphisms

$$\omega_{n,m}\simeq(\omega_{n,0})_{S^{2m}}\simeq(\omega_{n,0})\otimes_B B_{S^{2m}}=\omega_{n,0}\otimes_B\omega_{0,m}$$

and (as a special case of (18))

$$\omega_{l,0}\otimes_B\omega_{n,0}\simeq\omega_{l+n,0}.$$

Finally, the B-bimodule isomorphism $S^{2m}:{}_{S^{-2m}}A\to A_{S^{2m}}$ restricts to a B-bimodule isomorphism

$${}_{S^{-2m}}(\omega_{n,0})\simeq\omega_{n,m},$$

and combining these three equations we see that as B-bimodules we have

$$\omega_{n,m}\otimes_B\omega_{i,j}\simeq\omega_{n+i,m+j}. \tag{24}$$

Furthermore, we obtain by direct computation:

Lemma 12. *One has*

$$H^0(B, \omega_{i,j}) \simeq \begin{cases} k & i = 2(m-j) \text{ for some } 0 \le m \le 2j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The defining relations of A imply that the monomials

$$f_{lmn} := \begin{cases} a^l b^m c^n & l \ge 0, \\ d^{-l} b^m c^n & l < 0, \end{cases} \quad l \in \mathbb{Z}, m, n \in \mathbb{N}_0$$

form a vector space basis, that A is a $\mathbb{Z}$-graded B-bimodule,

$$A = \bigoplus_{l \in \mathbb{Z}} A_l, \quad B_i A_l B_j \subset A_{i+j+l}, \quad A_l := \mathrm{span}_k\{f_{lmn} \,|\, m, n \ge 0\},$$

and that

$$A_l = \{f \in A \,|\, y_0 f = q^{2l} f y_0\}.$$

Furthermore, the explicit formulas for the coproduct of the generators a, b, c, d given in Section 3.1 and the fact that $\pi(f_{lmn}) = \delta_{m,0}\delta_{n,0} z^l$ show that

$$\omega_{i,j} = \mathrm{span}_k\{f_{lmn} \,|\, i = l + m - n\} \subset A_{S^{2j}}.$$

The zeroth Hochschild cohomology is by very definition isomorphic to the centre of the coefficient bimodule (identify $\varphi \in \mathrm{Hom}_{B^e}(B, M)$ with $\varphi(1) \in M$), and since $S^2(y_0) = y_0$, the last two equations imply

$$H^0(B, \omega_{i,j}) \subset \omega_{i.j} \cap A_0 = \mathrm{span}_k\{b^m c^n \,|\, m - n = i\}.$$

And from $S^2(y_{\pm 1}) = q^{\mp 2} y_{\pm 1}$ and

$$y_{\pm 1} b^m c^n = q^{\mp(m+n)} b^m c^n y_{\pm 1}$$

we deduce now that

$$H^0(B, \omega_{i,j}) = \mathrm{span}_k\{b^m c^n \,|\, m - n = i, m + n = 2j\}.$$

The claim follows by elementary arithmetics. □

Now we can finish the proof of Theorem 3:

End of proof of Theorem 3. By Theorem 2 and Hadfield's explicit computation of $H_\bullet(B, B_{S^2})$ [16] we have

$$\begin{aligned} H^0(B, (\omega^{-1})_{S^2}) &\simeq H^0(B, \omega^{-1} \otimes_B B_{S^2}) \\ &\simeq H_2(B, \omega \otimes_B \omega^{-1} \otimes_B B_{S^2}) \\ &\simeq H_2(B, B_{S^2}) \simeq k. \end{aligned} \tag{25}$$

If $\omega = \omega_{n,1}$, then by (24) we have

$$(\omega^{-1})_{S^2} \simeq \omega_{-n,0},$$

and inserting this into Lemma 12 yields $n = 0$, so we have $\omega \simeq \omega_{0,1} = B_{S^2}$ as claimed in Theorem 3. □

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