Counting to Infinity
The Principle of Mathematical Induction

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Masterclass funded by

The Edinburgh Mathematical Society and

The Royal Institution of Great Britain

The natural numbers

- Counting is one of our fundamental intellectual abilities, and historically, humans did count before they started to write.
- It seems even many animals can count, see e.g.

http://www.scientificamerican.com/article.cfm?id=how-animals-have-the-ability-to-count

  However, they seem to rather have a direct feeling for the **cardinality of a set**, that is, for how many elements it contains, rather than the ability to count from 1 to some number n.
- In any case, counting is the starting point of mathematics and leads to the definition of the **(natural) numbers**

$$1, 2, 3, \ldots$$

  We denote the set of all these by the symbol $\mathbb{N}$.
- Remark: There are also other numbers such as the fractions $\frac{p}{q}$ of two natural numbers $p, q \in \mathbb{N}$, but we won't need them today.

On theorems and proofs

- Main difference between school, physics, engineering etc. versus pure maths: Elsewhere it is usually sufficient to get the right answer. But here we in addition want to **prove** that we got the right one. Compare this with the problem of catching criminals and then proving they actually have committed a crime.
- So usually in my daily research I try to state certain statements ("**theorems**") that I think are true under certain assumptions, and then I spend weeks, months or years trying to prove them.
- Example: Fermat's famous last theorem in the version $n = 3$:

Theorem

There are no (natural) numbers a, b, c such that

$$a \times a \times a + b \times b \times b = c \times c \times c.$$

Infinity

- Why is it difficult to prove theorems like the one above? Because the set $\mathbb{N}$ of natural numbers is **infinite** so we can not simply try out all a, b, c.
- Saying what this actually means is not that easy. Philosophers and mathematicians have thought about this already in ancient Greece and India. Here is one attempt:

Definition (Yajurveda)

If you remove a part from infinity or add a part to infinity, still what remains is infinity.

Giving a more precise definition would require some preparation and take a while, so let us just move on.

The principle of induction

- Is it impossible to prove theorems for infinitely many cases? The opposite is true - usually the finite ones are not really interesting because they can be checked case by case e.g. using a computer.
- Mathematical induction is the most fundamental method that allows us to prove theorems for infinitely many cases. The condition is, however, that there are only **countably many** cases, meaning that we can number them.
- Assuming this the grand idea is the following:

Mathematical Induction

1. **Show the theorem is true for the case having the number 1.**
2. **Show that if the theorem is true for some case, then it is is also true for the case with the next number.**

Then you have proven the theorem for all cases!!!.

A legend about Gauss

- The legend tells that the famous mathematician Gauss annoyed his mathematics teacher as a wee boy a lot since he simply always knew all the answers to all the questions. But once the teacher thought he could silence him at least for a while and asked him to add up the numbers from 1 to 100:

  $$1 + 2 + 3 + 4 + \cdots + 99 + 100$$

  However, Gauss laughed and said this is obviously 5050.
- I guess what Gauss in fact had proven (knowing that his teacher would anyway not understand it) is the following:

Theorem

For all natural numbers n we have $1 + 2 + 3 + 4 + \cdots + n = \frac{n \times (n+1)}{2}$.

Two checks

- First some checks: For $n = 3$ we have $n + 1 = 4$ and indeed

$$1 + 2 + 3 = 6 = \frac{3 \times 4}{2}$$

- One more, let's be more bold and do $n = 7$:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 = \frac{7 \times 8}{2}.$$

Surprise surprise, it seems to work!

The rigorous proof

- The cases we have to check are already nicely numbered.
- And for the first case $n = 1$ the formula is definitely correct since

$$1 = \frac{1 \times 2}{2}.$$

- Now assume the formula is true for some case, say $n = 37$, so we assume

$$1 + \cdots + 37 = \frac{37 \times 38}{2} = 703.$$

Well, if this is true, then we clearly have

$$1 + \cdots + 37 + 38 = (1 + \cdots + 37) + 38 = \frac{37 \times 38}{2} + 38.$$

The rigorous proof

- And then some awkward computation gives

$$\begin{aligned}\frac{37 \times 38}{2} + 38 &= \frac{37 \times 38}{2} + \frac{2 \times 38}{2} = \frac{37 \times 38 + 2 \times 38}{2} \\ &= \frac{(37+2) \times 38}{2} = \frac{39 \times 38}{2} = \frac{38 \times 39}{2}\end{aligned}$$

  which is what the formula gives for the next case $n = 38$.
- So we indeed have shown if the claim is true for the case 37, then it is true for the next case. Well, we actually should do this step for an **arbitrary** case and not for $n = 37$, but if you look carefully into the above, the same computation works for any number n:

The rigorous proof

- If we assume the formula holds for the case n,

$$1 + 2 + 3 + \cdots + n = \frac{n \times (n+1)}{2}$$

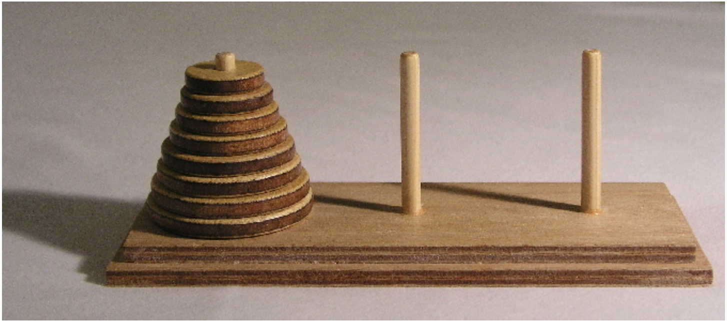
then we indeed can derive the formula for the case $n + 1$:

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + n + 1 &= \frac{n \times (n+1)}{2} + n + 1 \\ &= \frac{n \times (n+1) + 2 \times (n+1)}{2} \\ &= \frac{(n+1) \times (n+2)}{2} \end{aligned}$$

which finishes the formal proof.

The towers of Hanoi

- I guess you have seen that one already somewhere. There are three pegs and some number n (in the picture 8) of discs of different radius with a hole so one can stick them onto the pegs:



- Aim: Move all discs to the second peg but there must never be a larger disc placed on top of a smaller one, and of course in each move you must put only one single disc onto a different peg.

The theorem

- We will prove now the following

Theorem

The game has a solution using $2^n - 1$ moves ($n =$ number of discs).

Recall that $2^n = 2 \times 2 \times 2 \times \cdots \times 2$ (n factors). For example, I claim that with 3 discs you need $2 \times 2 \times 2 - 1 = 7$ moves.

- Step 1 of the proof: The case $n = 1$ is clear - if there is one disk to move I just put it in one move to the second peg, and we indeed have
$$2^1 - 1 = 2 - 1 = 1.$$
- Step 2 of the proof: Assume we have proven the theorem for some case n, that is, we know how to move n discs from one peg to the next using $2^n - 1$ moves. And now we want to show the theorem is also true for the next case $n + 1$.

The induction step

- Well, if we are given the game with $n + 1$ discs we can use our solution for n discs to move the top n of our discs to the third peg within $2^n - 1$ moves. In all this, the largest disc remains untouched on the first peg.
- Then we can use 1 more move to put the largest disc onto the second peg.
- Finally we again use our solution for n to move the n discs back from the third to the second, not moving the largest disc. We are done!
- And how many moves have we used?

$$(2^n - 1) + 1 + (2^n - 1) = 2^n + 2^n - 1 = 2 \times 2^n - 1 = 2^{n+1} - 1$$

which is exactly the number we have claimed (don't forget, we are now in the case $n + 1$).

Exercise session 1

- Work out a solution for the towers of Hanoi with $n = 4$ discs.
- Given a rectangular chocolate bar with r rows of s pieces of chocolate in each row, is there an optimal strategy to break this into single pieces? I claim: No, all that matters is the total number of pieces:

Theorem

It takes $n - 1$ breaks to break a chocolate bar into its n pieces.

Things can go wrong

- Now we prove the following:

Theorem

Any n Glaswegians are of the same age.

- We prove this by induction on the number of Glaswegians. If there is only one there is nothing to show.
- Now assume we have proven the theorem for some number n of Glaswegians.
- If there are $n+1$ Glaswegians, ask one to wait for a moment outside. Let's say her name is Fiona.
- By the assumption made, all the others must be of the same age.

Things can go wrong

- Now send someone else away, so we are left with $n-1$ Glaswegians, all of the same age. If we invite Fiona back in, then we again have a group of n Glaswegians, so they must be of the same age.
- That is, Fiona also had the same age as all the others! The theorem is proved.
- What is wrong here??
- It is the case $n=2$ where the argument doesn't work at all! So as you see, one has to be rather careful when proving theorems...

The story about the green and the red eyes

- This story is about an ancient tribe living in a village in a desert in Siberia. A small village in fact, there were only 42 people left.
- And here is what is so special about them: Some of them have red eyes and the rest green ones. However, no one ever even only mentions the topic of the colour of their eyes, it is completely forbidden to talk about anything that has to do with this. They of course all see each other, so the only eye colour no one is certain about is the one of their own eyes.
- More weirdly, a law says that if someone knew he or she had red eyes, this person would have to commit suicide the next morning, at 5 AM. So far this law never had to be applied.
- Clearly, there are no mirrors or so in that desert...

The story about the green and the red eyes

- One day a visitor came along and stayed for a while. And when he left, he made a small speech, thanking them for their hospitality.
- Unfortunately he could not control himself and mentioned at the very end how fascinating this whole story about the eyes was, and how incredible it is that no red-eyed person has ever found out about their eye colour. And off he went.
- Needless to say, the villagers were really nervous since their taboo had been broken and someone had openly talked about the colour of their eyes. Well, first they thought the situation wasn't that bad - the only thing that had been revealed was that there exists at least one person with red eyes, but that had been clear to anyone anyway.
- Still, they went home and thought about the whole issue very very carefully.

The story about the green and the red eyes

- And I claim this is what happened as a result of their considerations:

Theorem

If there were n people with red eyes in the village, they all committed suicide on the n-th morning after the speech.

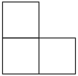
- I'll prove this by induction on n. If $n = 1$, then this poor chap had always only seen green-eyed people but could never be sure whether he might not be the one and only red-eyed in the village.
- When the speech happened the situation was clear to him, the red-eyed that had never killed himself could only be him! So the next morning...

The induction step

- Now assume we have shown the theorem for n red-eyed people but there are actually $n + 1$.
- Let's think like one of them. Well, there are only 42 people in the village, so of course our hero can count the number of red-eyed people he can see (namely n).
- But he always wondered whether there are in fact $n + 1$ (he has red eyes) or only the n he can see (he has green eyes).
- Now, by our induction hypothesis, if there were n they would commit suicide after n days.
- So our hero waits patiently. And when on the n-th day the others are still there, he finally knows the truth, and on the $n + 1$st morning, the story comes to its sad ending.

Exercise session 2

- Remove one field from a (generalised) chessboard of $2^n \times 2^n$ fields, n some natural number. Prove that the board now can be completely covered with small L-shaped patterns formed out of three fields each:



- You have proved the following theorem!

Theorem

$2^n - 1$ *is for all numbers n divisible by 3.*

I suggest to check out next week whether your math teacher knows that.

Fermat numbers

- The **Fermat numbers** are

$$F_n = 2^{2^{n-1}} + 1, \quad n = 1, 2, 3, \ldots$$

so explicitly,

$$F_1 = 3, \quad F_2 = 5, \quad F_3 = 17, \quad F_4 = 257, \ldots$$

- We claim:

Theorem

$F_1 \times F_2 \times \cdots \times F_{n-1} = F_n - 2$ *for all* n.

- For $n = 2$ this is clear.

The induction step

- Assume the formula holds for n. Then we get

$$
\begin{aligned}
F_1 \times \cdots \times F_n &= F_1 \times \cdots \times F_{n-1} \times F_n \\
&= (F_n - 2) \times F_n \\
&= (2^{2^{n-1}} - 1)(2^{2^{n-1}} + 1) \\
&= (2^{2^{n-1}})^2 - 1 \\
&= 2^{2^n} - 1
\end{aligned}
$$

as we had to show.

Exercise session 3

- Use the theorem to show that for two numbers $n < m$, there is no prime number p that divides both F_n and F_m (you will **not** need an induction argument for this, only basic arithmetics).
- Note the difference to the theorem proven in the previous session: $2^n - 1$ is always divisible by 3!
- There are clearly infinitely many Fermat numbers, each of which has a prime factorisation that we now know involves different primes for each F_n. So we have proven a deep theorem:

Theorem

There are infinitely many prime numbers.

References

- I will put the slides of this talk on my web page

  www.maths.gla.ac.uk/˜ukraehmer

- I took most examples from lecture notes of Victor Adamchik that you can find on

  www.cs.cmu.edu/˜adamchik/21 − 127/index.html

- The other source I used is

  www.cut − the − knot.org/induction.shtml

- A nice The Tower of Hanoi applet can be found at

  www.mathsnet.net/puzzles/hanoi/