Knots as mathematical objects

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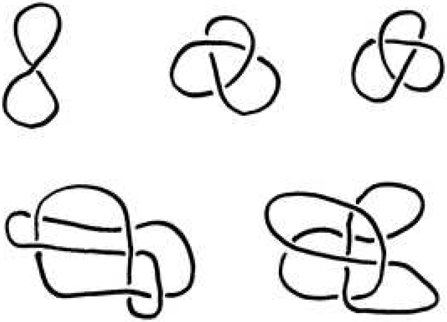


Why I decided to speak about knots

- They are a good example to show that mathematics deals with much more than boring numbers, functions and triangles.
- They are a good example to show that numbers, functions and triangles help studying interesting things...
- They are good for illustrating some general patterns of university mathematics.
- They are a Scottish story! Just search the web for Lord Kelvin, Peter Guthrie Tait and the Tait conjectures that were formulated in 18something but solved only 20 years ago!

What is a knot?

- Stupid question? Obvious? Who cares? Mathematicians do!
- Formulating a precise definition forces one to think first about what one wants to do, often leads to a good terminology or notation or even to the relevant problems to solve.

Definition

For two points x_1, x_2 in $\mathbb{R}^3$ let $\overline{x_1 x_2}$ be the straight line connecting the two points. A knot is a finite union

$$\overline{x_1 x_2} \cup \overline{x_2 x_3} \cup \overline{x_3 x_4} \cup \ldots \cup \overline{x_{n-1} x_n} \cup \overline{x_n x_1}$$

of such lines each of which intersects exactly two other ones, and this only in the end points. A link is a finite union of disjoint knots (called the components of the link).

And what does this mean?

- A picture says more than 10^3 words:

- One reason for "discretising" knots is to avoid monsters like

Knot diagrams

- Despite this abstract stuff we will usually draw knots in the obvious way using smooth arcs in the plane and leaving small gaps to denote what happens on crossings.
- One can prove rigorously that every knot can be displayed in this way without losing information, but we shall not go into that.

Equivalent knots

- The following knots are different but in some sense "the same":

- To formalise this one defines on the set (?) of all knots what mathematicians call an equivalence relation.

The precise definition

Definition

If $x_1, \dots, x_n \in \mathbb{R}^3$ define a knot K they are a sequence of vertices for K if there is no subsequence that defines the same knot.

Definition

A knot is an elementary deformation of another knot if there is a sequence $x_0, x_1, \dots, x_n \in \mathbb{R}^3$ such that

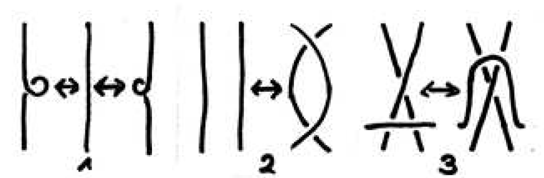
1. *$x_1, \dots, x_n$ is a sequence of vertices for one of the two and $x_0, x_1, \dots, x_n$ is a sequence of vertices for the other and*
2. *the triangle spanned by x_0, x_1, x_n intersects the knot given by $x_1, \dots, x_n$ only in $\overline{x_1 x_n}$.*

Two knots are equivalent if you can get from one to the other by a finite number of elementary deformations.

Reidemeister moves

Theorem

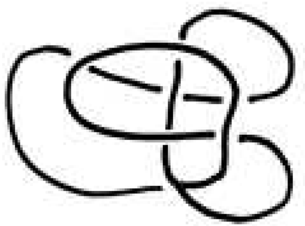
Two knot diagrams define equivalent knots if and only if they can be transformed into each other by a finite number of the so-called Reidemeister moves:



Note this is neither very precise nor true yet, one has to allow also some wiggling around that does not change the crossings, but to make this precise takes a while.

The first exercises

- Find a sequence of Reidemeister moves that transforms



into the standard form of the trivial knot (a circle). What is the minimal number of Reidemeister moves needed?

- Prove that all knots with three vertices are equivalent i.e. just different forms of the same knot. This is called the trivial knot.
- Is every knot with exactly four vertices trivial, that is, equivalent to the one from the previous exercise? Prove your answer!

Invariants and the classification problem

- The problem: if we suspect two given knots are not equivalent, how can one prove this?
- Invariants: define a recipe that attaches to any knot some datum (called the value of the invariant on the knot, could be e.g. a number but also totally different things) in such a way that the invariant has the same value on equivalent knots. The recipe should be so simple that one can compute the invariant more or less easily and if one then gets different values for two knots one knows for sure they can not be equivalent.
- The simpler the invariant is to compute, the less efficient will it usually be in distinguishing knots (meaning that it will have the same value on many nonequivalent knots).

Colouring knots

- Here is an invariant with two possible values: YES or NO.

Definition

A knot diagram is colourable if its arcs can be drawn with three different colours such that

1. *at least two colours get actually used and*
2. *at each crossing where two colours occur all three colours occur.*

- This is indeed an invariant:

Theorem

If one diagram of a knot can be coloured, then all diagrams of all equivalent knots can be coloured.

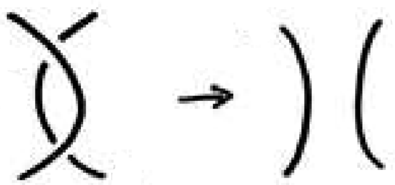
Example

- The trivial knot can not be coloured (one would use only one colour).
- But both the left- and the right-handed trefoil knot can be:

  So we have proved that these are nontrivial!
- However, one can not decide in this way whether the left-handed and the right-handed trefoil knot are equivalent.
- The tecnique can be generalised to labellings with values in a mathematical object called group, see the literature.

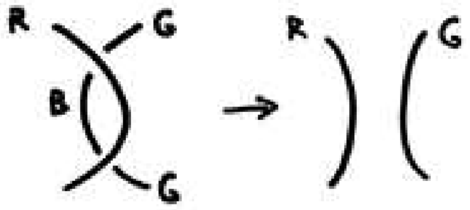
The proof I

- We have to show that Reidemeister moves do not alter the colourability of a diagram. We do move number 2, the other moves can be done similarly as exercises.
- We only consider the part of the diagram affected by the move, first the "resolving" move.



The proof of the theorem I

- We have to show that Reidemeister moves do not alter the colourability of a diagram. We do move number 2, the other moves can be done similarly as exercises.
- We only consider the part of the diagram affected by the move, first the "resolving" move. There are two possible cases, the first being that we have three different colours before the move, and then we do this:



Here R, G, B stands for the three colours, say red, green, blue.

The proof of the theorem II

- On the connections to the rest of the diagram nothing has changed, and in the considered part of the diagram there are already two colours used, so the new diagram is colourable.
- Case two is that there was only colour used before the move and then we can keep this.
- For the inverse Reidemeister move one reverses the whole.

More exercises

- Is the figure eight knot shown below colourable? Is it trivial?

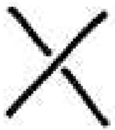
- Finish the proof of the theorem by considering the other two Reidemeister moves.

The Jones polynomial

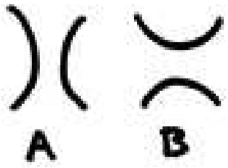
- As the name says, the value of this invariant will be a polynomial.
- Polynomial knot invariants have a long history. The first one was introduced by Alexander in 1928. But the Jones polynomial is much better than all the ones that existed before.
- Jones' original construction was rather complicated, but Kauffman then found a much simpler one, search the web for more historical information.
- It will be necessary to work with links, even if at the end we only want to deal with knots.

Resolving a crossing

- Given a diagram D of a link, fix one crossing and rotate the diagram such that the crossing looks like this:



- The crossing can be resolved in two ways:



- Pick one. We obtain a link diagram with one less crossing.

Continue to produce a state

- Pick again any crossing of the new link diagram, rotate and choose a resolution as before. Do this again and again until there are no crossings left.
- You end up with a trivial link S. The trivial links that arise in this way from D are called the states of D.
- Now comes the hammer: define

$$f_S(x) := x^{a-b} \cdot (-x^2 - x^{-2})^{|S|-1},$$

where $|S|$ is the number of components in S and a, b are the numbers of resolutions of type A and B made on our way from D to S. This is a Laurent polynomial, i.e. a polynomial in which also negative powers of the variable x occur.

The Kauffman bracket

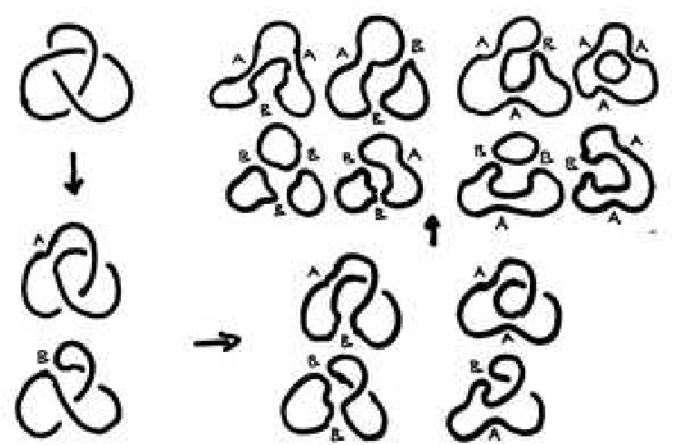
- A diagram with n crossings has 2^n states, and we take the sum of all the resulting f_S,

$$\langle D \rangle := \sum_S f_S.$$

  This is called the Kauffman bracket of D.
- How about an example, hm? We'll do the trefoil knot which has $2^3 = 8$ states.

The states of the trefoil knot

- Here are the resolutions:



The Kauffman bracket of the trefoil knot

- The corresponding polynomials are:

$$\begin{array}{cccc} x, & -x-x^{-3}, & x, & -x^5-x, \\ x+2\cdot x^{-3}+x^{-7}, & -x-x^{-3}, & -x-x^{-3}, & x \end{array}$$

- The Kauffman bracket is

$$\langle D\rangle(x) = -x^5 - x^{-3} + x^{-7}.$$

We are not done yet...

- Bad news: the Kauffman bracket is *not* an invariant. It is invariant under Reidemeister moves 2 and 3 but not under 1.
- But: this can be rectified. To this end we choose in any component of the original D an orientation, i.e. fix a direction. This is usually depicted by putting small arrows on the arcs.
- To any crossing one assigns a sign ± 1 as follows:

- Let $w(D)$ be the sum over all these ± 1's.
- Exercise: $w(D)$ does not change if one reverts all the orientations. In particular, it does not depend on the choice of the orientation if D is a knot.

The Kauffman polynomial

Definition

The Kauffman polynomial of D is $p_K(x) := (-x)^{-3 \cdot w(D)} \langle D \rangle (x)$.

Theorem

p_K is an invariant of the oriented link D. For a knot it does not depend on the chosen orientation.

- Proof: Exercise!!!
- The following is less trivial, you'll need to read a bit to understand this:

Theorem

If D has an odd number of components, then each exponent of p_K is divisible by 4. Otherwise they are of the form $4m + 2$ for integers m.

The Jones polynomial

- Obviously we get rid of the redundant factors 4 in the exponents:

Definition

The Jones polynomial of a knot D is $p_J(x) := p_K(x^{-1/4})$.

- So this is it. I want to finish with two more exercises:
- If K is a knot and K^* is its mirror image (like the left-handed vs. the right-handed trefoil knot) and p_J, p_J^* are their Jones polynomials, then $p_J(x) = p_J^*(x^{-1})$.
- The left-handed trefoil knot is not equivalent to the right-handed one.

Literature

- Livingston, “Knot theory”
- Prasolov, Sossinsky: “Knots, links, braids and 3-manifolds”
- Sossinsky: “Knots: mathematics with a twist”
- If you need more suggestions for further reading or have any question about this lecture or a more general one, please just write me under `ukraehmer@maths.gla.ac.uk`