Structure and Symmetry

Ulrich Krähmer (University of Glasgow)

Alfred-Krupp Stiftung, Greifswald 2016

Structure and Symmetry

Ulrich Krähmer (University of Glasgow)

Alfred-Krupp Stiftung, Greifswald 2016

Groups

Structure and Symmetry

Ulrich Krähmer (University of Glasgow)

Alfred-Krupp Stiftung, Greifswald 2016

Groups+
Quantum

Structure and Symmetry

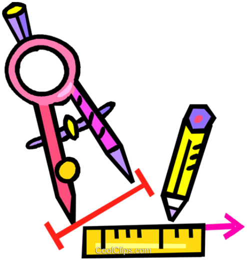
Ulrich Krähmer (University of Glasgow)

Alfred-Krupp Stiftung, Greifswald 2016

$$\left.\begin{array}{l}\text{Groups}+\\ \text{Quantum}\end{array}\right\} = \text{Quantum groups}$$

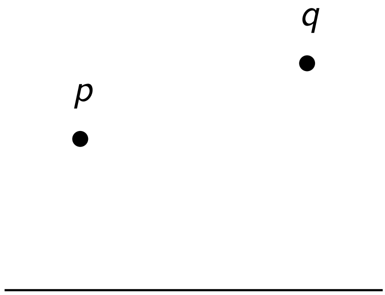
Geometry

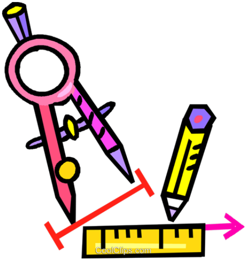
- Geometry = constructions with ruler and compass.



Geometry

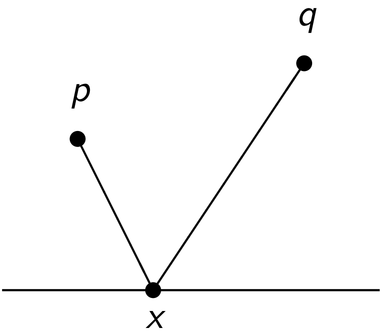
- Geometry = constructions with ruler and compass.
- Problem: Given points p, q in the upper half plane, construct the point x on the horizontal axis for which the sum of the lengths px and xq is minimaal.

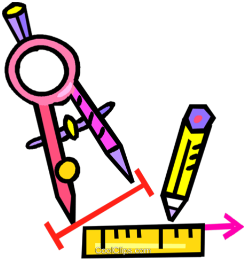




Geometry

- Geometry = constructions with ruler and compass.
- Problem: Given points p, q in the upper half plane, construct the point x on the horizontal axis for which the sum of the lengths px and xq is minimaal.





Algebra

- Algebra $=$ solving systems of polynomial equations.

Algebra

- Algebra = solving systems of polynomial equations.
- Problem: For which numbers a, b do the equations

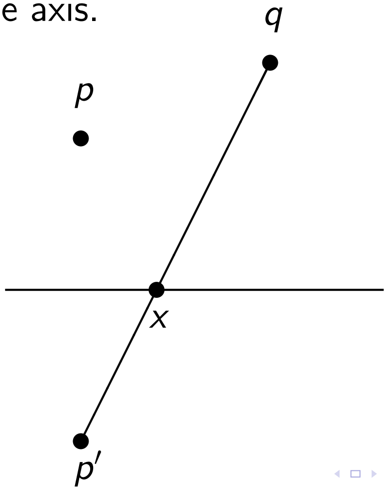
$$x + y + z = 3,$$

$$x^2 + y^2 + z^2 = a,$$
$$x^3 + y^3 + z^3 = b$$

admit exactly one solution?

Solution: symmetry

- Reflect p at the axis and connect the resulting point p' with q. Now take x to be the intersection of this line with the axis.



Solution: symmetry

- If (x, y, z) is a solution, then so are (y, x, z) and (x, z, y). So a unique solution must be of the form

$$x = y = z.$$

Solution: symmetry

- If (x, y, z) is a solution, then so are (y, x, z) and (x, z, y). So a unique solution must be of the form

$$x = y = z.$$

- Inserting this into the equations yields

$$x + x + x = 3x = 3,$$

$$x^2 + x^2 + x^2 = 3x^2 = a,$$

$$x^3 + x^3 + x^3 = 3x^3 = b.$$

The first equation implies $x = 1$. Inserting this into the remaining two implies $a = b = 3$.

What is a symmetry?

- Each problem involved certain objects: the points on the plane or the variables x, y, z.

What is a symmetry?

- Each problem involved certain objects: the points on the plane or the variables x, y, z.
- A symmetry is given by a **map** (or **function**)

$$f : m \mapsto f(m)$$

that permutes these objects. Think of a machine that takes an object m as an input and produces a new object $f(m)$ as output.

What is a symmetry?

- Each problem involved certain objects: the points on the plane or the variables x, y, z.
- A symmetry is given by a **map** (or **function**)

$$f : m \mapsto f(m)$$

  that permutes these objects. Think of a machine that takes an object m as an input and produces a new object $f(m)$ as output.
- The rôles of inputs and outputs should be symmetric in that there is an **inverse map** f^{-1} such that $f^{-1}(n) = m$ if and only if $f(m) = n$.

Composition

- Applying one symmetry after another yields a new one called the **composition**

$$f \circ g \colon m \mapsto f(g(m)).$$

Composition

- Applying one symmetry after another yields a new one called the **composition**

$$f \circ g \colon m \mapsto f(g(m)).$$

- We can also build $f \circ g \circ f$, and $g \circ f \circ g$, and then $f \circ g \circ f \circ g$... Note: $f \circ g \neq g \circ f$ in general!

Groups

Definition

A nonempty set G of permutations of some objects is a **group** if it contains for all f, g in G also $f \circ g$ and f^{-1}.

Groups

Definition

A nonempty set G of permutations of some objects is a **group** if it contains for all f, g in G also $f \circ g$ and f^{-1}.

Example: If r_α is the anti-clockwise rotation of the Euclidean plane by the angle α, then

$$r_0, \quad r_{90}, \quad r_{180}, \quad r_{270}$$

form a group. For example, we have

$$r_{180} \circ r_{270} = r_{90}, \quad r_{90}^{-1} = r_{270}.$$

Isomorphic groups

- Example: On $\{0, 1, 2, 3\}$, define the symmetries

$$a_n(m) = m + n \text{ modulo } 4.$$

Then

$$a_0, \quad a_1, \quad a_2, \quad a_3$$

form a group.

Isomorphic groups

- Example: On $\{0, 1, 2, 3\}$, define the symmetries

$$a_n(m) = m + n \text{ modulo } 4.$$

  Then

$$a_0, \quad a_1, \quad a_2, \quad a_3$$

  form a group.

- This can be identified with the group of rotations by multiples of 90°; the two groups are **isomorphic**:

$$a_0 \mapsto r_0, \quad a_1 \mapsto r_{90}, \quad a_2 \mapsto r_{180}, \quad a_3 \mapsto r_{270}.$$

Group theory

- A mathematician would ask: How many pairwise nonisomorphic groups with n elements exist?

$$
\begin{gathered}
0, 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14, \\
1, 5, 1, 5, 2, 2, 1, 15, 2, 2, 5, 4, 1, 4, 1, 51, 1, \\
2, 1, 14, 1, 2, 2, 14, 1, 6, 1, 4, 2, 2, 1, 52, 2, \\
5, 1, 5, 1, 15, 2, 13, 2, 2, 1, 13, 1, 2, 4, 267, \\
1, 4, 1, 5, 1, 4, 1, 50, 1, 2, 3, 4, 1, 6, 1, 52, 15, \\
2, 1, 15, 1, 2, 1, 12, 1, 10, 1, 4, 2, \ldots
\end{gathered}
$$

Group theory

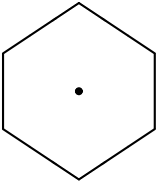
- A mathematician would ask: How many pairwise nonisomorphic groups with n elements exist?

$$0, 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14,$$
$$1, 5, 1, 5, 2, 2, 1, 15, 2, 2, 5, 4, 1, 4, 1, 51, 1,$$
$$2, 1, 14, 1, 2, 2, 14, 1, 6, 1, 4, 2, 2, 1, 52, 2,$$
$$5, 1, 5, 1, 15, 2, 13, 2, 2, 1, 13, 1, 2, 4, 267,$$
$$1, 4, 1, 5, 1, 4, 1, 50, 1, 2, 3, 4, 1, 6, 1, 52, 15,$$
$$2, 1, 15, 1, 2, 1, 12, 1, 10, 1, 4, 2, \ldots$$

- A normal person would ask: So what?

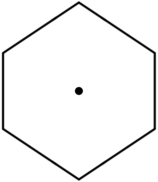
The Gauss-Wantzel theorem

- Problem: Construct the **regular** n-**gon**



The Gauss-Wantzel theorem

- Problem: Construct the **regular** n-**gon**



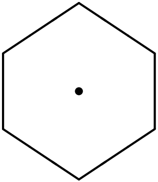
- Solution: Possible if and only if n is of the form

$$n = 2^k \cdot p_1 \cdot p_2 \cdots p_i$$

where the p_i are pairwise different **Fermat primes**, i.e., prime numbers of the form $2^{2^j} + 1$ for some j.

The Gauss-Wantzel theorem

- Problem: Construct the **regular** *n***-gon**



- Solution: Possible if and only if n is of the form

$$n = 2^k \cdot p_1 \cdot p_2 \cdots p_i$$

  where the p_i are pairwise different **Fermat primes**, i.e., prime numbers of the form $2^{2^j} + 1$ for some j.
- We believe there are only five of them:

$$3, \quad 5, \quad 17, \quad 257, \quad 65537.$$

Complex numbers

- In **Cartesian coordinates** (x, y), intersecting lines and circles means solving polynomial equations.

Complex numbers

- In **Cartesian coordinates** (x, y), intersecting lines and circles means solving polynomial equations.
- Helpful tool: Invent the field $\mathbb{C}$ of **complex numbers** by imagining a number i satisfying

  $$i^2 = -1$$

  and identify a point (x, y) with $u = x + iy$.

Complex numbers

- In **Cartesian coordinates** (x, y), intersecting lines and circles means solving polynomial equations.
- Helpful tool: Invent the field $\mathbb{C}$ of **complex numbers** by imagining a number i satisfying

$$i^2 = -1$$

  and identify a point (x, y) with $u = x + iy$.
- We denote by $\|u\|$ the distance of (x, y) from $(0, 0)$, so the **unit circle** S^1 consists of all u with $\|u\| = 1$. The corners of the regular n-gon are the solutions of

$$u^n - 1 = 0.$$

Galois theory

- Every polynomial equation has a solution in $\mathbb{C}$.
- Problem: Can it be expressed in **radicals** as in

$$u^2 + pu + q = 0 \quad \Leftrightarrow \quad u = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}?$$

Galois theory

- Every polynomial equation has a solution in $\mathbb{C}$.
- Problem: Can it be expressed in **radicals** as in

$$u^2 + pu + q = 0 \quad \Leftrightarrow \quad u = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}?$$

- Solution: Possible if and only if the group of symmetries of the equation is **soluble**.
- Example: Not the case for

$$u^5 - 6u + 3 = 0.$$

Quantum mechanics

- In **quantum mechanics**, a particle is modelled by a $\mathbb{C}$-valued **wave function** ψ on space-time, and

$$\int_M \|\psi(t,x,y,z)\|^2 dxdydz$$

represents the probability of detecting the particle at time t in the region $M \subseteq \mathbb{R}^3$.

Quantum mechanics

- In **quantum mechanics**, a particle is modelled by a $\mathbb{C}$-valued **wave function** ψ on space-time, and

$$\int_M \|\psi(t,x,y,z)\|^2 dxdydz$$

represents the probability of detecting the particle at time t in the region $M \subseteq \mathbb{R}^3$.

- If u is in the unit circle, then

$$\|u\psi\| = \|u\|\|\psi\| = \|\psi\|$$

Electromagentism can be derived from the resulting S^1-symmetry of quantum mechanics.

Gauge theory

- Cartan, Engel, Klein, Killing, Lie: Study **compact Lie groups** - classification possible and fascinating:

$$S^1, \; A_n, \; B_n, \; C_n, \; D_n, \; E_6, \; E_7, \; E_8, \; F_4, \; G_2.$$

Gauge theory

- Cartan, Engel, Klein, Killing, Lie: Study **compact Lie groups** - classification possible and fascinating:

$$S^1, \ A_n, \ B_n, \ C_n, \ D_n, \ E_6, \ E_7, \ E_8, \ F_4, \ G_2.$$

- **Standard model** of elementary particles: A_2- and A_3-symmetries describe weak and strong interaction.
- **GUT**: Explain all forces as one using a bigger symmetry group (**ToE** $=$ GUT $+$ gravity).

Quantum groups

- **Observables** are modelled by linear operators acting on wave functions. That they do not commute leads to the **uncertainty principle**.
- Space-time itself is not affected, the quantum versions of the Cartesian coordinates commute.

Quantum groups

- **Observables** are modelled by linear operators acting on wave functions. That they do not commute leads to the **uncertainty principle**.
- Space-time itself is not affected, the quantum versions of the Cartesian coordinates commute.
- **Noncommutative geometry**: Quantise space-time itself.

Quantum groups

- **Observables** are modelled by linear operators acting on wave functions. That they do not commute leads to the **uncertainty principle**.
- Space-time itself is not affected, the quantum versions of the Cartesian coordinates commute.
- **Noncommutative geometry**: Quantise space-time itself.
- **Quantum groups**: Quantise even the syemmtries. Just like a quantum particle does not have a position, quantum groups do not have elements but only an algebra of their "observables".

Some applications

- Analysis: Nonabelian **Pontryagin duality**.

Some applications

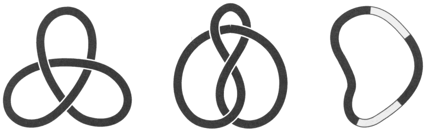
- Analysis: Nonabelian **Pontryagin duality**.
- Algebra: New examples of **Hopf algebras**; algebraic structure of **(co)homology theories**.

Some applications

- Analysis: Nonabelian **Pontryagin duality**.
- Algebra: New examples of **Hopf algebras**; algebraic structure of **(co)homology theories**.
- Physics: The integrability of **spin chains** in a magnetic field is due to a quantum group symmetry.

Some applications

- Analysis: Nonabelian **Pontryagin duality**.
- Algebra: New examples of **Hopf algebras**; algebraic structure of **(co)homology theories**.
- Physics: The integrability of **spin chains** in a magnetic field is due to a quantum group symmetry.
- Topology: The representations (realisations as linear symmetries of vector spaces) of many quantum groups form **braided monoidal categories** which yield polynomial invariants of knots.



Carbon Oxygen Hydrogen

Homogeneous spaces

Definition

A set M is a **homogeneous space** for a group G of symmetries if there is an element m such that for any other element n there exists some f in G with $f(m) = n$.

Homogeneous spaces

Definition

A set M is a **homogeneous space** for a group G of symmetries if there is an element m such that for any other element n there exists some f in G with $f(m) = n$.

M can be described entirely in terms of G and the subgroup H of the symmetries f with $f(m) = m$.

Homogeneous spaces

Definition

A set M is a **homogeneous space** for a group G of symmetries if there is an element m such that for any other element n there exists some f in G with $f(m) = n$.

M can be described entirely in terms of G and the subgroup H of the symmetries f with $f(m) = m$. Here

structure = symmetry

Homogeneous spaces

Definition

A set M is a **homogeneous space** for a group G of symmetries if there is an element m such that for any other element n there exists some f in G with $f(m) = n$.

M can be described entirely in terms of G and the subgroup H of the symmetries f with $f(m) = m$. Here

structure = symmetry

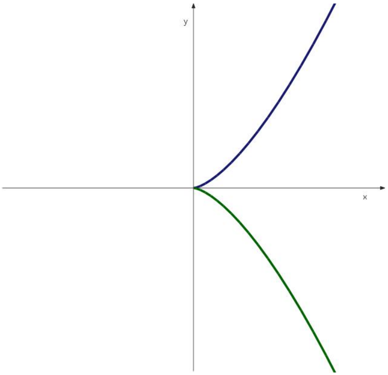
Definition

A **quantum homogeneos space** is a right coideal subalgebra B of a Hopf algebra A such that A is a faithfully flat B-module.

An unexpected example

- The **plane curve** given by the equation

$$x^3 - y^2 = 0$$

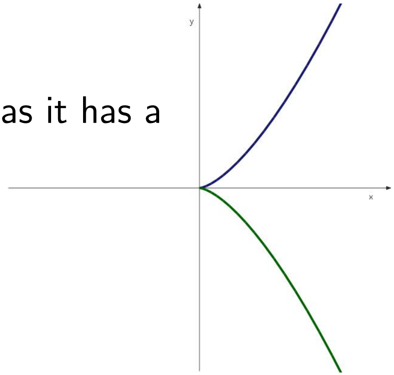


An unexpected example

- The **plane curve** given by the equation

$$x^3 - y^2 = 0$$

is not a homogeneous space as it has a **singularity** at $x = y = 0$.

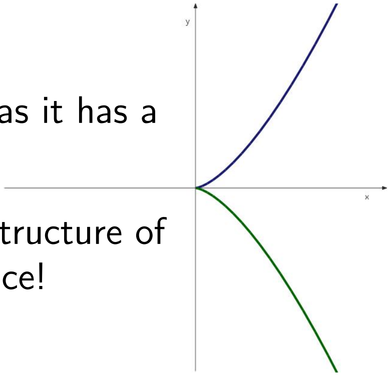


An unexpected example

- The **plane curve** given by the equation

$$x^3 - y^2 = 0$$

  is not a homogeneous space as it has a **singularity** at $x = y = 0$.



- But: It turns out it has the structure of a quantum homogeneous space!

Future

- Which varieties, schemes, orbifolds etc are quantum homogeneous spaces?

Future

- Which varieties, schemes, orbifolds etc are quantum homogeneous spaces?
- Can we use quantum group symmetries to solve classical problems in geometry, for example the study of singularities and of their resolutions?

Future

- Which varieties, schemes, orbifolds etc are quantum homogeneous spaces?
- Can we use quantum group symmetries to solve classical problems in geometry, for example the study of singularities and of their resolutions?
- Applications in representation theory?

Future

- Which varieties, schemes, orbifolds etc are quantum homogeneous spaces?
- Can we use quantum group symmetries to solve classical problems in geometry, for example the study of singularities and of their resolutions?
- Applications in representation theory?
- Lots of interesting analytic questions.

Some references

- Krähmer, Tabiri, *The nodal cubic is a quantum homogeneous space*
- Kassel, *Quantum Groups*
- Artin, *Galois Theory*
- Weyl, *Group Theory and Quantum Mechanics*
- Paramanov, *Symmetries in mathematics*
- Or if you want to ask me a question later:

`ulrich.kraehmer@glasgow.ac.uk`