A Lie-Rinehart algebra with no antipode

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Abstract

The aim of this note is to communicate a simple example of a Lie-Rinehart algebra whose enveloping algebra is not a Hopf algebroid in the sense of Böhm and Szlachányi.

1 Introduction

The enveloping algebra of a Lie algebra is a classical example of a Hopf algebra. Hence it is natural to ask whether the enveloping algebra of a Lie algebroid [Pra67] or more generally of a Lie-Rinehart algebra [Rin63] carries the structure of a Hopf algebroid. It turns out that they always are *left bialgebroids* (introduced under the name $\times_R$-bialgebras by Takeuchi [Tak77]), see [Xu01], and in fact *left Hopf algebroids* (introduced under the name $\times_R$-Hopf algebras by Schauenburg [Sch00]), see [KK10, Example 2]; see also [Hue08, MM10].

However, the question whether these left Hopf algebroids are *full Hopf algebroids* in the sense of [Böh09] (generalising the notion from [Lu96]) remained open. In the light of [KP11, Proposition 3.11], this is known to be true for Lie algebroids [ELW99] and for the Lie-Rinehart algebras associated to Poisson algebras [Hue98, Section (3.2)]. A counterexample was announced by Kowalzig and the first author, see [KP11, Remark 3.12], but the construction contained a gap. To our knowledge, the literature still contains no example of a left Hopf algebroid that is not a full one. Hence the aim of the present note is to communicate such an example:

Theorem 1.1. *Let K be a field, $R := K[x,y]/\langle x \cdot y, x^2, y^2 \rangle$, L be the 1-dimensional Lie algebra with basis $\{\alpha\}$ and $E \in \mathrm{Der}_K(R)$ be the derivation with $E(x) = y, E(y) = 0$.*

1. *There is a Lie-Rinehart algebra structure on (R, L) with R-module structure on L given by $x \cdot \alpha = y \cdot \alpha = 0$ and anchor map given by $\rho(\alpha) = E$.*
2. *There is no right $V(R, L)$-module structure on R that extends right multiplication in R. In particular, $V(R, L)$ is not a full Hopf algebroid.*

The note is structured as follows: in Section 2 we recall some basic definitions. In Section 3 we provide a construction method of Lie-Rinehart algebras whose enveloping algebras do not admit an antipode. The simplest example of these is the one in our theorem. Lastly, Section 4 illustrates the result by giving an explicit presentation of $V(R, L)$ for our example in which the nonexistence of an antipode becomes evident.

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2 Background

This section contains background on Lie-Rinehart algebras [Rin63], see also [Hue98, Kow09, KP11, MM10] for more information. For the corresponding differential geometric notion of a Lie algebroid see [Pra67] and for example [Mac87] for further details.

We fix a field K. An unadorned $\otimes$ denotes the tensor product of K-vector spaces.

Definition 2.1. *A Lie-Rinehart algebra consists of*

1. *a commutative K-algebra $(R, \cdot)$,*
2. *a Lie algebra $(L, [-,-]_L)$ over K,*
3. *a left R-module structure $R \otimes L \to L$, $r \otimes \xi \mapsto r \cdot \xi$, $r \in R, \xi \in L$, and*
4. *an R-linear Lie algebra homomorphism $\rho : L \to \mathrm{Der}_K(R)$ satisfying*

$$[\xi, r \cdot \zeta]_L = r \cdot [\xi, \zeta]_L + \rho(\xi)(r) \cdot \zeta, \quad r \in R, \xi, \zeta \in L. \tag{2.1}$$

The map ρ is referred to as the anchor map.

There are two fundamental examples: if R is any commutative algebra, one can take L to be $\mathrm{Der}_K(R)$ with its usual Lie algebra and R-module structure, and $\rho = \mathrm{id}$. The other extreme is $R = K$ and $\rho = 0$, L being any Lie algebra.

In his paper [Rin63], Rinehart generalised the construction of the universal enveloping algebra of a Lie algebra to Lie-Rinehart algebras, see Section 2 therein for the precise construction. The result is an associative K-algebra $V(R, L)$ that is generated by the (sum of the) images of a K-algebra map

$$R \longrightarrow V(R, L)$$

and a Lie algebra map

$$(L, [-,-]_L) \longrightarrow (V(R, L), [-,-]), \quad \xi \longmapsto \bar{\xi}$$

where $[-,-]$ denotes the commutator in $V(R, L)$. As Rinehart, we do not distinguish between an element in R and its image in $V(R, L)$ which is justified as the first map is always injective. The construction is such that in $V(R, L)$ one has for all $r \in R, \xi \in L$

$$[\bar{\xi}, r] = \rho(\xi)(r), \quad r\bar{\xi} = \overline{r \cdot \xi}, \tag{2.2}$$

where the product in $V(R, L)$ is denoted by concatenation.

As indicated in the introduction, $V(R, L)$ has the structure of a left Hopf algebroid. Its counit endows R with the structure of a left $V(R, L)$-module, in such a way that the induced action of $r \in R$ is given by left multiplication, and the induced action of $\xi \in L$ is given by the anchor map. For a full Hopf algebroid, composing the counit with the antipode yields also a right $V(R, L)$-module structure on the base algebra R extending right multiplication in R, see [KP11, Proposition 3.11] for full details. Thus the nonexistence of such a right module structure on the base algebra R indeed implies the nonexistence of an antipode.

3 Proof of Theorem 1.1

We now prove Theorem 1.1. We begin by considering more generally Lie-Rinehart algebras (R, L) whose R-module structure on L is given by a character $\chi : R \to K$.

Proposition 3.1. *Let $(R, \cdot)$ be a commutative K-algebra, $(L, [-,-]_L)$ be a Lie algebra and $\rho : L \to \mathrm{Der}_K(R)$ be a Lie algebra map. Define an R-module structure on L by $r \cdot \xi := \chi(r)\xi$, where $\chi : R \to K$ is a character on R. Then (R, L) is a Lie-Rinehart algebra if and only if ρ is R-linear and $\rho(\xi)(r) \in \ker \chi$ for all $r \in R, \xi \in L$.*

Proof. This follows as the Leibniz rule (2.1) takes the form

$$[\xi, \chi(r)\zeta]_L = \chi(r)[\xi, \zeta]_L + \chi(\rho(\xi)(r))\zeta$$

and hence by the K-linearity of the bracket becomes equivalent to $\rho(\xi)(r) \in \ker\chi$. □

Note that for these examples, $[-.-]_L$ is even R-linear, so L is a Lie algebra over R. However, in general we have $\rho \neq 0$.

Assume now that (R, L) is a Lie-Rinehart algebra as in the above proposition, and that right multiplication in R can be extended to a right $V(R, L)$-module structure on R. Denote by $\partial(\xi) \in R$ the element obtained by acting with $\xi \in L$ on $1 \in R$ under this right action. This defines a K-linear map $\partial : L \to R$, and in $V(R, L)$ we have

$$\rho(\xi)(r) = [\bar{\xi}, r] = \bar{\xi}r - r\bar{\xi} = \bar{\xi}r - \overline{r \cdot \xi} = \bar{\xi}r - \chi(r)\bar{\xi},$$

so by acting with this element on $1 \in R$, one sees that this map ∂ satisfies

$$\rho(\xi)(r) = \partial(\xi) \cdot (r - \chi(r)). \tag{3.1}$$

A K-linear map ∂ with this property defines a right $V(R, L)$-module structure extending multiplication on R if and only if it satisfies the condition $\partial([\xi, \zeta]_L) = \rho(\xi)(\partial(\zeta)) - \rho(\zeta)(\partial(\xi))$. It also corresponds to a generator of the Gerstenhaber bracket on $\Lambda_R L$, see [Hue98], but we shall not need these facts:

Proof of Theorem 1.1. The first part is verified by explicit computation; the Lie-Rinehart algebra is of the form as in Proposition 3.1 with χ given by $\chi(x) = \chi(y) = 0$.

For 2., take $r = x$ and $\xi = \alpha$ in (3.1). One obtains $y = E(x) = \rho(\alpha)(x) = \partial(\alpha) \cdot x$. However, there is no element $z \in R$ such that $y = z \cdot x$. □

4 A Hopf algebroid without antipode

Carrying out Rinehart's construction explicitly yields a presentation of the associative K-algebra $V(R, L)$ in terms of generators $x, y, \bar{\alpha}$ satisfying the relations

$$\bar{\alpha}x = y, \quad \bar{\alpha}y = x\bar{\alpha} = y\bar{\alpha} = x^2 = y^2 = xy = yx = 0.$$

Hence $V(R, L)$ has a K-linear basis given by $\{\bar{\alpha}^n, x, y\}_{n \in \mathbb{N}}$.

In view of Axiom (iii) in [Böh09, Definition 4.1], the antipode S of any Hopf algebroid H over R satisfies $S(t(r)) = s(r)$ where $s, t : R \to V(R, L)$ are the source and the target map of the underlying left bialgebroid, respectively. For the left bialgebroid $V(R, L)$, these are both the inclusion of R into $V(R, L)$, hence an antipode on $V(R, L)$ would satisfy $S(x) = x$, $S(y) = y$.

However, the antipode of a Hopf algebroid is an algebra antihomomorphism, $S(gh) = S(h)S(g)$ for all $g, h \in H$, see e.g. [Böh09, Proposition 4.4 (i)]. So in $V(R, L)$, one would have

$$y = S(y) = S(\bar{\alpha}x) = S(x)S(\bar{\alpha}) = xS(\bar{\alpha}).$$

This illustrates directly that $V(R, L)$ admits no antipode, since there is no element $z \in V(R, L)$ such that $y = xz$.

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