Twisted homology of the quantum $SL(2)$

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Abstract

We calculate the twisted Hochschild and cyclic homology of the quantum group $SL_q(2)$ relative to a specific family of automorphisms. Our calculations are based on the free resolution of $SL_q(2)$ due to Masuda, Nakagami and Watanabe.

1 Introduction

Cyclic homology and cohomology were discovered by Alain Connes (and independently by Boris Tsygan) in the early 1980's [Co85], and should be thought of as extensions of de Rham (co)homology to various categories of noncommutative algebras.

Quantum groups also appeared in various guises from the early 1980's onwards, with the first example of a "compact quantum group" in the C*-algebraic setting being Woronowicz's "quantum $SU(2)$" [Wo87a]. The noncommutative differential geometry (in the sense of Connes) of the quantum $SU(2)$ was thoroughly investigated by Masuda, Nakagami and Watanabe in their paper [MNW90]. They first calculated the Hochschild and cyclic homology and cohomology of the underlying algebra of the quantum $SL(2)$, and then extended this work to the topological setting of the unital C*-algebra of "continuous functions on the compact quantum group $SU_q(2)$", in addition finding the K-theory and K-homology of this C*-algebra. In particular they found an explicit free left resolution of quantum $SL(2)$, which we rely on for the main calculations of this paper.

Twisted cyclic cohomology was discovered by Kustermans, Murphy and Tuset [KMT03], arising naturally from the study of covariant differential calculi over compact quantum groups. Given an algebra $\mathcal{A}$, and an automorphism σ, they defined a cohomology theory relative to the pair $(\mathcal{A}, \sigma)$, which on taking σ to be the identity reduces to the ordinary cyclic cohomology of $\mathcal{A}$. Viewed this way, twisted cyclic cohomology generalizes the very simplest and most concrete formulation of cyclic cohomology (as described, for example, in [Co85] p. 317-323), however it was immediately recognised that it fits happily within Connes' much more general framework of cyclic objects [Co83].

The aim of the present paper is to compute these homologies for $SL_q(2)$ and all automorphisms of the form $x, y, u, v \mapsto \lambda x, \lambda^{-1} y, \rho u, \rho^{-1} v$, where x, y, u, v are the standard generators. It turns out that there exist values of λ, ρ for which the twisted Hochschild dimension becomes the classical dimension 3 of $SL(2)$ (but none for which it exceeds 3). That is, the twisted theory is able to avoid the 'dimension drop' of standard Hochschild homology. Similar effects were observed for the Podleś quantum sphere and quantum hyperplanes [Ha04], [Si04], [SW03a], [SW03b]. To give an overview of the results, we collect the dimensions of the twisted Hochschild homology groups $HH_n(\mathcal{A}, \sigma)$ as a k-vector space:

ρ, λ (with $a, b \geq 0$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n > 3$
$q^{\pm(a+1)}, q^{b+1}$	2	4	2	0	0
$1, q^{b+2}$	∞	∞	$b+1$	$b+1$	0
$q^{\pm(a+1)}, \notin q^{\mathbf{N}}$ or $\notin q^{\mathbf{Z}}, \neq 1$	0	0	0	0	0
otherwise	∞	∞	0	0	0

A summary of this paper is as follows. Section 2 contains preliminaries, and recalls the definitions of [KMT03] in a homological setting. In section 3 we define the underlying algebras of the quantum group $SL_q(2)$ and the compact quantum group $SU_q(2)$. Algebra automorphisms of $\mathcal{A}(SL_q(2))$ fall into two families, each of which is parameterised by two nonzero elements of the underlying field. In section 4 we use the free left resolution of $SL_q(2)$, due to Masuda, Watanabe and Nakagami [MNW90], to calculate the Hochschild homology $H_*(\mathcal{A}, {}_\sigma\mathcal{A})$ of $\mathcal{A}(SL_q(2))$ relative to the first family of automorphisms. We then prove that in this situation, these groups in fact coincide with the K-M-T twisted Hochschild homology groups $HH_*(\mathcal{A}, \sigma)$. This allows us to use the long exact S-B-I sequences of [KMT03] to calculate the twisted cyclic homology (section 5).

The calculations that appear here for the Masuda-Nakagami-Watanabe resolution were done by the first author. The same results were independently obtained by the second author using a simpler resolution based on [FT91], and the details of this will appear shortly.

2 Twisted Hochschild and cyclic homology

Although throughout this paper we work in the setting of homology, the motivation and definitions of [KMT03] arose in the cohomological setting.

2.1 Twisted cyclic cohomology

Twisted cyclic cohomology arose from the study of covariant differential calculi over compact quantum groups. This is very clearly explained in [KMT03].

Let $\mathcal{A}$ be an algebra over $\mathbf{C}$. Given a differential calculus (Ω, d) over $\mathcal{A}$, with $\Omega = \oplus_{n=0}^{N} \Omega_n$, Connes considered linear functionals $\int : \Omega_N \to \mathbf{C}$, which are closed and graded traces on Ω, meaning

$$\int d\omega = 0 \quad \forall\, \omega \in \Omega \tag{1}$$

$$\int \omega_m \omega_n = (-1)^{mn} \int \omega_n \omega_m \quad \forall\, \omega_m \in \Omega_m,\ \omega_n \in \Omega_n \tag{2}$$

Connes found that such linear functionals are in one to one correspondence with cyclic N-cocycles τ on the algebra, via

$$\tau(a_0, a_1, \ldots, a_N) = \int a_0\, da_1\, da_2 \ldots da_N \tag{3}$$

and this led directly to his simplest formulation of cyclic cohomology [Co85].

In the theory of differential calculi over compact quantum groups, as developed by Woronowicz [Wo87a], [Wo87b], the algebra $\mathcal{A}$ is now equipped with a comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, and the appropriate differential calculi to study are covariant. A left-covariant differential calculus over $(\mathcal{A}, \Delta)$ is a differential calculus (Ω, d) equipped with a left coaction

$$\Delta_L : \Omega \to \mathcal{A} \otimes \Omega \tag{4}$$

satisfying certain relations. For compact quantum groups the natural linear functionals $\int : \Omega_N \to \mathbf{C}$ are no longer graded traces, but instead twisted graded traces, meaning that

$$\int \omega_m \omega_n = (-1)^{mn} \int \sigma(\omega_n) \omega_m \quad \forall\, \omega_m \in \Omega_m,\ \omega_n \in \Omega_n \tag{5}$$

for some degree zero automorphism σ of Ω. In particular, σ restricts to an automorphism of $\mathcal{A}$, and, for any $a \in \mathcal{A}$, $\omega_N \in \Omega_N$ we have

$$\int \omega_N a = \int \sigma(a) \omega_N \tag{6}$$

Hence for each left covariant calculus there is a natural automorphism of $\mathcal{A}$.

Motivated by this observation, Kustermans, Murphy and Tuset defined "twisted" Hochschild and cyclic cohomology for any pair $(\mathcal{A}, \sigma)$ of an algebra $\mathcal{A}$ and automorphism σ. We will transpose their definitions to the setting of homology. We note that the definitions in [KMT03] were given over $\mathbf{C}$, however extend immediately to arbitrary fields k (we always assume characteristic zero). We also note that the definition of twisted Hochschild and cyclic homology we give was not explicitly written down in [KMT03], but was obviously well-understood.

2.2 Twisted Hochschild and cyclic homology

Let ${}_\sigma\mathcal{A}$ be the $\mathcal{A}$-bimodule which is $\mathcal{A}$ as a vector space with left and right action defined by $a \triangleright b \triangleleft c := \sigma(a)bc,\ a, b, c \in \mathcal{A}$.

We denote by $H_*(\mathcal{A}, {}_\sigma\mathcal{A})$ the Hochschild homology of $\mathcal{A}$ with coefficients in ${}_\sigma\mathcal{A}$, that is, the homology of the complex $\{C_n, b_\sigma\}_{n\geq 0}$, where

$$C_n := \mathcal{A}^{\otimes(n+1)}$$

and $b_\sigma : C_n \to C_{n-1}$ is the linear map defined by

$$\begin{aligned} b_\sigma(a_0, \cdots, a_n) &:= \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) \\ &\quad +(-1)^n (\sigma(a_n)a_0, \ldots, a_{n-1}) \\ &=: \sum_{i=0}^{n} (-1)^i b_i(a_0, \ldots, a_n), \end{aligned}$$

where we denote $a_0 \otimes \cdots \otimes a_n$ by $(a_0, \ldots, a_n)$.

Note that modifying as well the right multiplication of $\mathcal{A}$ by inserting another automorphism σ' yields a complex which is isomorphic to one of the above form with σ replaced by $(\sigma')^{-1} \circ \sigma$.

It was noticed in [KMT03] that in this situation, there is an analogue of the cyclic permuter of cyclic homology [Co85]. Indeed, if one defines the linear map

$$\lambda_\sigma : C_n \to C_n, \quad \lambda_\sigma(a_0, \ldots, a_n) := (-1)^n (\sigma(a_n), a_0, a_1, \ldots, a_{n-1}),$$

then

$$(1 - \lambda_\sigma) \circ b'_\sigma = b_\sigma \circ (1 - \lambda_\sigma), \quad b'_\sigma := \sum_{i=0}^{n-1} (-1)^i b_i.$$

Hence the cokernels of $1 - \lambda_\sigma$ form a well-defined quotient complex of $\{C_n, b_\sigma\}$ whose homology $HC_*(\mathcal{A}, \sigma)$ was called twisted cyclic homology in [KMT03].

The main difference to the standard theory with $\sigma = \mathrm{id}$ is that

$$\lambda_\sigma^{n+1} = \sigma \otimes \cdots \otimes \sigma \neq \mathrm{id}_{C_n}.$$

That is, $(C_*, b_\sigma, \lambda_\sigma)$ does not define a cyclic object. To obtain this, we have to replace C_n by the cokernel of $1 - \lambda_\sigma^{n+1}$. The homology of this quotient complex was called twisted Hochschild homology $HH_*(\mathcal{A}, \sigma)$ in [KMT03]. As a consequence of the general theory of cyclic homology theories, $HC_*(\mathcal{A}, \sigma)$ can be computed from $HH_*(\mathcal{A}, \sigma)$ by the direct analogue of Connes' S-B-I sequence.

In our application, we will consider only diagonalizable automorphisms of $\mathcal{A}$. For such ones, we have

$$C_n = C_n^{\mathrm{inv}} \oplus (1 - \sigma)C_n,$$

where $C_n^{\mathrm{inv}} := \ker(1 - \lambda_\sigma^{n+1})$ is the eigenspace of λ_σ^{n+1} corresponding to the eigenvalue 1 and $(1 - \sigma)C_n$ is the sum of the other eigenspaces.

Since λ_σ^{n+1} commutes with b_σ, we see that the above decomposition is a decomposition of complexes. This proves:

Proposition 2.1 *If σ acts diagonally, then there is an isomorphism $HH_*(\mathcal{A},\sigma) \simeq H_*(C_n^{\text{inv}}, b_\sigma)$. Furthermore, the resulting map $HH_*(\mathcal{A},\sigma) \to H_*(\mathcal{A}, {}_\sigma\mathcal{A})$ is an embedding of vector spaces.*

Recall [Lo98] that $H_*(\mathcal{A}, {}_\sigma\mathcal{A}) \simeq \mathrm{Tor}_*^{\mathcal{A}^e}({}_\sigma\mathcal{A}, \mathcal{A})$, where $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{op}$. Hence $H_*(\mathcal{A}, {}_\sigma\mathcal{A})$ can be computed from any projective resolution of the $\mathcal{A}^e$-module $\mathcal{A}$. This will be applied in section 4 for the standard quantum group $\mathcal{A} = \mathcal{A}(SL_q(2))$ and all automorphisms given by rescaling the standard generators by a nonzero scalar. The computations will be based on the free resolution of $\mathcal{A}$ found in [MNW90]. It turns out that the embedding $HH_*(\mathcal{A},\sigma) \to H_*(\mathcal{A}, {}_\sigma\mathcal{A})$ is in this case an isomorphism.

3 Quantum $SL(2)$ and quantum $SU(2)$

We follow the notation of Masuda, Nakagami and Watanabe [MNW90]. Let k be a field of characteristic zero, and $q \in k$ some nonzero parameter, which we assume is not a root of unity (this assumption is also made in [MNW90]).

We define the coordinate ring $\mathcal{A}(SL_q(2))$ of the quantum group $SL_q(2)$ over k to be the k-algebra generated by symbols x, y, u, v subject to the relations

$$ux = qxu, \quad vx = qxv, \quad yu = quy, \quad yv = qvy, \quad vu = uv \tag{7}$$

$$xy - q^{-1}uv = 1, \quad yx - quv = 1 \tag{8}$$

Hence a Poincaré-Birkhoff-Witt basis for $\mathcal{A}(SL_q(2))$ consists of the monomials

$$\{x^l u^m v^n\}_{l,m,n\geq 0} \quad , \quad \{y^{l+1} u^m v^n\}_{l,m,n\geq 0} \tag{9}$$

It is well-known how to equip this algebra with the structure of a Hopf algebra, but this will play no role in the sequel.

Specializing to the case $k = \mathbf{C}$, we define a *-structure:

$$x^* = y, \quad y^* = x, \quad v^* = -qu, \quad u^* = -q^{-1}v \tag{10}$$

where we now assume that q is real, and $0 < q < 1$. Writing $\alpha = y$, $\beta = u$, we find that the relations (7), (8) become

$$\alpha^*\alpha + \beta^*\beta = 1, \quad \alpha\alpha^* + q^2\beta^*\beta = 1 \tag{11}$$

$$\beta^*\beta = \beta\beta^*, \quad \alpha\beta = q\beta\alpha, \quad \alpha\beta^* = q\beta^*\alpha \tag{12}$$

We define $\mathcal{A}_f(SU_q(2))$ to be the unital *-algebra over $\mathbf{C}$ (algebraically) generated by elements α, β satisfying the relations (11), (12), and the unital C*-algebra $\mathcal{A}(SU_q(2))$ of "continuous functions on the quantum $SU(2)$", to be the C*-algebraic completion of $\mathcal{A}_f$.

Returning to $\mathcal{A} = \mathcal{A}(SL_q(2))$, we define $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{op}$, where $\mathcal{A}^{op}$ is the opposite algebra of $\mathcal{A}$. Masuda, Nakagami and Watanabe gave an explicit resolution of $\mathcal{A}$,

$$\ldots \to \mathcal{M}_{n+1} \to \mathcal{M}_n \to \ldots \to \mathcal{M}_2 \to \mathcal{M}_1 \to \mathcal{M}_0 \to \mathcal{A} \to 0 \tag{13}$$

by free left $\mathcal{A}^e$-modules $\mathcal{M}_n$, with

$$\mathrm{rank}(\mathcal{M}_0) = 1, \quad \mathrm{rank}(\mathcal{M}_1) = 4, \quad \mathrm{rank}(\mathcal{M}_2) = 7,$$
$$\mathrm{rank}(\mathcal{M}_n) = 8, \quad n \geq 3 \tag{14}$$

In section 4 we will use this resolution to calculate the twisted Hochschild homology of $\mathcal{A}(SL_q(2))$.

3.1 Comparison of the M-N-W and bar resolutions

Recall [Lo98], p12 the bar resolution

$$\ldots \to \mathcal{A}^{\otimes(n+2)} \to^{b'} \mathcal{A}^{\otimes(n+1)} \to \ldots \to \mathcal{A}^{\otimes 2} \to^{b'} \mathcal{A} \to 0 \tag{15}$$

which is a projective resolution of $\mathcal{A}$ as a left $\mathcal{A}^e$-module. We recall that each $\mathcal{A}^{\otimes(n+2)}$ is a left $\mathcal{A}^e$-module via

$$(x \otimes y^o)(a_0, a_1, \ldots a_{n+1}) = (xa_0, a_1, \ldots, a_{n+1}y) \tag{16}$$

and

$$b'(a_0, a_1, \ldots a_{n+1}) = \Sigma_{j=0}^{n} (-1)^j (a_0, \ldots a_j a_{j+1} \ldots, a_{n+1}) \tag{17}$$

We have a commutative diagram

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & \mathcal{M}_3 & \xrightarrow{d_3} & \mathcal{M}_2 & \xrightarrow{d_2} & \mathcal{M}_1 & \xrightarrow{d_2} & \mathcal{M}_0 & \xrightarrow{d_1} & \mathcal{A} & \longrightarrow & 0 \\
 & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & {\scriptstyle \cong}\downarrow & & \\
\ldots & \longrightarrow & \mathcal{A}^{\otimes 5} & \xrightarrow{b'} & \mathcal{A}^{\otimes 4} & \xrightarrow{b'} & \mathcal{A}^{\otimes 3} & \xrightarrow{b'} & \mathcal{A}^{\otimes 2} & \xrightarrow{b'} & \mathcal{A} & \longrightarrow & 0
\end{array}$$

The vertical maps $f_i : \mathcal{M}_i \to \mathcal{A}^{\otimes(i+2)}$ satisfy $b' f_{i+1} = d_{i+1} f_i$, and are given by:

$$f_0(a_1 \otimes {a_2}^o) = (a_1, a_2) \tag{18}$$

$\mathcal{M}_1$ is a free left $\mathcal{A}^e$-module of rank 4, with basis $\{e_v, e_u, e_x, e_y\}$. We have:

$$f_1(e_t) = (1, t, 1) \quad t = u, v, x, y \tag{19}$$

$\mathcal{M}_2$ is a free left $\mathcal{A}^e$-module of rank 7, with basis $\{(e_v \wedge e_u)$, $(e_v \wedge e_x)$, $(e_v \wedge e_y)$, $(e_u \wedge e_x)$, $(e_u \wedge e_y)$, $\vartheta_S^{(1)}$, $\vartheta_T^{(1)}\}$. We have:

$$\begin{aligned}
f_2(e_v \wedge e_u) &= (1, v, u, 1) - (1, u, v, 1) \\
f_2(e_v \wedge e_x) &= (1, v, x, 1) - q(1, x, v, 1) \\
f_2(e_v \wedge e_y) &= q(1, v, y, 1) - (1, y, v, 1) \\
f_2(e_u \wedge e_x) &= (1, u, x, 1) - q(1, x, u, 1) \\
f_2(e_u \wedge e_y) &= q(1, u, y, 1) - (1, y, u, 1) \\
f_2(\vartheta_S^{(1)}) &= (1, y, x, 1) - q(1, u, v, 1) + (1, 1, 1, 1) \\
f_2(\vartheta_T^{(1)}) &= (1, x, y, 1) - q^{-1}(1, u, v, 1) + (1, 1, 1, 1)
\end{aligned} \tag{20}$$

$\mathcal{M}_3$ is a free left $\mathcal{A}^e$-module of rank 8. We will only need:

$$\begin{aligned}
f_3(e_v \wedge \vartheta_T^{(1)}) &= (1, v, x, y, 1) + (1, x, y, v, 1) - q(1, x, v, y, 1) \\
&\quad -q^{-1}(1, v, u, v, 1) + (1, v, 1, 1, 1) + (1, 1, 1, v, 1) \\
f_3(e_v \wedge \vartheta_S^{(1)}) &= (1, v, y, x, 1) + (1, y, x, v, 1) - q(1, v, u, v, 1) \\
&\quad - q^{-1}(1, y, v, x, 1) + (1, v, 1, 1, 1) + (1, 1, 1, v, 1)
\end{aligned} \tag{21}$$

Applying the $A_\sigma \otimes_{\mathcal{A}^e} -$ functor to both resolutions allows us to identify the generators of twisted Hochschild homology found from the M-N-W resolution with explicit cycles in the bar resolution.

3.2 Automorphisms of $\mathcal{A}(SL_q(2))$

Given nonzero λ, $\rho \in k$, we define automorphisms σ, τ of $\mathcal{A}(SL_q(2))$ by

$$\sigma(x) = \lambda x, \quad \sigma(y) = \lambda^{-1} y, \quad \sigma(u) = \rho^{-1} u, \quad \sigma(v) = \rho v \tag{22}$$

$$\tau(x) = \lambda x, \quad \tau(y) = \lambda^{-1} y, \quad \tau(u) = \rho v, \quad \tau(v) = \rho^{-1} u \tag{23}$$

Proposition 3.1 *Every automorphism of $\mathcal{A}(SL_q(2))$ is of this form.*

In this paper we will work exclusively with the automorphisms σ. We note that:

1. From (6) the automorphism σ associated to Woronowicz's left-covariant three dimensional calculus over $\mathcal{A}(SU_q(2))$ is [KMT03], p22:

$$\sigma(\alpha) = q^{-2}\alpha, \quad \sigma(\alpha^*) = q^2\alpha^*, \quad \sigma(\beta) = q^{-4}\beta, \quad \sigma(\beta^*) = q^4\beta^* \tag{24}$$

   Here we are working over $\mathbf{C}$, with $0 < q < 1$, but the analogous automorphism of $\mathcal{A}(SL_q(2))$

$$\sigma(x) = q^2 x, \quad \sigma(y) = q^{-2} y, \quad \sigma(v) = q^4 v, \quad \sigma(u) = q^{-4} u \tag{25}$$

   makes sense for any field k and nonzero q.

2. Similarly, the automorphism associated to both the four dimensional calculi over $SU_q(2)$ is [K]:

$$\sigma(\alpha) = q^2\alpha, \quad \sigma(\alpha^*) = q^{-2}\alpha^*, \quad \sigma(\beta) = \beta, \quad \sigma(\beta^*) = \beta^* \tag{26}$$

   Again, we have an analogous automorphism of $\mathcal{A}(SL_q(2))$

$$\sigma(x) = q^{-2} x, \quad \sigma(y) = q^2 y, \quad \sigma(v) = v, \quad \sigma(u) = u \tag{27}$$

   that makes sense for any field k and nonzero q.

4 Twisted Hochschild homology of $\mathcal{A}(SL_q(2))$

We will now calculate the twisted Hochschild homology and cohomology of $\mathcal{A} = \mathcal{A}(SL_q(2))$ relative to the automorphisms σ (22). We consider twelve different cases:

1. $\rho = 1$, $\lambda = 1$ ($\sigma = \mathrm{id}$).
2. $\rho = 1$, $\lambda = q$.
3. $\rho = 1$, $\lambda = q^{b+2}$, $b \geq 0$.
4. $\rho = 1$, $\lambda \notin q^{\mathbf{N}}$.
5. $\rho = q^{a+1}$, $a \geq 0$, $\lambda = 1$.
6. $\rho = q^{a+1}$, $\lambda = q^{b+1}$, a, $b \geq 0$.
7. $\rho = q^{a+1}$, $a \geq 0$, $\lambda \notin q^{\mathbf{N}}$

8. $\rho = q^{-(a+1)}$, $a \geq 0$, $\lambda = 1$.

9. $\rho = q^{-(a+1)}$, $\lambda = q^{b+1}$, $a, b \geq 0$

10. $\rho = q^{-(a+1)}$, $\lambda \notin q^{\mathbf{N}}$.

11. $\rho \notin q^{\mathbf{Z}}$, $\lambda = 1$.

12. $\rho \notin q^{\mathbf{Z}}$, $\lambda \neq 1$.

4.1 $H_0(\mathcal{A}, {}_\sigma\mathcal{A})$

Proposition 4.1 *We give an explicit description of $H_0(\mathcal{A}, {}_\sigma\mathcal{A})$ for all possible σ. Following the scheme above, we find that:*

1. *In cases 7, 10 and 12 $H_0(\mathcal{A}, {}_\sigma\mathcal{A})$ is trivial,*

2. *In cases 6 and 9 $H_0(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k^2$,*

3. *In cases 1 - 5, 8 and 11 $H_0(\mathcal{A}, {}_\sigma\mathcal{A})$ is a countably infinite dimensional k-vector space.*

In each case we exhibit a basis.

Proof: We have $H_0(\mathcal{A}, {}_\sigma\mathcal{A}) = \{ [a] \ : \ a \in \mathcal{A}, \ [a_1 a_2] = [\sigma(a_2) a_1] \}$. So for all l, m, $n \geq 0$, we have

$$[x^l u^{m+1} v^n] = [\sigma(u) x^l u^m v^n] = \rho^{-1} q^l [x^l u^{m+1} v^n] \tag{28}$$

$$[x^l u^m v^{n+1}] = [\sigma(v) x^l u^m v^n] = \rho q^l [x^l u^m v^{n+1}] \tag{29}$$

$$[y^l u^{m+1} v^n] = [\sigma(u) y^l u^m v^n] = \rho^{-1} q^{-l} [y^l u^{m+1} v^n] \tag{30}$$

$$[y^l u^m v^{n+1}] = [\sigma(v) y^l u^m v^n] = \rho q^{-l} [y^l u^m v^{n+1}] \tag{31}$$

We note straight away that $[x^{l+1} u^{m+1} v^{n+1}] = 0 = [y^{l+1} u^{m+1} v^{n+1}]$.

If $\rho \notin q^{\mathbf{Z}}$, the only potentially nonzero classes are $[x^l]$, $[y^l]$ for all $l \geq 0$. Now,

$$[x^{l+1}] = [\sigma(x) x^l] = \lambda [x^{l+1}] \tag{32}$$

$$[y^{l+1}] = [\sigma(y) y^l] = \lambda^{-1} [y^{l+1}] \tag{33}$$

$$[1] = [xy - q^{-1} uv] = [xy] = [\sigma(y) x] = \lambda^{-1} [yx] = \lambda^{-1} [1 + quv] = \lambda^{-1} [1] \tag{34}$$

using the fact that $[uv] = [\sigma(v) u] = \rho [uv]$, hence $[uv] = 0$ since $\rho \neq 1$.

Case 12. $\rho \notin q^{\mathbf{Z}}$, $\lambda \neq 1$. Then $[x^l u^m v^n] = 0 = [y^l u^m v^n]$ for all l, m, $n \geq 0$. Hence

$$H_0(\mathcal{A}, {}_\sigma\mathcal{A}) = 0 \tag{35}$$

Case 11. $\rho \notin q^{\mathbf{Z}}$, $\lambda = 1$. Then

$$H_0(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k[1] \oplus (\Sigma^{\oplus}_{l>0} \ k[x^l]) \oplus (\Sigma^{\oplus}_{l>0} \ k[y^l]) \tag{36}$$

Now, if $\rho = 1$, then

$$[x^{l+1} u^m v^n] = 0 = [y^{l+1} u^m v^n] \quad \forall \, l \geq 0, \ m + n \geq 1 \tag{37}$$

$$
\begin{aligned}
[u^m v^n] = [(xy - q^{-1}uv)u^m v^n] &= q^{m+n}[\sigma(y)xu^m v^n] - q^{-1}[u^{m+1}v^{n+1}] \\
&= \lambda^{-1}q^{m+n}[yxu^m v^n] - q^{-1}[u^{m+1}v^{n+1}] \\
&= \lambda^{-1}q^{m+n}[(1+quv)u^m v^n] - q^{-1}[u^{m+1}v^{n+1}]
\end{aligned}
$$

Hence

$$
(\lambda - q^{m+n})[u^m v^n] = (-q^{-1})(\lambda - q^{m+n+2})[u^{m+1}v^{n+1}] \tag{38}
$$

Define $f(t) = \lambda - q^t$, for $t \in \mathbf{Z}$. Then

$$
f(m+n)[u^m v^n] = (-q^{-1})^s f(m+n+2s)[u^{m+s}v^{n+s}] \quad \forall\, m,n,s \geq 0 \tag{39}
$$

So if $m \geq n$, we have

$$
f(m-n)[u^{m-n}] = (-q^{-1})^n f(m+n)[u^m v^n] \tag{40}
$$

whereas if $m \leq n$, we have

$$
f(n-m)[v^{n-m}] = (-q^{-1})^m f(m+n)[u^m v^n] \tag{41}
$$

It follows from (32), (33), (37), (40), (41) that:

Case 4. If $\rho = 1$, $\lambda \notin q^{\mathbf{N}}$,

$$
H_0(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k[1] \oplus (\Sigma^{\oplus}_{m>0}\, k[u^m]) \oplus (\Sigma^{\oplus}_{n>0}\, k[v^n]) \tag{42}
$$

Case 1. $\rho = 1$, $\lambda = 1$ ($\sigma = \mathrm{id}$).

$$
H_0(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k[1] \oplus (\Sigma^{\oplus}_{l>0}\, k[x^l]) \oplus (\Sigma^{\oplus}_{l>0}\, k[y^l]) \oplus (\Sigma^{\oplus}_{m>0}\, k[u^m]) \oplus (\Sigma^{\oplus}_{n>0}\, k[v^n]) \tag{43}
$$

Cases 2 and 3. $\rho = 1$, $\lambda = q^{b+1}$, $b \geq 0$. For m, $n \geq 0$, the classes $[1]$, $[u^{m+1}]$, $[v^{n+1}]$ are all independent and nonvanishing, apart from (40), (41) those of the form $[u^{b+1-2s}]$, $[v^{b+1-2s}]$ for $s \geq 1$, $b+1-2s \geq 0$. Hence just as in (43), $H_0(\mathcal{A}, {}_\sigma\mathcal{A})$ is a countable direct sum of copies of k indexed by these classes.

Now suppose that $\rho = q^{a+1}$, for some $a \geq 0$, and λ is arbitrary. The only potentially nonzero classes are $[1]$, $[x^{l+1}]$, $[y^{l+1}]$, $[x^{a+1}u^m]$ and $[y^{a+1}v^n]$, for l, m, $n \geq 0$. We have

$$
[x^{a+1}u^m] = q^{-m}[\sigma(x)x^a u^m] = \lambda q^{-m}[x^{a+1}u^m] \tag{44}
$$

$$
[y^{a+1}v^n] = q^n[\sigma(y)y^a v^n] = \lambda^{-1}q^n[y^a v^n] \tag{45}
$$

Using also (32), (33), (34) it follows that:

Case 7. If $\rho = q^{a+1}$, $a \geq 0$, $\lambda \notin q^{\mathbf{N}}$, then $H_0(\mathcal{A}, {}_\sigma\mathcal{A}) = 0$.

Case 5. $\rho = q^{a+1}$, $a \geq 0$, $\lambda = 1$.

$$
H_0(\mathcal{A}, {}_\sigma\mathcal{A}) = k[1] \oplus (\Sigma^{\oplus}_{l>0}\, k[x^l]) \oplus (\Sigma^{\oplus}_{l>0}\, k[y^l]) \tag{46}
$$

Case 6. If $\rho = q^{a+1}$, $\lambda = q^{b+1}$, some a, $b \geq 0$, then

$$H_0(\mathcal{A}, {}_\sigma\mathcal{A}) = k[x^{a+1}u^{b+1}] \oplus k[y^{a+1}v^{b+1}] \tag{47}$$

Now suppose $\rho = q^{-(a+1)}$, for some $a \geq 0$. The only potentially nonzero classes are $[1]$, $[x^{l+1}]$, $[y^{l+1}]$, $[x^{a+1}v^n]$, $[y^{a+1}u^m]$, for l, m, $n \geq 0$. Now,

$$[x^{a+1}v^n] = q^{-n}[\sigma(x)x^a v^n] = \lambda q^{-n}[x^{a+1}v^n] \tag{48}$$

$$[y^{a+1}u^m] = q^m[\sigma(y)y^a u^m] = \lambda^{-1}q^m[y^{a+1}u^m] \tag{49}$$

So we have:

Case 10. If $\rho = q^{-(a+1)}$, $a \geq 0$, and $\lambda \notin q^{\mathbf{N}}$, then $H_0(\mathcal{A}, {}_\sigma\mathcal{A}) = 0$.

Case 8. If $\rho = q^{-(a+1)}$, $a \geq 0$, and $\lambda = 1$, then

$$H_0(\mathcal{A}, {}_\sigma\mathcal{A}) = k[1] \oplus (\Sigma^{\oplus}_{l>0}\, k[x^l]) \oplus (\Sigma^{\oplus}_{l>0}\, k[y^l]) \tag{50}$$

Case 9. If $\rho = q^{-(a+1)}$, and $\lambda = q^{b+1}$, some a, $b \geq 0$,

$$H_0(\mathcal{A}, {}_\sigma\mathcal{A}) = k[x^{a+1}v^{b+1}] \oplus k[y^{a+1}u^{b+1}] \tag{51}$$

□

Corollary 4.2 *$H_0(\mathcal{A}, {}_\sigma\mathcal{A}) \cong HH_0(\mathcal{A}, \sigma)$ for every σ.*

Proof: Every basis element $[a]$ that we have written above is σ-invariant. Hence Proposition 2.1 gives the isomorphism. □

4.2 $H_1(\mathcal{A}, {}_\sigma\mathcal{A})$

Proposition 4.3 *We give an explicit description of $H_1(\mathcal{A}, {}_\sigma\mathcal{A})$ for all σ.*

1. *In cases 7, 10 and 12, $H_1(\mathcal{A}, {}_\sigma\mathcal{A})$ is trivial.*
2. *In cases 6 and 9 $H_1(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k^4$.*
3. *In cases 1 - 5, 8 and 11, $H_1(\mathcal{A}, {}_\sigma\mathcal{A})$ is a countably infinite dimensional k-vector space.*

In each case we exhibit a basis.

Proof: For the Masuda-Nakagami-Watanabe resolution, we have

$$d_1 : \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_1 \to \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_0 \cong \mathcal{A} \tag{52}$$

given by

$$d_1(a \otimes e_v) = a.(v \otimes 1 - 1 \otimes v) = av - \sigma(v)a = av - \rho va,$$

$$d_1(a\otimes e_u) = a.(u\otimes 1 - 1\otimes u) = au - \sigma(u)a = au - \rho^{-1}ua,$$
$$d_1(a\otimes e_x) = a.(x\otimes 1 - 1\otimes x) = ax - \sigma(x)a = ax - \lambda xa,$$
$$d_1(a\otimes e_y) = a.(y\otimes 1 - 1\otimes y) = ay - \sigma(y)a = av - \lambda^{-1}ya, \tag{53}$$

Here $\{e_v, e_u, e_x, e_y\}$ is the given basis of $\mathcal{M}_1$ as a free left $\mathcal{A}^e$-module of rank 4, and we are treating $\mathcal{A}$ as a right $\mathcal{A}^e$-module with module structure given by

$$t.(a_0 \otimes a_1{}^o = \sigma(a_1)ta_0 \tag{54}$$

Hence

$$\ker(d_1) \cong \{(a_1, a_2, a_3, a_4) \in \mathcal{A}^4 :$$
$$(a_1v - \rho va_1) + (a_2u - \rho^{-1}ua_2) + (a_3x - \lambda xa_3) + (a_4y - \lambda^{-1}ya_4) = 0\} \tag{55}$$

We also have

$$d_2 : \mathcal{A}\otimes_{\mathcal{A}^e} \mathcal{M}_2 \to \mathcal{A}\otimes_{\mathcal{A}^e} \mathcal{M}_1 \tag{56}$$
$$d_2(b\otimes(e_v\wedge e_u)) = (bv - \rho vb)\otimes e_u - (bu - \rho^{-1}ub)\otimes e_v,$$
$$d_2(b\otimes(e_v\wedge e_x)) = (bv - q\rho vb)\otimes e_x - (qbx - \lambda xb)\otimes e_v,$$
$$d_2(b\otimes e_v\wedge e_y) = (qbv - \rho vb)\otimes e_y - (by - q\lambda^{-1}yb)\otimes e_v,$$
$$d_2(b\otimes(e_u\wedge e_x)) = (bu - q\rho^{-1}ub)\otimes e_x - (qbx - \lambda xb)\otimes e_u,$$
$$d_2(b\otimes(e_u\wedge e_y)) = (qbu - \rho^{-1}ub)\otimes e_y - (by - q\lambda^{-1}yb)\otimes e_u,$$
$$d_2(b\otimes\vartheta_S^{(1)}) = by\otimes e_x + \lambda xb\otimes e_y - qbu\otimes e_v - q\rho vb\otimes e_u,$$
$$d_2(b\otimes\vartheta_T^{(1)}) = \lambda^{-1}yb\otimes e_x + bx\otimes e_y - q^{-1}bu\otimes e_v - q^{-1}\rho vb\otimes e_u, \tag{57}$$

where $\{(e_v\wedge e_u), (e_v\wedge e_x), (e_v\wedge e_y), (e_u\wedge e_x), (e_u\wedge e_y), \vartheta_S^{(1)}, \vartheta_T^{(1)}\}$ is the given basis of $\mathcal{M}_2$ as a free left $\mathcal{A}^e$-module of rank 7. So

$$d_2[b_1\otimes(e_v\wedge e_u) + b_2\otimes(e_v\wedge e_x) + b_3\otimes(e_v\wedge e_y)$$
$$+b_4\otimes(e_u\wedge e_x) + b_5\otimes(e_u\wedge e_y) + b_6\otimes\vartheta_S^{(1)} + b_7\otimes\vartheta_T^{(1)}] =$$
$$[(\rho^{-1}ub_1 - b_1u) + (\lambda xb_2 - qb_2x) + (q\lambda^{-1}yb_3 - b_3y) - qb_6u - q^{-1}b_7u]\otimes e_v$$
$$+[(b_1v - \rho vb_1) + (\lambda xb_4 - qb_4x) + (\lambda^{-1}qyb_5 - b_5y) - q\rho vb_6 - q^{-1}\rho vb_7]\otimes e_u$$
$$+[(b_2v - q\rho vb_2) + (b_4u - q\rho^{-1}ub_4) + b_6y + \lambda^{-1}yb_7]\otimes e_x$$
$$+ [(qb_3v - \rho vb_3) + (qb_5u - \rho^{-1}ub_5) + \lambda xb_6 + b_7x]\otimes e_y \tag{58}$$

We will now use (55) and (58) to calculate $\ker(d_1)/\mathrm{im}(d_2)$.

If $\rho = 1$, then all $[u^mv^n\otimes e_v]$, and $[u^mv^n\otimes e_u]$ are potentially nonvanishing. We find that

$$f(m+n)[u^mv^n\otimes e_t] = (-q^{-1})^s f(m+n+2s)[u^{m+s}v^{n+s}\otimes e_t] \tag{59}$$

for $t = v$, u, and once again $f(n) = \lambda - q^{n+1}$. It follows that

$$(n\geq m)\ :\quad f(n-m)[v^{n-m}\otimes e_t] = (-q^{-1})^m f(m+n)[u^mv^n\otimes e_t] \tag{60}$$

$$(m\geq n)\ :\quad f(m-n)[u^{m-n}\otimes e_t] = (-q^{-1})^n f(m+n)[u^mv^n\otimes e_t] \tag{61}$$

for $t = v,\, u$. We also note that for any ρ, λ we have

$$(q^{m+2} - \lambda)([u^{m+1} \otimes e_v] + \rho[u^m v \otimes e_u]) = 0 \tag{62}$$

$$(q^{n+2} - \lambda)([v^{n+1} \otimes e_u] + \rho^{-1}[uv^n \otimes e_v]) = 0 \tag{63}$$

For $\lambda \neq q^2$, define

$$[\omega_1] = [u \otimes e_v] = -\rho[v \otimes e_u] \tag{64}$$

(For $\lambda = q^2$ the equality need not hold).

Case 1. $\rho = 1$, $\lambda = 1$ ($\sigma = \mathrm{id}$). Then

$$H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = (\Sigma^{\oplus}_{n\geq 0}\, k[v^n \otimes e_v]) \oplus k[\omega_1] \oplus (\Sigma^{\oplus}_{m\geq 0}\, k[u^m \otimes e_u]) \oplus$$
$$\oplus\, (\Sigma^{\oplus}_{l\geq 0}\, k[x^l \otimes e_x]) \oplus (\Sigma^{\oplus}_{l\geq 0}\, k[y^l \otimes e_y]) \tag{65}$$

where $[\omega_1] = [u \otimes e_v] = -[v \otimes e_u]$. This is in agreement with [MNW90], apart from the slight sign change in $[\omega_1]$.

Case 2. $\rho = 1$, $\lambda = q$.

$$H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = (\Sigma^{\oplus}_{n\geq 0}\, k[v^n \otimes e_v]) \oplus k[\omega_1] \oplus (\Sigma^{\oplus}_{m\geq 0}\, k[u^m \otimes e_u]) \tag{66}$$

where $[\omega_1] = [u \otimes e_v] = -[v \otimes e_u]$.

Now suppose $\rho = 1$ and $\lambda = q^{b+2}$, some $b \geq 0$. Then $f(n+2m) = \lambda - q^{n+2m+1} = 0$ if and only if $n+2m = b+1$ for some $m \geq 1$. So $[v^n \otimes e_v] = 0$ if there exists some $m \geq 1$ such that $n + 2m = b+1$, i.e. if $n \in \{b+1-2m\}_{m\geq 1}$. Similarly for $[u^m \otimes e_u]$. Hence:

Case 3. $\rho = 1$, $\lambda = q^{b+2}$.

$$H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = (\Sigma^{\oplus}_{n\in S}\, k[v^n \otimes e_v]) \oplus k[\omega_1] \oplus (\Sigma^{\oplus}_{m\in S}\, k[u^m \otimes e_u])$$
$$\oplus\, [u^{b+1} \otimes e_v] \oplus [v^{b+1} \otimes e_u] \tag{67}$$

where $[\omega_1] = [u \otimes e_v] = -[v \otimes e_u]$ (provided $\lambda \neq q^2$), and $S \subseteq \mathbf{N}$ is given by $S = \{n \geq b\} \cup \{b-2, b-4, \ldots\}$. We note that if $b = 0$, i.e. $\rho = 1$, $\lambda = q^2$, then we have:

$$H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = (\Sigma^{\oplus}_{n\geq 0}\, k[v^n \otimes e_v]) \oplus (\Sigma^{\oplus}_{m\geq 0}\, k[u^m \otimes e_u]) \oplus k[u \otimes e_v] \oplus k[v \otimes e_u] \tag{68}$$

Case 4. $\rho = 1$, $\lambda \notin q^{\mathbf{N}}$.

$$H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = (\Sigma^{\oplus}_{n\geq 0}\, k[v^n \otimes e_v]) \oplus k[\omega_1] \oplus (\Sigma^{\oplus}_{m\geq 0}\, k[u^m \otimes e_u]) \tag{69}$$

where $[\omega_1] = [u \otimes e_v] = -[v \otimes e_u]$.

Case 5. $\rho = q^{a+1}$, $a \geq 0$, $\lambda = 1$. Then

$$H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = (\Sigma^{\oplus}_{l\geq 0}\, k[x^l \otimes e_x]) \oplus (\Sigma^{\oplus}_{l\geq 0}\, k[y^l \otimes e_y]) \oplus k[\omega_1] \tag{70}$$

where $[\omega_1] = (\rho^{-1} - 1)[y \otimes e_x] + (q - q^{-1})[v \otimes e_u]$.

Case 6. $\rho = q^{a+1}$, $\lambda = q^{b+1}$, $a, b \geq 0$.

$$H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = k[y^{a+1}v^b \otimes e_v] \oplus k[x^{a+1}u^b \otimes e_u] \oplus$$
$$\oplus k[x^a u^{b+1} \otimes e_x] \oplus k[y^a v^{b+1} \otimes e_y] \tag{71}$$

Case 7. $\rho = q^{a+1}$, $\lambda \notin q^{\mathbf{N}}$. Then $H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = 0$.

Case 8. $\rho = q^{-(a+1)}$, $a \geq 0$, $\lambda = 1$. Then

$$H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = (\Sigma^{\oplus}_{l \geq 0}\, k[x^l \otimes e_x]) \oplus (\Sigma^{\oplus}_{l \geq 0}\, k[y^l \otimes e_y]) \oplus k[\omega_1] \tag{72}$$

where $[\omega_1] = (\rho^{-1} - 1)[y \otimes e_x] + (q - q^{-1})[v \otimes e_u]$.

Case 9. $\rho = q^{-(a+1)}$, $\lambda = q^{b+1}$, $a, b \geq 0$.

$$H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = k[x^{a+1}v^b \otimes e_v] \oplus k[y^{a+1}u^b \otimes e_u] \oplus$$
$$\oplus k[x^a v^{b+1} \otimes e_x] \oplus k[y^a u^{b+1} \otimes e_y] \tag{73}$$

Case 10. $\rho = q^{-(a+1)}$, $\lambda \notin q^{\mathbf{N}}$. Then $H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = 0$.

Case 11. $\rho \notin q^{\mathbf{Z}}$, $\lambda = 1$. Then

$$H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = (\Sigma^{\oplus}_{l \geq 0}\, k[x^l \otimes e_x]) \oplus (\Sigma^{\oplus}_{l \geq 0}\, k[y^l \otimes e_y]) \oplus k[\omega_1] \tag{74}$$

where $[\omega_1] = (\rho^{-1} - 1)[y \otimes e_x] + (q - q^{-1})[v \otimes e_u]$.

Case 12. $\rho \notin q^{\mathbf{Z}}$, $\lambda \neq 1$. Then $H_1(\mathcal{A}, {}_\sigma\mathcal{A}) = 0$.

The generators can be translated into Hochschild cycles in $\mathcal{A}^{\otimes 2}$ using (19). Concretely, we have

$$[\alpha \otimes e_t] \mapsto (\alpha, t) \quad \alpha \in \mathcal{A}, \quad t = u, v, x, y \tag{75}$$

□

Corollary 4.4 *$H_1(\mathcal{A}, {}_\sigma\mathcal{A}) \cong HH_1(\mathcal{A}, \sigma)$ for every σ.*

Proof: Every basis element that we have written above is already σ-invariant, hence Proposition 2.1 gives the isomorphism. □

4.3 $H_2(\mathcal{A}, {}_\sigma\mathcal{A})$

Proposition 4.5 *We give an explicit description of $H_2(\mathcal{A}, {}_\sigma\mathcal{A})$ for all σ. There are four cases to consider:*

1. $\rho = 1$, $\lambda = q^{b+2}$, $b \geq 0$. *Then* $H_2(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k^{b+1}$
2. $\rho = q^{a+1}$, $\lambda = q^{b+1}$, $a, b \geq 0$. *Then* $H_2(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k^2$
3. $\rho = q^{-(a+1)}$, $\lambda = q^{b+1}$, $a, b \geq 0$. *Then* $H_2(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k^2$
4. *Otherwise,* $H_2(\mathcal{A}, {}_\sigma\mathcal{A}) = 0$.

Proof: The map $d_2 : \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_2 \to \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_1$ was given in (58). The map $d_3 : \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_3 \to \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_2$ in the M-N-W resolution is explicitly given by:

$$d_3[a_1 \otimes (e_v \wedge e_u \wedge e_x) + a_2 \otimes (e_v \wedge e_u \wedge e_y) + a_3 \otimes (e_v \wedge \vartheta_S^{(1)}) + a_4 \otimes (e_v \wedge \vartheta_T^{(1)}) +$$

$$\begin{aligned} a_5 \otimes (e_u \wedge \vartheta_S^{(1)}) + a_6 \otimes (e_u \wedge \vartheta_T^{(1)}) + a_7 \otimes (e_x \wedge \vartheta_S^{(1)}) + a_8 \otimes (e_y \wedge \vartheta_T^{(1)})] = \\ [(q^2 a_1 x - \lambda x a_1) + (a_2 y - \lambda^{-1} q^2 y a_2) + q\rho v a_3 + \\ + q^{-1}\rho v a_4 - q a_5 u - q^{-1} a_6 u] \otimes (e_v \wedge e_u) \\ + [(q\rho^{-1} u a_1 - a_1 u) - q^{-1} a_3 y - \lambda^{-1} y a_4 - q^{-1} a_7 u] \otimes (e_v \wedge e_x) \\ + [(\rho^{-1} u a_2 - q a_2 u) - q^{-1}\lambda x a_3 - a_4 x - a_8 u] \otimes (e_v \wedge e_y) \\ + [(a_1 v - q\rho v a_1) - q^{-1} a_5 y - \lambda^{-1} y a_6 - \rho v a_7] \otimes (e_u \wedge e_x) \\ + [(q a_2 v - \rho v a_2) - q^{-1}\lambda x a_5 - a_6 x - q^{-1}\rho v a_8] \otimes (e_u \wedge e_y) \\ + [(a_3 v - \rho v a_3) + (a_5 u - \rho^{-1} u a_5) + a_7 x - \lambda^{-1} y a_8] \otimes \vartheta_S^{(1)} \\ + [(a_4 v - \rho v a_4) + (a_6 u - \rho^{-1} u a_6) - \lambda x a_7 + a_8 y] \otimes \vartheta_T^{(1)} \end{aligned} \tag{76}$$

where $\{(e_v \wedge e_u \wedge e_x), (e_v \wedge e_u \wedge e_y), (e_v \wedge \vartheta_S^{(1)}), (e_v \wedge \vartheta_T^{(1)}), (e_u \wedge \vartheta_S^{(1)}), (e_u \wedge \vartheta_T^{(1)}), (e_x \wedge \vartheta_S^{(1)}), (e_y \wedge \vartheta_T^{(1)})\}$ is the given basis of $\mathcal{M}_3$ as a free left $\mathcal{A}^e$-module of rank 8.

Case 1. $\rho = 1$, $\lambda = q^{b+2}$, $b \geq 0$. Then $H_2(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k^{b+1}$, with basis

$$\begin{aligned} \omega_2(b, i) &= q^2[u^i v^{b-i} \otimes \vartheta_T^{(1)}] - [u^i v^{b-i} \otimes \vartheta_S^{(1)}], \quad 0 \leq i \leq b \\ &= q^2(u^i v^{b-i}, x, y) - (u^i v^{b-i}, y, x) + (q^2 - 1)(u^i v^{b-i}, 1, 1) \end{aligned} \tag{77}$$

where we are also writing the corresponding Hochschild cycles in $\mathcal{A}^{\otimes 3}$.

Case 2. $\rho = q^{a+1}$, $\lambda = q^{b+1}$, $a, b \geq 0$. Then $H_2(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k^2$, with basis

$$\omega_2 = [x^a u^b \otimes (e_u \wedge e_x)] = (x^a u^b, u, x) - q(x^a u^b, x, u) \tag{78}$$

$$\omega_2' = [y^a v^b \otimes (e_v \wedge e_y)] = (y^a v^b, v, y) - q^{-1}(y^a v^b, y, v) \tag{79}$$

Case 3. $\rho = q^{-(a+1)}$, $\lambda = q^{b+1}$, $a, b \geq 0$. Then $H_2(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k^2$, with basis given by the elements

$$\omega_2 = [x^a v^b \otimes (e_v \otimes e_x)] = (x^a v^b, v, x) - q(x^a v^b, x, v) \tag{80}$$

$$\omega_2^{'} = [y^a u^b \otimes (e_u \wedge e_y)] = (y^a u^b, u, y) - q^{-1}(y^a u^b, y, u) \tag{81}$$

Case 4. For all other values of ρ and λ, $H_2(\mathcal{A}, {}_\sigma\mathcal{A})$ vanishes. $\square$

Corollary 4.6 *$H_2(\mathcal{A}, {}_\sigma\mathcal{A}) \cong HH_2(\mathcal{A}, \sigma)$ for every σ.*

Proof: All the Hochschild cycles given above (77), (78), (79), (80), (81) are σ-invariant. Hence the result. $\square$

4.4 $H_n(\mathcal{A}, {}_\sigma\mathcal{A})$, $n \geq 3$

Proposition 4.7 *For $\rho = 1$, $\lambda = q^{b+2}$, we have $H_3(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k^{b+1}$. For all other values of ρ and λ, $H_3(\mathcal{A}, {}_\sigma\mathcal{A}) = 0$.*

Proof: The map $d_4 : \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_4 \to \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_3$ in the M-N-W resolution is explicitly given by:

$$\begin{gathered}
d_4[b_1 \otimes (e_v \wedge e_u \wedge \vartheta_S^1) + b_2 \otimes (e_v \wedge e_u \wedge \vartheta_T^1) + \\
b_3 \otimes (e_v \wedge e_x \wedge \vartheta_S^1) + b_4 \otimes (e_v \wedge e_y \wedge \vartheta_T^1) + b_5 \otimes (e_u \wedge e_x \wedge \vartheta_S^1) + \\
b_6 \otimes (e_u \wedge e_y \wedge \vartheta_T^1) + b_7 \otimes \vartheta_S^2 + b_8 \otimes \vartheta_T^2] = \\
[q^{-2} b_1 y + \rho^{-1} y b_2 + \rho v b_3 - q^{-1} b_5 u] \otimes (e_v \wedge e_u \wedge e_x) \\
+ [\lambda q^{-2} x b_1 + b_2 x + q^{-1} \rho v b_4 - b_6 u] \otimes (e_v \wedge e_u \wedge e_y) \\
+ [(\rho^{-1} u b_1 - b_1 u) - q b_3 x + q\lambda^{-1} y b_4 - q b_7 u] \otimes (e_v \wedge \vartheta_S^{(1)}) \\
+ [(\rho^{-1} u b_2 - b_2 u) + \lambda x b_3 - b_4 y - q^{-1} b_8 u] \otimes (e_v \wedge \vartheta_T^{(1)}) \\
+ [(b_1 v - \rho v b_1) - q b_5 x + q\lambda^{-1} y b_6 - q\rho v b_7] \otimes (e_u \wedge \vartheta_S^{(1)}) \\
+ [(b_2 v - \rho v b_2) + \lambda x b_5 - b_6 y - q^{-1} \rho v b_8] \otimes (e_u \wedge \vartheta_T^{(1)}) \\
+ [(b_3 v - q\rho v b_3) + (b_5 u - q\rho^{-1} u b_5) + b_7 y + \lambda^{-1} y b_8] \otimes (e_x \wedge \vartheta_S^{(1)}) \\
+ [(q b_4 v - \rho v b_4) + (q b_6 u - \rho^{-1} u b_6) + \lambda x b_7 + b_8 x] \otimes (e_y \wedge \vartheta_T^{(1)})
\end{gathered} \tag{82}$$

Using the new resolution of the second author, we find that, for $\rho = 1$, $\lambda = q^{b+2}$, we have $H_3(\mathcal{A}, {}_\sigma\mathcal{A}) \cong k^{b+1}$.

For $\lambda = q^{b+3}$, $b+1$ of the $b+2$ generators are given by the M-N-W elements

$$\omega_3(b, i) = q^2 u^i v^{b-i} \otimes (e_v \wedge \vartheta_T^{(1)}) - u^i v^{b-i} \otimes (e_v \wedge \vartheta_S^{(1)}), \quad 0 \leq i \leq b \tag{83}$$

For each $\lambda = q^{b+2}$ all the generating Hochschild cycles are given explicitly as follows. For $0 \leq i \leq b$, define

$$\omega_3(b,i) = A(b,i) - B(b,i) \tag{84}$$

where

$$A(b,i) = (xu^i v^{b-i}) \otimes (y \wedge u \wedge v) + (vu^i v^{b-i}) \otimes (u \wedge y \wedge x) \tag{85}$$

$$\begin{aligned} B(b,i) = {} & (-qxuu^i v^{b-i}) \otimes (1 \wedge y \wedge v) + (-q^{-1} vyu^i v^{b-i}) \otimes (1 \wedge u \wedge x) \\ & +(xyu^i v^{b-i}) \otimes (1 \wedge u \wedge v) + (uvu^i v^{b-i}) \otimes (1 \wedge y \wedge x) \\ & + (q - q^{-1})(uvu^i v^{b-i}) \otimes ((v,u,1) - (1,v,u) + (v,1,u)) \end{aligned} \tag{86}$$

where the terms $''a_0 \wedge a_1 \wedge a_2''$ are explicitly given by:

$$y \wedge u \wedge v = (y,u,v) - (y,v,u) + q(v,y,u) - q^2(v,u,y) + q^2(u,v,y) - q(u,y,v)$$

$$\begin{aligned} u \wedge y \wedge x &= (u,y,x) - (u,x,y) + q(x,u,y) - (x,y,u) + (y,x,u) - q^{-1}(y,u,x) \\ 1 \wedge y \wedge v &= (1,y,v) - q(1,v,y) + q(v,1,y) - q(v,y,1) + (y,v,1) - (y,1,v) \\ 1 \wedge u \wedge x &= (1,u,x) - q(1,x,u) - (u,1,x) + (u,x,1) - q(x,u,1) + q(x,1,u) \\ 1 \wedge u \wedge v &= (1,u,v) - (1,v,u) - (u,1,v) + (u,v,1) + (v,1,u) - (v,u,1) \\ 1 \wedge y \wedge x &= (1,y,x) - (1,x,y) + (x,1,y) - (x,y,1) + (y,x,1) - (y,1,x) \end{aligned} \tag{87}$$

and throughout we denote $a_0 \otimes a_1 \otimes a_2$ by (a_0, a_1, a_2). □

Corollary 4.8 *$H_3(\mathcal{A}, {}_\sigma\mathcal{A}) \cong HH_3(\mathcal{A}, \sigma)$ for every σ.*

Proof: Once again, all the given Hochschild cycles (84) are σ-invariant. □

Just as in the untwisted case, all the higher twisted Hochschild homology groups vanish:

Proposition 4.9 *$H_n(\mathcal{A}, {}_\sigma\mathcal{A}) = 0$ for $n > 3$.*

5 Twisted cyclic homology of quantum $SL(2)$

Connes' long exact S-B-I sequence relates twisted Hochschild and cyclic homology [KMT03]. We have:

$$\rightarrow HH_{n+1}(\mathcal{A}, \sigma) \rightarrow^I HC_{n+1}(\mathcal{A}, \sigma) \rightarrow^S HC_{n-1}(\mathcal{A}, \sigma) \rightarrow^B HH_n(\mathcal{A}, \sigma) \rightarrow \tag{88}$$

As an immediate consequence, $HC_0(\mathcal{A}, \sigma) \cong HH_0(\mathcal{A}, \sigma)$ for all σ.

In the following we will denote by $[(a_0, a_1, \dots, a_n)]$ the equivalence class under λ_σ of $(a_0, a_1, \dots, a_n) \in \mathcal{A}^{\otimes(n+1)}$.

Cases 1, 2, 4, 5, 8, 11: In each of these cases $HC_0(\mathcal{A}, \sigma)$ is infinite-dimensional, while

$$HC_{2n+1}(\mathcal{A}, \sigma) = k[\omega_1] \tag{89}$$

$$HC_{2n+2}(\mathcal{A}, \sigma) = k[1] \tag{90}$$

where in each case ω_1 is the distinguished generator of $H_1(\mathcal{A}, {}_\sigma\mathcal{A})$.

Case 3a: $\rho = 1$, $\lambda = q^2$. $HC_0(\mathcal{A},\sigma)$ is infinite-dimensional. Under the mapping $I : HH_1(\mathcal{A},\sigma) \to HC_1(\mathcal{A},\sigma)$, the two distinct Hochschild cycles (v,u) and (u,v) are both mapped to the (class of the) cyclic cycle $[(u,v)]$. Hence,

$$HC_1(\mathcal{A},\sigma) = k[(u,v)] \tag{91}$$

$$HC_{2n+2}(\mathcal{A},\sigma) = k[1] \oplus k[\omega_2] \tag{92}$$

$$HC_{2n+3}(\mathcal{A},\sigma) = k[(u,v)] \oplus k[\omega_3] \tag{93}$$

where ω_2, ω_3 were defined in (77), (84) respectively.

Case 3b: $\rho = 1$, $\lambda = q^{b+3}$, $b \geq 0$. $HC_0(\mathcal{A},\sigma)$ is infinite-dimensional. We have

$$HC_1(\mathcal{A},\sigma) = k[(u^{b+2},v)] \oplus k[(v^{b+2},u)] \oplus k[(u,v)] \tag{94}$$

$$HC_{2n+2}(\mathcal{A},\sigma) \cong k^{b+3} = k[1] \oplus (\ \Sigma^{\oplus}_{0\leq i\leq b+1}\ k[\omega_2(b,i)]\ ) \tag{95}$$

$$HC_{2n+3}(\mathcal{A},\sigma) \cong k^{b+5} = (94) \oplus (\ \Sigma^{\oplus}_{0\leq i\leq b+1}\ k[\omega_3(b,i)]\ ) \tag{96}$$

where the $\omega_2(b,i)$, $\omega_3(b,i)$ were defined in (77), (84) respectively.

Cases 7, 10, 12: $\rho = q^{\pm(a+1)}$, $a \geq 0$, $\lambda \notin q^{\mathbf{N}}$, and $\rho \notin q^{\mathbf{Z}}$, $\lambda \neq 1$. Since $H_n(\mathcal{A}, {}_\sigma\mathcal{A}) = 0$ for all $n \geq 0$, we have

$$HC_n(\mathcal{A},\sigma) = 0 \quad \forall\, n \geq 0 \tag{97}$$

Case 6: $\rho = q^{a+1}$, $\lambda = q^{b+1}$. We have from (47):

$$HC_0(\mathcal{A},\sigma) = k[x^{a+1}u^{b+1}] \oplus k[y^{a+1}v^{b+1}] \tag{98}$$

$$HC_{2n+1}(\mathcal{A},\sigma) = k[(y^{a+1}v^b,v)] \oplus k[(x^{a+1}u^b,u)] \oplus k[(x^a u^{b+1},x)] \oplus k[(y^a v^{b+1},y)] \tag{99}$$

$$HC_{2n+2}(\mathcal{A},\sigma) = k[x^{a+1}u^{b+1}] \oplus k[y^{a+1}v^{b+1}] \oplus k[\omega_2] \oplus k[\omega_2^{'}] \tag{100}$$

where ω_2, $\omega_2^{'}$ were defined in (78), (79).

Case 9: $\rho = q^{-(a+1)}$, $\lambda = q^{b+1}$. We have from (51):

$$HC_0(\mathcal{A},\sigma) = k[x^{a+1}v^{b+1}] \oplus k[y^{a+1}u^{b+1}] \tag{101}$$

$$HC_{2n+1}(\mathcal{A},\sigma) = k[(x^{a+1}v^b,v)] \oplus k[(y^{a+1}u^b,u)] \oplus k[(x^a v^{b+1},x)] \oplus k[(y^a u^{b+1},y)] \tag{102}$$

$$HC_{2n+2}(\mathcal{A},\sigma) = k[x^{a+1}v^{b+1}] \oplus k[y^{a+1}u^{b+1}] \oplus k[\omega_2] \oplus k[\omega_2^{'}] \tag{103}$$

where ω_2, $\omega_2^{'}$ were defined in (80), (81).

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