TWISTED HOMOLOGY OF QUANTUM $SL(2)$ - PART II

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Abstract. In this article, the cup and cap product in Hochschild (co)-homology are studied for the quantised coordinate ring $\mathcal{A} = \mathbb{K}_q[SL(2)]$ of $SL(2)$, with coefficients being $\mathcal{A}$ with the canonical left action twisted to $a \triangleright b = \sigma(a)b$, $\sigma \in \mathrm{Aut}(\mathcal{A})$. The results are used to complete the calculation of the twisted cyclic homology that we began in [12]. In particular, a nontrivial cyclic 3-cocycle is constructed which remains nontrivial when considered in Hochscild cohomology.

Dedicated to Prof. K. Schmüdgen on the occasion of his 60th birthday

1. Introduction

1.1. **Background.** In [12], we computed the Hochschild homology $H_\bullet(\mathcal{A}, {}_\sigma\mathcal{A})$ of the quantised coordinate ring $\mathcal{A} = \mathbb{K}_q[SL(2)]$, where $q \in \mathbb{K}$ is not a root of unity and ${}_\sigma\mathcal{A}$ is the $\mathcal{A}$-bimodule arising from $\mathcal{A}$ by twisting the canonical left action to $a \triangleright b = \sigma(a)b$, $\sigma \in \mathrm{Aut}(\mathcal{A})$. It had been pointed out by Kustermans, Murphy and Tuset [20] that the cyclic homology $HC^\sigma_\bullet(\mathcal{A})$ built on $H_\bullet(\mathcal{A}, {}_\sigma\mathcal{A})$ as a special case of Hopf-cyclic homology [6, 14] is intimately related to Woronowicz's theory of covariant differential calculi, and this connection was the original motivation for our work. However, we realised in the subsequent paper [13] that the coefficients ${}_\sigma\mathcal{A}$ also arise naturally from Poincaré duality in Hochschild (co)homology: using the general theory developed by van den Bergh in [28] we showed that

$$H^n(\mathcal{A}, \mathcal{A}) \simeq H_{3-n}(\mathcal{A}, {}_{\sigma_{q^{-2},1}}\mathcal{A}), \tag{1}$$

where for $\lambda, \mu \in \mathbb{K} \setminus \{0\}$, $\sigma_{\lambda,\mu} \in \mathrm{Aut}(\mathcal{A})$ is determined by its values λa, μb, $\mu^{-1} c$, $\lambda^{-1} d$ on the standard generators a, b, c, d of $\mathcal{A}$. In particular, $\sigma_{q^{-2},1}$ reduces for $q \in \mathbb{R}$ to the modular automorphism of the Haar state of the compact quantum group generated by $\mathcal{A}$.

As a consequence of (1) there is a fundamental class $\mathsf{d}\mathcal{A} \in H_3(\mathcal{A}, {}_{\sigma_{q^{-2},1}}\mathcal{A})$ which corresponds under the isomorphism to $1 \in H^0(\mathcal{A}, \mathcal{A})$, the centre of $\mathcal{A}$. Explicitly, $\mathsf{d}\mathcal{A}$ is represented in the normalised Hochschild complex $\bar{C}_\bullet(\mathcal{A}, {}_{\sigma_{q^{-2},1}}\mathcal{A}) = \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes \bullet}$, $\bar{\mathcal{A}} := \mathcal{A}/\mathbb{K}$, as

$$\mathsf{d}\mathcal{A} = [-qd \otimes (b \wedge a \wedge c) + c \otimes (b \wedge a \wedge d)].$$

Here $\wedge$ is defined using the braiding

$$\Psi : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad x \otimes y \mapsto \mathbf{r}(y_{(1)}, x_{(1)}) y_{(2)} \otimes x_{(2)} \mathbf{r}(S(y_{(3)}), x_{(3)})$$

in which $\mathbf{r}$ is the universal r-form of the standard coquasitriangular Hopf algebra structure on $\mathcal{A}$, $S : \mathcal{A} \to \mathcal{A}$ is the antipode and $x \mapsto x_{(1)} \otimes x_{(2)}$ is the coproduct in Sweedler notation.

More generally, a Poincaré-type duality as in (1) holds for formal deformation quantisations of smooth Poisson varieties as shown by Etingof and Dolgushev [8] and also, as shown by Brown and Zhang, for a large class of Noetherian Hopf algebras that includes in particular the quantised coordinate rings $\mathbb{K}_q[G]$ for all simple algebraic groups G [2]. See also [9, 18] for more examples and background. It thus seems the rule rather than the exception that a well-behaved algebra admits a fundamental class $\mathsf{d}\mathcal{A}$ in Hochschild homology, but in general this unavoidably involves noncanonical coefficient bimodules.

1.2. **Results.** In the present article, we finish the computation of the cyclic theory $HC^\sigma_\bullet(\mathcal{A})$ for the case $\sigma = \sigma_{q^{-N},1}$, $N \in \mathbb{Z}$, which was left open in [12]. We begin by recalling the results of our computations from [12] with some simplifications and corrections, see Section 2. Inspired by the work of Nest and Tsygan (see e.g. [25]), we investigate the cup and cap product

$$\begin{aligned} \smile &: H^m(\mathcal{A}, {}_\sigma\mathcal{A}) \otimes H^n(\mathcal{A}, {}_\tau\mathcal{A}) \to H^{m+n}(\mathcal{A}, {}_{\tau\circ\sigma}\mathcal{A}), \\ \frown &: H_n(\mathcal{A}, {}_\sigma\mathcal{A}) \otimes H^m(\mathcal{A}, {}_\tau\mathcal{A}) \to H_{n-m}(\mathcal{A}, {}_{\tau^{-1}\circ\sigma}\mathcal{A}). \end{aligned}$$

For coordinate rings of smooth varieties, the Hochschild-Kostant-Rosenberg theorem identifies the (untwisted) Hochschild cohomology $(H^\bullet(\mathcal{A},\mathcal{A}), \smile)$ with the algebra of multivector fields on the variety. For $\mathcal{A} = \mathbb{K}_q[SL(2)]$, there are twisted derivations $\partial^+_H, \partial^+_E, \partial^+_F$ and $\partial^-_H, \partial^-_E, \partial^-_F$ that arise respectively from the left and right action of the Hopf dual $\mathcal{A}^\circ$ and deform the action of left- and right-invariant vector fields on $SL(2)$ (see Section 3.1.5 for their definition). In Section 3.2.3 we show that these generate a q-deformed exterior algebra inside the twisted Hochschild cohomology ring of $\mathcal{A}$. Its cap product action allows us to identify nontrivial 2- and 3-cycles (Sections 3.2.2 and 3.2.4), and using this tool we then compute $HC^\sigma_\bullet(\mathcal{A})$ in Section 4.

As shown by Connes [4], any cyclic homology theory rests upon a simplicial one which for $HC^\sigma_\bullet(\mathcal{A})$ is denoted by $HH^\sigma_\bullet(\mathcal{A})$. There is a map

$$H_\bullet(\mathcal{A}, {}_\sigma\mathcal{A}) \to HH^\sigma_\bullet(\mathcal{A}),$$

and when σ is diagonalisable, this is an isomorphism (see Section 4 for details). Recall further that there is the Connes spectral sequence

$$E^1_{pq} = HH^\sigma_{q-p}(\mathcal{A}) \Rightarrow HC^\sigma_{p+q}(\mathcal{A}),$$

and that $HH^\sigma_n(\mathcal{A}) = 0$ for all $n > 3$ (which follows from Poincaré duality) implies $HC^\sigma_n(\mathcal{A}) \simeq HC^\sigma_{n+2}(\mathcal{A})$ for $n > 3$. In this way, one obtains the two periodic cyclic homology groups $HP^\sigma_0(\mathcal{A})$ and $HP^\sigma_1(\mathcal{A})$ as the limit of the $HC^\sigma_{2n}(\mathcal{A})$ and $HC^\sigma_{2n+1}(\mathcal{A})$, respectively.

The following theorem summarises the result of our computations. For the sake of clarity, we only describe here the periodic cyclic homology, see Section 4.2.4 for the additional classes in the nonperiodic $HC^\sigma_\bullet(\mathcal{A})$.

THEOREM 1.1. *Let $\mathcal{A}$ be the quantised coordinate ring $\mathbb{K}_q[SL(2)]$, q not a root of unity, and $\sigma_{q^{-N},1} \in \mathrm{Aut}(\mathcal{A})$ be determined by*

$$\sigma_{q^{-N},1}(a) = q^{-N}a, \quad \sigma_{q^{-N},1}(b) = b, \quad N \in \mathbb{Z},$$

where a, b, c, d are the standard generators of $\mathcal{A}$. Then the $\sigma_{q^{-N},1}$-twisted periodic cyclic homology of $\mathcal{A}$ is given by

$$HP_0^{\sigma_{q^{-N},1}} = HC_4^{\sigma_{q^{-N},1}} = \begin{cases} \mathbb{K}[b^r c^r (\mathsf{d}\mathcal{A} \frown \partial_H^-)] & N = 2r+2, r \geq 0, \\ \mathbb{K}[1] & \textit{otherwise} \end{cases}$$

$$HP_1^{\sigma_{q^{-N},1}} = HC_3^{\sigma_{q^{-N},1}} = \begin{cases} \mathbb{K}[b^r c^r \mathsf{d}\mathcal{A}] & N = 2r+2, r \geq 0, \\ \mathbb{K}[b \otimes c] & \textit{otherwise.} \end{cases}$$

Here, classes in $HC_n^\sigma(\mathcal{A})$ are represented by classes in $HH_{n-2p}^\sigma(\mathcal{A})$, $p \geq 0$ using the spectral sequence $E_{pq}^1 = HH_{q-p}^\sigma(\mathcal{A}) \Rightarrow HC_{p+q}^\sigma(\mathcal{A})$.

Finally, we construct in Sections 3.2.4 and 5.2.2 a cyclic cocycle that pairs nontrivially with the fundamental class $\mathsf{d}\mathcal{A}$, that is, a linear functional

$$\varphi : \bar{C}_\bullet(\mathcal{A}, {}_\sigma\mathcal{A}) \to \mathbb{K}$$

which is a Hochschild cocycle (i.e, vanishes on the image of the Hochschild boundary b and thus induces a functional on $H_\bullet(\mathcal{A}, {}_\sigma\mathcal{A})$) and futher satisfies

$$\varphi(a_0, \bar{a}_1, \ldots, \bar{a}_n) = (-1)^n \varphi(\sigma(a_n), \bar{a}_0, \ldots, \bar{a}_{n-1})$$

for all $a_i \in \mathcal{A}$, where $\bar{a}$ is the class of $a \in \mathcal{A}$ in $\bar{\mathcal{A}}$:

THEOREM 1.2. *Define for all $j, k \geq 0$ a functional $\int_{[b^j c^k]} : \mathcal{A} \to \mathbb{K}$ by*

$$\int_{[b^j c^k]} e_{r,s,t} := \delta_{0,r}\delta_{j,s}\delta_{k,t}, \quad e_{i,j,k} := \begin{cases} a^i b^j c^k & i \geq 0 \\ d^{-i} b^j c^k & i < 0 \end{cases}$$

and two linear functionals $\bar{C}_3(\mathcal{A}, {}_{\sigma_{q^{-2},1}}\mathcal{A}) \to \mathbb{K}$ by

$$\varphi_{\mathcal{A}}(\cdot) := \int_{[1]} \cdot \frown (\partial_H^+ \smile \partial_E^+ \smile \partial_F^+),$$

$$\eta_{\mathcal{A}}(\cdot) := 2 \int_{[bc]} \cdot \frown (\partial_H^+ \smile (\sigma_{1,1/2} - \mathrm{id}) \smile (\sigma_{1,2} - \mathrm{id})).$$

Then $\varphi_{\mathcal{A}}$ and $\eta_{\mathcal{A}}$ are $\sigma_{q^{-2},1}$-twisted Hochschild cocycles, $\varphi_{\mathcal{A}} + \eta_{\mathcal{A}}$ is a $\sigma_{q^{-2},1}$-twisted cyclic cocycle, and $(\varphi_{\mathcal{A}} + \eta_{\mathcal{A}})(\mathsf{d}\mathcal{A}) = \varphi_{\mathcal{A}}(\mathsf{d}\mathcal{A}) = 1$.

The cocycle $\varphi_{\mathcal{A}} + \eta_{\mathcal{A}}$ gives an explicit description of $HC^3_{\sigma_{q^{-2},1}}(\mathcal{A}) = (HC_3^{\sigma_{q^{-2},1}}(\mathcal{A}))^* \simeq \mathbb{K}$ in terms of Connes' λ-complex, and one should expect that there is a spectral triple that realises this cocycle via a twisted variant of the Connes-Moscovici local index formula, e.g. as in [7, 24, 26].

We stress that most computations in this paper have been verified with the help of the computer algebra system FELIX [1] in order to exclude as far as possible computational mistakes. The FELIX output is available in electronic form [19], and it can easily be adapted to perform similar computations with other algebras given in terms of generators and relations.

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2. Hochschild homology

2.1. **Background.** In Section 2.1, we fix notations and conventions concerning Hochschild homology and the quantised coordinate ring of $SL(2)$. For proofs and details see for example [3, 23, 29] and [16, 17], respectively.

2.1.1. *Algebras and bimodules.* Throughout this paper, $\mathbb{K}$ is an algebraically closed field of characteristic 0, and "algebra" means unital associative $\mathbb{K}$-algebra. An unadorned $\otimes$ denotes the tensor product of $\mathbb{K}$-vector spaces. We denote for a bimodule $\mathcal{M}$ and two automorphisms σ, τ of an algebra $\mathcal{A}$ by ${}_\sigma\mathcal{M}_\tau$ the bimodule which is $\mathcal{M}$ as vector space with bimodule structure $x \blacktriangleright y \blacktriangleleft z := \sigma(x) \triangleright y \triangleleft \tau(z)$, $x, z \in \mathcal{A}, y \in \mathcal{M}$, where $\triangleleft, \triangleright$ are the original actions on $\mathcal{M}$. Note that one has bimodule isomorphisms

$$ {}_{\sigma'}({}_\sigma\mathcal{M}_\tau)_{\tau'} \simeq {}_{\sigma\circ\sigma'}\mathcal{M}_{\tau\circ\tau'}, \quad \mathcal{M} \otimes_\mathcal{A} {}_\sigma\mathcal{N} \simeq \mathcal{M}_{\sigma^{-1}} \otimes_\mathcal{A} \mathcal{N}, \quad \mathcal{A}_{\sigma^{-1}} \simeq {}_\sigma\mathcal{A}. $$

2.1.2. *Hochschild homology.* The Hochschild homology groups of an algebra $\mathcal{A}$ with coefficients in an $\mathcal{A}$-bimodule $\mathcal{M}$ are

$$ H_n(\mathcal{A}, \mathcal{M}) := \mathrm{Tor}_n^{\mathcal{A}^e}(\mathcal{M}, \mathcal{A}), $$

where $\mathcal{A}^e := \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ is the enveloping algebra of $\mathcal{A}$ (so $\mathcal{A}^e$-modules are just $\mathcal{A}$-bimodules). They can be computed using the canonical bar resolution of $\mathcal{A}$ as an $\mathcal{A}^e$-module, and then are realised as the simplicial homology of the simplicial $\mathbb{K}$-vector space $C_\bullet(\mathcal{A}, \mathcal{M}) := \mathcal{M} \otimes \mathcal{A}^{\otimes\bullet}$ whose structure maps are

$$ \begin{aligned} &\mathsf{b}_0 : a_0 \otimes \ldots \otimes a_n \mapsto a_0 \triangleleft a_1 \otimes a_2 \otimes \ldots \otimes a_n, \\ &\mathsf{b}_i : a_0 \otimes \ldots \otimes a_n \mapsto a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \quad 0 < i < n, \\ &\mathsf{b}_n : a_0 \otimes \ldots \otimes a_n \mapsto a_n \triangleright a_0 \otimes \ldots \otimes a_{n-1}, \\ &\mathsf{s}_i : a_0 \otimes \ldots \otimes a_n \mapsto a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_n. \end{aligned} $$

That is, $H_\bullet(\mathcal{A}, \mathcal{M})$ is (isomorphic to) the homology of the chain complex $C_\bullet(\mathcal{A}, \mathcal{M})$ whose boundary map is given by

$$ \mathsf{b} := \sum_{i=0}^{n} (-1)^i \mathsf{b}_i. $$

In the sequel, we will often write $\mathsf{b}(a_0, \ldots, a_n)$ instead of $\mathsf{b}(a_0 \otimes \ldots \otimes a_n)$.

If $\mathcal{A}$ is the coordinate ring $\mathbb{K}[X]$ of a smooth affine variety, then $H_\bullet(\mathcal{A}, \mathcal{A})$ can be identified canonically with the Kähler differentials (algebraic differential forms) on X (Hochschild-Kostant-Rosenberg theorem).

2.1.3. *The normalised complex.* For any simplicial $\mathbb{K}$-vector space,

$$ D := \mathrm{span}\{\mathrm{im}\, \mathsf{s}_i\} \subset C $$

is a contractible subcomplex with respect to b, so the canonical map

$$ C \to \bar{C} := C/D $$

is a quasi-isomorphism of complexes, and to work with the so-called normalised complex $(\bar{C}_\bullet, \mathsf{b})$ simplifies many computations.

For $C = C(\mathcal{A}, \mathcal{M})$ from the previous section, we have $\bar{C}_n = \mathcal{M} \otimes \bar{A}^{\otimes n}$, where $\bar{A} := \mathcal{A}/\mathbb{K}$. So informally speaking one can neglect in the computation of Hochschild homology all elementary tensors with a tensor component being equal to a multiple of $1 \in \mathcal{A}$.

2.1.4. *Quantum $SL(2)$.* For the remainder of Section 2, $q \in \mathbb{K}$ denotes a fixed nonzero parameter, which we assume is not a root of unity. Furthermore, $\mathcal{A}$ is throughout the quantised coordinate algebra $\mathbb{K}_q[SL(2)]$ of $SL(2)$, that is, the algebra generated by symbols a, b, c, d with relations

$$ab = qba, \quad ac = qca, \quad bc = cb, \quad bd = qdb, \quad cd = qdc,$$

$$ad - qbc = 1, \quad da - q^{-1}bc = 1.$$

It follows from the defining relations that the elements

$$e_{i,j,k} := \begin{cases} a^i b^j c^k & i \geq 0 \\ d^{-i} b^j c^k & i < 0 \end{cases} \qquad i \in \mathbb{Z},\ j,k \in \mathbb{N} \tag{2}$$

form a vector space basis of $\mathcal{A}$.

For $\lambda, \mu \in \mathbb{K}^\times$ there are unique automorphisms $\sigma_{\lambda,\mu}, \tau_{\lambda,\mu}$ of $\mathcal{A}$ with

$$\sigma_{\lambda,\mu}(a) = \lambda a, \quad \sigma_{\lambda,\mu}(b) = \mu b, \quad \sigma_{\lambda,\mu}(c) = \mu^{-1} c, \quad \sigma_{\lambda,\mu}(d) = \lambda^{-1} d,$$

$$\tau_{\lambda,\mu}(a) = \lambda a, \quad \tau_{\lambda,\mu}(b) = \mu^{-1} c, \quad \tau_{\lambda,\mu}(c) = \mu b, \quad \tau_{\lambda,\mu}(d) = \lambda^{-1} d,$$

and all automorphisms of $\mathcal{A}$ are of this form (see [16]). For later use, we compute for all $\sigma_{\lambda,\mu}$ the twisted commutators

$$\begin{aligned}
e_{i,j,k}a - \lambda a e_{i,j,k} &= (q^{-j-k} - \lambda) e_{i+1,j,k} \\
&\quad + \begin{cases} 0 & i \geq 0 \\ (q^{-j-k-1} - \lambda q^{-1-2i}) e_{i+1,j+1,k+1} & i < 0, \end{cases} \\
e_{i,j,k}b - \mu b e_{i,j,k} &= (1 - \mu q^{-i}) e_{i,j+1,k}, \\
e_{i,j,k}c - \mu^{-1} c e_{i,j,k} &= (1 - \mu^{-1} q^{-i}) e_{i,j,k+1}, \\
e_{i,j,k}d - \lambda^{-1} d e_{i,j,k} &= (q^{j+k} - \lambda^{-1}) e_{i-1,j,k} \\
&\quad + \begin{cases} 0 & i \leq 0 \\ (q^{j+k+1} - \lambda^{-1} q^{1-2i}) e_{i-1,j+1,k+1} & i > 0 \end{cases}.
\end{aligned} \tag{3}$$

Finally, we recall (see [17]) that the standard Hopf algebra structure on $\mathcal{A}$ admits a so-called universal r-form $\mathbf{r} : \mathcal{A} \otimes \mathcal{A} \to \mathbb{K}$. This can be used in particular to define a braiding

$$\Psi : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad x \otimes y \mapsto \mathbf{r}(y_{(1)}, x_{(1)}) y_{(2)} \otimes x_{(2)} \mathbf{r}(S(y_{(3)}), x_{(3)}),$$

where $x \mapsto x_{(1)} \otimes x_{(2)}$ is the coproduct in Sweedler notation and $S : \mathcal{A} \to \mathcal{A}$ is the antipode. This braiding should be considered as a quantum analogue of the tensor flip and is used in the standard way to define

$$x \wedge y := (\mathrm{id} - \Psi)(x \otimes y) \tag{4}$$

and

$$\begin{aligned} x \wedge y \wedge z \quad := \quad & (\mathrm{id} - \Psi_{1,2} - \Psi_{2,3} + \Psi_{2,3} \circ \Psi_{1,2} + \Psi_{1,2} \circ \Psi_{2,3} \\ & - \Psi_{1,2} \circ \Psi_{2,3} \circ \Psi_{1,2})(x \otimes y \otimes z), \end{aligned}$$

where $\Psi_{2,3} := \mathrm{id} \otimes \Psi$ and $\Psi_{1,2} := \Psi \otimes \mathrm{id}$. On generators, one has

$$\begin{bmatrix} \mathbf{r}(a,a) & \mathbf{r}(a,b) & \mathbf{r}(a,c) & \mathbf{r}(a,d) \\ \mathbf{r}(b,a) & \mathbf{r}(b,b) & \mathbf{r}(b,c) & \mathbf{r}(b,d) \\ \mathbf{r}(c,a) & \mathbf{r}(c,b) & \mathbf{r}(c,c) & \mathbf{r}(c,d) \\ \mathbf{r}(d,a) & \mathbf{r}(d,b) & \mathbf{r}(d,c) & \mathbf{r}(d,d) \end{bmatrix} = q^{-1/2} \begin{bmatrix} q & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & q - q^{-1} & 0 & 0 \\ 1 & 0 & 0 & q \end{bmatrix}$$

and

$$\begin{aligned}
&\Psi(a\otimes a)=a\otimes a,\quad \Psi(a\otimes b)=q^{-1}b\otimes a+(1-q^{-2})a\otimes b,\\
&\Psi(a\otimes c)=qc\otimes a,\quad \Psi(a\otimes d)=d\otimes a+(q-q^{-1})c\otimes b,\\
&\Psi(b\otimes a)=q^{-1}a\otimes b,\quad \Psi(b\otimes b)=b\otimes b,\quad \Psi(b\otimes c)=c\otimes b,\\
&\Psi(b\otimes d)=qd\otimes b,\quad \Psi(c\otimes a)=qa\otimes c+(1-q^{2})c\otimes a,\\
&\Psi(c\otimes b)=b\otimes c-(q-q^{-1})^2c\otimes b+(q-q^{-1})a\otimes d+(q^{-1}-q)d\otimes a,\\
&\Psi(c\otimes c)=c\otimes c,\quad \Psi(c\otimes d)=q^{-1}d\otimes c+(1-q^{-2})c\otimes d,\\
&\Psi(d\otimes a)=a\otimes d-(q-q^{-1})c\otimes b,\quad \Psi(d\otimes b)=qb\otimes d+(1-q^{2})d\otimes b,\\
&\Psi(d\otimes c)=q^{-1}c\otimes d,\quad \Psi(d\otimes d)=d\otimes d.
\end{aligned}$$

2.2. **Results.** Here we recall from [12] the description of $H_n(\mathcal{A},{}_\sigma\mathcal{A})$ for $\mathcal{A}=\mathbb{K}_q[SL(2)]$ and $\sigma=\sigma_{\lambda,\mu}$, but we simplify the presentation and also correct some errors. Throughout, elements of $H_n(\mathcal{A},{}_\sigma\mathcal{A})$ are represented in the normalised Hochschild complex $\bar{C}_\bullet(\mathcal{A},{}_\sigma\mathcal{A})$.

2.2.1. $n=0$. $H_0(\mathcal{A},{}_\sigma\mathcal{A})$ is easily computed directly, using the canonical complex $C_\bullet(\mathcal{A},{}_\sigma\mathcal{A})$. Since

$$x\otimes yz=xy\otimes z+\sigma(z)x\otimes y-\mathsf{b}(x,y,z), \tag{5}$$

the boundary operator b on $C_1(\mathcal{A},{}_\sigma\mathcal{A})$ satisfies

$$\mathsf{b}(x,yz)=\mathsf{b}(xy,z)+\mathsf{b}(\sigma(z)x,y),$$

so its image is spanned by $\mathsf{b}(e_{i,j,k},a)$, $\mathsf{b}(e_{i,j,k},b)$, $\mathsf{b}(e_{i,j,k},c)$ and $\mathsf{b}(e_{i,j,k},d)$, that is, by the twisted commutators (3). This yields a vector space basis of $H_0(\mathcal{A},{}_\sigma\mathcal{A})$ consisting of the homology classes

$$\begin{aligned}
&\{[a^i],[d^i]\,|\,i\geq 0,\lambda=1\}\cup\{[b^j],[c^j]\,|\,j\in S,\mu=1\}\\
\cup\quad&\{[\omega_{N,i}]\,|\,i=0,\dots,N,\lambda=q^{-N},N>0,\mu=1\}\\
\cup\quad&\{[e_{i,N,0}],[e_{-i,0,N}]\,|\,\lambda=q^{-N},\mu=q^i,i\neq 0,N>0\},
\end{aligned}\tag{6}$$

where

$$S:=\mathbb{N}\setminus\{N-2r\,|\,\lambda=q^{-N},N>0,r\geq 0\},\quad \omega_{r,i}:=b^ic^{r-i}, \tag{7}$$

and [1] is counted only once when appearing repeatedly in (6).

Note that although this computation was carried out correctly in [12], the overview table on p. 328-329 therein claimed that $H_0(\mathcal{A},{}_\sigma\mathcal{A})$ is infinite-dimensional only for $\mu=1$ whereas it should correctly read $\mu=1$ or $\lambda=1$.

2.2.2. $n=1$. We also used $C_\bullet(\mathcal{A},{}_\sigma\mathcal{A})$ to compute $H_1(\mathcal{A},{}_\sigma\mathcal{A})$. By (5), $H_1(\mathcal{A},{}_\sigma\mathcal{A})$ is generated by the classes of linear combinations of $e_{i,j,k}\otimes a$, $e_{i,j,k}\otimes b$, $e_{i,j,k}\otimes c$ and $e_{i,j,k}\otimes d$. These are mapped by b to the twisted commutators (3), and it is straightforward to compute which linear combination of these tensors defines a cycle and which of these are homologous to each other (see [12] for the details). As a result, one obtains the following vector space basis of $H_1(\mathcal{A}, {}_{\sigma_{\lambda,\mu}}\mathcal{A})$:

$$
\begin{array}{rl}
 & \{[a^i \otimes a], [d^i \otimes d] \,|\, i \geq 0, \lambda = 1\} \\
\cup & \{[b^{j-1} \otimes b], [c^{j-1} \otimes c] \,|\, j \in S, \mu = 1\} \\
\cup & \{[\omega_{N-1,i} \otimes b], [\omega_{N-1,i} \otimes c] \,|\, 0 \leq i \leq N-1, \lambda = q^{-N}, N > 0, \mu = 1\} \\
\cup & \{[a^{i-1}b^N \otimes a], [a^i b^{N-1} \otimes b] \,|\, \lambda = q^{-N}, \mu = q^i, i, N > 0\} \\
\cup & \{[d^i c^{N-1} \otimes c], [d^{i-1}c^N \otimes d] \,|\, \lambda = q^{-N}, \mu = q^i, i, N > 0\} \\
\cup & \{[d^{i-1}b^N \otimes d], [d^i b^{N-1} \otimes b] \,|\, \lambda = q^{-N}, \mu = q^{-i}, i, N > 0\} \\
\cup & \{[a^i c^{N-1} \otimes c], [a^{i-1}c^N \otimes a] \,|\, \lambda = q^{-N}, \mu = q^{-i}, i, N > 0\},
\end{array}
\tag{8}
$$

where S and $\omega_{r,i}$ are as in (7) and we write with abuse of notation

$$[c^{-1} \otimes c] := [b \otimes c], \quad [b^{-1} \otimes b] := [c \otimes b]$$

which appears in the above set except when $\lambda = q^{-N}$ with $N = 2r$, $r > 0$, but should be counted only once:

LEMMA 2.1. *For $\lambda \neq q^{-2}$, one has $[b \otimes c] = -[c \otimes b]$.*

Proof. This follows from $[1 \otimes bc] = [b \otimes c] + [c \otimes b]$ (as a special case of (5)) in combination with $\mathsf{b}(1 \otimes (a \otimes d - \lambda^{-1} d \otimes a + (1-\lambda^{-1}) \otimes 1)) = (\lambda^{-1}q^{-1} - q) \otimes bc$. □

2.2.3. $n = 2$. In higher degrees, working with the canonical complex is no longer feasible, but we showed in [12], Proposition 4.1 that the trival left $\mathcal{A}$-module $\mathbb{K}$ (on which $\mathcal{A}$ acts by the counit ε of the standard Hopf algebra structure) admits a noncommutative Koszul resolution of the form

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^3 \longrightarrow \mathcal{A}^3 \longrightarrow \mathcal{A} \longrightarrow 0 \tag{9}$$

with morphisms given by the matrices

$$\begin{pmatrix} c & -b & q^{-2}a - 1 \end{pmatrix}, \quad \begin{pmatrix} b & 1 - q^{-1}a & 0 \\ c & 0 & 1 - q^{-1}a \\ 0 & c & -b \end{pmatrix}, \quad \begin{pmatrix} a - 1 \\ b \\ c \end{pmatrix}$$

that operate by right multiplication on row vectors. Then we computed $H_n(\mathcal{A}, {}_\sigma\mathcal{A})$ for $n \geq 2$, using the alternative derived functor description of $H_\bullet(\mathcal{A}, \mathcal{M})$ for Hopf algebras due to Feng and Tsygan [10].

However, for $n = 2$ the computations in [12] contain a mistake: The second half of the first sentence after Proposition 4.10 on page 349 in [12] is wrong, the generators given in Proposition 4.10 for $\mu = 1$ all remain linearly independent in homology. The correct result is that $H_2(\mathcal{A}, {}_{\sigma_{\lambda,\mu}}\mathcal{A}) = 0$ except when $\lambda = q^{-N}$, $N > 0$, and in this case

$$\dim_{\mathbb{K}} H_2(\mathcal{A}, {}_{\sigma_{\lambda,\mu}}\mathcal{A}) \cong \begin{cases} 2(N-1) & \mu = 1 \\ 2 & \mu = q^{\pm i}, i > 0 \\ 0 & \mu \notin q^{\mathbb{Z}} \end{cases}$$

with a linear basis given by

$$
\begin{array}{rl}
 & \{[\omega_2(N-2,i)], [\omega_2'(N-2,i)] \,|\, i = 0, \ldots, N-2, \mu = 1\} \\
\cup & \{[a^{i-1}b^{N-1} \otimes (b \wedge a)], [d^{i-1}c^{N-1} \otimes (d \wedge c)] \,|\, \mu = q^i, i > 0\} \\
\cup & \{[a^{i-1}c^{N-1} \otimes (a \wedge c)], [d^{i-1}b^{N-1} \otimes (b \wedge d)] \,|\, \mu = q^{-i}, i > 0\},
\end{array}
$$

where $x \wedge y$ is as defined in (4) and

$$\begin{aligned} \omega_2(r,i) &:= \omega_{r,i}\big(bc \otimes (a \wedge d) - bd \otimes (a \wedge c) + \\ &\quad da \otimes (b \wedge c) - q^{-1} ca \otimes (b \wedge d)\big), \\ \omega_2'(r,i) &:= \omega_{r,i} \otimes (b \wedge c). \end{aligned}$$

2.2.4. $n = 3$. One has $H_3(\mathcal{A}, {}_{\sigma_{\lambda,\mu}}\mathcal{A}) = 0$ except when $\lambda = q^{-N}$, $N > 1$, $\mu = 1$, and in this case

$$\dim_{\mathbb{K}} H_3(\mathcal{A}, {}_{\sigma_{q^{-N},1}}\mathcal{A}) = N - 1$$

with a basis given in the normalised complex $\bar{C}_3(\mathcal{A}, {}_{\sigma_{q^{-N},1}})$ by the classes of

(10) $$\omega_3(N-2,i) := \omega_{r,i}\big(-qd \otimes (b \wedge a \wedge c) + c \otimes (b \wedge a \wedge d)\big), \quad 0 \leq i \leq N-2.$$

Again, there was a misprint in [12], where in the explicit formula for $b \wedge a \wedge d$ an additional term $-(q^{-1} - q) b^i c^{N+1-i} \otimes b \otimes c \otimes b$ was missing.

2.2.5. $n > 3$. Since the resolution (9) has length 3, $H_n(\mathcal{A}, {}_\sigma\mathcal{A}) = 0$ for $n > 3$.

3. Hochschild cohomology

3.1. **Background.** Section 3.1 recalls products and Poincaré duality in Hochschild (co)homology (see e.g. [3, 25, 28]), as well as twisted derivations of $\mathbb{K}_q[SL(2)]$ that arise from the left and right action of its Hopf dual.

3.1.1. *Hochschild cohomology.* The Hochschild cohomology $H^n(\mathcal{A}, \mathcal{M}) := \mathrm{Ext}^n_{\mathcal{A}^e}(\mathcal{A}, \mathcal{M})$ of $\mathcal{A}$ with coefficients in an $\mathcal{A}$-bimodule $\mathcal{M}$ can be computed as the cohomology of the cochain complex $C^\bullet(\mathcal{A}, \mathcal{M})$ of $\mathbb{K}$-linear maps $\varphi : \mathcal{A}^{\otimes \bullet} \to \mathcal{M}$ with coboundary map given by

$$\begin{aligned} (\mathsf{b}\varphi)(a_1, \ldots, a_{n+1}) &:= a_1 \triangleright \varphi(a_2, \ldots, a_{n+1}) - \varphi(a_1 a_2, \ldots, a_{n+1}) + \ldots \\ &\quad + (-1)^{n+1} \varphi(a_1, \ldots, a_n) \triangleleft a_{n+1}. \end{aligned}$$

This presentation of cocycles yields for example the standard identification

$$H^0(\mathcal{A}, \mathcal{M}) \simeq \{m \in \mathcal{M} \mid a \triangleright m = m \triangleleft a \text{ for all } a\},$$

and of $H^1(\mathcal{A}, \mathcal{M})$ with the space of derivations

(11) $$\partial : \mathcal{A} \to \mathcal{M}, \quad \partial(xy) = x \triangleright \partial(y) + \partial(x) \triangleleft y$$

modulo inner ones (those of the form $x \mapsto x \triangleright m - m \triangleleft x$, $m \in \mathcal{M}$).

3.1.2. *The cup product.* The cup product

$$\smile : H^m(\mathcal{A}, \mathcal{M}) \otimes H^n(\mathcal{A}, \mathcal{N}) \to H^{m+n}(\mathcal{A}, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$$

is given on the level of cochains by

$$(\varphi \smile \psi)(a_1, \ldots, a_{m+n}) := \varphi(a_1, \ldots, a_m) \otimes \psi(a_{m+1}, \ldots, a_{m+n}),$$

where $\varphi \in C^m(\mathcal{A}, \mathcal{M}), \psi \in C^n(\mathcal{A}, \mathcal{N})$. Since

$$\mathsf{b}(\varphi \smile \psi) = (\mathsf{b}\varphi) \smile \psi + (-1)^m \varphi \smile (\mathsf{b}\psi)$$

(for $\varphi \in C^m(\mathcal{A}, \mathcal{M})$), the cup product is well-defined on the level of cohomology. As a special case, we obtain for $\sigma, \tau \in \mathrm{Aut}(\mathcal{A})$ a map

(12) $$H^m(\mathcal{A}, {}_\sigma\mathcal{A}) \otimes H^n(\mathcal{A}, {}_\tau\mathcal{A}) \to H^{m+n}(\mathcal{A}, {}_{\tau \circ \sigma}\mathcal{A})$$

given on cochains by

$$(\varphi \smile \psi)(a_1, \ldots, a_{m+n}) = \tau(\varphi(a_1, \ldots, a_m))\psi(a_{m+1}, \ldots, a_{m+n}). \tag{13}$$

Thus for any monoid $G \subset \mathrm{Aut}(\mathcal{A})$ we obtain an $\mathbb{N} \times G$-graded algebra

$$\Lambda^\bullet_G(\mathcal{A}) := \bigoplus_{n \in \mathbb{N}, \sigma \in G} H^n(\mathcal{A}, {}_\sigma\mathcal{A}).$$

We call the subalgebra $\Lambda^0_G(\mathcal{A})$ the G-twisted centre of $\mathcal{A}$ and the elements of the $\Lambda^0_G(\mathcal{A})$-bimodule $\Lambda^1_G(\mathcal{A})$ the G-twisted derivations of $\mathcal{A}$. Obviously, G could be replaced by any monoid of bimodules, but we will not need this in the present paper.

Note that in degree 0, (13) reduces to the *opposite* product of $\mathcal{A}$,

$$x \smile y = yx, \quad x \in H^0(\mathcal{A}, {}_\sigma\mathcal{A}), \quad y \in H^0(\mathcal{A}, {}_\tau\mathcal{A}), \tag{14}$$

and that for $z \in H^0(\mathcal{A}, {}_\sigma\mathcal{A})$ and $\partial \in C^1(\mathcal{A}, {}_\tau\mathcal{A})$ we have

$$\begin{aligned} & (z \smile \partial)(x) = \sigma(\partial(x))z = z\partial(x), \\ & (\partial \smile z)(x) = \tau(z)\partial(x), \\ \Rightarrow \quad & \partial \smile z = \tau(z) \smile \partial. \end{aligned} \tag{15}$$

Finally, we have for twisted derivations $\partial \in C^1(\mathcal{A}, {}_\sigma\mathcal{A}), \partial' \in C^1(\mathcal{A}, {}_\tau\mathcal{A})$

$$\begin{aligned} & (\partial \smile \partial' + (\sigma^{-1} \circ \partial' \circ \sigma) \smile \partial)(x, y) \\ = \; & \tau(\partial(x))\partial'(y) + \partial'(\sigma(x))\partial(y) \\ = \; & \partial'(\partial(x)y) - \partial'(\partial(x))y + \partial'(\sigma(x)\partial(y)) - \tau(\sigma(x))\partial'(\partial(y)) \\ = \; & \partial'(\partial(xy)) - \partial'(\partial(x))y - \tau(\sigma(x))\partial'(\partial(y)) \\ = \; & -\mathsf{b}(\partial' \circ \partial)(x, y), \end{aligned}$$

so in cohomology one has

$$[\partial] \smile [\partial'] = -[\sigma^{-1} \circ \partial' \circ \sigma] \smile [\partial] \in H^2(\mathcal{A}, {}_{\tau \circ \sigma}\mathcal{A}). \tag{16}$$

3.1.3. *The cap product.* The duality between Hochschild homology and cohomology results from the cap product pairing

$$\frown : H_n(\mathcal{A}, \mathcal{M}) \otimes H^m(\mathcal{A}, \mathcal{N}) \to H_{n-m}(\mathcal{A}, \mathcal{M} \otimes_\mathcal{A} \mathcal{N}), \quad m \leq n$$

defined on (co)chains simply by evaluation,

$$(a_0 \otimes \ldots \otimes a_n) \frown \varphi = a_0 \otimes \varphi(a_1, \ldots, a_m) \otimes a_{m+1} \otimes \ldots \otimes a_n.$$

For $m = n$, the pairing $\frown$ becomes the duality pairing from [23], Section 1.5.9 after identifying

$$H_0(\mathcal{A}, \mathcal{M} \otimes_\mathcal{A} \mathcal{N}) = \mathcal{M} \otimes_\mathcal{A} \mathcal{N} \otimes_{\mathcal{A}^e} \mathcal{A} \simeq \mathcal{M} \otimes_{\mathcal{A}^e} \mathcal{N}.$$

Taking $\mathcal{N} = \mathcal{M}^* = \mathrm{Hom}_\mathbb{K}(\mathcal{M}, \mathbb{K})$ and composing with the canonical evaluation map $\mathcal{M} \otimes_{\mathcal{A}^e} \mathcal{M}^* \to \mathbb{K}$ gives the duality pairing

$$H_n(\mathcal{A}, \mathcal{M}) \otimes H^n(\mathcal{A}, \mathcal{M}^*) \to \mathbb{K}.$$

By the universal coefficient theorem, this yields an isomorphism $H^n(\mathcal{A}, \mathcal{M}^*) \simeq (H_n(\mathcal{A}, \mathcal{M}))^*$. In this way a Hochschild cocycle $\varphi \in C^n(\mathcal{A}, \mathcal{M}^*)$ will usually be viewed as a $\mathbb{K}$-linear map $\mathcal{M} \otimes \mathcal{A}^{\otimes n} \to \mathbb{K}$.

For any $G \subset \mathrm{Aut}(\mathcal{A})$, the cap product endows

$$\Omega^G_\bullet(\mathcal{A}) := \bigoplus_{n \in \mathbb{N}, \sigma \in G} H_n(\mathcal{A}, {}_{\sigma^{-1}}\mathcal{A})$$

with the structure of an $\mathbb{N} \times G$-graded (right) module over $\Lambda^\bullet_G(\mathcal{A})$ (here we use the convention that a homogeneous element of a graded ring *lowers* the degree of an element of a homogeneous module by its degree). Be aware that one has to take into account the identification ${}_\sigma\mathcal{A} \otimes_\mathcal{A} {}_\tau\mathcal{A} \to {}_{\tau\circ\sigma}\mathcal{A}$: explicitly, the action of $\varphi \in C^m(\mathcal{A}, {}_\tau\mathcal{A})$ on $a_0 \otimes \ldots \otimes a_n \in C_n(\mathcal{A}, {}_\sigma\mathcal{A})$ is given by

$$(a_0 \otimes \ldots \otimes a_n) \frown \varphi = \tau(a_0)\varphi(a_1, \ldots, a_m) \otimes a_{m+1} \otimes \ldots \otimes a_n \in C_{n-m}(\mathcal{A}, {}_{\tau\circ\sigma}\mathcal{A}).$$

In particular, the cap product with a twisted central element $z \in H^0(\mathcal{A}, {}_\sigma\mathcal{A})$ is simply given by multiplication from the left,

$$(a_0 \otimes \ldots \otimes a_n) \frown z = \sigma(a_0)z \otimes \ldots \otimes a_n = za_0 \otimes \ldots \otimes a_n. \tag{17}$$

3.1.4. *Poincaré duality.* The cap product is also the source of Poincaré-type dualities in Hochschild (co)homology. As pointed out by van den Bergh [28], for well-behaved algebras $\mathcal{A}$ there exists a specific invertible bimodule $\mathcal{D}$ and a fixed number d such that

$$H^n(\mathcal{A}, \mathcal{N}) \simeq H_{d-n}(\mathcal{A}, \mathcal{D} \otimes_\mathcal{A} \mathcal{N}) \tag{18}$$

for all bimodules $\mathcal{N}$. This applies in particular to $\mathcal{A} = \mathbb{K}_q[SL(2)]$ [13]:

THEOREM 3.1. *$\mathcal{A} = \mathbb{K}_q[SL(2)]$ satisfies (18) with $n = 3$ and $\mathcal{D} = {}_{\sigma_{q^{-2},1}}\mathcal{A}$.*

3.1.5. *Twisted primitive elements in the Hopf dual of* $\mathbb{K}_q[SL(2)]$. For the rest of Section 3, we focus again on $\mathcal{A} = \mathbb{K}_q[SL(2)]$ and recall (see e.g. [15, 17] and the references therein) that the Hopf dual of the standard Hopf algebra structure on $\mathcal{A}$ contains a Hopf subalgebra $\mathcal{U}$ that has generators H, K, K^{-1}, E, F having relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \quad [H, K] = 0, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

This is the standard Drinfeld-Jimbo quantised universal enveloping algebra $U_q(\mathfrak{sl}(2))$ extended by the unquantised functional H (so when working with formal deformations one would have $K = e^{\hbar H}$, $q = e^\hbar$).

The dual pairing of $\mathcal{U}$ and $\mathcal{A}$ gives two commuting left and right actions of $\mathcal{U}$ on $\mathcal{A}$, and the operators assigned by these actions to H, EK^{-1} and F are (twisted) derivations which we denote by

$$\partial^+_H : a, b, c, d \mapsto -a, b, -c, d, \quad \partial^-_H : a, b, c, d \mapsto -a, -b, c, d \in C^1(\mathcal{A}, \mathcal{A}),$$
$$\partial^+_E : a, b, c, d \mapsto qb, 0, qd, 0, \quad \partial^+_F : a, b, c, d \mapsto 0, a, 0, c \in C^1(\mathcal{A}, {}_{\sigma_{q,q^{-1}}}\mathcal{A}),$$
$$\partial^-_E : a, b, c, d \mapsto 0, 0, q^{-1}a, q^{-1}b, \quad \partial^-_F : a, b, c, d \mapsto c, d, 0, 0 \in C^1(\mathcal{A}, {}_{\sigma_{q,q}}\mathcal{A}).$$

Because of the Leibniz rule (11), this determines the derivations uniquely.

3.2. **Results.** Here we compute the twisted centre and the twisted derivations of $\mathbb{K}_q[SL(2)]$. Then we show how their cap product action on the twisted Hochschild homology groups can be used to determine the homology class of a given 2- or 3-cycle.

3.2.1. *Twisted central elements.* The twisted centre $\Lambda^0 := \Lambda^0_{\mathrm{Aut}(\mathcal{A})}(\mathcal{A})$ of $\mathcal{A}$ is a (commutative) polynomial ring in two indeterminates:

LEMMA 3.2. *There is an isomorphism of graded algebras*

$$\Lambda^0 = \bigoplus_{N \geq 0} H^0(\mathcal{A}, {}_{\sigma_{q^{-N},1}}\mathcal{A}) \simeq \mathbb{K}[b, c].$$

Proof. By Poincaré duality (Theorem 3.1) and the results of the computation of $H_3(\mathcal{A}, {}_\sigma\mathcal{A})$ recalled in Section 2.2.4, $H^0(\mathcal{A}, {}_\sigma\mathcal{A})$ vanishes except when $\sigma = \sigma_{q^{-N},1}$, $N \geq 0$, and in this case it has dimension $N+1$ over $\mathbb{K}$. Obviously, the monomials $\omega_{N,i} = b^i c^{N-i}$, $i = 0, \ldots, N$, form $N+1$ linearly independent elements of $H^0(\mathcal{A}, {}_{\sigma_{q^{-N},1}}\mathcal{A})$ and hence a vector space basis. Finally, (14) gives $\omega_{r,i} \smile \omega_{s,j} = \omega_{r,i}\omega_{s,j}$ since b and c commute. □

The cap product action of Λ^0 gives additional structure to the results of our computations of $H_\bullet(\mathcal{A}, {}_\sigma\mathcal{A})$. For example, the direct sum of the nontrivial $H_3(\mathcal{A}, {}_\sigma\mathcal{A})$ forms a free module of rank one over Λ^0 with generator $\mathsf{d}\mathcal{A}$. Similarly we will see below that applying twisted derivations to $\mathsf{d}\mathcal{A}$ leads for example to the 2-dimensional $H_2(\mathcal{A}, {}_{\sigma_{q^{-N},q^{\pm i}}}\mathcal{A})$.

3.2.2. *Detecting nontrivial 2-cycles.* For large n, m it is typically difficult to decide whether or not a given cycle $c \in C_n(\mathcal{A}, {}_\sigma\mathcal{A})$ and cocycle $\varphi \in C^m(\mathcal{A}, {}_\tau\mathcal{A})$ have trivial classes in (co)homology. A sufficient criterion is that $c \frown \varphi \in C_{n-m}(\mathcal{A}, {}_{\tau\circ\sigma}\mathcal{A})$ is nontrivial in homology, and this might be easy to verify for small $n - m$. Here we give an example of this kind for $\mathcal{A} = \mathbb{K}_q[SL(2)]$ whose result is used below in the computation of twisted cyclic homology, and also in order to determine $\Lambda^1_{\mathrm{Aut}(\mathcal{A})}(\mathcal{A})$.

LEMMA 3.3. *Abbreviate* $\partial := \frac{1}{2}(\partial^+_H + \partial^-_H)$ *and* $\partial' := -\partial^-_H$. *Then one has*

$$[\omega_2(N-2,i)] \frown [\partial] = [\omega'_2(N-2,i)] \frown [\partial'] = [\omega_{N-1,i+1} \otimes c] + [\omega_{N-1,i} \otimes b],$$
$$[\omega'_2(N-2,i)] \frown [\partial] = [\omega_2(N-2,i)] \frown [\partial'] = 0.$$

Proof. For $\sigma = \sigma_{q^{-2},1}$ one has in $\bar{C}_1(\mathcal{A}, {}_\sigma\mathcal{A})$

$$\begin{aligned}
&\mathsf{b}(bc \otimes a \otimes d) = q^{-2}abc \otimes d - qbc \otimes bc + q^2 dbc \otimes a,\\
&\mathsf{b}(bc \otimes b \otimes c) = b^2c \otimes c - bc \otimes bc + bc^2 \otimes b,\\
&\mathsf{b}(ca \otimes (d \wedge b)) = (q^3 - q)bc^2 \otimes b,\\
&\mathsf{b}(ba \otimes (d \wedge c)) = (q - q^{-1})b^2c \otimes c.
\end{aligned}$$

Using this, one computes directly that

$$\begin{aligned}
[\omega_2(0,0)] \frown [\partial^+_H] &= 2[-q^{-2}abc \otimes d - q^2 dbc \otimes a + qbc^2 \otimes b\\
&\quad +q^{-1}b^2c \otimes c + c \otimes b + b \otimes c]\\
&= 2(q^{-1} - q)[\omega_{3,2} \otimes c] + 2[\omega_{1,1} \otimes c] + 2[\omega_{1,0} \otimes b]\\
&= 2[\omega_{1,1} \otimes c] + 2[\omega_{1,0} \otimes b],
\end{aligned}$$

that $[\omega_2(0,0)] \frown [\partial^-_H] = 0$, and that

$$[\omega'_2(0,0)] \frown [\partial^+_H] = -[\omega'_2(0,0)] \frown [\partial^-_H] = [c \otimes b + b \otimes c].$$

The claim follows by $\Lambda^0_{\mathrm{Aut}(\mathcal{A})}(\mathcal{A})$-linearity of the products. □

COROLLARY 3.4. *The $2(N+1)$ cohomology classes*

$$[\omega_{N,i}] \smile [\partial], \quad [\omega_{N,i}] \smile [\partial'], \quad i = 0, \dots, N,$$

are linearly independent in $H^1(\mathcal{A}, {}_{\sigma_{q^{-N},1}}\mathcal{A})$.

3.2.3. *Twisted derivations.* We now describe the twisted derivations (modulo inner derivations) $\Lambda^1 := \Lambda^1_{\mathrm{Aut}(\mathcal{A})}(\mathcal{A})$ as a bimodule over the twisted centre Λ^0, and study their relations under the cup product.

Note first that for all $i > 0$, the cochains

$$\begin{aligned}
C^1(\mathcal{A}, {}_{\sigma_{q,q^{-i}}}\mathcal{A}) &\ni \partial^+_E \smile d^{i-1} : a, b, c, d \mapsto q d^{i-1} b, 0, q d^i, 0,\\
C^1(\mathcal{A}, {}_{\sigma_{q,q^{-i}}}\mathcal{A}) &\ni \partial^+_F \smile a^{i-1} : a, b, c, d \mapsto 0, a^i, 0, a^{i-1} c,\\
C^1(\mathcal{A}, {}_{\sigma_{q,q^{i}}}\mathcal{A}) &\ni \partial^-_E \smile a^{i-1} : a, b, c, d \mapsto 0, 0, q^{-1} a^i, q^{-1} a^{i-1} b,\\
C^1(\mathcal{A}, {}_{\sigma_{q,q^{i}}}\mathcal{A}) &\ni \partial^-_F \smile d^{i-1} : a, b, c, d \mapsto d^{i-1} c, d^i, 0, 0,
\end{aligned}$$

are (twisted) derivations, although $a^{i-1}, d^{i-1} \notin \Lambda^0$. The point is that for example $\partial^+_E(\mathcal{A}) \subset \mathcal{B}$, where $\mathcal{B} \subset \mathcal{A}$ is the subalgebra generated by b, d, and one has $d^{i-1} \in \Lambda^0_{\mathrm{Aut}(\mathcal{B})}(\mathcal{B})$ with twisting automorphism extending to the whole of $\mathcal{A}$. This implies the claim which of course can also be verified directly. We denote the classes of these derivations in cohomology by $\partial^+_i, \partial^+_{-i}, \partial^-_i, \partial^-_{-i}$, respectively, and put $\partial^\pm_0 := [\partial^\pm_H]$.

LEMMA 3.5. *As a left Λ^0-module, Λ^1 is generated by the $\{\partial^\pm_i, i \in \mathbb{Z}\}$, and*

$$\begin{aligned}
&[b] \smile \partial^\pm_{-i} = 0, \quad [c] \smile \partial^\pm_i = 0, \quad i > 1,\\
&\partial^\pm_i \smile [b^j c^k] = q^{\pm i(j-k)} [b^j c^k] \smile \partial^\pm_i, \quad i \in \mathbb{Z},\ j, k \in \mathbb{N}, \qquad (19)\\
&\partial^\varepsilon_i \smile \partial^\delta_j = -q^{\delta j + \varepsilon |i| \mathrm{sgn}(j)} \partial^\delta_j \smile \partial^\varepsilon_i, \quad i, j \in \mathbb{Z},\ \varepsilon, \delta \in \{-1, +1\}.
\end{aligned}$$

Proof. The second and third lines in (19) are computed directly from (15) and (16). Next, notice that Section 2.2.3 and (18) give

$$\dim_{\mathbb{K}} H^1(\mathcal{A}, {}_{\sigma_{\lambda,\mu}}\mathcal{A}) = \begin{cases} 2(N+1) & \lambda = q^{-N}, \mu = 1,\\ 2 & \lambda = q^{-j}, \mu = q^i, i \neq 0, j > 0,\\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.4 implies that $\{[\omega_{N,i}] \smile \partial^\pm_0 \mid i = 0, \dots, N\}$ is a linearly independent subset of $H^1(\mathcal{A}, {}_{\sigma_{q^{-N},1}}\mathcal{A})$, so it is a basis for dimension reasons. The $\partial^\pm_i$, $i \neq 0$, are treated similarly. For example, one can use that for $i, j > 0$

$$[\omega'_2(0,0)] \frown (\partial^\pm_i \smile \partial^\pm_{-j}) = -q^{\pm(1-i)} [e_{\pm(j-i),0,0}] \in H_0(\mathcal{A}, {}_{\sigma_{1,q^{\mp(i+j)}}}\mathcal{A}),$$

and that by the third line in (15),

$$\partial^\varepsilon_i \smile \partial^\delta_j = q^{(\mathrm{sgn}(i)+\mathrm{sgn}(j))(\varepsilon|i|+\delta|j|)} \partial^\varepsilon_i \smile \partial^\delta_j,$$

so $\partial^\varepsilon_i \smile \partial^\delta_j = 0$ except when $\mathrm{sgn}(j) = -\mathrm{sgn}(i)$ or $\delta = -\varepsilon, |j| = |i|$. $\square$

3.2.4. *Detecting nontrivial 3-cycles.* Here we present computations similar to those in Section 3.2.2, but this time we act on $\omega_3(0,0)$ and want to decide whether 3-cycles are trivial.

LEMMA 3.6. *In $H_1(\mathcal{A}, {}_{\sigma_{q^{-N},1}}\mathcal{A})$, one has*

$$[\omega_3(N-2,i)] \frown ([\partial_H^+] \smile [\partial_H^-]) = 2([\omega_{N-1,i+1} \otimes c] + [\omega_{N-1,i} \otimes b]).$$

In particular, $\cdot \frown ([\partial_H^+] \smile [\partial_H^-]) : \Omega_3^{\mathrm{Aut}(\mathcal{A})}(\mathcal{A}) \to \Omega_1^{\mathrm{Aut}(\mathcal{A})}(\mathcal{A})$ is injective.

Proof. By direct computation, one obtains $[\omega_3(0,0)] \frown [\partial_H^-] = -[\omega_2(0,0)]$, so the result follows from Lemma 3.3 and Lemma 3.5. □

As we shall see below, this lemma is in practice strong enough to detect nontrivial homology classes. However, it would be obviously more easy (and standard) to apply a further twisted derivation to obtain 0-cycles whose classes in homology are even simpler to control than those of 1-cycles. While this works very nicely for the fundamental class $\mathsf{d}\mathcal{A} = [\omega_3(0,0)]$, it is unfortunately not possible for all the other generators $\omega_3(r,i)$, $r > 0$: The orbit of $\mathsf{d}\mathcal{A}$ under the cap product action of Λ^1 is determined completely by Lemma 3.5 and the straightforwardly computed relations

$$\begin{aligned}
&\omega_3(0,0) \frown (\partial_H^+ \smile \partial_E^+ \smile \partial_F^+) = (q^{-1}-q)bc + 1,\\
&\omega_3(0,0) \frown (\partial_H^+ \smile \partial_E^+ \smile \partial_H^-) = 2(q^4-1)db^2c - 2qdb,\\
&\omega_3(0,0) \frown (\partial_H^+ \smile \partial_E^+ \smile \partial_E^-) = (q-q^{-3})b^3c - q^{-2}b^2,\\
&\omega_3(0,0) \frown (\partial_H^+ \smile \partial_E^+ \smile \partial_F^-) = (q-q^5)d^2bc + d^2,\\
&\omega_3(0,0) \frown (\partial_H^+ \smile \partial_F^+ \smile \partial_H^-) = 2(q-q^{-3})abc^2 - 2ac,\\
&\omega_3(0,0) \frown (\partial_H^+ \smile \partial_F^+ \smile \partial_E^-) = (q^{-1}-q^{-5})a^2bc - a^2,\\
&\omega_3(0,0) \frown (\partial_H^+ \smile \partial_F^+ \smile \partial_F^-) = (q^{-1}-q^3)bc^3 + (2-q^2)c^2,\\
&\omega_3(0,0) \frown (\partial_H^+ \smile \partial_H^- \smile \partial_E^-) = 2(q^{-4}-q^{-2})ab^2c + 2q^{-1}ab,\\
&\omega_3(0,0) \frown (\partial_H^+ \smile \partial_H^- \smile \partial_F^-) = 2(q-q^3)dbc^2 + 2dc,\\
&\omega_3(0,0) \frown (\partial_H^+ \smile \partial_E^- \smile \partial_F^-) = 2(1-q^2)b^2c^2 + 2q^{-1}bc + 1,\\
&\omega_3(0,0) \frown (\partial_E^+ \smile \partial_F^+ \smile \partial_H^-) = 2(q^{-4}-q^2)b^2c^2 + (2q^{-3}-q+q^{-1})bc + 1,\\
&\omega_3(0,0) \frown (\partial_E^+ \smile \partial_F^+ \smile \partial_E^-) = (q^{-7}-q^{-1})ab^2c + q^{-4}ab,\\
&\omega_3(0,0) \frown (\partial_E^+ \smile \partial_F^+ \smile \partial_F^-) = (q^4-q^{-2})dbc^2 - q^{-1}dc,\\
&\omega_3(0,0) \frown (\partial_E^+ \smile \partial_H^- \smile \partial_E^-) = (q-q^{-3})b^3c - q^{-2}b^2,\\
&\omega_3(0,0) \frown (\partial_E^+ \smile \partial_H^- \smile \partial_F^-) = (q^5-q)d^2bc - d^2,\\
&\omega_3(0,0) \frown (\partial_E^+ \smile \partial_E^- \smile \partial_F^-) = (q^5-q)db^2c - db,\\
&\omega_3(0,0) \frown (\partial_F^+ \smile \partial_H^- \smile \partial_E^-) = (q^{-5}-q^{-1})a^2bc + a^2,\\
&\omega_3(0,0) \frown (\partial_F^+ \smile \partial_H^- \smile \partial_F^-) = (q^{-1}-q^3)bc^3 + (2-q^2)c^2,\\
&\omega_3(0,0) \frown (\partial_F^+ \smile \partial_E^- \smile \partial_F^-) = (q^{-2}-q^2)abc^2 + q^{-1}ac,\\
&\omega_3(0,0) \frown (\partial_H^- \smile \partial_E^- \smile \partial_F^-) = 1,
\end{aligned}$$

which hold on the level of chains in the normalised Hochschild complex. By inspection one now obtains:

LEMMA 3.7. *One has*

$$\mathsf{d}\mathcal{A} \frown [\partial_H^+ \smile \partial_E^+ \smile \partial_F^+] = [1] \in H_0(\mathcal{A}, {}_{\sigma_{1,q^{-2}}}\mathcal{A}),$$
$$[\omega_3(r,r)] \frown [\partial_H^+ \smile \partial_E^- \smile \partial_E^+] = [b^{r+2}] \in H_0(\mathcal{A}, {}_{\sigma_{q^{-r},1}}\mathcal{A}),$$
$$[\omega_3(r,0)] \frown [\partial_H^- \smile \partial_F^- \smile \partial_F^+] = [c^{r+2}] \in H_0(\mathcal{A}, {}_{\sigma_{q^{-r},1}}\mathcal{A}),$$

but for all $\partial_1, \partial_2, \partial_3 \in \Lambda^1$, $z \in \Lambda^0$, *and* $0 < i < r$ *one has*

$$[\omega_3(r,i)] \frown [z \smile \partial_1 \smile \partial_2 \smile \partial_3] = 0.$$

Proof. The first part is obtained by direct computation. For the second note that we can assume by Lemma 3.5 that $\partial_i \in \{\partial_H^+, \partial_H^-, \partial_E^+, \partial_E^-, \partial_F^+, \partial_F^-\}$, and we know that there is z' with $[z \smile \partial_1 \smile \partial_2 \smile \partial_3] = [\partial_1 \smile \partial_2 \smile \partial_3 \smile z']$ (z, z' might be monomials in a, b, c or d, b, c). The homology class of $\omega_3(0,0) \frown (\partial_1 \smile \partial_2 \smile \partial_3)$ can be read off for these ∂_i from the list above (using the commutation relations (15) of the ∂_i), and whenever the result is nonzero, then Lemma 3.5 gives $[\omega_{r,i} \smile \partial_1 \smile \partial_2 \smile \partial_3] = 0$ for $0 < i < r$, which implies the claim. □

To complete the picture, we compose the action of $[\partial_H^+ \smile \partial_E^+ \smile \partial_F^+]$ with a twisted trace

$$\int : \mathcal{A} \to \mathbb{K}, \qquad \int xy = \int \sigma_{1,q^{-2}}(y)x.$$

to obtain a numerical invariant of $\mathsf{d}\mathcal{A}$. For $\mathcal{A} = \mathbb{K}_q[SL(2)]$, the complete list of twisted traces can be given as follows: for any element $[e_{i,j,k}]$ of our basis (6) of $H_0(\mathcal{A}, {}_\sigma\mathcal{A})$ define a linear functional $\int_{[e_{i,j,k}]}$ by

$$\int_{[e_{i,j,k}]} e_{r,s,t} := \delta_{i,r} \cdot \begin{cases} \sum_{n=0}^{\infty} \delta_{s,j+n}\delta_{t,k+n}(-q)^n \frac{1-q^{j+k}\lambda}{1-q^{j+k+2n}\lambda} & i=0,\ jk=0 \\ \delta_{s,j}\delta_{t,k} & \text{otherwise.} \end{cases}$$

Then these descend to linearly independent functionals on $H_0(\mathcal{A}, {}_\sigma\mathcal{A})$ that are dual to the basis (6), $\int_{[e_{i,j,k}]}[e_{r,s,t}] = \delta_{i,r}\delta_{j,s}\delta_{k,t}$ for $[e_{i,j,k}], [e_{r,s,t}] \in (6)$.

If $\varphi \in C^n(\mathcal{A}, {}_\sigma\mathcal{A})$ is an n-cocycle and $\int \in C^0(\mathcal{A}, ({}_\tau\mathcal{A})^*)$ is a twisted trace, then, using ${}_\sigma\mathcal{A} \otimes ({}_\tau\mathcal{A})^* \simeq {}_\sigma({}_\tau\mathcal{A})^* \simeq ({}_\tau\mathcal{A}_\sigma)^* \simeq ({}_{\sigma^{-1}\circ\tau}\mathcal{A})^*$, $\varphi \smile \int$ can be identified with a functional on $H_n(\mathcal{A}, {}_{\sigma^{-1}\circ\tau}\mathcal{A})$. In particular, we get for $\mathcal{A} = \mathbb{K}_q[SL(2)]$ from the above Lemma 3.7:

COROLLARY 3.8. *Let* $\int_{[1]} : \mathcal{A} \to \mathbb{K}$ *be the* $\sigma_{1,q^{-2}}$*-twisted trace given by*

$$\int_{[1]} e_{r,s,t} := \delta_{0,r}\delta_{0,s}\delta_{0,t}.$$

Then

$$\int_{[1]} \mathsf{d}\mathcal{A} \frown [\partial_H^+ \smile \partial_E^+ \smile \partial_F^+] = 1.$$

In other words, the linear functional

$$\varphi_{\mathcal{A}}(a_0, a_1, a_2, a_3) := \int_{[1]} \sigma_{q^2,q^{-2}}(a_0\partial_H^+(a_1))\sigma_{q,q^{-1}}(\partial_E^+(a_2))\partial_F^+(a_3) \tag{20}$$

is a twisted Hochschild 3-cocycle with a nontrivial class in $H^3(\mathcal{A}, ({}_{\sigma_{q^{-2},1}}\mathcal{A})^*)$ that is dual to the fundamental class $\mathsf{d}\mathcal{A}$ in the sense that $\varphi_{\mathcal{A}}(\mathsf{d}\mathcal{A}) = 1$.

4. Cyclic homology

4.1. **Background.** In Section 4.1 we recall definitions and results used in the computation of the twisted cyclic homology of quantum $SL(2)$, namely. the definition of (para)cyclic objects, of cyclic homology, Connes' spectral sequence and the SBI-sequence. For background and details, see [23, 29].

4.1.1. *Paracyclic objects.* Paracyclic objects [11] (say in an abelian category) slightly generalise Connes' cyclic objects [4]:

DEFINITION 4.1. *A paracyclic object is a simplical object* $(C_\bullet, \mathsf{b}_\bullet, \mathsf{s}_\bullet)$ *equipped with morphisms* $\mathsf{t} : C_n \to C_n$ *that satisfy (on* C_n*)*

$$\mathsf{b}_i\mathsf{t} = -\mathsf{t}\mathsf{b}_{i-1},\ \mathsf{s}_i\mathsf{t} = -\mathsf{t}\mathsf{s}_{i-1},\ \mathsf{b}_0\mathsf{t} = (-1)^n\mathsf{b}_n,\ \mathsf{s}_0\mathsf{t} = (-1)^n\mathsf{t}^2\mathsf{s}_n,\ 1 \leq i \leq n.$$

The difference with cyclic objects is that $\mathsf{T} := \mathsf{t}^{n+1}$ is not required to be the identity id. However, one directly verifies that T commutes with all the paracyclic generators $\mathsf{t}, \mathsf{b}_i, \mathsf{s}_j$. As a consequence, one can attach to any paracyclic object a cyclic one by passing to the coinvariants $C/\mathrm{im}\,(\mathrm{id} - \mathsf{T})$ of T. In well-behaved cases, there is no loss of homological information in this step - for example, one has ([12], Proposition 2.1):

LEMMA 4.2. *If C is a paracyclic $\mathbb{K}$-vector space and T is diagonalisable, then* $(C, \mathsf{b}) \to (C/\mathrm{im}\,(\mathrm{id} - \mathsf{T}), \mathsf{b})$ *is a quasi-isomorphism.*

As for cyclic objects, one puts

$$\mathsf{N} := \sum_{i=0}^{n} \mathsf{t}^i, \quad \mathsf{s} := (-1)^{n+1}\mathsf{t}\mathsf{s}_n, \quad \mathsf{B} := (\mathrm{id} - \mathsf{t})\mathsf{s}\mathsf{N},$$

all acting on C_n, and as in the cyclic case one has

$$\mathsf{b}(\mathrm{id} - \mathsf{t}) = (\mathrm{id} - \mathsf{t})\mathsf{b}', \quad \mathsf{b}'\mathsf{N} = \mathsf{N}\mathsf{b}, \quad \mathsf{s}\mathsf{b}' + \mathsf{b}'\mathsf{s} = \mathrm{id}, \tag{21}$$

where $\mathsf{b}' := \sum_{i=0}^{n-1}(-1)^i\mathsf{b}_i$. But B is in general not a differential anticommuting with b - instead, one has $\mathsf{B}\mathsf{B} = (\mathrm{id} - \mathsf{T})(\mathrm{id} - \mathsf{t})\mathsf{s}\mathsf{s}\mathsf{N}$ and $\mathsf{b}\mathsf{B} + \mathsf{B}\mathsf{b} = \mathrm{id} - \mathsf{T}$.

Note that $\mathsf{B}(D) \subset D$, where $D := \mathrm{span}\{\mathrm{im}\,\mathsf{s}_i\}$ is the degenerate part of C. Therefore, B descends to the normalised complex $\bar{C}_\bullet$, where it takes the simplified form $\mathsf{B} = \mathsf{s}\mathsf{N}$ since $\mathsf{t}\mathsf{s} = (-1)^{n+1}\mathsf{t}\mathsf{t}\mathsf{s}_n = -\mathsf{s}_0\mathsf{t}$. Therefore, we work in the normalised complex throughout our explicit computations below.

4.1.2. *Cyclic homology.* The cyclic homology $HC_\bullet(C)$ of a cyclic object is the total homology of the bicomplex $E^0_{pq} := C_{q-p}$, $p, q \geq 0$, $C_n := 0$ for $n < 0$, whose differentials are given by b and B:

$$\begin{tikzcd}
 & \vdots \arrow[d, "\mathsf{b}"'] & \vdots \arrow[d, "\mathsf{b}"'] & \vdots \arrow[d, "\mathsf{b}"'] & \vdots \arrow[d] \\
0 & C_2 \arrow[l] \arrow[d, "\mathsf{b}"'] & C_1 \arrow[l, "\mathsf{B}"'] \arrow[d, "\mathsf{b}"'] & C_0 \arrow[l, "\mathsf{B}"'] \arrow[d] & 0 \arrow[l] \\
0 & C_1 \arrow[l] \arrow[d, "\mathsf{b}"'] & C_0 \arrow[l, "\mathsf{B}"'] \arrow[d] & 0 \arrow[l] & \\
0 & C_0 \arrow[l] \arrow[d] & 0 \arrow[l] & & \\
 & 0 & & &
\end{tikzcd}$$

Similarly one defines the periodic cyclic homology $HP_\bullet(C)$ as the (direct product) total homology of the bicomplex in which p, q take arbitrary values in $\mathbb{Z}$. For a paracyclic object, cyclic homology is defined in terms of the associated cyclic object $C/\mathrm{im}\,(\mathrm{id} - \mathsf{T})$.

4.1.3. *Connes' spectral sequence.* The main tool for computing cyclic homology is the spectral sequence $E^\bullet_{pq} \Rightarrow HC_{p+q}(C)$ arising from filtering the total complex of the (b, B)-bicomplex by columns. Its first page consists of the simplicial homologies $E^1_{pq} = H_{q-p}(C, \mathsf{b})$, between which B induces a boundary map whose homology fills the second page $E^2_{pq} = H_{q-p}(H(C, \mathsf{b}), \mathsf{B})$.

4.1.4. *The* SBI*-sequence.* If one views (C, b) (C a cyclic object) as a bicomplex with trivial horizontal differential, then the identity map $C_n \to C_n = E^0_{0q}$ is a bicomplex embedding of (C, b) into the (b, B)-bicomplex as its first column. The quotient bicomplex is the (b, B)-bicomplex again, but shifted in both degrees by 1. The resulting short exact sequence of total complexes gives a long exact homology sequence

$$\cdots \to H_n(C, \mathsf{b}) \xrightarrow{\mathsf{I}} HC_n(C) \xrightarrow{\mathsf{S}} HC_{n-2}(C) \xrightarrow{\mathsf{B}} H_{n-1}(C, \mathsf{b}) \to \cdots \tag{22}$$

in which B appears as the connecting homomorphism.

In particular, $H_n(C, \mathsf{b}) = 0$ for all $n > d$ implies that $\mathsf{S} : HC_{n+1}(C) \to HC_{n-1}(C)$ is an isomorphism for $n > d$. In this case, one obtains the periodic cyclic homology groups $HP_0(C)$ and $HP_1(C)$ as the limit of $HC_{2n}(C)$ and $HC_{2n+1}(C)$ respectively (see [23], Section 5.1.10).

4.1.5. *Twisted cyclic homology of an algebra* $\mathcal{A}$. If $\mathcal{A}$ is an algebra and $\sigma \in \mathrm{Aut}(\mathcal{A})$, then the simplicial object $C_\bullet(\mathcal{A}, {}_\sigma\mathcal{A})$ is in fact paracyclic [20] with

$$\mathsf{t} : a_0 \otimes \ldots \otimes a_n \mapsto (-1)^n \sigma(a_n) \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}.$$

For this paracyclic object, B is given on $\bar{C}_\bullet(\mathcal{A}, {}_\sigma\mathcal{A})$ by

$$\mathsf{B} : a_0 \otimes \ldots \otimes a_n \mapsto 1 \otimes \sum_{i=0}^{n} (-1)^{ni} \sigma(a_{n-i+1}) \otimes \ldots \otimes \sigma(a_n) \otimes a_0 \otimes \ldots \otimes a_{n-i}.$$

Following [20], we denote by $C^\sigma_\bullet(\mathcal{A}) := C_\bullet(\mathcal{A}, {}_\sigma\mathcal{A})/\mathrm{im}\,(\mathrm{id} - \mathsf{T})$ the associated cyclic object and by $HH^\sigma_\bullet(\mathcal{A})$ and $HC^\sigma_\bullet(\mathcal{A})$ its simplicial and cyclic homology, respectively. By Lemma 4.2 one has $H_\bullet(\mathcal{A}, {}_\sigma\mathcal{A}) \simeq HH^\sigma_\bullet(\mathcal{A})$ if σ is diagonalisable. This is crucial in the computation of $HC^\sigma_\bullet(\mathcal{A})$ since $H_\bullet(\mathcal{A}, {}_\sigma\mathcal{A})$ is computable via its derived functor description, while $HH^\sigma_\bullet(\mathcal{A})$ is the first page of the Connes spectral sequence $E \Rightarrow HC^\sigma(\mathcal{A})$.

For $\sigma = \mathrm{id}$, $HC^\sigma_\bullet(\mathcal{A})$ reduces to the standard cyclic homology $HC_\bullet(\mathcal{A})$ [4, 27]. If $\mathcal{A} = \mathbb{K}[X]$ for a smooth affine variety, then the Hochschild-Kostant-Rosenberg isomorphism identifies B with Cartan's exterior differential, and the Connes spectral sequence stabilises at E^2, giving

$$HP_n(\mathbb{K}[X]) \simeq \bigoplus_{i \geq 0} H^{2i+n}_{\mathrm{deRham}}(X), \tag{23}$$

where the right hand side is the even and odd algebraic de Rham cohomology of X with coefficients in $\mathbb{K}$, see e.g. [5, 23].

4.2. **Results.** Here we compute for $\mathcal{A} = \mathbb{K}_q[SL(2)]$ and

$$\sigma = \sigma_{q^{-N},1}, \quad N \in \mathbb{Z},$$

the second page of the spectral sequence $E \Rightarrow HC^\sigma(\mathcal{A})$. Lemma 4.2 gives $HH^\sigma(\mathcal{A}) \simeq H(\mathcal{A}, {}_\sigma\mathcal{A})$ which is reflected by the fact that all the generators of $H(\mathcal{A}, {}_\sigma\mathcal{A})$ listed in Section 2.2 are invariant under $\mathsf{T} = \sigma \otimes \ldots \otimes \sigma$. From now on we suppress the distinction between $HH^\sigma(\mathcal{A})$ and $H(\mathcal{A}, {}_\sigma\mathcal{A})$. We will compute the action of $\mathsf{B} : HH^\sigma_n(\mathcal{A}) \to HH^\sigma_{n+1}(\mathcal{A})$ on the vector space bases from Section 2.2. It will then be possible to read off directly the (co)homology E^2, and it is then immediate that $E^\bullet$ stabilises at E^2.

Recall that we work throughout in the normalised complex $\bar{C}_\bullet(\mathcal{A}, {}_\sigma\mathcal{A})$.

4.2.1. *E^2_{pp} and $HH^\sigma_1(\mathcal{A})/\operatorname{im}\mathsf{B}$.* For $n = 0$, the action of B on the basis (6) of $HH^\sigma_0(\mathcal{A})$ is given by

$$\begin{aligned}
&[1] \mapsto 0, \quad 0 \in S, \quad [a^i], [d^i] \mapsto i[a^{i-1} \otimes a], i[d^{i-1} \otimes d] \quad i \geq 1, N = 0,\\
&[b^j], [c^j] \mapsto j[b^{j-1} \otimes b], j[c^{j-1} \otimes c], \quad j > 0, j \in S,\\
&[\omega_{N,i}] \mapsto i[\omega_{N-1,i-1} \otimes b] + (N-i)[\omega_{N-1,i} \otimes c] \quad i = 0, \ldots, N, N > 0,
\end{aligned} \tag{24}$$

where $S := \mathbb{N} \setminus \{N - 2r \mid N > 0, r \geq 0\}$ as defined in (7).

Comparing (24) with the basis (8) of $HH^\sigma_1(\mathcal{A})$ gives:

LEMMA 4.3. *We have (for $p > 0$)*

$$\begin{aligned}
E^2_{pp} &= \begin{cases} \mathbb{K}[1] & 0 \in S, \\ 0 & 0 \notin S, \end{cases}\\
HH^\sigma_1(\mathcal{A})/\operatorname{im}\mathsf{B} &= \bigoplus_{i=1}^{N-2} \mathbb{K}[\omega_{N-1,i} \otimes b] \oplus \begin{cases} \mathbb{K}[b \otimes c] & 0 \in S, \\ 0 & 0 \notin S, \end{cases}
\end{aligned}$$

where we identify elements of $HH^\sigma_1(\mathcal{A})$ with their classes in $HH^\sigma_1(\mathcal{A})/\operatorname{im}\mathsf{B}$.

4.2.2. *E^2_{pp+1} and $HH^\sigma_2(\mathcal{A})/\operatorname{im}\mathsf{B}$.* The basis (8) is mapped by B to

$$\begin{aligned}
&[a^i \otimes a], [d^i \otimes d] \mapsto 0, \quad i \geq 0,\ N = 0,\\
&[b^{j-1} \otimes b], [c^{j-1} \otimes c] \mapsto 0, \quad j > 0,\ j \in S,\\
&[b \otimes c] \mapsto [1 \otimes (b \wedge c)], \quad 0 \in S,\\
&[\omega_{N-1,i} \otimes b] \mapsto -(N-1-i)[\omega_{N-2,i} \otimes (b \wedge c)], \quad 0 \leq i \leq N-1,\\
&[\omega_{N-1,i} \otimes c] \mapsto i[\omega_{N-2,i-1} \otimes (b \wedge c)], \quad 0 \leq i \leq N-1.
\end{aligned}$$

Now note that Lemma 3.3 and Lemma 2.1 imply $[1 \otimes (b \wedge c)] = 0$ for $0 \in S$, because in this case $[1 \otimes (b \wedge c)] \frown [\partial] = 0$ and $[1 \otimes (b \wedge c)] \frown [\partial'] = -[b \otimes c] - [c \otimes b] = 0$. Comparing with our descriptions of $HH^\sigma_1(\mathcal{A})/\operatorname{im}\mathsf{B}$ from Lemma 4.3 and of $HH^\sigma_2(\mathcal{A})$ given in Section 2.2.3 above, this yields

LEMMA 4.4. *We have (for $p > 1$)*

$$\begin{aligned}
E^2_{pp+1} &= \begin{cases} \mathbb{K}[b \otimes c] & 0 \in S, \\ 0 & 0 \notin S, \end{cases}\\
HH^\sigma_2(\mathcal{A})/\operatorname{im}\mathsf{B} &= \bigoplus_{i=0}^{N-2} \mathbb{K}[\omega_2(N-2, i)].
\end{aligned}$$

4.2.3. *E^2_{pp+2} and $E^2_{pp+3} = HH^\sigma_3(\mathcal{A})/\mathrm{im}\,\mathsf{B}$.* This involves a lengthier computation, so we state the result first:

LEMMA 4.5. *In $HH^\sigma_3(\mathcal{A})$, we have*

$$\mathsf{B}([\omega_2(r,i)]) = (2i-r)[\omega_3(r,i)], \quad r \geq 0, i = 0,\dots,r, \tag{25}$$

so (for $p > 2$)

$$\begin{aligned} E^2_{pp+2} &= \begin{cases} 0 & 0 \in S, \\ \mathbb{K}[\omega_2(2r,r)] & N = 2r+2, r \geq 0, \end{cases} \\ E^2_{pp+3} &= HH^\sigma_3(\mathcal{A})/\mathrm{im}\,\mathsf{B} = \begin{cases} 0 & 0 \in S, \\ \mathbb{K}[\omega_3(2r,r)] & N = 2r+2, r \geq 0. \end{cases} \end{aligned}$$

Proof. We show (25), the second part is then immediate. We have

$$\begin{aligned} & \mathsf{B}(\omega_2(r,i)) \\ = \ & \mathsf{B}(\omega_{r,i}(bc \otimes (a \otimes d - d \otimes a - (q-q^{-1})c \otimes b) - bd \otimes (a \otimes c - qc \otimes a) \\ & + da \otimes (b \otimes c - c \otimes b) - q^{-1}ca \otimes (b \otimes d - qd \otimes b))) \\ = \ & 1 \otimes (\omega_{r+2,i+1} \otimes a \otimes d + q^{r+2} d \otimes \omega_{r+2,i+1} \otimes a + a \otimes d \otimes \omega_{r+2,i+1} \\ & - \omega_{r+2,i+1} \otimes d \otimes a - q^{-r-2} a \otimes \omega_{r+2,i+1} \otimes d - d \otimes a \otimes \omega_{r+2,i+1} \\ & - (q-q^{-1})(\omega_{r+2,i+1} \otimes c \otimes b + b \otimes \omega_{r+2,i+1} \otimes c + c \otimes b \otimes \omega_{r+2,i+1}) \\ & - \omega_{r,i} bd \otimes a \otimes c - c \otimes \omega_{r,i} bd \otimes a - q^{-r-2} a \otimes c \otimes \omega_{r,i} bd \\ & + q\omega_{r,i} bd \otimes c \otimes a + qq^{-r-2} a \otimes \omega_{r,i} bd \otimes c + qq^{-r-2} c \otimes a \otimes \omega_{r,i} bd \\ & + \omega_{r,i} da \otimes b \otimes c + c \otimes \omega_{r,i} da \otimes b + b \otimes c \otimes \omega_{r,i} da - \omega_{r,i} da \otimes c \otimes b \\ & - b \otimes \omega_{r,i} da \otimes c - c \otimes b \otimes \omega_{r,i} da - q^{-1} \omega_{r,i} ca \otimes b \otimes d \\ & - q^{-1} q^{r+2} d \otimes \omega_{r,i} ca \otimes b - q^{-1} q^{r+2} b \otimes d \otimes \omega_{r,i} ca + \omega_{r,i} ca \otimes d \otimes b \\ & + b \otimes \omega_{r,i} ca \otimes d + q^{r+2} d \otimes b \otimes \omega_{r,i} ca \end{aligned}$$

We apply $\frown \partial$, where $\partial = \frac{1}{2}(\partial^+_H + \partial^-_H)$. This gives

$$\begin{aligned} & (\mathsf{B}(\omega_2(r,i))) \frown \partial \\ = \ & q^{r+2} d \otimes \omega_{r+2,i+1} \otimes a - a \otimes d \otimes \omega_{r+2,i+1} + q^{-r-2} a \otimes \omega_{r+2,i+1} \otimes d \\ & - d \otimes a \otimes \omega_{r+2,i+1} - \omega_{r,i} bd \otimes a \otimes c + q^{-r-2} a \otimes c \otimes \omega_{r,i} bd \\ & + q\omega_{r,i} bd \otimes c \otimes a - q^{-r-1} a \otimes \omega_{r,i} bd \otimes c + q^{-1} \omega_{r,i} ca \otimes b \otimes d \\ & - q^{r+1} d \otimes \omega_{r,i} ca \otimes b - \omega_{r,i} ca \otimes d \otimes b + q^{r+2} d \otimes b \otimes \omega_{r,i} ca. \end{aligned}$$

Now apply $\frown \frac{1}{2}(\partial^+_H - \partial^-_H)$. This gives

$$\begin{aligned} & ((\mathsf{B}(\omega_2(r,i))) \frown \partial) \frown \frac{1}{2}(\partial^+_H - \partial^-_H) \\ = \ & (2i-r) q^{r+2} d\omega_{r+2,i+1} \otimes a + (2i-r) q^{-r-2} a\omega_{r+2,i+1} \otimes d \\ & - q^{-r-2} ac \otimes \omega_{r+1,i+1} d - q\omega_{r+1,i+1} dc \otimes a \\ & - (2i-r+1) q^{-r-1} a\omega_{r+1,i+1} d \otimes c + q^{-1} \omega_{r+1,i} ab \otimes d \\ & - q^{r+1}(2i-r-1) d\omega_{r+1,i} a \otimes b + q^{r+2} db \otimes \omega_{r+1,i} a \\ = \ & (2i-r-1) q^{r+2} d\omega_{r+2,i+1} \otimes a + (2i-r+1) q^{-r-2} a\omega_{r+2,i+1} \otimes d \\ & - q^{-r-2} ac \otimes \omega_{r+1,i+1} d - (2i-r+1) ad\omega_{r+1,i+1} \otimes c \\ & - (2i-r-1) da\omega_{r+1,i} \otimes b + q^{r+2} db \otimes \omega_{r+1,i} a. \end{aligned}$$

Lemma 3.5 gives $[\partial] \smile \frac{1}{2}[\partial_H^+ - \partial_H^-] = \frac{1}{2}[\partial_H^-] \smile [\partial_H^+]$, so by subtracting

$$\mathsf{b}((2i-r-1)\omega_{r+2,i+1} \otimes d \otimes a + b \otimes \omega_{r+1,i} a \otimes d + q^{-r-2} ac \otimes \omega_{r+1,i+1} \otimes d)$$

we get in homology

$$\begin{aligned}
& \frac{1}{2}[\mathsf{B}(\omega_2(r,i))] \frown ([\partial_H^-] \smile [\partial_H^+]) \\
= \quad & [(2i-r-1)\omega_{r+2,i+1} \otimes bc + b \otimes \omega_{r+1,i} - c \otimes \omega_{r+1,i+1} \\
& -q^{-1} bc^2 \otimes \omega_{r+1,i+1} - (2i-r+1)\omega_{r+1,i+1} \otimes c \\
& -(2i-r+1) q \omega_{r+3,i+2} \otimes c - (2i-r-1)\omega_{r+1,i} \otimes b \\
& -(2i-r-1) q^{-1} \omega_{r+3,i+1} \otimes b].
\end{aligned}$$

Using the calculus of differential forms over $\mathbb{K}[b,c]$, that is, using the fact that $[f \otimes b^j c^k] = [j f b^{j-1} c^k \otimes b] + [k f b^j c^{k-1} \otimes c]$ for $f \in \mathbb{K}[b,c]$, we obtain

$$\begin{aligned}
& \frac{1}{2}[\mathsf{B}(\omega_2(r,i))] \frown ([\partial_H^-] \smile [\partial_H^+]) \\
= \quad & -(2i-r)[\omega_{r+1,i} \otimes b + \omega_{r+1,i+1} \otimes c] \\
& +[((2i-r-1) - q^{-1}(r-i) - (2i-r+1)q)\omega_{r+3,i+2} \otimes c \\
& +((2i-r-1) - (3i-r)q^{-1})\omega_{r+3,i+1} \otimes b].
\end{aligned}$$

And with $\mathsf{b}(\omega_{r+1,i} a \otimes b \wedge d) = (1-q^2)\omega_{r+3,i+1} \otimes b$ and $\mathsf{b}(\omega_{r+1,i+1} a \otimes d \wedge c) = (q-q^{-1})\omega_{r+3,i+2} \otimes c$ the above finally simplifies to

$$\frac{1}{2}[\mathsf{B}(\omega_2(r,i))] \frown ([\partial_H^-] \smile [\partial_H^+]) = -(2i-r)[\omega_{r+1,i} \otimes b + \omega_{r+1,i+1} \otimes c].$$

The claim now follows from Lemma 3.6. □

4.2.4. *Stabilisation of the spectral sequence.* There is no further page of the spectral sequence to be computed - the differential on E^2 maps E^2_{pq} to E^2_{p-2q+1}, and for all p,q either one space or the other is zero:

LEMMA 4.6. *For $\sigma = \sigma_{q^{-N},1}$, $N \in \mathbb{Z}$, we have $HC_n^\sigma(\mathcal{A}) \simeq \bigoplus_{p+q=n} E^2_{pq}$.*

Proof. In case $0 \in S$, E^2 looks as follows (the lines are for the reader's orientation and depict the $p=0$ and $q=0$ axes):

$$\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & \mathbb{K}[b \otimes c] & \cdots \\
0 & \bigoplus_{i=0}^{N-2} \mathbb{K}[\omega_2(N-2,i)] & \mathbb{K}[b \otimes c] & \mathbb{K}[1] & \cdots \\
0 & \begin{array}{c}\bigoplus_{i=1}^{N-2} \mathbb{K}[\omega_{N-1,i} \otimes b] \\ \oplus \mathbb{K}[b \otimes c]\end{array} & \mathbb{K}[1] & 0 & \\
0 & HH_0^\sigma(\mathcal{A}) & 0 & & \\
 & 0 & & &
\end{array}$$

And otherwise, it looks like that:

$$
\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
0 & 0 & \mathbb{K}[\omega_3(N-2,N/2-1)] & \mathbb{K}[\omega_2(N-2,N/2-1)] & \cdots \\
0 & \mathbb{K}[\omega_3(N-2,N/2-1)] & \mathbb{K}[\omega_2(N-2,N/2-1)] & 0 & \cdots \\
0 & \bigoplus_{i=0}^{N-2} \mathbb{K}[\omega_2(N-2,i)] & 0 & 0 & \cdots \\
0 & \bigoplus_{i=1}^{N-2} \mathbb{K}[\omega_{N-1,i} \otimes b] & 0 & 0 & \\
0 & HH_0^\sigma(\mathcal{A}) & 0 & & \\
 & 0 & & &
\end{array}
$$

$\square$

5. Cyclic cohomology

5.1. **Background.** In this final section we construct a cyclic cocycle that pairs nontrivially with $\mathsf{d}\mathcal{A}$ and hence represents a generator of $HC^3_{\sigma_{q^{-2},1}}(\mathcal{A}) \simeq \mathbb{K}$ in Connes' λ-complex.

5.1.1. *Cyclic cocycles.* In view of (21), $\mathrm{im}(\mathrm{id} - \mathsf{t}) \subset C$ is for any cyclic $\mathbb{K}$-vector space a subcomplex with respect to b, and as Connes showed, cyclic homology can be realised for $\mathbb{Q} \subset \mathbb{K}$ as the homology of the quotient,

$$HC_\bullet(C) \simeq H_\bullet(C/\mathrm{im}(\mathrm{id} - \mathsf{t}), \mathsf{b}).$$

For $C = C^\sigma(\mathcal{A})$ one can dually consider Hochschild cochains $\varphi \in C^n(\mathcal{A}, ({}_\sigma\mathcal{A})^*)$ which are cyclic, that is, satisfy

$$\varphi(a_0, \ldots, a_n) = (-1)^n \varphi(\sigma(a_n), a_0, \ldots, a_{n-1}) \tag{26}$$

for all $a_0, \ldots, a_n \in \mathcal{A}$. These form a subcomplex of $(C^\bullet(\mathcal{A}, ({}_\sigma\mathcal{A})^*), \mathsf{b})$ whose cohomology is twisted cyclic cohomology $HC^\bullet_\sigma(\mathcal{A}) \simeq (HC^\sigma_\bullet(\mathcal{A}))^*$.

If one works with the normalised complex, then $\varphi(a_0, \ldots, a_n) = 0$ whenever $a_i \in \mathbb{K}$ for some $i > 0$. For cyclic φ this property obviously extends to $i = 0$. Conversely, a Hochschild cocycle that vanishes on $1 \otimes a_1 \otimes \ldots \otimes a_n$ is cyclic as follows by applying it to $\mathsf{b}(1 \otimes a_0 \otimes \ldots \otimes a_n)$.

5.2. **Results.** First we show that $\varphi_{\mathcal{A}}$ from (20) itself is not cyclic. Then we construct a coboundary $\eta_{\mathcal{A}}$ by which it differs from a cyclic cocycle.

5.2.1. $\varphi_{\mathcal{A}}$ *is not cyclic.* The natural question is whether the twisted Hochschild 3-cocycle $\varphi_{\mathcal{A}}$ from (20) is already cyclic. As remarked at the end of Section 5.1.1, this is equivalent to the condition that

$$\varphi_{\mathcal{A}}(1, a_1, a_2, a_3) = 0$$

for all $a_1, a_2, a_3 \in \mathcal{A}$. And since one has for all $i, j \geq 0$

$$\sigma_{q,q^{-1}}(\partial_E^+(d^j c))\partial_F^+(a^i b) = q^{i-2j} d^{j+1} a^{i+1},$$
$$\sigma_{q^2,q^{-2}}(\partial_H^+(e_{j-i,0,0})) = (i-j)q^{2(j-i)} e_{j-i,0,0},$$

and therefore

$$\varphi_{\mathcal{A}}(1, e_{j-i,0,0}, d^j c, a^i b) = q^{-i}(i-j)\int_{[1]} e_{j-i,0,0} d^{j+1} a^{i+1} = q^{-i}(i-j),$$

we see that $\varphi_{\mathcal{A}}$ is not a cyclic cocycle.

5.2.2. *The correction term.* We make the ansatz

$$\eta_{\mathcal{A}}(\cdot) := \int_? \cdot \frown (\partial_H^+ \smile (\sigma_{\lambda_1,\mu_1} - \mathrm{id}) \smile (\sigma_{\lambda_2,\mu_2} - \mathrm{id})),$$

where $\int_?$ is a suitable twisted trace to be determined.

LEMMA 5.1. *$\varphi_{\mathcal{A}} + \eta_{\mathcal{A}}$ is a cyclic cocycle which is as a Hochschild cocycle cohomologous to $\varphi_{\mathcal{A}}$, provided that $\lambda_1 = \lambda_2 = 1, \mu_2 = \mu_1^{-1} \neq 1$ and*

$$\int_? = -\frac{\mu_2}{(\mu_2 - 1)^2}\int_{[bc]} \in H^0(\mathcal{A}, ({}_{\sigma_{q^{-2},1}}\mathcal{A})^*).$$

Proof. For any automorphism σ of an algebra, $\sigma - \mathrm{id}$ is a σ-twisted derivation which is inner - it is simply the twisted commutator with $1 \in \mathcal{A}$. Therefore, its cohomology class vanishes, and as a consequence, $\eta_{\mathcal{A}}$ is automatically a coboundary. Furthermore, one has for all $i, j \geq 0$

$$\begin{aligned} & \sigma_{\lambda_2,\mu_2}((\sigma_{\lambda_1,\mu_1} - \mathrm{id})(d^j c))(\sigma_{\lambda_2,\mu_2} - \mathrm{id})(a^i b) \\ = \; & q^{-i}\lambda_2^{-j}\mu_2^{-1}(\lambda_1^{-j}\mu_1^{-1} - 1)(\lambda_2^i \mu_2 - 1) d^j a^i bc, \\ & \sigma_{\lambda_1\lambda_2,\mu_1\mu_2}(\partial_H^+(e_{j-i,0,0})) \\ = \; & (i-j)(\lambda_1\lambda_2)^{j-i} e_{j-i,0,0}, \end{aligned}$$

so

$$\begin{aligned} & \eta_{\mathcal{A}}(1, e_{j-i,0,0}, d^j c, a^i b) \\ = \; & q^{-i}\lambda_2^{-j}\mu_2^{-1}(\lambda_1^{-j}\mu_1^{-1} - 1)(\lambda_2^i\mu_2 - 1)(i-j)(\lambda_1\lambda_2)^{j-i}\int_? e_{j-i,0,0} d^j a^i bc. \end{aligned}$$

Therefore, we need $\int_?$ to be a $\sigma_{q^{-2}\lambda_1\lambda_2,\mu_1\mu_2}$-twisted trace for which

$$\int_? e_{j-i,0,0} d^j a^i bc = -\frac{\lambda_1^{i-j}\lambda_2^i \mu_2}{(\lambda_1^{-j}\mu_1^{-1} - 1)(\lambda_2^i\mu_2 - 1)}.$$

One easily checks that $\varphi_{\mathcal{A}}(1, e_{i,j,k}, e_{l,m,n}, e_{r,s,t}) = \eta_{\mathcal{A}}(1, e_{i,j,k}, e_{l,m,n}, e_{r,s,t}) = 0$ for all other $i, j, k, l, m, n, r, s, t$, using that b, c are twisted central. $\square$

References

[1] J. Apel, K. Uwe, *FELIX – an assistant for algebraists*, see http://felix.hgb-leipzig.de
[2] K. Brown, J. Zhang: *Dualising complexes and twisted Hochschild (co)homology for noetherian Hopf algebras*, to appear in J. Algebra
[3] H. Cartan, S. Eilenberg, *Homological algebra*, Princeton University Press (1956).
[4] A. Connes, *Cohomologie cyclique et foncteurs* Ext^n*,* C. R. Acad. Sci. Paris I Math. **296**, no. 23, 953-958 (1983).
[5] A. Connes, *Noncommutative differential geometry*, Inst. Hautes Etudes Sci. Publ. Math. **62**, 257-360 (1985).
[6] A. Connes, H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*, Comm. Math. Phys. **198**, no. 1, 199-246 (1998).
[7] L. Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom, J. Varilly, *The local index formula for* $\mathrm{SU}_q(2)$, K-Theory **35**, no. 3-4, 375-394 (2005).
[8] P. Etingof, V. Dolgushev, *Hochschild cohomology of quantized symplectic orbifolds and the Chen-Ruan cohomology*, Int. Math. Res. Not. **27** 1657-1688 (2005).
[9] M. Farinati, *Hochschild duality, localization, and smash products*, J. Algebra **284**, no. 1, 415-434 (2005).
[10] P. Feng, B. Tsygan, *Hochschild and cyclic homology of quantum groups*, Comm. Math. Phys. **140**, no. 3, 481-521 (1991).
[11] E. Getzler, J. Jones, *The cyclic homology of crossed product algebras*, J. Reine Angew. Math., **445**, 161-174 (1993).
[12] T. Hadfield, U. Krähmer, *Twisted homology of quantum* $SL(2)$, K-theory, **34**, no. 4, 327-360 (2005)
[13] T. Hadfield, U. Krähmer, *On the Hochschild homology of quantum SL(N)*, Comptes Rendus Acad. Sci. Paris, Ser. I **343**, 9-13 (2006)
[14] P. Hajac, M. Khalkhali, B. Rangipour, Y. Sommerhäuser, *Hopf-cyclic homology and cohomology with coefficients*,C. R. Math. Acad. Sci. Paris **338**, no. 9, 667-672 (2004).
[15] I.. Heckenberger, *Classification of left-covariant differential calculi on the quantum group* $SL_q(2)$, J. Algebra **237**, 203-237 (2001).
[16] A. Joseph, *Quantum groups and their primitive ideals*, Springer (1995).
[17] A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*, Springer (1997).
[18] U. Krähmer, *Poincare duality in Hochschild (co)homology*, to appear in the proceedings of "New techniques in Hopf algebras and graded ring theory", Brussels 9/2006
[19] U. Krähmer, *twisted2.flx*, http://www.maths.gla.ac.uk/˜ukraehmer/twisted2.flx
[20] J. Kustermans, G. Murphy, L. Tuset, *Differential calculi over quantum groups and twisted cyclic cocycles,* J. Geom. Phys. **44**, no. 4, 570-594 (2003).
[21] S. Launois, T. Lenagan, *The first Hochschild cohomology group of quantum matrices and the quantum special linear group*, J. Noncommut. Geom. **1**, no. 3, 281-309 (2007).
[22] T. Masuda, Y. Nakagami, J. Watanabe, *Noncommutative differential geometry on the quantum SU(2), Part I*, K-theory **4**, no. 2, 157180 (1990).
[23] J.-L. Loday, *Cyclic homology*, Springer-Verlag (1998).
[24] S. Neshveyev, L. Tuset, *A local index formula for the quantum sphere*, Comm. Math. Phys. **254**, no. 2, 323-341 (2005).
[25] R. Nest, B. Tsygan, *On the cohomology ring of an algebra*, Progr. Math. **172**, 337-370 (1999).
[26] K. Schmüdgen, E. Wagner, *Dirac operator and a twisted cyclic cocycle on the standard Podleś quantum sphere*, J. Reine Angew. Math. **574**, 219-235 (2004).
[27] B. Tsygan, *Homology of matrix Lie algebras over rings and the Hochschild homology*, Uspekhi Mat. Nauk **38**, no. 2 (230), 217-218 (1983).
[28] M. van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proc. Amer. Math. Soc. **126**, no. 5, 1345-1348 (1998). Erratum: Proc. Amer. Math. Soc. **130**, no. 9, 2809-2810 (electronic) (2002).
[29] C. Weibel, *An introduction to homological algebra*, Cambridge Univ. Press (1995).

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