Real Polynomials with Definite Determinantal Representation

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Contents

Preface

A homogeneous polynomial $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ is said to be hyperbolic with respect to $e \in \mathbb{R}^\mathrm{n}$, if h does not vanish in e and if for every $v \in \mathbb{R}^\mathrm{n}$ the univariate polynomial $h(\mathrm{t}\, e + v) \in \mathbb{R}[\mathrm{t}]$ has only real roots. The hyperbolicity cone of h at e is the set of all $v \in \mathbb{R}^\mathrm{n}$ such that the zeros of $h(\mathrm{t}\, e + v)$ are all negative. Hyperbolicity cones are semi-algebraic convex cones. On the other hand, a spectrahedral cone is a set defined by some homogeneous linear matrix inequalities. Spectrahedral cones are of interest since they are feasible sets of semi-definite programming, which is an effective generalization of linear programming. It is not hard to check that every spectrahedral cone is the hyperbolicity cone of an appropriate hyperbolic polynomial. If $n = 3$, the converse is true: Every three dimensional hyperbolicity cone is also a spectrahedral cone. Helton and Vinnikov verified this by proving that every hyperbolic polynomial $h \in \mathbb{R}[\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]$ has a definite determinantal representation, i.e. there are symmetric matrices A_1, A_2, A_3 with real entries, such that $h = \det(\mathrm{x}_1 A_1 + \mathrm{x}_2 A_2 + \mathrm{x}_3 A_3)$, where $v_1 A_1 + v_2 A_2 + v_3 A_3$ is positive definite for some $v \in \mathbb{R}^3$ (note that every polynomial with a definite determinantal representation is hyperbolic). This result, previously known as the Lax Conjecture, gave rise to several further conjectures, for example:

Conjecture. *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be a hyperbolic polynomial. Then there is a positive integer N, such that h^N has a definite determinantal representation.*

This conjecture turned out to be false, as Brändén came up with a counterexample, using matroid theory. After some preliminaries the first goal of this thesis will be to present the construction of this counterexample and to discuss the connection of matroid theory with hyperbolic polynomials (section 2.3, in particular Proposition 2.3.15). The next, weaker conjecture is still unsettled:

Conjecture. *Every hyperbolicity cone is a spectrahedral cone.*

This conjecture is commonly referred to as the Generalized Lax Conjecture. We will show that it is not possible to disprove this conjecture analogously to the previous conjecture, providing some kind of a discrete version of the Generalized Lax Conjecture (section 2.4, in particular Theorem 2.4.6).

We will often consider stable polynomials, instead of hyperbolic polynomials. A homogeneous polynomial $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ is stable if and only if it is hyperbolic with respect to every vector in the positive orthant. Since, after a linear change of variables, every hyperbolic polynomial is stable, this is rather a normalization than a restriction. In the last third, we will focus on multiaffine polynomials, i.e. polynomials where every variable occurs at most of degree one. We will see that it would suffice to proof the Generalized Lax Conjecture in the case of multiaffine polynomials (Remark 3.2.15). After presenting a characterization of multiaffine stable polynomials due to Brändén (Theorem 3.2.7), we will give a very convenient criterion to decide whether a multiaffine stable polynomial has a definite determinantal representation (Theorem 3.4.7). In the last section, extending these two results partially to the non-multiaffine case, we will outline a connection between hyperbolicity and non-negativity on the one hand and determinantal representability and sum of squares on the other hand (Theorem 3.5.6).

1 Basic Notions and Theorems

1.1 Some Matrix Identities

Let $n \in \mathbb{Z}_{>0}$. Let $[n] = \{1, \ldots, n\}$ and let $\binom{[n]}{k}$ denote the set of all $k-$element subsets of $[n]$. Let K be a field and let $\mathrm{Mat}_K(n, m)$ denote the set of $n \times m$ matrices with entries in K. We write $\mathrm{Sym}_n(K)$ for the set of symmetric $n \times n$ matrices over K. If a matrix $A \in \mathrm{Sym}_n(\mathbb{R})$ is positive semi-definite, we write $A \succeq 0$. If it is positive definite, we write $A \succ 0$. Given a $n \times m$ matrix $A \in \mathrm{Mat}_K(n, m)$ and subsets $S \subseteq [n], T \subseteq [m]$ of the same size, we write $A(S, T)$ for the determinant of the matrix obtained from matrix A by deleting all rows except these indexed by S and all columns except these indexed by T. Further, if A is a $n \times n$ square matrix, we write $A_{ij} = A([n] \setminus \{i\}, [n] \setminus \{j\})$. Let I_n denote the identity matrix of size n and let $\delta_i \in K^n$ denote the ith unit vector.

1.1.1 Theorem (Leibniz formula, cf. [8], 3.2.5)**.** *Consider an $n \times n$ square matrix $A = (a_{ij})_{i,j=1,\ldots,n} \in \mathrm{Mat}_K(n, n)$. Then*

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i)i}$$

where sgn *is the sign function of permutations in the permutation group $\mathfrak{S}_n$, which returns $+1$ and -1 for even and odd permutations, respectively.*

1.1.2 Theorem (Laplace formula, cf. [1], Section 35)**.** *Let $k < n$, let $S \in \binom{[n]}{k}$ and let $A \in \mathrm{Mat}_K(n, n)$ be a $n \times n$ square matrix. For all $T \in \binom{[n]}{k}$, we denote $e(T) = \sum\limits_{s \in S} s + \sum\limits_{t \in T} t$. Then*

$$\det(A) = \sum_{T \in \binom{[n]}{k}} A(S, T) A([n] \setminus S, [n] \setminus T)(-1)^{e(T)}.$$

1.1.3 Corollary. *Let $I, J \in \binom{[n]}{k}$ and let $X = (\mathrm{x}_{ij})_{i,j=1,\ldots,n}$ be a $n \times n$ square matrix of distinct variables. We write $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_k\}$ and $m = \sum_{l=1}^{k}(i_l + j_l)$. Then we have*

$$\frac{\partial^k \det(X)}{\partial\, \mathrm{x}_{i_1 j_1} \cdots \partial\, \mathrm{x}_{i_k j_k}} = (-1)^m X([n] \setminus I, [n] \setminus J).$$

Proof. For all $T \in \binom{[n]}{k}$, let $e(T) = \sum\limits_{s \in I} s + \sum\limits_{t \in T} t$. By the Laplace formula, we have

$$\det(X) = \sum_{T \in \binom{[n]}{k}} X(I, T) X([n] \setminus I, [n] \setminus T)(-1)^{e(T)}.$$

For all $T \in \binom{[n]}{k}$, the polynomial $X([n] \setminus I, [n] \setminus T)$ does not depend on $\mathrm{x}_{i_l j_l}$ for $l \in [k]$. If we have in mind the Leibniz formula, we realize that $\frac{\partial^k X(I,T)}{\partial\, \mathrm{x}_{i_1 j_1} \cdots \partial\, \mathrm{x}_{i_k j_k}} = 0$ if $T \neq J$ and $\frac{\partial^k X(I,J)}{\partial\, \mathrm{x}_{i_1 j_1} \cdots \partial\, \mathrm{x}_{i_k j_k}} = 1$. This implies the claim. $\square$

1.1.4 Theorem (Cauchy-Binet, cf. [8], 3.3.7)**.** *Let $m \leq n$, let $A \in \mathrm{Mat}_K(m,n)$ and let $B \in \mathrm{Mat}_K(n,m)$. Then*

$$\det(AB) = \sum_{S \in \binom{[n]}{m}} A([m],S)B(S,[m]).$$

Proof. Let $A = (a_{ij})_{ij}$ and $B = (b_{jk})_{jk}$. Then

$$AB = \begin{pmatrix} \sum_{k_1=1}^n a_{1k_1}b_{k_1 1} & \cdots & \sum_{k_m=1}^n a_{1k_m}b_{k_m m} \\ \vdots & & \vdots \\ \sum_{k_1=1}^n a_{mk_1}b_{k_1 1} & \cdots & \sum_{k_m=1}^n a_{mk_m}b_{k_m m} \end{pmatrix}.$$

Because the determinant is linear in each column, it follows

$$\begin{aligned} \det(AB) &= \sum_{k_1,\ldots,k_m=1}^{n} \det\begin{pmatrix} a_{1k_1}b_{k_1 1} & \cdots & a_{1k_m}b_{k_m m} \\ \vdots & & \vdots \\ a_{mk_1}b_{k_1 1} & \cdots & a_{mk_m}b_{k_m m} \end{pmatrix} \\ &= \sum_{k_1,\ldots,k_m=1}^{n} \det\begin{pmatrix} a_{1k_1} & \cdots & a_{1k_m} \\ \vdots & & \vdots \\ a_{mk_1} & \cdots & a_{mk_m} \end{pmatrix} b_{k_1 1}\cdots b_{k_m m}. \end{aligned}$$

If $k_s = k_p$ for some $s \neq p$, then $\det(a_{ik_j})_{ij} = 0$, since in this case two columns are equal. So we have:

$$\begin{aligned} \det(AB) &= \sum_{1 \leq k_1 < \ldots < k_m \leq n} \left(\sum_{\sigma \in \mathfrak{S}_m} \det((a_{ik_{\sigma(j)}})_{ij}) \cdot b_{k_{\sigma(1)}1} \cdots b_{k_{\sigma(m)}m} \right) \\ &= \sum_{1 \leq k_1 < \ldots < k_m \leq n} \left(\det((a_{ik_j})_{ij}) \sum_{\sigma \in \mathfrak{S}_m} \mathrm{sgn}(\sigma) \cdot b_{k_{\sigma(1)}1} \cdots b_{k_{\sigma(m)}m} \right). \end{aligned}$$

Using the Leibniz formula we get the desired equation.

$$\det(AB) = \sum_{1 \leq k_1 < \ldots < k_m \leq n} \det((a_{ik_j})_{ij}) \det((b_{k_i j})_{ij}).$$

□

If $A \in \mathrm{Mat}_K(n,n)$, we denote the adjugate by $\mathrm{adj}(A) = ((-1)^{i+j}A_{ji})_{i,j\in[n]}$. The following proposition is a well-known result of linear algebra.

1.1.5 Proposition (cf. [8], 3.3.1)**.** *Let $A \in \mathrm{Mat}_K(n,n)$. Then we have the identity $A \cdot \mathrm{adj}(A) = \det(A) \cdot \mathrm{I}_n$ and for the rank of $\mathrm{adj}(A)$ holds:*

$$\mathrm{rank}(\mathrm{adj}(A)) = \begin{cases} n, & \textit{if } \mathrm{rank}(A) = n, \\ 1, & \textit{if } \mathrm{rank}(A) = n-1, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The identity $A \cdot \mathrm{adj}(A) = \det(A) \cdot \mathrm{I}_n$ follows directly from the Laplace formula. If $\det(A) \neq 0$, the matrix $\mathrm{adj}(A)$ is invertible, thus $\mathrm{rank}(\mathrm{adj}(A)) = n$. If $\mathrm{rank}(A) \leq n-2$, every $(n-1)\times(n-1)$ minor of A vanishes, thus $\mathrm{adj}(A) = 0$. Now let $\mathrm{rank}(A) = n-1$. We have $A \cdot \mathrm{adj}(A) = 0$, thus $\mathrm{im}(\mathrm{adj}(A)) \subseteq \ker(A)$. Therefore $\mathrm{adj}(A)$ has rank at most one, but since there is a $(n-1)\times(n-1)$ minor of A that does not vanish, we have $\mathrm{adj}(A) \neq 0$. □

1.2 Results of Real Algebraic Geometry

From time to time, we have to use some results of real algebraic geometry. Here we give an overview of the theorems we need. Most of these results can be looked up in [2]. Let $f_1, \dots, f_r \in \mathbb{R}[\mathrm{x_1}, \dots, \mathrm{x_n}]$. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We write $\mathcal{V}_{\mathbb{K}}(f_1, \dots, f_r) = \{x \in \mathbb{K}^n : \ f_i(x) = 0, \ i = 1, \dots r\}$. If we have $V \subseteq \mathbb{C}^n$, we write $V(\mathbb{R}) = V \cap \mathbb{R}^{\mathrm{n}}$ for the set of real points. A *semi-algebraic subset* of $\mathbb{R}^{\mathrm{n}}$ is the subset of points in $\mathbb{R}^{\mathrm{n}}$ satisfying a boolean combination of polynomial equations and inequalities with real coefficients.

1.2.1 Proposition ([22], Corollary 1.4)**.** *Let $f_1, \dots, f_r \in \mathbb{R}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ and let $V = \mathcal{V}_{\mathbb{C}}(f_1, \dots, f_r)$ be irreducible. The real points $V(\mathbb{R})$ lie Zariski dense in V if and only if V has a smooth, real point $p \in V(\mathbb{R})$.*

1.2.2 Theorem ([2], Théorème 2.4.5)**.** *Every semi-algebraic $A \subseteq \mathbb{R}^{\mathrm{n}}$ has a finite number of connected components, which are semi-algebraic too.*

1.2.3 Proposition ([2], Proposition 2.5.11)**.** *Let $A \subseteq \mathbb{R}^{\mathrm{n}}$ be a connected, semi-algebraic set. Then A is path-connected, i.e. for all $x, y \in A$, there is a continuous function $\varphi : [0, 1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$.*

1.2.4 Definition. Let $S \subseteq \mathbb{R}^{\mathrm{n}}$ be a semi-algebraic set and let $V = \mathrm{clos_Z}(S)$ be the Zariski closure of S. The *dimension* $\dim(S)$ of S is defined to be the dimension of V as an algebraic set.

1.2.5 Proposition ([2], Proposition 2.8.4)**.** *Let $U \subseteq \mathbb{R}^{\mathrm{n}}$ be an open, non-empty, semi-algebraic set. Then we have $\dim(U) = n$.*

1.2.6 Definition. Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ be semi-algebraic sets. A function $f : A \to B$ is called semi-algebraic, if its graph is a semi-algebraic subset of $\mathbb{R}^{m+n}$.

1.2.7 Proposition ([2], Proposition 2.2.7)**.** *Let A, B be semi-algebraic sets and let $f : A \to B$ be a semi-algebraic function. If $S \subseteq A$ is semi-algebraic, then its image $f(S)$ is also semi-algebraic. If $T \subseteq B$ is semi-algebraic, then its preimage $f^{-1}(T)$ is also semi-algebraic.*

1.2.8 Theorem ([2], Théorème 2.8.8)**.** *Let $A \subseteq \mathbb{R}^m$ be a semi-algebraic set and let $f : A \to \mathbb{R}^{\mathrm{n}}$ be a semi-algebraic function. Then $\dim(A) \geq \dim(f(A))$. If f is injective, it holds $\dim(A) = \dim(f(A))$.*

1.3 Hyperbolic Polynomials

The concept of hyperbolic polynomials is essential for most of the parts of this thesis. In this section we introduce them and study some of their properties. For more information about hyperbolic polynomials and their applications, see for example [9, 10, 20].

1.3.1 Definition. Let $h \in \mathbb{R}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ be a homogeneous polynomial. We say that h is *hyperbolic with respect to $e \in \mathbb{R}^n$*, if $h(e) \neq 0$ and if for all $v \in \mathbb{R}^{\mathrm{n}}$ and $\mu \in \mathbb{C}$ we have

$$h(v + \mu e) = 0 \ \Rightarrow \ \mu \in \mathbb{R}\,.$$

1.3.2 Definition. Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be hyperbolic with respect to $e \in \mathbb{R}^\mathrm{n}$. The *hyperbolicity cone* $\mathrm{C}_h(e)$ of h at e is the set of all $v \in \mathbb{R}^\mathrm{n}$, such that the univariate polynomial $h(v + \mathrm{t}\, e) \in \mathbb{R}[\mathrm{t}]$ has only negative roots.

1.3.3 Remark. Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be hyperbolic with respect to $e \in \mathbb{R}^\mathrm{n}$ with $\deg(h) = d$. Here are some basic facts.

1. Consider the semi-algebraic set

$$M = \{(t, v) \in \mathbb{R}^{n+1} :\ h(te + v) = 0,\ t \geq 0\}.$$

   The projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^\mathrm{n}$, $(t, v) \mapsto v$ clearly is semi-algebraic. Thus, by Proposition 1.2.7 the hyperbolicity cone $\mathrm{C}_h(e) = \mathbb{R}^\mathrm{n} \setminus \pi(M)$ is a semi-algebraic set.

2. The only root of $h(\mathrm{t}\, e + e)$ is -1, therefore $e \in \mathrm{C}_h(e)$.

3. If $v \in \mathrm{C}_h(e)$ then $h(v) \neq 0$, since otherwise 0 would be a root of $h(\mathrm{t}\, e + v)$.

4. Let $v \in \mathrm{C}_h(e)$ and let $s > 0$. Then $sv \in \mathrm{C}_h(e)$, because the univariate polynomial $h(\mathrm{t}\, e + sv) = s^d h((s^{-1}\, \mathrm{t})e + v)$ has only negative roots.

5. Let $p, q \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ and let $e \in \mathbb{R}$. The product $p \cdot q$ is hyperbolic with respect to e, if and only if p and q both are hyperbolic with respect to e. In this case we have $\mathrm{C}_{pq}(e) = \mathrm{C}_p(e) \cap \mathrm{C}_q(e)$. This follows immediately from the definitions.

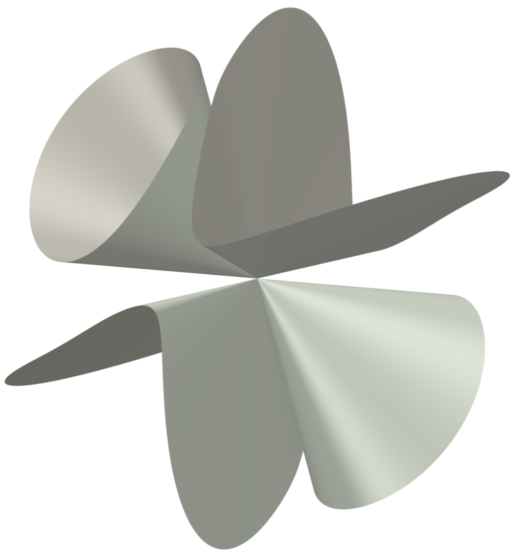

Figure 1: The zero set of a hyperbolic polynomial in three variables of degree four.

1.3.4 Example. The homogeneous polynomial $h = \mathrm{x}_1^2 - \mathrm{x}_2^2$ is hyperbolic with respect to all $e \in \mathbb{R}^2$ with $h(e) \neq 0$, because

$$h(\mathrm{t}\, e + v) = (\mathrm{t}(e_1 + e_2) + v_1 + v_2) \cdot (\mathrm{t}(e_1 - e_2) + v_1 - v_2).$$

Let, for example, $e = (1, 0)^\mathrm{T}$, then we have $\mathrm{C}_h(e) = \{v \in \mathbb{R}^2 :\ -v_1 < v_2 < v_1\}$.

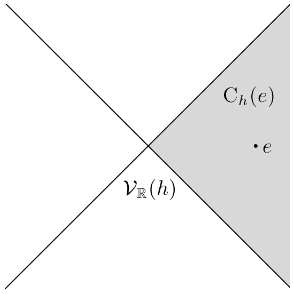

Figure 2: The hyperbolicity cone of $h = \mathrm{x}_1^2 - \mathrm{x}_2^2$ at $e = (1,0)^{\mathrm{T}}$.

1.3.5 Example. Let $h \in \mathbb{R}[\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}]$ be hyperbolic with respect to $e \in \mathbb{R}^\mathrm{n}$. Let $\mathrm{D}_e h$ be the directional derivative of h along e. For all $v \in \mathbb{R}^\mathrm{n}$ the univariate polynomial $h(\mathrm{t}\, e + v)$ has only real roots. By the intermediate value theorem, the derivative $\frac{\partial h(\mathrm{t}\, e+v)}{\partial\, \mathrm{t}} = \mathrm{D}_e h(\mathrm{t}\, e + v)$ has only real roots too. Therefore $\mathrm{D}_e h$ is hyperbolic with respect to e.

1.3.6 Example. Consider the symmetric matrix

$$X = \begin{pmatrix} \mathrm{x}_{11} & \cdots & \mathrm{x}_{1n} \\ \vdots & \ddots & \vdots \\ \mathrm{x}_{1n} & \cdots & \mathrm{x}_{nn} \end{pmatrix}.$$

The polynomial $h = \det(X) \in \mathbb{R}[\mathrm{x}_{ij} : i,j \in [n],\, i \le j]$ is hyperbolic with respect to the identity matrix I_n, because for all symmetric matrices $A \in \mathrm{Sym}_n(\mathbb{R})$

$$h(A + \mathrm{t}\, \mathrm{I}_n)$$

is the characteristic polynomial of $-A$ and has thus only real roots. The hyperbolicity cone contains exactly all positive definite matrices.

1.3.7 Example. Consider the polynomial

$$h = \det(\mathrm{x}_1\, A_1 + \ldots + \mathrm{x}_n\, A_n)$$

with symmetric matrices $A_1, \dots, A_n \in \mathrm{Sym}_d(\mathbb{R})$. Then let $e \in \mathbb{R}^\mathrm{n}$ and let $B = e_1 A_1 + \ldots + e_n A_n$. If B is positive definite, then h is hyperbolic with respect to e. Because then there exists an invertible matrix $S \in \mathrm{GL}_d(\mathbb{R})$ with $B = S^T S$. For all $v \in \mathbb{R}^\mathrm{n}$ and $\mu \in \mathbb{C}$ it holds that

$$h(v + \mu e) = \det(\mu B + v_1 A_1 + \ldots + v_n A_n) = \det(S)^2 \det(\mu I + T),$$

for the symmetric matrix $T = (S^{-1})^{\mathrm{T}}(v_1 A_1 + \ldots + v_n A_n) S^{-1}$. If $h(v + \mu e) = 0$, then $-\mu$ is an eigenvalue of this symmetric matrix T and thus is real. The hyperbolicity cone of h is the set of $v \in \mathbb{R}^\mathrm{n}$, such that $v_1 A_1 + \ldots + v_n A_n$ is positive definite.

1.3.8 Definition. Let $h \in \mathbb{R}[\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}]$ be hyperbolic with respect to $e \in \mathbb{R}^\mathrm{n}$. We say that h has a *definite determinantal representation*, if there are symmetric matrices $A_1, \dots, A_n \in \mathrm{Sym}_d(\mathbb{R})$ such that $e_1 A_1 + \ldots + e_n A_n$ is positive definite and

$$h = \lambda \det(\mathrm{x}_1\, A_1 + \ldots + \mathrm{x}_n\, A_n)$$

for some $\lambda \in \mathbb{R}$.

The next Theorem is crucial, but we leave out the proof nevertheless, since it is technical and lengthy.

1.3.9 Theorem ([9], Theorem 2)**.** *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be hyperbolic with respect to $\mathrm{e} \in \mathbb{R}^\mathrm{n}$. Then the hyperbolicity cone $\mathrm{C}_h(e)$ is convex and h is hyperbolic with respect to all $v \in \mathrm{C}_h(e)$.*

1.3.10 Corollary. *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be hyperbolic with respect to $e \in \mathbb{R}^\mathrm{n}$. The hyperbolicity cone $\mathrm{C}_h(e)$ is an open convex cone and $\mathrm{C}_h(e)$ is the connected component of $\mathbb{R}^\mathrm{n} \smallsetminus \mathcal{V}_\mathbb{R}(h)$ that contains e.*

Proof. The fact that $\mathrm{C}_h(e)$ is a convex cone follows from the preceding Theorem and Remark 1.3.3. The fact that $\mathrm{C}_h(e)$ is open follows from the second claim. Let C be the connected component of $\mathbb{R}^\mathrm{n} \smallsetminus \mathcal{V}_\mathbb{R}(h)$ that contains e. Because h does not vanish on $\mathrm{C}_h(e)$ and because $\mathrm{C}_h(e)$ is connected, since convex, the cone $\mathrm{C}_h(e)$ is a subset of C. Suppose there is a $x \in C \smallsetminus \mathrm{C}_h(e)$. Since C is semi-algebraic by Proposition 1.2.2 and therefore path-connected by Proposition 1.2.3, there is a path $\alpha : [0,1] \to C$ such that $\alpha(0) = e$ and $\alpha(1) = x$. Then $h(\mathrm{t}\,e + \alpha(0))$ has only negative roots and $h(\mathrm{t}\,e + \alpha(1))$ has at least one non-negative root. But since the zeros of a polynomial depend continuously on its coefficients, there is a $t_0 \in [0,1]$, such that $h(\alpha(t_0)) = 0$ by the intermediate value theorem. But this is a contradiction to $\alpha(t_0) \in C \subseteq \mathbb{R}^\mathrm{n} \smallsetminus \mathcal{V}_\mathbb{R}(h)$. ☐

1.3.11 Lemma (cf. the proof of [12], Lemma 2.1)**.** *Let $U_1, U_2 \subseteq \mathbb{R}^\mathrm{n}$ be two disjoint, non-empty, semi-algebraic open subsets. Then we have*

$$\dim(\mathbb{R}^\mathrm{n} \smallsetminus (U_1 \cup U_2)) \geq n - 1.$$

Proof. Let $F = \mathbb{R}^\mathrm{n} \smallsetminus (U_1 \cup U_2)$ and let $x \in U_1$ and $y \in U_2$. Let $H \subseteq \mathbb{R}^\mathrm{n}$ be a hyperplane, that contains y, but not x. Let $H' \subseteq \mathbb{R}^\mathrm{n}$ be the hyperplane, that is parallel to H, such that $x \in H'$. Let $\pi : \mathbb{R}^\mathrm{n} \smallsetminus H' \to H$ be the linear projection of center x. We will proof that the set $\pi(F \smallsetminus H')$ contains $H \cap U_2$. Let $z \in H \cap U_2$ and let l be the line through x and z. Then $l \not\subseteq U_1 \cup U_2$, since l is connected and $U_1 \cap U_2 = \emptyset$, thus $l \cap F \neq \emptyset$. Therefore $H \cap U_2 \subseteq \pi(F \smallsetminus H')$. Since $\dim(H \cap U_2) = n - 1$ by Proposition 1.2.5, this implies $\dim(F) \geq n - 1$ by Proposition 1.2.8. ☐

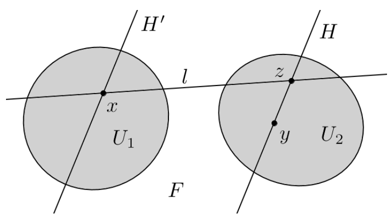

Figure 3: Illustration of the proof of Lemma 1.3.11.

1.3.12 Proposition (cf. the proof of [12], Lemma 2.1)**.** *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be hyperbolic with respect to $e \in \mathbb{R}^\mathrm{n}$ and let $V = \mathcal{V}_\mathbb{C}(h)$. The real points $V(\mathbb{R})$ of V are Zariski dense in V. If h is irreducible, then the boundary of the hyperbolicity cone $\partial\,\mathrm{C}_h(e)$ is Zariski dense in V.*

Proof. Let h not be constant, since otherwise the claim is clear. For a start, let h be irreducible. Since $\mathrm{C}_h(e)$ is open, we have

$$\partial\,\mathrm{C}_h(e) = \overline{\mathrm{C}_h(e)} \smallsetminus \mathrm{C}_h(e) = \mathbb{R}^{\mathrm{n}} \smallsetminus ((\mathbb{R}^{\mathrm{n}} \smallsetminus \overline{\mathrm{C}_h(e)}) \cup \mathrm{C}_h(e)).$$

Thus, the preceding Lemma implies $\dim \partial\,\mathrm{C}_h(e) \geq n-1$. On the other hand, we have by Corollary 1.3.10 that $\partial\,\mathrm{C}_h(e) \subseteq V(\mathbb{R})$. Since $\dim(V) = n-1$ and because V is irreducible, we get the claim.

In the case where h is not irreducible we get nevertheless, that the real points of every irreducible component V_i of V lie Zariski dense in V_i, because every irreducible factor of h is also hyperbolic. Therefore the Lemma holds for reducible h as well. □

1.4 Hyperbolicity Cones and Spectrahedra

A famous open question is about the connection between hyperbolicity cones and so-called spectrahedral cones, i.e. convex cones, that can be described as the solution set of a linear matrix inequality.

1.4.1 Definition. Let $d \geq 0$. Given symmetric matrices $A_1 \ldots, A_n \in \mathrm{Sym}_d(\mathbb{R})$ and $v \in \mathbb{R}^{\mathrm{n}}$. We denote the *homogeneous linear matrix pencil* defined by $A_1 \ldots, A_n$ in v by

$$A(v) = v_1 A_1 + \ldots + v_n A_n.$$

A *spectrahedral cone* is the set of all points in $\mathbb{R}^{\mathrm{n}}$ that satisfy a certain linear matrix inequality, i.e. a set of the form

$$\{v \in \mathbb{R}^{\mathrm{n}} :\ A(v) \succeq 0\} = \{v \in \mathbb{R}^{\mathrm{n}} :\ v_1 A_1 + \ldots + v_n A_n \text{ is positive semi-definite}\},$$

for some symmetric matrices $A_1, \ldots, A_n$ with real entries.

The following Proposition sums up some basic properties of spectrahedral cones, which are well-known results of convex geometry.

1.4.2 Proposition. *Let $d \geq 0$, let $A_1, \ldots, A_n \in \mathrm{Sym}_d(\mathbb{R})$ and consider the spectrahedral cone $C = \{v \in \mathbb{R}^{\mathrm{n}} :\ A(v) \succeq 0\}$ defined by these matrices. Then, the following holds:*

(i) C is semi-algebraic, closed and a convex cone.

(ii) If there is a $w \in \mathbb{R}^{\mathrm{n}}$, such that $A(w) \succ 0$, then the boundary of C is $\partial C = \{v \in \mathbb{R}^{\mathrm{n}} :\ A(v) \succeq 0,\ \det A(v) = 0\}$

(iii) If there is a $w \in \mathbb{R}^{\mathrm{n}}$, such that $A(w) \succ 0$, then the interior of C is $C^{\circ} = \{v \in \mathbb{R}^{\mathrm{n}} :\ A(v) \succ 0\}$.

(iv) The intersection of two spectrahedral cones is again a spectrahedral cone.

Proof. Since a symmetric matrix is positive semi-definite if and only if all of its 2^n principal minors are non-negative, it is clear that C is semi-algebraic closed. In order to show that C is a convex cone, let $v, w \in C$ and let $\lambda, \mu \geq 0$. For all $x \in \mathbb{R}^{\mathrm{n}}$, we have

$$x^{\mathrm{T}} A(\lambda v + \mu w) x = \lambda x^{\mathrm{T}} A(v) x + \mu x^{\mathrm{T}} A(w) x \geq 0,$$

since $A(v), A(w) \succeq 0$. Hence we have $A(\lambda v + \mu w) \succeq 0$. Thus, we have (i).

In (ii) we show both inclusions. Let $v \in \mathbb{R}^n$ such that $A(v) \succeq 0$ and $\det A(v) = 0$. Since $A(w) \succ 0$, there is an invertible matrix $S \in \mathrm{GL}_d(\mathbb{R})$, such that $S^{\mathrm{T}} A(w) S = \mathrm{I}_d$. Let $u \in \ker(S^{\mathrm{T}} A(v) S)$, $u \neq 0$. Then we have for all $\epsilon > 0$:

$$(Su)^{\mathrm{T}} A(v - \epsilon w)(Su) = u^{\mathrm{T}}(S^{\mathrm{T}} A(v) S - \epsilon\, \mathrm{I}_d)u = -\epsilon \|u\|^2 < 0.$$

Therefore, $v - \epsilon w$ does not lie in C, for all $\epsilon > 0$. Thus, $v \in \partial C$. Conversely, let $v \in \mathbb{R}^n$ such that $A(v) \succeq 0$ and $\det A(v) \neq 0$. This means, that the characteristic polynomial $\det(\mathrm{t}\, \mathrm{I}_d - A(v))$ has only strictly positive roots. Since the zeros of a polynomial depend continuously on its coefficients, there is some neighborhood U of v, such that for all $w \in U$ the matrix $A(w)$ has only strictly positive eigenvalues as well. Thus, $v \notin \partial C$. The claim (iii) follows directly from (ii).

In order to show (iv), let $C' = \{v \in \mathbb{R}^n : \; B(v) \succeq 0\}$ be the spectrahedral cone defined by $B_1, \ldots, B_n \in \mathrm{Sym}_d(\mathbb{R})$. The block diagonal matrix

$$M(v) = \begin{pmatrix} A(v) & 0 \\ 0 & B(v) \end{pmatrix}$$

is positive semi-definite if and only if $A(v)$ and $B(v)$ are. Therefore we have $v \in C \cap C' \Leftrightarrow M(v) \succeq 0$. □

1.4.3 Lemma. *Let $C \subseteq \mathbb{R}^n$ be a spectrahedral cone that has non-empty interior. Then we can find $d \geq 0$, $B_1, \ldots, B_n \in \mathrm{Sym}_d(\mathbb{R})$ and $w \in \mathbb{R}^n$ such that $B(w) \succ 0$ and $C = \{v \in \mathbb{R}^n : \; B(v) \succeq 0\}$.*

Proof. There are $N \geq 0$ and symmetric matrices $A_1, \ldots, A_n \in \mathrm{Sym}_N(\mathbb{R})$ such that $C = \{v \in \mathbb{R}^n : \; A(v) \succeq 0\}$. Let $w \in C$ such that $d = \operatorname{rank} A(w)$ is maximal and let $V = \operatorname{im}(A(w))$. If there would be a $v \in C$ with $\operatorname{im}(A(v)) \not\subseteq V$, then $\operatorname{rank}(A(v + w)) = \operatorname{rank}(A(v) + A(w)) > \operatorname{rank} A(w)$, since $A(v), A(w) \succeq 0$. This contradicts the assumed maximality. Thus $\operatorname{im}(A(v)) \subseteq V$ for all $v \in C$. Since C has non-empty interior, it follows that $\operatorname{im}(A(v)) \subseteq V$ for all $v \in \mathbb{R}^n$. In particular, $\operatorname{im}(A_i) \subseteq V$ for all $i \in [n]$. Thus there exists an invertible matrix $S \in \mathrm{GL}_d(\mathbb{R})$ such that

$$S^{\mathrm{T}} A_i S = \begin{pmatrix} B_i & 0 \\ 0 & 0 \end{pmatrix},$$

with $B_i \in \mathrm{Sym}_d(\mathbb{R})$ for $i \in [n]$. Then we have $C = \{v \in \mathbb{R}^n : \; B(v) \succeq 0\}$ and since

$$\operatorname{rank}(B(w)) = \operatorname{rank}(S^{\mathrm{T}} A(w) S) = \operatorname{rank}(A(w)) = d,$$

we also have $B(w) \succ 0$. □

Let $d \in \mathbb{Z}_{\geq 0}$, let $A_1, \ldots, A_n \in \mathrm{Sym}_d(\mathbb{R})$ and let $C = \{v \in \mathbb{R}^n : \; A(v) \succeq 0\}$. If $A(e) \succ 0$ at some point $e \in \mathbb{R}^n$, we can realize C° as the hyperbolicity cone of some hyperbolic polynomial. Namely, let $h = \det(\mathrm{x}_1\, A_1 + \ldots + \mathrm{x}_n\, A_n)$. As we have seen in Example 1.3.7, h is hyperbolic with respect to e and $C^\circ = \mathrm{C}_h(e)$. It is still a fundamental open question, if the converse is also true, cf. [12], section 6.

1.4.4 Conjecture (Generalized Lax Conjecture)**.** *The closure of every hyperbolicity cone C is a spectrahedral cone, i.e. there are real symmetric matrices $A_1, \ldots, A_n$, such that*

$$\overline{C} = \{v \in \mathbb{R}^{\mathrm{n}} : \ v_1 A_1 + \ldots + v_n A_n \succeq 0\}.$$

1.4.5 Remark. Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be a hyperbolic polynomial. If h has a definite determinantal representation, then the closure of the hyperbolicity cone of h is a spectrahedral cone by Example 1.3.7. Since taking powers does not change the hyperbolicity cone by Remark 1.3.3, it even suffices that h^N has a definite determinantal representation for some $N \in \mathbb{Z}_{>0}$. In fact, Helton and Vinnikov conjectured that some power of every hyperbolic polynomial has a definite determinantal representation. This would imply Conjecture 1.4.4. But actually, this is not true, see section 2.3.

The next Lemma provides a more convenient criterion for being a spectrahedral cone.

1.4.6 Lemma. *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be an irreducible polynomial that is hyperbolic with respect to $e \in \mathbb{R}^{\mathrm{n}}$. The closure of the hyperbolicity cone $\overline{\mathrm{C}_h(e)}$ is a spectrahedral cone, if and only if there is a $q \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ that is hyperbolic with respect to $e \in \mathbb{R}^{\mathrm{n}}$, such that $\mathrm{C}_h(e) \subseteq \mathrm{C}_q(e)$ and such that $q \cdot h$ has a definite determinantal representation.*

Proof. Let $\overline{\mathrm{C}_h(e)}$ be a spectrahedral cone. By Lemma 1.4.3, we can find $d \geq 0$ and $A_1, \ldots, A_n \in \mathrm{Sym}_d(\mathbb{R})$ such that $A(w) \succ 0$ for some $w \in \mathbb{R}^{\mathrm{n}}$ and

$$\overline{\mathrm{C}_h(e)} = \{v \in \mathbb{R}^{\mathrm{n}} : \ v_1 A_1 + \ldots + v_n A_n \succeq 0\}.$$

By Proposition 1.4.2(*iii*), we have $A(e) \succ 0$. Furthermore, the polynomial $p = \det(\mathrm{x}_1\, A_1 + \ldots + \mathrm{x}_n\, A_n)$ vanishes on $\partial\, \mathrm{C}_h(e)$ by Proposition 1.4.2(*ii*), thus by Proposition 1.3.12, the polynomial p vanishes $\mathcal{V}_{\mathbb{C}}(h)$. Note that p does not vanish identically, since $p(e) \neq 0$. Therefore, there is some $q \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ such that $q \cdot h = p$. Since p is hyperbolic with respect to e, the polynomial q is also hyperbolic with respect to e. For the hyperbolicity cones holds

$$\mathrm{C}_h(e) = \mathrm{C}_p(e) = \mathrm{C}_{qh}(e) = \mathrm{C}_q(e) \cap \mathrm{C}_h(e).$$

Therefore $\mathrm{C}_h(e) \subseteq \mathrm{C}_q(e)$.

The other direction is easy. We have

$$\overline{\mathrm{C}_h(e)} = \overline{\mathrm{C}_h(e) \cap \mathrm{C}_q(e)} = \overline{\mathrm{C}_{qh}(e)}$$

and $\overline{\mathrm{C}_{qh}(e)}$ is a spectrahedral cone by assumption. □

1.4.7 Remark. By Proposition 1.4.2(*iv*), it would be sufficient to proof the Generalized Lax Conjecture for irreducible polynomials.

1.5 Stable Polynomials

In this section we study stable polynomials. These are a class of polynomials, that are closely related to hyperbolic polynomials. An important aspect of stable polynomials is the fact that their support has a nice structure, which we will see in 2.3. First, we present basic results from [7], mostly section 2. In this section, let $H \subseteq \mathbb{C}$ always denote an open half-plane, whose boundary contains the origin, i.e. it is of the form $H = \{\exp(\mathrm{i}\,\varphi) \cdot \xi : \ \mathrm{Im}(\xi) > 0\}$ for some $\varphi \in \mathbb{R}$.

1.5.1 Definition. We say that a polynomial $p \in \mathbb{C}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ is $H-$*stable*, if $p(\xi) \neq 0$ for all $\xi \in H^n$ or if p is identically zero. If H is the open upper half-plane, i.e. $H = \{\xi \in \mathbb{C} : \ \mathrm{Im}(\xi) > 0\}$, we call p *stable*.

1.5.2 Remark. Let $H' \subseteq \mathbb{C}$ be an other open half-plane, whose boundary contains the origin. H' can be written as $H' = \{\exp(\mathrm{i}\,\varphi) \cdot \xi : \ \xi \in H\}$ for some $\varphi \in \mathbb{R}$. If $p \in \mathbb{C}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ is a $H-$stable polynomial that is homogeneous of degree d, then

$$p(\exp(\mathrm{i}\,\varphi)\,\mathrm{x}) = \exp(\mathrm{i}\,\varphi \cdot d)p(\mathrm{x})$$

is non-vanishing on H^n and therefore, p is also $H'-$stable.

1.5.3 Example. Let $h \in \mathbb{R}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ be hyperbolic with respect to the vector $e = (1, \dots, 1)^{\mathrm{T}}$, such that the hyperbolicity cone $\mathrm{C}_h(e)$ contains the positive orthant $(\mathbb{R}_{>0})^n$. Let $\xi \in \mathbb{C}^n$ with $\mathrm{Im}(\xi_j) > 0$, $j = 1, \dots, n$. By Theorem 1.3.9 the polynomial h is hyperbolic with respect to $\mathrm{Im}(\xi) = (\mathrm{Im}(\xi_1), \dots, \mathrm{Im}(\xi_n))^{\mathrm{T}}$. Thus $h(\xi) = h(\mathrm{Re}(\xi) + \mathrm{i}\,\mathrm{Im}(\xi)) \neq 0$. Therefore h is stable. Later we will see, that every polynomial $h \in \mathbb{R}[\mathrm{x_1}, \dots, \mathrm{x_n}]$, that is homogeneous and stable, is of this form, see Corollary 1.5.11.

1.5.4 Example. Let $f \in \mathbb{R}[\mathrm{t}]$ be a polynomial that has only real roots. Then f is stable. Conversely, an univariate stable polynomial $f \in \mathbb{R}[\mathrm{t}]$ has only real roots: Let $f(\xi) = 0$ for some $\xi \in \mathbb{C}$. Because the complex conjugate of ξ is also a zero of f, we can assume $\mathrm{Im}\,\xi \geq 0$. But since f is stable, we see $\mathrm{Im}\,\xi = 0$.

As we will see in the next Lemma, we can reduce stability to the univariate case.

1.5.5 Lemma. *Let $h \in \mathbb{C}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ be a non-zero polynomial. Then the following are equivalent:*

(i) h is stable.

(ii) For all $v \in \mathbb{R}^n$ and $e \in (\mathbb{R}_{>0})^n$ the univariate polynomial $h(v + \mathrm{t}\,e)$ is stable.

Proof. Let h be stable. Let $v \in \mathbb{R}^n$ and $e \in (\mathbb{R}_{>0})^n$. Finally let $\xi \in \mathbb{C}$ with $\mathrm{Im}\,\xi > 0$, then every coordinate of $v + \xi e$ lies in the upper open half-plane. Thus $h(v + \xi e) \neq 0$.

In order to show the converse, let $\mu \in \mathbb{C}^n$, such that $\mathrm{Im}\,\mu_i > 0$ for all $i \in [n]$. Let $v = \mathrm{Re}\,\mu \in \mathbb{R}^n$ and $e = \mathrm{Im}\,\mu \in (\mathbb{R}_{>0})^n$. Then *(ii)* implies that $h(\mu) = h(v + \mathrm{i}\,e)$ cannot be zero. ☐

1.5.6 Definition. Let $G \subseteq \mathbb{C}^n$ be a domain, i.e. non-empty, open and connected. We call a function $f : G \to \mathbb{C}$ *real-part-positive* on G, if $\mathrm{Re}(f(\xi)) \geq 0$ for all $\xi \in G$. If we actually have $\mathrm{Re}(f(\xi)) > 0$ for all $\xi \in G$, then we call f *strictly real-part-positive* on G.

1.5.7 Lemma ([7], Lemma 2.4)**.** *Let $G \subseteq \mathbb{C}^n$ be a domain and let $f : G \to \mathbb{C}$ be analytic and real-part-positive on G. Then f is strictly real-part-positive on G or f is a constant.*

Proof. If f is not constant, then $f(G)$ is open by the open mapping theorem. Thus, the set $f(G)$ is by assumption contained in the interior of the set

$$\{\xi \in \mathbb{C} : \mathrm{Re}(\xi) \geq 0\}.$$

This implies that f is strictly real-part-positive. □

1.5.8 Lemma ([7], Lemma 2.6)**.** *Let $p, q \in \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ and let p be $H-$stable. If $\frac{q}{p}$ is real-part-positive on H^n, then q is also $H-$stable.*

Proof. The rational function $\frac{q}{p}$ is either constant or strictly real-part-positive on H^n by Lemma 1.5.7. In the case $\frac{q}{p} = c \in \mathbb{C}$ constant, $q = cp$ is $H-$stable. If $\frac{q}{p}$ is strictly real-part-positive on H^n, then q is also $H-$stable, since $\mathrm{Re}(\frac{q}{p})$ does not vanish on H^n. □

1.5.9 Proposition (cf. [7], Lemma 2.8)**.** *Let $p \in \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be $H-$stable and let $k \in [n]$. Then the partial derivative $\frac{\partial p}{\partial \mathrm{x}_k}$ is also $H-$stable.*

Proof. Without loss of generality, we can assume that $H = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ is the right open half-plane: If $H = \{\exp(\mathrm{i}\,\varphi)z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, then replace p by $p(\exp(-\mathrm{i}\,\varphi)\,\mathrm{x})$. Further, let $q = \frac{\partial p}{\partial \mathrm{x}_k}$. By Lemma 1.5.8 it suffices to show that $\frac{q}{p}$ is real-part-positive on H^n.

First we consider the case $n = 1$. If p is constant, then $q = 0$. Let therefore $d = \deg(p) \geq 1$. We have $p = \beta \cdot \prod_{i=1}^{d}(x - \alpha_i)$ with $\beta, \alpha_1, \ldots, \alpha_d \in \mathbb{C}$, $\beta \neq 0$ and $\mathrm{Re}(\alpha_i) \leq 0$. Thus, $\frac{q}{p} = \sum_{i=1}^{d} \frac{1}{x - \alpha_i}$ is strictly real-part-positive on H.

In order to proof the case $n > 1$, let $z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n \in H$. Then

$$\tilde{p} = p(z_1, \ldots, z_{k-1}, \mathrm{t}, z_{k+1}, \ldots, z_n) \in \mathbb{C}[\mathrm{t}]$$

is $H-$stable. Further, let

$$\tilde{q} = \frac{\partial \tilde{p}}{\partial\,\mathrm{t}} = q(z_1, \ldots, z_{k-1}, \mathrm{t}, z_{k+1}, \ldots, z_n).$$

As we have seen above, $\frac{\tilde{q}}{\tilde{p}}$ is real-part-positive on H. Therefore $\frac{q}{p}$ is also real-part-positive on H^n, since the z_i have been chosen arbitrarily. □

The next Corollary is originally from [7, Theorem 6.1], but we present a shorter proof found in [3].

1.5.10 Corollary (cf. [3], Lemma 4.2)**.** *Let $p \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be homogeneous and stable, and let $p(1, \ldots, 1) \geq 0$. Then all coefficients of p are non-negative.*

Proof. We can assume that p is not identically zero. We proceed by induction on $d = \deg(p)$. The claim is clear if $d = 0$. Thus, let $d \geq 1$. Because p is homogeneous, p is also $H'-$stable, where $H' = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ is the right half-plane, see Remark 1.5.2. According to Euler's formula we have

$$dp = \sum_{k=1}^{n} \mathrm{x}_k \frac{\partial p}{\partial\,\mathrm{x}_k}.$$

By Proposition 1.5.9, the derivative $\frac{\partial p}{\partial \mathrm{x}_k}$ is $H'-$stable for all $k \in [n]$. By the induction hypothesis, it suffices therefore to show that

$$\frac{\partial p(1, \ldots, 1)}{\partial\,\mathrm{x}_k} \geq 0.$$

Consider the univariate polynomial

$$\tilde{p} = p(1, \dots, 1, \underbrace{\mathrm{t}}_{k\text{th argument}}, 1, \dots, 1).$$

Because p is non-vanishing on $(H')^n$, the polynomial $\tilde{p}$ is not identically zero and for the zeros $\alpha_1, \dots, \alpha_m$ of $\tilde{p}$ holds $\mathrm{Re}(\alpha_i) \leq 0$, $i \in [m]$. Thus

$$\frac{\frac{\partial p(1,\dots,1)}{\partial \mathrm{x}_k}}{p(1,\dots,1)} = \mathrm{Re}\left(\frac{\tilde{p}'(1)}{\tilde{p}(1)}\right) = \mathrm{Re}\left(\sum_{i=1}^{m} \frac{1}{1-\alpha_i}\right) \geq 0.$$

□

In Example 1.5.3, we have seen, that a hyperbolic polynomial, whose hyperbolicity cone contains the positive orthant, is stable. Now we can proof that some kind of converse is also true.

1.5.11 Corollary. *Let $0 \neq h \in \mathbb{R}[\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}]$ be homogeneous and stable. Then h is hyperbolic with respect to the vector $e = (1, \dots, 1)^{\mathrm{T}}$ and the hyperbolicity cone $\mathrm{C}_h(e)$ contains the positive orthant $(\mathbb{R}_{>0})^n$.*

Proof. Let $v \in \mathbb{R}^\mathrm{n}$ and $a, b \in \mathbb{R}$. Then we have

$$h(v + (a + b\,\mathrm{i})e) = h(v_1 + a + \mathrm{i}\,b, \dots, v_n + a + \mathrm{i}\,b).$$

If $b \neq 0$, the $(v_i + a) + \mathrm{i}\,b$ are either all lying in the upper open half-plane or all in the lower open half-plane. Since h is stable, $h(v + (a + b\,\mathrm{i})e)$ therefore cannot be zero. Thus, $h(v + \mu e) = 0$ implies $\mu \in \mathbb{R}$. Therefore h is hyperbolic with respect to e. Now let $v \in (\mathbb{R}_{>0})^n$. Because all coefficients of h are non-negative by Corollary 1.5.10, the polynomial $h(v + \mathrm{t}\,e)$ does not have non-negative zeros. □

1.5.12 Remark. Let $0 \neq h \in \mathbb{R}[\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}]$ be homogeneous and stable. Let $e = (1, \dots, 1)^{\mathrm{T}}$. As from now we will refer to the hyperbolicity cone $\mathrm{C}_h(e)$ of h with respect to e just as the hyperbolicity cone C_h of h.

2 Combinatorial Aspects

In this chapter, we will examine the support of a stable polynomial, i.e. the set of monomials that appear with non-zero coefficient. Every such set has some nice combinatorial properties, namely it is a so-called jump system. If a stable polynomial has a definite determinantal representation, its support fulfills even stronger conditions. This theory leads to the famous example of a hyperbolic polynomial h, the Vámos polynomial, where no power h^N has a definite determinantal representation, found by Brändén in 2010. In the last section of this chapter, we consider the possibility of disproving the Generalized Lax Conjecture 1.4.4 with combinatorial methods. Actually we will proof a discrete version of the Generalized Lax Conjecture, claiming that it is not possible to falsify the Generalized Lax Conjecture only by looking at the support of polynomials.

2.1 Matroids and Polymatroids

First of all, we have to introduce basic notions from the field of matroid theory. All of this section can be looked up in any standard reference book, for example [17, Sections 1.1, 1.2].

2.1.1 Definition. A *polymatroid* is a pair $\mathcal{M} = (E, r)$ where E is a finite set with power set 2^E and where $r : 2^E \to \mathbb{Z}_{\geq 0}$ is a function, that has for all $S, T \subseteq E$ the following properties:

(1) $r(\emptyset)$=0,

(2) $S \subseteq T \subseteq E$ implies $r(S) \leq r(T)$ (heredity),

(3) $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$ (submodularity).

We call r the *rank function* of $\mathcal{M}$. If additionally

(4) $r(\{i\}) \leq 1$ for all $i \in E$,

we call r a *matroid*.

2.1.2 Remark.

1. Let (E, r) denote a matroid (resp. a polymatroid) and $S \subseteq E$. Then $(S, r|_{2^S})$ is also a matroid (resp. a polymatroid).

2. Let (E, r) denote a matroid. Let $S \subseteq E$ and $e \in E$, then it holds that $r(S \cup \{e\}) \leq r(S) + 1$ by submodularity. Thus, we have $r(S) \leq |S|$ for all $S \subseteq E$.

2.1.3 Example. Because letting $S, T \subseteq E$, $U = \sum_{i \in S} V_i$ and $W = \sum_{i \in T} V_i$ the submodularity follows from

$$\begin{aligned}
r_{\mathcal{V}}(S \cup T) + r_{\mathcal{V}}(S \cap T) &= \dim(\underbrace{\sum_{i \in S \cup T} V_i}_{=U+W}) + \dim(\underbrace{\sum_{i \in S \cap T} V_i}_{\subseteq U \cap W}) \\
&\leq \dim(U + W) + \dim(U \cap W) \\
&= \dim(U) + \dim(W) \\
&= r_{\mathcal{V}}(S) + r_{\mathcal{V}}(T).
\end{aligned}$$

We call a polymatroid $\mathcal{M}$ *representable over* K, if there is such a description of $\mathcal{M}$ by a collection of subspaces.

2.1.4 Example. Consider the collection of subspaces $\mathcal{V} = (V_k)_{k\in[4]}$ where $V_k \subseteq \mathbb{R}^2$ is the subspace spanned by (k, k^2). Let $(r_{\mathcal{V}}, [4])$ be the corresponding matroid. By a Vandermonde argument, we have

$$r_{\mathcal{V}}(S) = \begin{cases} |S|, & \text{if } |S| \leq 2, \\ 2, & \text{otherwise.} \end{cases}$$

This matroid is however not representable over the field $\mathbb{F}_2$ with two elements. Suppose there are subspaces $W_1, \ldots, W_4 \subseteq V$ of some finite dimensional vector space V over $\mathbb{F}_2$, such that $r_{\mathcal{V}}(S) = \dim(\sum_{i\in S} W_i)$ for all $S \subseteq [4]$. Since $r_{\mathcal{V}}([4]) = 2$, we can assume $V = \mathbb{F}_2^2$. But $\mathbb{F}_2^2$ has only three subspaces of dimension one, thus we have $W_i = W_j$ and therefore $r_{\mathcal{V}}(\{i,j\}) = 1$ for some distinct $i, j \in [4]$.

Representable polymatroids satisfy an inequality, which plays a crucial role in Brändén's proof, that no power of the Vámos polynomial has a definite determinantal representation. This inequality goes back to Ingleton [13].

2.1.5 Proposition (Ingleton inequalities, [13], Section 4)**.** *Let K be a field and let (E, r) be a polymatroid which is representable over K. Then for all $S_1, \ldots, S_4 \subseteq E$ it holds*

$$\begin{aligned} & r(S_1) + r(S_2) + r(S_1 \cup S_2 \cup S_3) + r(S_1 \cup S_2 \cup S_4) + r(S_3 \cup S_4) \\ \leq\; & r(S_1 \cup S_2) + r(S_1 \cup S_3) + r(S_1 \cup S_4) + r(S_2 \cup S_3) + r(S_2 \cup S_4). \end{aligned}$$

Proof. Let V be a vector space over K and $A, B, C, D \subseteq V$ subspaces. The dimension formula says

$$\dim(A + B) + \dim(A \cap B) = \dim(A) + \dim(B).$$

Applying this to three subspaces, we obtain

$$\begin{aligned} \dim((A \cap B) \cap C) &= \dim(A \cap B) + \dim(C) - \dim((A \cap B) + C) \\ &\geq \dim(A \cap B) + \dim(C) - \dim((A + C) \cap (B + C)) \\ &= \dim(A \cap B) + \dim(C) - (\dim(A + C) \\ &\quad + \dim(B + C) - \dim(A + B + C)), \quad (*) \end{aligned}$$

and to four subspaces

$$\begin{aligned} \dim(A \cap B \cap C \cap D) &= \dim(A \cap B \cap C) + \dim(A \cap B \cap D) \\ &\quad - \dim((A \cap B \cap C) + (A \cap B \cap D)) \\ &\geq \dim(A \cap B \cap C) + \dim(A \cap B \cap D) - \dim(A \cap B). \end{aligned}$$

We apply $(*)$ twice to the last line and use again the dimension formula:

$$\begin{aligned}
\dim(A \cap B \cap C \cap D) &\geq \dim(C) - \dim(A+C) - \dim(B+C) \\
&\quad + \dim(A+B+C) + \dim(A \cap B \cap D) \\
&\geq \dim(C) - \dim(A+C) - \dim(B+C) \\
&\quad + \dim(A+B+C) + \dim(A \cap B) \\
&\quad + \dim(D) - \dim(A+D) - \dim(B+D) \\
&\quad + \dim(A+B+D) \\
&= \dim(C) - \dim(A+C) - \dim(B+C) \\
&\quad + \dim(A+B+C) \\
&\quad + \dim(A) + \dim(B) - \dim(A+B) \\
&\quad + \dim(D) - \dim(A+D) - \dim(B+D) \\
&\quad + \dim(A+B+D). \qquad (**)
\end{aligned}$$

On the other side it holds

$$\dim(A \cap B \cap C \cap D) \leq \dim(C \cap D) = \dim(C) + \dim(D) - \dim(C+D).$$

Finally we apply this to $(**)$, and we obtain

$$\begin{aligned}
\dim(C) + \dim(D) - \dim(C+D) &\geq \dim(C) - \dim(A+C) - \dim(B+C) \\
&\quad + \dim(A+B+C) \\
&\quad + \dim(A) + \dim(B) - \dim(A+B) \\
&\quad + \dim(D) - \dim(A+D) - \dim(B+D) \\
&\quad + \dim(A+B+D),
\end{aligned}$$

thus

$$\begin{aligned}
&\dim(A) + \dim(B) + \dim(A+B+C) + \dim(A+B+D) + \dim(C+D) \\
\leq\ &\dim(A+B) + \dim(A+C) + \dim(A+D) + \dim(B+C) + \dim(B+D).
\end{aligned}$$

□

2.1.6 Definition. Let $\mathcal{M} = (E, r)$ be a matroid. A subset $S \subseteq E$ is called *independent*, if $r(S) = |S|$. A subset which is not independent is called *dependent*. We call a maximal independent subset of E a *base* of $\mathcal{M}$. We denote by $\mathcal{B}(\mathcal{M})$ the set of all bases of $\mathcal{M}$. Finally we call a minimal dependent subset of E a *circuit* of $\mathcal{M}$.

2.1.7 Remark. $\emptyset$ is always independent by definition.

We initially defined a matroid via his rank function. Another approach would be to give the set of bases. We will see, that both definitions are equivalent, if correct stated. But this will need some work.

2.1.8 Lemma. *Let (E, r) be a polymatroid and let $X, Y \subseteq E$ be subsets such that $r(X \cup \{y\}) = r(X)$ for all $y \in Y \smallsetminus X$. Then we have $r(X \cup Y) = r(X)$.*

Proof. Let $Y \setminus X = \{y_1, \ldots, y_n\}$. We proceed by induction on n. In the case of $n = 1$ we have

$$r(X \cup Y) = r(X \cup \{y_1\}) = r(X).$$

Letting $n \geq 2$ we have by induction hypothesis

$$r(X) = r(X \cup \{y_1, \ldots, y_{n-1}\}) = r(X \cup \{y_n\}).$$

Let $S = X \cup \{y_1, \ldots, y_{n-1}\}$ and $T = X \cup \{y_n\}$. Applying the submodularity of r to this, we obtain

$$\begin{aligned} r(X) + r(X) &= r(S) + r(T) \\ &\geq r(S \cup T) + r(S \cap T) \\ &\geq r(X \cup Y) + r(X) \\ &\geq r(X) + r(X). \end{aligned}$$

This implies $r(X \cup Y) = r(X)$. □

2.1.9 Proposition. *Let (E, r) be a matroid. Then we have for all $I_1, I_2 \subseteq E$:*

(i) If $I_1 \subseteq I_2$ and I_2 is independent, then I_1 is independent.

(ii) If I_1 and I_2 are independent and $|I_1| < |I_2|$, then there is an element $e \in I_2 \setminus I_1$, such that $I_1 \cup \{e\}$ is independent.

Proof. (i) follows directly from the submodularity:

$$|I_2| = r(I_2) = r(I_1 \cup (I_2 \setminus I_1)) \leq r(I_2 \setminus I_1) + r(I_1) \leq |I_2 \setminus I_1| + |I_1| = |I_2|.$$

Suppose that (ii) is wrong, i.e. I_1 and I_2 are independent with $|I_1| < |I_2|$ and for all $e \in I_2 \setminus I_1$ we have $r(I_1 \cup \{e\}) = r(I_1)$. Then Lemma 2.1.8 implies

$$|I_1| = r(I_1) = r(I_1 \cup I_2) \geq r(I_2) = |I_2|.$$

This contradicts $|I_1| < |I_2|$. □

2.1.10 Corollary. *If B_1 and B_2 are bases of a matroid $\mathcal{M}$, then $|B_1| = |B_2|$.* □

2.1.11 Corollary. *A subset of E is independent if and only if it is contained in a base and it is dependent if and only if it contains a circuit.* □

2.1.12 Lemma. *Let C_1 and C_2 be two distinct circuits of a matroid $\mathcal{M}$ and let $e \in C_1 \cap C_2$. Then there is a circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.*

Proof. Suppose that $(C_1 \cup C_2) \setminus \{e\}$ does not contain a circuit, i.e. $(C_1 \cup C_2) \setminus \{e\}$ is independent. Clearly the set $C_2 \setminus C_1$ is non-empty. Let $f \in C_2 \setminus C_1$, then $C_2 \setminus \{f\}$ is independent. Now choose a subset I of $C_1 \cup C_2$ which is maximal with the properties that it contains $C_2 \setminus \{f\}$ and that it is independent. We observe that $f \notin I$ and that we can find an element $g \in C_1$ that is not in I, since otherwise I would be dependent. Clearly we have $f \neq g$, thus

$$|I| \leq |(C_1 \cup C_2) \setminus \{f, g\}| = |C_1 \cup C_2| - 2 < |(C_1 \cup C_2) \setminus \{e\}|.$$

Applying Proposition 2.1.9(ii) to I and $(C_1 \cup C_2) \setminus \{e\}$ gives a contradiction to the maximality of I. □

2.1.13 Lemma. *Let B_1 and B_2 be bases of a matroid $\mathcal{M}$.*

(i) Let $x \in B_1 \setminus B_2$, then there exists an element $y \in B_2 \setminus B_1$, such that $(B_1 \setminus \{x\}) \cup \{y\}$ is a base.

(ii) Let $x \in B_2 \setminus B_1$, then there exists an element $y \in B_1 \setminus B_2$, such that $(B_1 \cup \{x\}) \setminus \{y\}$ is a base.

Proof. In order to proof (i) it suffices by Corollary 2.1.10 to show, that there is an element $y \in B_2 \setminus B_1$, such that the set $(B_1 \setminus \{x\}) \cup \{y\}$ is independent. But because $B_1 \setminus \{x\}$ is independent and $|B_1 \setminus \{x\}| < |B_2|$, this follows directly from Proposition 2.1.9(ii).

Now we proof (ii). Like above it suffices to show, that there is an element $y \in B_1 \setminus B_2$, such that $(B_1 \cup \{x\}) \setminus \{y\}$ is independent. The set $B_1 \cup \{x\}$ is dependent and thus contains a circuit C. By Proposition 2.1.9(i) we have $C \not\subseteq B_1$, so there is an element $y \in C \setminus B_1$. Assume $(B_1 \cup \{x\}) \setminus \{y\}$ is dependent, then it contains a circuit C'. Because $y \notin C'$ we have $C \neq C'$. Further we have $x \in C \cap C'$, because $C, C' \not\subseteq B_1$ and $C, C' \subseteq B_1 \cup \{x\}$. By Lemma 2.1.12 $(C \cup C') \setminus \{x\} \subseteq B_1$ contains a circuit. But this is a contradiction. □

2.1.14 Proposition. *Let $\mathcal{M} = (E, r)$ be a matroid. For all $S \subseteq E$ we have*

$$r(S) = \max\{|S \cap B| : B \in \mathcal{B}(\mathcal{M})\}.$$

Proof. For all $B \in \mathcal{B}(\mathcal{M})$ we have $|S \cap B| = r(S \cap B) \leq r(S)$. Therefore it holds that

$$r(S) \geq \max\{|S \cap B| : B \in \mathcal{B}(\mathcal{M})\}.$$

Conversely, let $I \subseteq S$ be a maximal independent subset of S. Then, for all $e \in S$, we have $r(I \cup \{e\}) = r(I)$. Thus Lemma 2.1.8 implies $|I| = r(I) = r(S)$. Letting B be a bases of $\mathcal{M}$ which contains I, we obtain $|S \cap B| = r(S)$. □

2.1.15 Corollary. *Let $\mathcal{M} = (E, r)$ be a matroid. A subset $I \subseteq E$ is a base if and only if I independent and $r(I) = r(E)$.*

Proof. Let $I \subseteq E$ be a base. Then we have by Proposition 2.1.14

$$r(E) = \max\{|B| : B \in \mathcal{B}(\mathcal{M})\}.$$

Thus Corollary 2.1.10 implies $r(E) = r(I)$. Now let $I \subseteq E$ be an independent subset such that $r(E) = r(I)$. Then I is contained in a base B, thus we have

$$|I| \leq |B| = r(B) \leq r(E) = r(I) = |I|.$$

Therefore we have $I = B$. □

Proposition 2.1.14 claims that the set of bases of a given matroid completely describes this matroid. In the remainder of this section, we will characterize the families of sets, that can occur as the set of bases of a matroid. It will turn out, that these are exactly the non-empty families of sets, that have the property (i) of Lemma 2.1.13.

2.1.16 Definition. Let E be a finite set and $\mathfrak{B} \subseteq 2^E$ be a non-empty system of subsets. If there is for all $B_1, B_2 \in \mathfrak{B}$ and every $x \in B_1 \setminus B_2$ an element $y \in B_2 \setminus B_1$, such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathfrak{B}$ (*exchange axiom*), we call $\mathfrak{B}$ an *exchange system*.

2.1.17 Remark. The set of bases of a matroid is an exchange system by Lemma 2.1.13(i).

2.1.18 Lemma. *Let $\mathfrak{B}$ be an exchange system and $B_1, B_2 \in \mathfrak{B}$. Then we have* $|B_1| = |B_2|$.

Proof. Let B_1 and B_2 be two distinct elements of $\mathfrak{B}$ such that $|B_1| > |B_2|$ and such that $|B_1 \smallsetminus B_2|$ is minimal. Clearly $B_1 \smallsetminus B_2 \neq \emptyset$. Let $x \in B_1 \smallsetminus B_2$, then there is an element $y \in B_2 \smallsetminus B_1$ with $(B_1 \smallsetminus \{x\}) \cup \{y\} \in \mathfrak{B}$. But then we have $|(B_1 \smallsetminus \{x\}) \cup \{y\}| = |B_1| > |B_2|$ and $|((B_1 \smallsetminus \{x\}) \cup \{y\}) \smallsetminus B_2| < |B_1 \smallsetminus B_2|$. □

2.1.19 Lemma. *Let E be a finite set, let $\mathfrak{B} \subseteq 2^E$ be an exchange system and let I_1, I_2 be two subsets of E, each contained in some element of $\mathfrak{B}$. If $|I_1| < |I_2|$, then there is an element $e \in I_2 \smallsetminus I_1$, such that $I_1 \cup \{e\}$ is contained in some element of $\mathfrak{B}$.*

Proof. Suppose that for all $e \in I_2 \smallsetminus I_1$, the set $I_1 \cup \{e\}$ is not contained in any element of $\mathfrak{B}$. Let $B_1, B_2 \in \mathfrak{B}$ with $I_1 \subseteq B_1$ and $I_2 \subseteq B_2$, such that $|B_2 \smallsetminus (I_2 \cup B_1)|$ is minimal. Then we have

$$I_2 \smallsetminus B_1 = I_2 \smallsetminus I_1,$$

because $e \in I_2 \smallsetminus I_1$ and $e \notin I_2 \smallsetminus B_1$ would imply $I_1 \cup \{e\} \subseteq B_1$.

Suppose $B_2 \smallsetminus (I_2 \cup B_1) \neq \emptyset$ and let $x \in B_2 \smallsetminus (I_2 \cup B_1)$. Because $\mathfrak{B}$ is an exchange system, there exists some $y \in B_1 \smallsetminus B_2$, such that $(B_2 \smallsetminus \{x\}) \cup \{y\} \in \mathfrak{B}$. But then we have $I_2 \subseteq (B_2 \smallsetminus \{x\}) \cup \{y\}$ and

$$|((B_2 \smallsetminus \{x\}) \cup \{y\}) \smallsetminus (I_2 \cup B_1)| < |B_2 \smallsetminus (I_2 \cup B_1)|.$$

This is a contradiction to our choice of B_2. Therefore we have

$$B_2 \smallsetminus (I_2 \cup B_1) = \emptyset,$$

thus $B_2 \smallsetminus B_1 = I_2 \smallsetminus B_1 = I_2 \smallsetminus I_1$.

Further it holds that $B_1 \smallsetminus (I_1 \cup B_2) = \emptyset$. Because otherwise, there would be elements $x \in B_1 \smallsetminus (I_1 \cup B_2)$ and $y \in B_2 \smallsetminus B_1$, such that $(B_1 \smallsetminus \{x\}) \cup \{y\} \in \mathfrak{B}$. But then we have $I_1 \cup \{y\} \subseteq (B_1 \smallsetminus \{x\}) \cup \{y\}$ and $y \in B_2 \smallsetminus B_1 = I_2 \smallsetminus I_1$. This is a contradiction to our first assumption.

Thus we have $B_1 \smallsetminus (I_1 \cup B_2) = \emptyset$ and therefore $B_1 \smallsetminus B_2 = I_1 \smallsetminus B_2 \subseteq I_1 \smallsetminus I_2$. According to Lemma 2.1.18 we have $|B_1| = |B_2|$, thus $|B_1 \smallsetminus B_2| = |B_2 \smallsetminus B_1|$. Summing up we obtain

$$|I_1 \smallsetminus I_2| \geq |B_1 \smallsetminus B_2| = |B_2 \smallsetminus B_1| = |I_2 \smallsetminus I_1|,$$

thus $|I_1| \geq |I_2|$, which contradicts the assumption $|I_1| < |I_2|$. □

2.1.20 Proposition. *Let E be a finite set and let $\mathfrak{B} \subseteq 2^E$ be an exchange system. Further, let*

$$r : 2^E \to \mathbb{Z}_{\geq 0},\ S \mapsto \max\{|S \cap B| : B \in \mathcal{B}\}.$$

Then $\mathcal{M} = (E, r)$ is a matroid and $\mathfrak{B} = \mathfrak{B}(\mathcal{M})$.

Proof. Clearly $\mathcal{M}$ has the properties (1),(2) and (4). It remains to proof that for all $X, Y \subseteq E$

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

For that purpose let $B_1 \in \mathfrak{B}$, such that $|X \cap Y \cap B_1| = r(X \cap Y)$. Further let $S_1 = X \cap Y \cap B_1$. Analogously let $B' \in \mathfrak{B}$, such that $|(X \cup Y) \cap B'| = r(X \cup Y)$. In addition let $S' = (X \cup Y) \cap B'$. By Lemma 2.1.19 we can extend S_1 with elements of S' to a set S_2, such that $|S_2| = |S'|$ and $S_2 \subseteq B_2$ for some $B_2 \in \mathfrak{B}$. Then we have:

$$\begin{aligned} r(X) + r(Y) &\geq |B_2 \cap X| + |B_2 \cap Y| \\ &\geq |S_2 \cap X| + |S_2 \cap Y| \\ &= |(S_2 \cap X) \cup (S_2 \cap Y)| + |(S_2 \cap X) \cap (S_2 \cap Y)| \\ &= |S_2 \cap (X \cup Y)| + |S_2 \cap (X \cap Y)|. \end{aligned}$$

Since $S_2 \subseteq X \cup Y$ and $S_1 \subseteq S_2 \cap (X \cap Y)$, this implies:

$$\begin{aligned} r(X) + r(Y) &\geq |S_2| + |S_1| \\ &= r(X \cup Y) + r(X \cap Y). \end{aligned}$$

Thus $\mathcal{M}$ is a matroid. Obviously, the independent sets of $\mathcal{M}$ are exactly those contained in some element of $\mathfrak{B}$. Therefore we have $\mathfrak{B} = \mathfrak{B}(\mathcal{M})$. $\square$

2.1.21 Example. Let $E = [n]$ for some $n \in \mathbb{Z}_{>0}$, let $k \leq n$ and let $\mathfrak{B} = \binom{[n]}{k}$. Clearly, $\mathfrak{B}$ satisfies the exchange axiom. The corresponding matroid $U_{k,n}$ is called the *uniform matroid*.

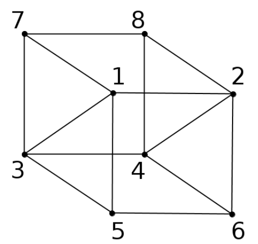

Figure 4: A geometric representation of the Vámos cube of Example 2.1.22: The 4-element circuits are those that form a square without diagonal.

2.1.22 Example. Let $E = \{1, \ldots, 8\}$ and let

$$\mathfrak{A} = \{\{1,2,3,4\}, \{1,2,5,6\}, \{1,2,7,8\}, \{3,4,5,6\}, \{3,4,7,8\}\}.$$

Then the set $\mathfrak{B} = \{B \in 2^E : |B| = 4 \text{ and } B \notin \mathfrak{A}\}$ is an exchange system: Let $B_1, B_2 \in \mathfrak{B}$ be distinct. Clearly, if $|B_1 \cap B_2| = 3$, then the exchange axiom is fulfilled. If $|B_1 \cap B_2| < 3$ then the exchange axiom is fulfilled too, because $\mathfrak{A}$ does not contain two subsets that have three elements in common. Thus we obtain an unique matroid on E which has $\mathfrak{B}$ as set of bases. It is called the *Vámos cube* V_8. The Vámos cube is not representable over any field, because it fails to satisfy Ingleton's inequalities. This is seen by choosing

$$S_1 = \{1,2\},\ S_2 = \{3,4\},\ S_3 = \{5,6\},\ S_4 = \{7,8\}.$$

Because we have

$$\begin{aligned}
& r_{V_8}(S_1) + r_{V_8}(S_2) + r_{V_8}(S_1 \cup S_2 \cup S_3) + \\
& r_{V_8}(S_1 \cup S_2 \cup S_4) + r_{V_8}(S_3 \cup S_4) \\
= \;& r_{V_8}(\{1,2\}) + r_{V_8}(\{3,4\}) + r_{V_8}(\{5,6,7,8\}) \\
& + r_{V_8}(\{1,2,3,4,5,6\}) + r_{V_8}(\{1,2,3,4,7,8\}) \\
= \;& 2+2+4+4+4 \\
= \;& 16,
\end{aligned}$$

but

$$\begin{aligned}
& r_{V_8}(S_1 \cup S_2) + r_{V_8}(S_1 \cup S_3) + r_{V_8}(S_1 \cup S_4) \\
& + r_{V_8}(S_2 \cup S_3) + r_{V_8}(S_2 \cup S_4) \\
= \;& r_{V_8}(\{1,2,3,4\}) + r_{V_8}(\{1,2,5,6\}) + r_{V_8}(\{1,2,7,8\}) \\
& + r_{V_8}(\{3,4,5,6\}) + r_{V_8}(\{3,4,7,8\}) \\
= \;& 3+3+3+3+3 \\
= \;& 15.
\end{aligned}$$

2.2 Jump Systems

Jump systems, first introduced in [4], are special sets of integral vectors, that are closely connected to the theory of polymatroids.

2.2.1 Definition. Let $\alpha, \beta \in \mathbb{Z}^n$ and define $|\alpha| = \sum_{i=1}^{n} |\alpha_i|$. The set of *steps from α to β* is defined by

$$\mathrm{St}(\alpha, \beta) = \{\sigma \in \mathbb{Z}^n : |\sigma| = 1,\ |\alpha + \sigma - \beta| = |\alpha - \beta| - 1\}.$$

A *jump system* is a finite set $J \subseteq \mathbb{Z}^n$, that respects the *two-step axiom*, i.e. for all $\alpha, \beta \in J$ and $\sigma \in \mathrm{St}(\alpha, \beta)$ with $\alpha + \sigma \notin J$, there is a step $\tau \in \mathrm{St}(\alpha + \sigma, \beta)$, such that $\alpha + \sigma + \tau \in J$.

2.2.2 Example. The set $\{1, 3, 4\} \subseteq \mathbb{Z}$ is a jump system, while $\{1, 4\} \subseteq \mathbb{Z}$ is not. Another example of a jump system is illustrated in Figure 5.

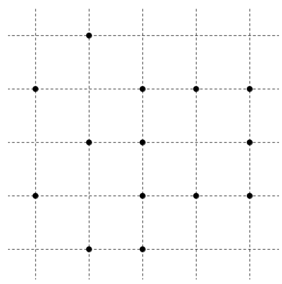

Figure 5: A jump sytem in $\mathbb{Z}^2$.

2.2.3 Definition. Let $J \subseteq \mathbb{Z}^n$ be a jump system. If we have for all $\alpha, \beta \in J$ that $|\alpha| = |\beta|$, then we call J a *constant sum jump system*.

2.2.4 Example. Let $\mathcal{M} = ([n], r)$ be a matroid. Consider the set

$$J = \{(\chi_B(i))_{i=1,\ldots,n} : B \in \mathfrak{B}(\mathcal{M})\} \subseteq \mathbb{Z}^{\mathrm{n}},$$

where χ_B is the characteristic function on B, i.e.

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases}$$

J is a jump system by Lemma 2.1.13. Actually J is a constant sum jump system by Corollary 2.1.10.

The next technical Lemma states that the property of being a jump system is preserved under some kind of affine transformations.

2.2.5 Lemma. *Let $c \in \mathbb{Z}^{\mathrm{n}}$ and let A be a $n \times n$ diagonal matrix with diagonal entries $e_1, \ldots, e_n \in \{-1, 1\}$. Consider the affine transformation*

$$\psi : \mathbb{Z}^{\mathrm{n}} \to \mathbb{Z}^{\mathrm{n}},\ \alpha \mapsto A\alpha + c.$$

Then it holds for all $\alpha, \beta \in \mathbb{Z}^{\mathrm{n}}$:

(i) $\psi(\alpha) + \psi(\beta) - c = \psi(\alpha + \beta)$.

(ii) $|\psi(\alpha) - \psi(\beta)| = |\alpha - \beta|$.

(iii) If $\sigma \in \mathrm{St}(\alpha, \beta)$, then $(\psi(\sigma) - c) \in \mathrm{St}(\psi(\alpha), \psi(\beta))$.

(iv) A subset $J \subseteq \mathbb{Z}^{\mathrm{n}}$ is a jump system if and only if $\psi(J)$ is a jump system.

Proof. (i) and (ii) are trivial. We observe, that ψ is bijective with inverse function

$$\alpha \mapsto A\alpha - Ac.$$

Let $\sigma \in \mathrm{St}(\alpha, \beta)$. Then $|\psi(\sigma) - c| = |\psi(\sigma) - \psi(0)| = |\sigma| = 1$ and

$$\begin{aligned} |\psi(\alpha) + \psi(\sigma) - c - \psi(\beta)| &= |\psi(\alpha + \sigma) - \psi(\beta)| \\ &= |\alpha + \sigma - \beta| \\ &= |\alpha - \beta| - 1 \\ &= |\psi(\alpha) - \psi(\beta)| - 1. \end{aligned}$$

In order to show (iv) let $\psi(J)$ be a jump system, let $\alpha, \beta \in J$, $\sigma \in \mathrm{St}(\alpha, \beta)$ and let $\alpha + \sigma \notin J$. It follows that $(\psi(\sigma) - c) \in \mathrm{St}(\psi(\alpha), \psi(\beta))$ and we have $\psi(\alpha) + (\psi(\sigma) - c) = \psi(\alpha + \sigma) \notin \psi(J)$. Thus, there is a $\tau \in \mathrm{St}(\psi(\alpha + \sigma), \psi(\beta))$, such that

$$\psi(\alpha + \sigma + \psi^{-1}(\tau + c)) = \psi(\alpha + \sigma) + \tau \in \psi(J).$$

Therefore we have $\alpha + \sigma + \psi^{-1}(\tau + c) \in J$. Applying (iii) on ψ^{-1}, it follows from $\tau \in \mathrm{St}(\psi(\alpha + \sigma), \psi(\beta))$, that $\psi^{-1}(\tau) + Ac \in \mathrm{St}(\alpha + \sigma, \beta)$. Therefore we have $\psi^{-1}(\tau + c) = A(\tau + c) - Ac = (A\tau - Ac) + Ac = \psi^{-1}(\tau) + Ac \in \mathrm{St}(\alpha + \sigma, \beta)$. Thus, J satisfies the two-step axiom. The converse direction follows, if we transpose ψ and ψ^{-1}. $\square$

2.2.6 Definition. Let $J \subseteq \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ be a jump system. The *rank function* r of J is defined by

$$r : 2^{[n]} \to \mathbb{Z}_{\geq 0},$$
$$S \mapsto \max\{\sum_{i \in S} \alpha_i : \alpha \in J\}.$$

2.2.7 Lemma (cf. [5], section 4)**.** *Let $J \subseteq \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ be a jump system with rank function r. Let $S \subseteq [n]$ and let $\beta \in J$. Then there is some $\alpha \in J$ so that $\alpha_i \geq \beta_i$ for all $i \in S$ and $\sum_{i \in S} \alpha_i = r(S)$.*

Proof. Let $\alpha \in J$ such that $\sum_{i \in S} \alpha_i = r(S)$ and such that

$$d = \sum_{i \in S, \beta_i > \alpha_i} (\beta_i - \alpha_i)$$

is minimal. If $d = 0$, then we are finished. Thus we assume that $d > 0$. There exists some $k \in S$, such that $\alpha_k < \beta_k$. Then δ_k is an element of $\mathrm{St}(\alpha, \beta)$. We have $\alpha + \delta_k \notin J$, because of

$$\sum_{i \in S} \alpha_i = r(S) = \max\{\sum_{i \in S} \gamma_i : \gamma \in J\}.$$

Since J is a jump system, there is a $\tau \in \mathrm{St}(\alpha+\delta_k, \beta)$ such that $\alpha' = \alpha+\delta_k+\tau \in J$. Again, because of $\sum_{i \in S} \alpha_i = r(S)$, we have $\tau = -\delta_j$ for some $j \in S$. Then $-\delta_j \in \mathrm{St}(\alpha + \delta_k, \beta)$ implies $\alpha_j > \beta_j$, thus $\alpha'_j \geq \beta_j$. Finally we have

$$\sum_{i \in S} \alpha_i = \sum_{i \in S} \alpha'_i$$

and

$$\sum_{i \in S, \beta_i > \alpha'_i} (\beta_i - \alpha'_i) = \sum_{i \in S, \beta_i > \alpha_i} (\beta_i - \alpha_i) - 1.$$

This is a contradiction to the minimality of d. □

2.2.8 Proposition (cf. [5], section 4)**.** *Let $J \subseteq \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ be a jump system with rank function r. Then $([n], r)$ is a polymatroid.*

Proof. All properties but the submodularity are obvious. In order to show the submodularity of r, let $S, T \subseteq [n]$ and let $\beta \in J$ such that

$$r(S \cap T) = \sum_{i \in S \cap T} \beta_i.$$

By Lemma 2.2.7 there is a $\alpha \in J$ with $\alpha_i \geq \beta_i$ for all $i \in S \cup T$ and with $\sum_{i \in S \cup T} \alpha_i = r(S \cup T)$. Then holds

$$\begin{aligned} r(S \cap T) + r(S \cup T) &= \sum_{i \in S \cap T} \beta_i + \sum_{i \in S \cup T} \alpha_i \\ &\leq \sum_{i \in S \cap T} \alpha_i + \sum_{i \in S \cup T} \alpha_i \\ &= \sum_{i \in S} \alpha_i + \sum_{i \in T} \alpha_i \\ &\leq r(S) + r(T). \end{aligned}$$

□

2.2.9 Corollary. *Let $J \subseteq \{0,1\}^n$ be a jump system with rank function r. Then $([n], r)$ is a matroid.*

Proof. Let $i \in [n]$. Then we clearly have $r(\{i\}) = \max_{\alpha \in J} \alpha_i \leq 1$. $\square$

We will need some notion of polarization for jump systems. This was first introduced by Kabadi and Sridhar, cf. [14, Section 4].

2.2.10 Definition. We define on $\mathbb{Z}^{\mathrm{n}}$ the partial order:

$$\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i \text{ for all } i = 1, \ldots, n.$$

If we have $\gamma \in \mathbb{Z}^{\mathrm{n}}$ and $A \subseteq \mathbb{Z}^{\mathrm{n}}$, such that $\gamma \geq \alpha$ for all $\alpha \in A$, we write $\gamma \geq A$.

2.2.11 Definition. Let $A \subseteq \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ be a finite set and let $d \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ with $d \geq A$. We define

$$V_d = \{(i,j) : i \in [n],\ 0 \leq j \leq d_i\}.$$

Consider the projection

$$\mathrm{Pr}_d : \{0,1\}^{V_d} \to \mathbb{Z}^{\mathrm{n}}_{\geq 0},$$
$$\sigma \mapsto (\sum_{j=0}^{d_1} \sigma(1,j), \ldots, \sum_{j=0}^{d_n} \sigma(n,j)).$$

The preimage

$$\mathcal{P}_d(A) = \mathrm{Pr}_d^{-1}(A) = \{\sigma \in \{0,1\}^{V_d} : \mathrm{Pr}_d(\sigma) \in A\}$$

is the *polarization with respect to d of A.*

2.2.12 Remark. The condition $d \geq A$ in the preceding Definition ensures that $\mathrm{Pr}_d(\mathcal{P}_d(A)) = A$.

2.2.13 Lemma ([14], Theorem 4.1)**.** *Let $J \subseteq \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ be a jump system and let $d \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ with $d \geq J$. Then $\mathcal{P}_d(J)$ is a jump system too.*

Proof. Let $J \subseteq \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ be a jump system. Let $e \in \mathbb{Z}_{\geq 0}$, such that $e \geq \alpha_n$ for all $\alpha \in J$. Let $A = \{\alpha \in (\mathbb{Z}_{\geq 0})^{n+1} :\ \alpha_n \leq e-1,\ \alpha_{n+1} \leq 1\}$. Consider the map:

$$\pi : A \to \mathbb{Z}^{\mathrm{n}}_{\geq 0},\ (\alpha_1, \ldots, \alpha_{n+1}) \mapsto (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n + \alpha_{n+1}).$$

We will proof that $\pi^{-1}(J)$ is a jump system too. Iterating this result proofs the claim. We have to show, that $\pi^{-1}(J)$ satisfies the two-step axiom. Let $\alpha, \beta \in \pi^{-1}(J)$ and $\sigma \in \mathrm{St}(\alpha, \beta)$ such that $\alpha + \sigma \notin \pi^{-1}(J)$. Then we have $|\alpha_i + \sigma_i - \beta_i| \leq |\alpha_i - \beta_i|$ for all $i \in [n+1]$ and the inequality is strict in the case $i = k$ for an unique $k \in [n+1]$.

Case 1: $\sigma_n = \sigma_{n+1} = 0$. We have $\pi(\sigma) \in \mathrm{St}(\pi(\alpha), \pi(\beta))$, because

$$\begin{aligned} |\pi(\alpha) + \pi(\sigma) - \pi(\beta)| &= \sum_{i=1}^{n-1} |\alpha_i + \sigma_i - \beta_i| + |\alpha_n - \beta_n + \alpha_{n+1} - \beta_{n+1}| \\ &< \sum_{i=1}^{n-1} |\alpha_i - \beta_i| + |\alpha_n - \beta_n + \alpha_{n+1} - \beta_{n+1}| \\ &= |\pi(\alpha) - \pi(\beta)|. \end{aligned}$$

And we clearly have $\pi(\alpha) + \pi(\sigma) = \pi(\alpha + \sigma) \notin J$. Thus, there exists some $\tau \in \mathrm{St}(\pi(\alpha) + \pi(\sigma), \pi(\beta))$, such that $\pi(\alpha) + \pi(\sigma) + \tau \in J$.

Case 1A: $\tau_n = 0$**.** Let $\tau' = (\tau_1, \dots, \tau_n, 0)$. We have $\tau' \in \mathrm{St}(\alpha + \sigma, \beta)$, because

$$\begin{aligned} |\alpha + \sigma + \tau' - \beta| &= \sum_{i=1}^{n-1} |\alpha_i + \sigma_i + \tau_i - \beta_i| \\ &\quad + |\alpha_n - \beta_n| + |\alpha_{n+1} - \beta_{n+1}| \\ &< \sum_{i=1}^{n-1} |\alpha_i + \sigma_i - \beta_i| + |\alpha_n - \beta_n| + |\alpha_{n+1} - \beta_{n+1}| \\ &= |\alpha + \sigma - \beta|. \end{aligned}$$

We have $\pi(\alpha+\sigma+\tau') = \pi(\alpha)+\pi(\sigma)+\tau \in J$, thus $\alpha+\sigma+\tau' \in \pi^{-1}(J)$.

Case 1B: $\tau_n \neq 0$**.** Without loss of generality we may assume $\tau_n = 1$. Thus we have $\alpha_n+\alpha_{n+1} < \beta_n+\beta_{n+1}$, hence $\alpha_n < \beta_n$ or $\alpha_{n+1} < \beta_{n+1}$. Without loss of generality let $\alpha_n < \beta_n$ and define

$$\tau' = (0, \dots, 0, 1, 0) \in \mathbb{Z}_{\geq 0}{}^{\mathrm{n}+1}.$$

Then $\tau' \in \mathrm{St}(\alpha+\sigma, \beta)$ and $\pi(\alpha+\sigma+\tau') = \pi(\alpha)+\pi(\sigma)+\tau \in J$, thus $\alpha + \sigma + \tau' \in \pi^{-1}(J)$.

Case 2: $\sigma_n \neq 0$ **or** $\sigma_{n+1} \neq 0$**.** Suppose first $\pi(\sigma) \in \mathrm{St}(\pi(\alpha), \pi(\beta))$. Then we have some $\tau \in \mathrm{St}(\pi(\alpha) + \pi(\sigma), \pi(\beta))$, such that $\pi(\alpha) + \pi(\sigma) + \tau \in J$. If $\tau_n = 0$, we can argue in the same way as in case 1A. If $\tau_n \neq 0$ the non-zero entry of σ has the same sign as τ_n, because otherwise we would have $\pi(\sigma) + \tau = 0$, which contradicts $\tau \in \mathrm{St}(\pi(\alpha) + \pi(\sigma), \pi(\beta))$. Thus we can argue analogously as in case 1B. Finally we consider the case where $\pi(\sigma) \notin \mathrm{St}(\pi(\alpha), \pi(\beta))$. Without loss of generality we may assume $\sigma_n = 1$. Then we have $\alpha_n < \beta_n$ and $\alpha_{n+1}+\alpha_n \geq \beta_{n+1}+\beta_n$, thus $\alpha_{n+1} > \beta_{n+1}$. Let $\tau' = (0, \dots, 0, -1)$, then $\tau' \in \mathrm{St}(\alpha + \sigma, \beta)$ and $\pi(\alpha + \sigma + \tau') = \pi(\alpha) \in J$, therefore $\alpha + \sigma + \tau' \in \pi^{-1}(J)$.

□

2.2.14 Corollary. *Let $d \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}$. If $J \subseteq \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ is a constant sum jump system with $d \geq J$, then $\mathcal{P}_d(J)$ is also a constant sum jump system.*

Proof. With the notation as in Definition 2.2.11 this is clear, since $|\operatorname{Pr}_d(\alpha)| = |\alpha|$ for all $\alpha \in \{0,1\}^{V_d}$. □

2.2.15 Definition. Let $\alpha \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}$. The *support* of α is

$$\mathrm{supp}(\alpha) = \{k \in [n] : \ \alpha_k \neq 0\}.$$

2.2.16 Lemma (cf. [4], Section 3)**.** *Let $J \subseteq \{0,1\}^n$ be a constant sum jump system with rank function r. Then $\mathcal{M} = ([n], r)$ is a matroid with set of bases*

$$\mathfrak{B}(\mathcal{M}) = \{\mathrm{supp}(\alpha) : \ \alpha \in J\}.$$

Proof. By Corollary 2.2.9 $\mathcal{M}$ is a matroid. We have $r([n]) = \max_{\alpha \in J} |\alpha|$. Because J is a constant sum jump system, we have $r([n]) = |\alpha|$ for arbitrary $\alpha \in J$. Let $B = \mathrm{supp}(\alpha)$ for some $\alpha \in J$. Since $J \subseteq \{0,1\}^n$, we have

$$r(B) = \max\{\sum_{i \in B} \alpha_i : \ \alpha \in J\} = |B| = |\alpha| = r([n]).$$

Therefore B is a basis by Corollary 2.1.15.

Conversely, let B be a basis of $\mathcal{M}$. Let $\alpha \in J$ such that $r(B) = \sum_{i \in B} \alpha_i$. Since B is a basis and $r([n]) = |\alpha|$, we actually have by Corollary 2.1.15 that $|\alpha| = \sum_{i \in B} \alpha_i$. Therefore $\operatorname{supp}(\alpha) \subseteq B$. But since $\operatorname{supp}(\alpha)$ is a basis, as we have seen above, we have $B = \operatorname{supp}(\alpha)$. □

We have seen, that the rank function of a jump system is a polymatroid. The aim is now to show, that this polymatroid completely describes the underlying jump system, in the case of a constant sum jump system. In order to show this, Petter Brändén's idea of using a notion of polarization for jump systems proved very helpful. We have not found this result in the literature, thus we think that the remain of this section is new.

2.2.17 Lemma. *Let $J \subseteq \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ be a jump system with rank function r. Let $S \subseteq [n-1]$. Let $\alpha \in J$ with $r(S \cup \{n\}) = \sum_{i \in S} \alpha_i + \alpha_n$ such that α_n is minimal. Then we have $\alpha_n = r(S \cup \{n\}) - r(S)$.*

Proof. By Lemma 2.2.7 there is a $\beta \in J$ with $\beta_i \geq \alpha_i$ for all $i \in S$ such that $r(S) = \sum_{i \in S} \beta_i$. This implies $r(S \cup \{n\}) - r(S) = \sum_{i \in S}(\alpha_i - \beta_i) + \alpha_n$. Thus $\alpha_n \geq r(S \cup \{n\}) - r(S)$. Again by Lemma 2.2.7, there is a $\gamma \in J$ with $\gamma_i \geq \beta_i$ for all $i \in S \cup \{n\}$ such that $r(S \cup \{n\}) = \sum_{i \in S} \gamma_i + \gamma_n$. This implies $r(S \cup \{n\}) - r(S) = \sum_{i \in S}(\gamma_i - \beta_i) + \gamma_n$. Thus $\gamma_n \leq r(S \cup \{n\}) - r(S)$. Since we have assumed α_n to be minimal, we get $\alpha_n = r(S \cup \{n\}) - r(S)$. □

2.2.18 Lemma. *Let $I \subseteq \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ resp. $J \subseteq \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ be a jump system with rank function r resp. s. Let $d \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ such that $d \geq I$ and $d \geq J$. Then let r_P resp. s_P be the rank function of $\mathcal{P}_d(I)$ resp. $\mathcal{P}_d(J)$. If $r = s$ then $r_P = s_P$.*

Proof. Let $r = s$ and let $e \in \mathbb{Z}_{>0}$, such that $e \geq \alpha_n$ for all $\alpha \in I \cup J$. Let $A = \{\alpha \in (\mathbb{Z}_{\geq 0})^{n+1} : \ \alpha_n \leq e-1, \ \alpha_{n+1} \leq 1\}$. Consider the map

$$\pi : A \to \mathbb{Z}^{\mathrm{n}}_{\geq 0}, \ (\alpha_1, \dots, \alpha_{n+1}) \mapsto (\alpha_1, \dots, \alpha_{n-1}, \alpha_n + \alpha_{n+1}).$$

We will show that the rank function r_0 of $\pi^{-1}(I)$ coincides with the rank function s_0 of $\pi^{-1}(J)$. Iterating this result shows the claim (note that $\pi^{-1}(I)$ and $\pi^{-1}(J)$ are jump sytems too, cf. the proof of Lemma 2.2.13).

Let $S \subseteq [n-1]$. Then we clearly have:

$$r_0(S) = r(S) = s(S) = s_0(S),$$

$$r_0(S \cup \{n, n+1\}) = r(S \cup \{n\}) = s(S \cup \{n\}) = s_0(S \cup \{n, n+1\}).$$

It remains to show that we also have that $r_0(S \cup \{n\}) = s_0(S \cup \{n\})$ and $r_0(S \cup \{n+1\}) = s_0(S \cup \{n+1\})$. Let $\alpha \in I$ with $r(S \cup \{n\}) = \sum_{i \in S} \alpha_i + \alpha_n$ such that α_n is minimal. Analogously, let $\beta \in J$ with $s(S \cup \{n\}) = \sum_{i \in S} \beta_i + \beta_n$ such that β_n is minimal. Since $r = s$, the preceding Lemma implies $\alpha_n = \beta_n$. First, we consider the case, where $\alpha_n < e$. We will show that $r_0(S \cup \{n\}) = r(S \cup \{n\})$. Clearly, we have:

$$r_0(S \cup \{n\}) \leq r_0(S \cup \{n, n+1\}) = r(S \cup \{n\}).$$

Since $(\alpha_1, \dots, \alpha_n, 0) \in \pi^{-1}(\{\alpha\})$, we get

$$r_0(S \cup \{n\}) \geq \sum_{i \in S \cup \{n\}} \alpha_i = r(S \cup \{n\}),$$

thus $r_0(S \cup \{n\}) = r(S \cup \{n\})$. In the other case, $\alpha_n = e$, we will show that $r_0(S \cup \{n\}) = r(S \cup \{n\}) - 1$. Since $(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n - 1, 1) \in \pi^{-1}(\{\alpha\})$, we get

$$r_0(S \cup \{n\}) \geq \sum_{i \in S} \alpha_i + \alpha_n - 1 = r(S \cup \{n\}) - 1.$$

Assume $r_0(S \cup \{n\}) = r(S \cup \{n\})$ and choose $\gamma \in \pi^{-1}(I)$ in such a way that $r_0(S \cup \{n\}) = \sum_{i \in S \cup \{n\}} \gamma_i$. Since $r_0(S \cup \{n\}) = r_0(S \cup \{n, n+1\})$, we have $\gamma_{n+1} = 0$. Letting

$$\gamma' = \pi(\gamma) = (\gamma_1, \ldots, \gamma_n) \in I,$$

we obtain $r(S \cup \{n\}) = \sum_{i \in S} \gamma'_i + \gamma'_n$, but since $\gamma \in A$, we have $\gamma'_n \leq e - 1 < \alpha_n$. This is a contradiction to our assumed minimality of α_n. Therefore we have $r_0(S \cup \{n\}) = r(S \cup \{n\}) - 1$. Applying these both cases to s, s_0 and β, we get that $r_0(S \cup \{n\}) = s_0(S \cup \{n\})$.

Now, it only remains to show that $r_0(S \cup \{n+1\}) = s_0(S \cup \{n+1\})$. This will be done analogously to the previous part of the proof. First, we consider the case, where $\alpha_n \leq 1$. We will show that $r_0(S \cup \{n+1\}) = r(S \cup \{n\})$. Clearly, we have:

$$r_0(S \cup \{n+1\}) \leq r_0(S \cup \{n, n+1\}) = r(S \cup \{n\}).$$

Since $(\alpha_1, \ldots, \alpha_{n-1}, 0, \alpha_n) \in \pi^{-1}(\{\alpha\})$, we get

$$r_0(S \cup \{n+1\}) \geq \sum_{i \in S} \alpha_i + \alpha_n = r(S \cup \{n\}),$$

thus $r_0(S \cup \{n+1\}) = r(S \cup \{n\})$. In the other case, $\alpha_n > 1$, we will show that $r_0(S \cup \{n+1\}) = r(S \cup \{n\}) - \alpha_n + 1$. Since $(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n - 1, 1) \in \pi^{-1}(\{\alpha\})$, we get

$$r_0(S \cup \{n+1\}) \geq \sum_{i \in S} \alpha_i + 1 = r(S \cup \{n\}) - \alpha_n + 1.$$

Assume $r_0(S \cup \{n+1\}) > r(S \cup \{n\}) - \alpha_n + 1$ and choose $\kappa \in \pi^{-1}(I)$ in such a way that $r_0(S \cup \{n+1\}) = \sum_{i \in S \cup \{n+1\}} \kappa_i$. By Lemma 2.2.7, there is a $\nu \in \pi^{-1}(I)$ with $\nu_i \geq \kappa_i$ for all $i \in S \cup \{n, n+1\}$ such that we have $r_0(S \cup \{n, n+1\}) = \sum_{i \in S \cup \{n, n+1\}} \nu_i$. Since

$$r_0(S \cup \{n+1\}) + \nu_n \leq r_0(S \cup \{n, n+1\}) = r(S \cup \{n\}) < r_0(S \cup \{n+1\}) + \alpha_n - 1,$$

we have $\nu_n < \alpha_n - 1$. Letting $\nu' = \pi(\nu) \in I$, we thus have

$$r(S \cup \{n\}) = \sum_{i \in S} \nu'_i + \nu'_n,$$

but $\nu'_n = \nu_n + \nu_{n+1} \leq \nu_n + 1 < \alpha_n$. This is a contradiction to our assumed minimality of α_n. Therefore we have $r_0(S \cup \{n+1\}) = r(S \cup \{n\}) - \alpha_n + 1$. Again, applying both cases to s, s_0 and β, we finally get that

$$r_0(S \cup \{n+1\}) = s_0(S \cup \{n+1\}).$$

□

The next Theorem is the desired result about the connection between constant sum jump systems and their rank functions.

2.2.19 Theorem. *If $I, J \subseteq \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ are two constant sum jump systems whose rank functions coincide, then $I = J$.*

Proof. First we show that the claim holds in the case when $I, J \subseteq \{0,1\}^n$. Let r be the rank function of I and J. By the preceding Lemma $\mathcal{M} = ([n], r)$ is a matroid with set of bases

$$\{\mathrm{supp}(\alpha) : \alpha \in I\} = \mathfrak{B}(\mathcal{M}) = \{\mathrm{supp}(\alpha) : \alpha \in J\}.$$

Since an element of $\{0,1\}^n$ is determined by its support, we have $I = J$.

In the general case, let $d \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ with $d \geq I, J$. Then $\mathcal{P}_d(I)$ and $\mathcal{P}_d(J)$ are constant sum jump systems by Corollary 2.2.14 whose rank functions coincide by Lemma 2.2.18. So we have $\mathcal{P}(I) = \mathcal{P}(J)$, which implies $I = J$. □

2.3 The Support of Stable Polynomials with and without Determinantal Representations

Now we have prepared the ground for examining the support of a stable polynomial and its combinatorial structure. The main results of this section are on the one hand Theorem 2.3.3, which states that the support of a stable polynomial is a jump system, and on the other hand Theorem 2.3.8, which characterizes the associated polymatroid of this jump system, in the case where the polynomial has a definite determinantal representation, in terms of the occurring matrices. Both are due to Brändén, see [6] and [5]. The latter will be used to construct a hyperbolic polynomial h from the Vámos matroid, cf. Example 2.1.22, such that no power h^N has a definite determinantal representation. This is also from [5].

2.3.1 Definition. The *support* of a polynomial

$$p = \sum_{\alpha \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}} a_\alpha \, \mathrm{x}^\alpha \in \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$$

is defined as

$$\mathrm{supp}(p) = \{\alpha \in \mathbb{Z}^{\mathrm{n}}_{\geq 0} : a_\alpha \neq 0\}.$$

Let $\alpha \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ and let $p \in \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$. In this section we will use the following notation for derivatives

$$\partial^\alpha p = \frac{\partial^{|\alpha|} p}{\partial^{\alpha_1} \mathrm{x}_1 \cdots \partial^{\alpha_n} \mathrm{x}_n}.$$

Recall our definition of the partial order $\leq$ on $\mathbb{Z}^{\mathrm{n}}$:

$$\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i \text{ for all } i = 1, \ldots, n.$$

When we write $\alpha < \beta$, we mean that $\alpha \leq \beta$ but $\alpha \neq \beta$. Further, let

$$[\alpha, \beta] = \{\gamma \in \mathbb{Z}^{\mathrm{n}} : \alpha \leq \gamma \leq \beta\}$$

and

$$(\alpha, \beta) = \{\gamma \in \mathbb{Z}^{\mathrm{n}} : \alpha < \gamma < \beta\}.$$

If we have $\gamma \in \mathbb{Z}^{\mathrm{n}}$ and $A \subseteq \mathbb{Z}^{\mathrm{n}}$, such that $\gamma \geq \alpha$ for all $\alpha \in A$, we write $\gamma \geq A$.

2.3.2 Lemma. *Let* $p = \sum_{0 \le \gamma \le \kappa} a_\gamma \, \mathrm{x}^\gamma \in \mathbb{C}[\mathrm{x}_1, \dots, \mathrm{x_n}]$ *be a* $H-$*stable polynomial. Let* $\alpha, \beta \in \operatorname{supp}(p)$ *such that* $\alpha \le \beta$ *and let* $q = \partial^{\kappa - \beta}(\mathrm{x}^\kappa \, p(\frac{1}{\mathrm{x}_1}, \dots, \frac{1}{\mathrm{x}_n}))$. *Then the polynomial* $p_{\alpha,\beta} = \partial^\alpha(\mathrm{x}^\beta \, q(\frac{1}{\mathrm{x}_1}, \dots, \frac{1}{\mathrm{x}_n}))$ *is also* $H-$*stable and we have* $\operatorname{supp}(p_{\alpha,\beta}) = \{\gamma - \alpha : \gamma \in \operatorname{supp}(p) \cap [\alpha, \beta]\}$.

Proof. First we proof the statements about the support:

$$\begin{array}{rrcl} & \operatorname{supp}(\mathrm{x}^\kappa \, p(\frac{1}{\mathrm{x}_1}, \dots, \frac{1}{\mathrm{x}_n})) & = & \{\kappa - \gamma : \gamma \in \operatorname{supp}(p)\} \\ \Rightarrow & \operatorname{supp}(q) & = & \{\beta - \gamma : \gamma \in \operatorname{supp}(p), \gamma \le \beta\} \\ \Rightarrow & \operatorname{supp}(\mathrm{x}^\beta \, q(\frac{1}{\mathrm{x}_1}, \dots, \frac{1}{\mathrm{x}_n})) & = & \{\gamma \in \operatorname{supp}(p) : \gamma \le \beta\} \\ \Rightarrow & \operatorname{supp}(p_{\alpha,\beta}) & = & \{\gamma - \alpha : \gamma \in \operatorname{supp}(p) \cap [\alpha, \beta]\}. \end{array}$$

Applying Proposition 1.5.9, we get that q is $H'-$stable, where

$$H' = \{\xi \in \mathbb{C} : \ \xi_i^{-1} \in H, \ \ i = 1, \dots, n\}.$$

Thus, again with Proposition 1.5.9, $p_{\alpha,\beta}$ is $H-$stable. $\square$

2.3.3 Theorem ([6], Theorem 3.2)**.** *The support of a* $H-$*stable polynomial is a jump system.*

Proof. We assume that the claim is wrong. Let $p \in \mathbb{C}[\mathrm{x}_1, \dots, \mathrm{x_n}]$ be $H-$stable and let $\alpha, \beta \in \operatorname{supp}(p)$ violate the two-step axiom. Let p and α, β with this properties be chosen such that $|\alpha - \beta|$ is minimal. Since for all $\varphi \in \mathbb{R}$ holds that $\operatorname{supp}(p(\mathrm{x})) = \operatorname{supp}(p(\exp(-\mathrm{i}\,\varphi)\,\mathrm{x}))$, we can assume, that

$$H = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}.$$

Let $\mu(\mathrm{x})$ be the change of variables defined by

$$\mathrm{x}_i \mapsto \begin{cases} \mathrm{x}_i^{-1}, & \text{if } \alpha_i > \beta_i, \\ \mathrm{x}_i, & \text{otherwise.} \end{cases}$$

Let $\gamma \in \mathbb{Z}^{\mathrm{n}}_{\ge 0}$, such that $q = \mathrm{x}^\gamma \, p(\mu(\mathrm{x}))$ is a polynomial. Since H is closed under inversion, p is $H-$stable if and only if q is $H-$stable. By Lemma 2.2.5 $\operatorname{supp}(p)$ is a jump system if and only if $\operatorname{supp}(q)$ is a jump system. Thus, without loss of generality we can assume $\alpha \le \beta$. Note that this does not change the value of $|\alpha - \beta|$. But then $p_{\alpha,\beta}$ with $0, \beta - \alpha \in \operatorname{supp}(p_{\alpha,\beta})$ is by Lemma 2.3.2 such a counterexample too. Therefore we can assume, that $p = \sum_{0 \le \gamma \le \beta} a_\gamma \, \mathrm{x}^\gamma$ with $a_0 a_\beta \ne 0$ and $\beta_i > 0$ for $i = 1, \dots, n$ (if $\beta_i = 0$ for some i, we restrict to a smaller number of variables). Since we have assumed that the two-step axiom is violated, there is a $\sigma \in \mathrm{St}(0, \beta)$ with $\sigma \notin \operatorname{supp}(p)$, such that for every $\tau \in \mathrm{St}(\sigma, \beta)$ we have $\sigma + \tau \notin \operatorname{supp}(p)$. By symmetry we can assume that $\sigma = \delta_1$. It follows, that $\delta_1, 2\delta_1, \delta_1 + \delta_2, \dots \delta_1 + \delta_n \notin \operatorname{supp}(p)$. If there was a $\xi \in \operatorname{supp}(p) \cap (\delta_1, \beta)$, then $p_{0,\xi}$ would yield a smaller counterexample. Thus, if we have a $\gamma \in \mathbb{Z}^{\mathrm{n}}_{\ge 0}$ with $\gamma_1 > 0$, then $a_\gamma = 0$ or $\gamma = \beta$. Now let $\lambda > 0$ and $r = \frac{1}{\beta_1} \sum_{i=2}^n \beta_i$. Then the univariate polynomial $p(\lambda^{-r}\,\mathrm{t}, \lambda\,\mathrm{t}, \dots, \lambda\,\mathrm{t})$ is $H-$stable. Because the zeros of a polynomial depend continuously on its coefficients, we obtain, letting $\lambda \to 0$, that the polynomial

$$a_0 + a_\beta \, \mathrm{t}^{|\beta|}$$

is $H-$stable. It is $|\beta| \ge 3$, because otherwise the two-step axiom would hold. But with $k \ge 3$ there is always a kth root of $-\frac{a_0}{a_\beta}$, that lies in H. Thus $a_0 + a_\beta \, \mathrm{t}^{|\beta|}$ cannot be $H-$stable. $\square$

2.3.4 Remark. Let $p \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be a real stable polynomial. The preceding Theorem implies that $J = \mathrm{supp}(p)$ is a jump system. Let r be the rank function of J, cf. Definition 2.2.6. By Proposition 2.2.8, $([n], r)$ is a polymatroid. There is another way to express this rank function. Let $S \subseteq [n]$ and let $e = (1, \ldots, 1)^\mathrm{T} \in \mathbb{R}^\mathrm{n}$. Then we have

$$r(S) = \max\{\sum_{i \in S} \alpha_i : \alpha \in J\} = \deg(p(e + \mathrm{t} \sum_{i \in S} \delta_i)).$$

To see the second equality, have in mind that the coefficients of p are all non-negative or all non-positive, by Corollary 1.5.10. This motivates our next definition.

2.3.5 Definition. Let $p \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$. The *rank function* $r_p : 2^{[n]} \to \mathbb{Z}$ of p is defined by

$$r_p(S) = \deg(p(e + \mathrm{t} \sum_{i \in S} \delta_i))$$

for all $S \subseteq [n]$.

2.3.6 Lemma (cf. the proof of [5], Theorem 2.2)**.** *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be homogeneous, stable with a definite determinantal representation, i.e.*

$$h = \lambda \cdot \det(\mathrm{x}_1 A_1 + \ldots + \mathrm{x}_n A_n)$$

with $A_1, \ldots, A_n \in \mathrm{Sym}_d(\mathbb{R})$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $B = A_1 + \ldots + A_n$ positive definite. Then $A_1, \ldots, A_n$ are positive semi-definite.

Proof. Let $S \in \mathrm{GL}_d(\mathbb{R})$ such that $S^T B S = \mathrm{I}_d$. By Corollary 1.5.11, h is hyperbolic with respect to $e = (1, \ldots, 1)^\mathrm{T}$ and the hyperbolicity cone $\mathrm{C}_h(e)$ contains the positive orthant $(\mathbb{R}_{>0})^n$. Thus the univariate polynomial

$$h(\delta_i - \mathrm{t}\, e) = \lambda \det(A_i - \mathrm{t}\, B) = \frac{\lambda}{\det(S)^2} \det(S^T A_i S - \mathrm{t}\, \mathrm{I}_d)$$

has only real, non-negative zeros. Therefore all eigenvalues of A_i are non-negative. □

At this point, recall how to define a polymatroid from a subspace arrangement: Let E be a finite set and $\mathcal{V} = (V_j)_{j \in E}$ a collection of subspaces of a finite dimensional vector space V over a field K. Let $r_\mathcal{V}(S) = \dim(\sum_{i \in S} V_i)$. Then $(E, r_\mathcal{V})$ is a polymatroid, cf. Example 2.1.3.

2.3.7 Lemma (cf. [5], Section 3)**.** *Let $A_1, \ldots, A_n \in \mathrm{Sym}_d(\mathbb{R})$ be positive semi-definite and let $\mathcal{V} = (V_1, \ldots, V_n)$ be the family of subspaces of $\mathbb{R}^d$ defined by $V_i = \mathrm{im}(A_i)$. Then we have*

$$r_\mathcal{V}(S) = \dim(\sum_{i \in S} V_i) = \deg(\det(\mathrm{I}_d + \mathrm{t} \sum_{i \in S} A_i))$$

for all $S \subseteq [n]$.

Proof. At first, we consider the case $S = [n]$ and $A_i = v_i v_i^{\mathrm{T}}$ with $v_i \in \mathbb{R}^d$. Let D be the $(d+n) \times (d+n)$ diagonal matrix, where the first d entries are 1 and the remaining are t. Let B the $d \times (d+n)$ matrix with $\delta_1, \ldots, \delta_d, v_1, \ldots, v_n$ as columns.

$$B = \begin{pmatrix} 1 & & 0 & | & & | \\ & \ddots & & v_1 & \cdots & v_n \\ 0 & & 1 & | & & | \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ & & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \mathrm{t} & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & \mathrm{t} \end{pmatrix}.$$

Then we have by the Cauchy-Binet theorem 1.1.4

$$\det(\mathrm{I}_d + \mathrm{t} \sum_{i \in S} A_i) = \det(BDB^T) = \sum_{S \in \binom{[d+n]}{d}} |B([d], S)|^2 \, \mathrm{t}^{|S \cap \{d+1, \ldots, d+n\}|}.$$

Thus, the degree of the polynomial on the right hand side is equal to the length of a maximal linearly independent subset of $\{v_1, \ldots, v_n\}$, thus equals $\dim(\sum_{i=1}^n V_i)$.

In order to see the general case, note that, because $A_1, \ldots, A_n$ are positive semi-definite, there are $w_{ij} \in \mathbb{R}^d$, such that

$$A_i = w_{i1} w_{i1}^T + \ldots + w_{id} w_{id}^T.$$

Then we have $V_i = \mathrm{Span}_{\mathbb{R}}\{w_{i1}, \ldots, w_{im}\}$. Thus the claim follows, after relabelling, from the case we have already shown. □

2.3.8 Theorem (cf. [5], Section 3)**.** *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be homogeneous, stable with a definite determinantal representation, i.e.*

$$h = \lambda \cdot \det(\mathrm{x}_1 A_1 + \ldots + \mathrm{x}_n A_n)$$

with $A_1, \ldots, A_n \in \mathrm{Sym}_d(\mathbb{R})$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $B = A_1 + \ldots + A_n$ positive definite. Then the rank function of h,

$$r_h : 2^{[n]} \to \mathbb{Z}_{\geq 0}, \; r_h(T) = \deg(h(e + \mathrm{t} \sum_{i \in T} \delta_i)),$$

is the rank function of a polymatroid, that is representable over $\mathbb{R}$ and the degree of h in x_i is exactly the rank of A_i. Conversely, if r is the rank function of a polymatroid, that is representable over $\mathbb{R}$, then there is a homogeneous, stable polynomial $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ with a definite determinantal representation, such that $r = r_h$ as defined above.

Proof. Let $S \in \mathrm{GL}_d(\mathbb{R})$ such that $S^{\mathrm{T}} B S = \mathrm{I}_d$ and let $T \subseteq [n]$. Then we have

$$\begin{aligned} r_h(T) &= \deg(h(e + \mathrm{t} \sum_{i \in T} \delta_i)) \\ &= \deg(\lambda \cdot \det(B + \mathrm{t} \sum_{i \in T} A_i)) \\ &= \deg(\frac{\lambda}{\det(S)^2} \det(\mathrm{I}_d + \mathrm{t} \sum_{i \in T} S^{\mathrm{T}} A_i S)). \end{aligned}$$

By Lemma 2.3.7 we have $r_h(T) = \dim(\sum_{i\in T}(\operatorname{im} S^{\mathrm{T}} A_i S))$, thus $([n], r_h)$ is a representable polymatroid. Note that by Corollary 1.5.10 all coefficients of h are either all non-negative or all non-positive, thus the degree of h in x_i is $r_h(\{i\}) = \operatorname{rank}(S^{\mathrm{T}} A_i S) = \operatorname{rank} A_i$.

In order to show the converse, let $V_1, \ldots, V_n$ be a collection of subspaces of $\mathbb{R}^N$, such that

$$r(S) = \dim(\sum_{i\in S} V_i)$$

for all $S \subseteq [n]$. Let $V_i = \operatorname{Span}_{\mathbb{R}}\{v_{i1}, \ldots, v_{im_i}\}$ and let $A_i = v_{i1}v_{i1}^{\mathrm{T}} + \ldots + v_{im_i}v_{im_i}^{\mathrm{T}}$. As we have seen above, the polynomial

$$h = \det(\mathrm{x}_1\, A_1 + \ldots + \mathrm{x}_n\, A_n)$$

has the desired properties. □

2.3.9 Example. We really do need the determinantal representation to be definite in the above Theorem. Let $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. As we can see, A_1 has rank one, but

$$\det(\mathrm{x}\, A_1 + \mathrm{y}\, A_2) = \det \begin{pmatrix} \mathrm{x}+\mathrm{y} & \mathrm{x} \\ \mathrm{x} & \mathrm{x}-\mathrm{y} \end{pmatrix} = -\,\mathrm{y}^2$$

has not degree one in x.

2.3.10 Definition. Let $\mathcal{M} = ([n], r_{\mathcal{M}})$ be a matroid with set of bases $\mathfrak{B}(\mathcal{M})$. The *bases generated* polynomial $h_{\mathcal{M}} \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ of $\mathcal{M}$ is defined by

$$h_{\mathcal{M}}(\mathrm{x}) = \sum_{B\in\mathfrak{B}(\mathcal{M})} \prod_{j\in B} \mathrm{x}_j\,.$$

2.3.11 Remark. There is a large class of matroids whose bases generated polynomials are stable, but there are also matroids that do not occur as the support of any stable polynomial. It is a question of current interest, how to characterize those matroids whose bases generated polynomials are stable. This has been discussed for example in [6, 7, 23].

2.3.12 Remark. Let $h_{\mathcal{M}}$ be the bases generated polynomial of some matroid $\mathcal{M}$ and let $S \subseteq [n]$. Then we have by Proposition 2.1.14

$$r_{\mathcal{M}}(S) = \max\{|S \cap B| : B \in \mathcal{B}(\mathcal{M})\} = \deg(h_{\mathcal{M}}(e + \mathrm{t}\sum_{i\in S}\delta_i)) = r_{h_{\mathcal{M}}}(S),$$

where $e = (1, \ldots, 1)$ and $r_{h_{\mathcal{M}}}$ is the rank function of $h_{\mathcal{M}}$ as defined above.

2.3.13 Example. Let V_8 be the Vámos cube and let h_{V_8} be its bases generated polynomial. We consider the polynomial

$$h_4 = h_{V_8}(\mathrm{x}_1, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_4).$$

We call h_4 the *specialized Vámos polynomial*. Let $e = (1, 1, 1, 1)$. The function

$$r_{h_4} : 2^{[4]} \to \mathbb{Z}_{\geq 0},\ S \mapsto \deg(h_4(e + \mathrm{t}\sum_{i\in S}\delta_i))$$

violates Ingleton's inequalities 2.1.5 choosing $S_i = \{i\}$ for $i \in [4]$. Compare Example 2.1.22. Anyhow, we will see that h_4 is stable.

Initially, the next Lemma was proved by Wagner and Wei [23], using an improved version of Theorem 3.2.7, but we will present a more elementary proof.

2.3.14 Lemma. *The specialized Vámos polynomial $h_4 \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_4]$ is stable.*

Proof. As we have seen in Example 1.5.3, it suffices to show that h_4 is hyperbolic with respect to $e = (1,1,1,1)$ and that its hyperbolicity cone contains the positive orthant. We have

$$h_4 = 4 \cdot (\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3 + \mathrm{x}_4) \cdot (\mathrm{x}_1\,\mathrm{x}_2\,\mathrm{x}_3 + \mathrm{x}_1\,\mathrm{x}_2\,\mathrm{x}_4 + \mathrm{x}_1\,\mathrm{x}_3\,\mathrm{x}_4 + \mathrm{x}_2\,\mathrm{x}_3\,\mathrm{x}_4) + \mathrm{x}_3^2\,\mathrm{x}_4^2\,.$$

Let $v \in \mathbb{R}^4$ and $p(\mathrm{t}) = h_4(\mathrm{t}\,e - v)$. We have to show, that p has only real zeros. That is obvious if $v_3 = v_4 = v_1 = v_2$, because then it holds $p(\mathrm{t}) = 65(t - v_1)^4$. Thus we can assume that $v_1, \ldots, v_4$ are not all equal to each other. By symmetry we can further assume $v_1 \leq v_2$ and $v_3 \leq v_4$. The remaining cases are:

1. Case: $v_3 \leq v_4 \leq v_1 \leq v_2$. A short calculation verifies that

$$p(-\infty) > 0,$$

$$\begin{aligned} p(\frac{v_3+v_4}{2}) &= -\frac{1}{16}(v_4 - v_3)^2 \cdot (4v_1 + 4v_2 - 5v_3 - 3v_4) \\ &\quad \cdot((4v_1 - 4v_4) + (4v_2 - 3v_3 - v_4)) \\ &\leq 0 \end{aligned}$$

(strict if $v_3 \neq v_4$),

$$\begin{aligned} p(\frac{v_1+v_2+v_3+v_4}{4}) &= \frac{1}{256}(v_1 + v_2 + v_3 - 3v_4)^2(v_1 + v_2 - 3v_3 + v_4)^2 \\ &\geq 0 \end{aligned}$$

(strict if $3v_4 \neq v_3 + v_1 + v_2$),

$$\begin{aligned} p(\frac{v_1+v_2+v_4}{3}) &= -\frac{1}{81}(v_1 + v_2 - 3v_3 + v_4) \\ &\quad \cdot(9(v_1 - v_2)^2(2v_1 + 2v_2 - 3v_3 - v_4) \\ &\quad +(v_1 + v_2 - 2v_4)^2(v_1 + v_2 - 6v_3 + 4v_4)) \leq 0 \end{aligned}$$

(strict if not $v_4 = v_1 = v_2$),

$$p(\infty) > 0.$$

And in any case at most one inequality is not strict. Thus by the intermediate value theorem p has only real zeros.

2. Case: $v_3 \leq v_1 \leq v_4 \leq v_2$. Then holds

$$p(-\infty) > 0,$$

$$\begin{aligned} p(\frac{v_1+v_3}{2}) &= -\frac{1}{16}(v_1 - v_3)^2 \cdot (-5v_1 + 4v_2 - 5v_3 + 6v_4) \\ &\quad \cdot(-3v_1 + 4v_2 - 3v_3 + 2v_4) \leq 0 \end{aligned}$$

(strict if $v_3 \neq v_4$),

$$\begin{aligned} p(\frac{v_1+v_2+v_3+v_4}{4}) &= \frac{1}{256}(v_1+v_2+v_3-3v_4)^2(v_1+v_2-3v_3+v_4)^2 \\ &\geq 0 \end{aligned}$$

(strict if $3v_4 \neq v_3+v_1+v_2$),

$$\begin{aligned} p(\frac{v_1+v_2+v_4}{3}) &= -\frac{1}{81}(v_1+v_2-3v_3+v_4) \\ &\cdot(9(v_1-v_2)^2((2v_1+v_2-3v_3)+(v_2-v_4)) \\ &+(v_1+v_2-2v_4)^2(v_1+v_2-6v_3+4v_4)) \leq 0 \end{aligned}$$

(strict if not $v_4 = v_1 = v_2$),

$$p(\infty) > 0.$$

In any case at most one inequality is not strict. Thus p has only real zeros.

3. Case: $v_3 \leq v_1 \leq v_2 \leq v_4$ or $v_1 \leq v_3 \leq v_4 \leq v_2$. We have

$$p(-\infty) > 0,$$

$$p(\frac{v_1+v_3}{2}) = \frac{1}{16}(v_1-v_3)^2(6v_4+4v_2-5v_3-5v_1)(3v_3+3v_1-2v_4-4v_2) < 0,$$

$$p(\frac{v_1+v_3+v_3+v_4}{4}) = \frac{1}{256}(v_1+v_2+v_3-3v_4)^2(v_1+v_2-3v_3+v_4)^2 \geq 0,$$

$$p(\frac{v_2+v_4}{2}) = \frac{1}{16}(v_2-v_4)^2(6v_3+4v_1-5v_4-5v_2)(3v_4+3v_2-2v_3-4v_1) < 0,$$

$$p(\infty) > 0.$$

Thus p has only real zeros.

4. We get the cases $v_1 \leq v_3 \leq v_2 \leq v_4$ and $v_1 \leq v_2 \leq v_3 \leq v_4$ from the cases above if we replace t by $-\mathrm{t}$.

Thus h_4 is hyperbolic with respect to e. Because all coefficients of h_4 are non-negative, we get also that its hyperbolicity cone contains the positive orthant. □

2.3.15 Proposition (cf. [5], Theorem 3.3)**.** *Consider the specialized Vámos polynomial as in Example 2.3.14:*

$$h_4 = 4\cdot(\mathrm{x}_1+\mathrm{x}_2+\mathrm{x}_3+\mathrm{x}_4)\cdot(\mathrm{x}_1\,\mathrm{x}_2\,\mathrm{x}_3+\mathrm{x}_1\,\mathrm{x}_2\,\mathrm{x}_4+\mathrm{x}_1\,\mathrm{x}_3\,\mathrm{x}_4+\mathrm{x}_2\,\mathrm{x}_3\,\mathrm{x}_4)+\mathrm{x}_3^2\,\mathrm{x}_4^2\,.$$

(i) h_4 is hyperbolic with respect to $e = (1,1,1,1)$.

(ii) There is no positive integer N, such that h_4^N has a definite determinantal representation.

Proof. (*i*) is Lemma 2.3.14. Assume that for some positive integer h_4^N has a definite determinantal representation. By Theorem 2.3.8 the function

$$r_N : 2^{[4]} \to \mathbb{Z}_{\geq 0},\ S \mapsto \deg(h_4(e + \mathrm{t}\sum_{i\in S}\delta_i)^N)$$

is the rank function of a representable polymatroid. But on the other hand we have for all $S \subseteq [4]$

$$r_N(S) = N\deg(h_4(e + \mathrm{t}\sum_{i\in S}\delta_i)) = N r_{h_4}(S),$$

where r_{h_4} is defined as in Example 2.3.13. But r_{h_4} violates Ingleton's inequalities and so does r_N. But then r_N cannot be the rank function of some representable polymatroid. $\square$

2.4 A Discrete Version of the Generalized Lax Conjecture

Since the theory of matroids provides a counterexample to the conjecture that every hyperbolic polynomial has a definite determinantal representation, after taking sufficient large powers, it is a natural question to ask, if there can be a combinatorial obstruction to the Generalized Lax Conjecture 1.4.4. Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be a hyperbolic polynomial. After a change of variables, we can assume that h is stable. Have in mind, that by Lemma 1.4.6 the hyperbolicity cone C_h of h is a spectrahedral cone, if and only if there is a stable and homogeneous polynomial $q \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ with hyperbolicity cone C_q, such that $q \cdot h$ has a definite determinantal representation and such that $\mathrm{C}_h \subseteq \mathrm{C}_q$. One way to disprove the Generalized Lax Conjecture thus could be to find a homogeneous, stable polynomial h, such that for all stable $q \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$, the rank function of $q \cdot h$ is not a representable polymatroid. This would imply that $q \cdot h$ does not have definite determinantal representation. A second way, suggested by Brändén, is more subtle. Let r_h be the rank function of h, i.e. $r_h(S) = \deg(h(e + \mathrm{t}\sum_{i\in S}\delta_i))$, where $e = (1, \ldots, 1)^\mathrm{T} \in \mathbb{R}^\mathrm{n}$ and $S \subseteq [n]$. Having in mind, that by Corollary 1.5.10 all coefficients of h have the same sign, it follows that for all $S \subseteq [n]$, h vanishes in $\delta_S = \sum_{i\in S}\delta_i$ if and only if $r_h(S) < r_h([n])$. Therefore we have $\delta_S \in \mathrm{C}_h$ if and only if $r_h(S) = r_h([n])$. If $q \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ is a homogeneous stable polynomial with rank function r_q such that $\mathrm{C}_h \subseteq \mathrm{C}_q$, it follows that $r_q(S) = r_q([n])$ whenever $r_h(S) = r_h([n])$, where $S \subseteq [n]$. Thus one could disprove the Generalized Lax Conjecture by finding a stable, homogeneous polynomial $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$, such that for all stable and homogeneous polynomials $q \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$, either the rank function r_{qh} of qh is not a representable polymatroid, or there is a $S \subseteq [n]$ such that $r_h(S) = r_h([n])$ but $r_q(S) < r_q([n])$. In this section, providing a discrete version of the Generalized Lax Conjecture, we will show that it is not possible to give such a counterexample. As far as we know, the results from this section are new.

Given $n \in \mathbb{Z}_{>0}$ let $\mathrm{J}_n = 2^{[n]}\backslash\{\emptyset\}$, and let $\mathfrak{A}_n = (a_{S,T})_{S,T\in\mathrm{J}_n}$ be the matrix whose entries are given by

$$a_{S,T} = \begin{cases} 1, & \text{if } S \cap T \neq \emptyset \\ 0, & \text{if } S \cap T = \emptyset. \end{cases}$$

2.4.1 Lemma. *For the determinant of* $\mathfrak{A}_n$ *holds*

$$\det(\mathfrak{A}_n) = \begin{cases} 1, & \textit{if } n = 1 \\ -1, & \textit{if } n \geq 2. \end{cases}$$

Proof. We will prove this by induction on n. If $n = 1$, we have $\mathfrak{A}_1 = (1)$. If $n > 1$, we are numbering the lines and columns of $\mathfrak{A}_n$ in the following way: The first $2^{n-1} - 1$ lines and columns are indexed by the non-empty subsets $S_1, \ldots, S_{2^{n-1}-1}$ of $[n-1]$. The 2^nth line and column is indexed by $\{n\}$ and the remaining by $S_1 \cup \{n\}, \ldots, S_{2^{n-1}} \cup \{n\}$. With this sorting, we have

$$\mathfrak{A}_n = \begin{pmatrix} & & & 0 & & & \\ & \mathfrak{A}_{n-1} & & \vdots & & \mathfrak{A}_{n-1} & \\ & & & 0 & & & \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ & & & 1 & 1 & \cdots & 1 \\ & \mathfrak{A}_{n-1} & & \vdots & \vdots & \ddots & \vdots \\ & & & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Applying elementary row and column operations, we obtain the matrix

$$\begin{pmatrix} & & & 0 & 0 & \cdots & 0 \\ & \mathfrak{A}_{n-1} & & \vdots & \vdots & \ddots & \vdots \\ & & & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & & & \\ \vdots & \ddots & \vdots & \vdots & & -\mathfrak{A}_{n-1} & \\ 0 & \cdots & 0 & 0 & & & \end{pmatrix}.$$

Therefore we have $\det(\mathfrak{A}_n) = -\det(\mathfrak{A}_{n-1})^2 = -1$. $\square$

Consider a map $r : 2^{[n]} \to \mathbb{Z}$ with $r(\emptyset) = 0$. Let $v_r \in \mathbb{Z}^{\mathrm{J}_n}$ be the vector defined by $(v_r)_S = r(S)$, for all $S \in \mathrm{J}_n$, and $w_r = {\mathfrak{A}_n}^{-1}\, v_r \in \mathbb{Z}^{\mathrm{J}_n}$. Then let

$$r^* : 2^{[n]} \to \mathbb{Z}, \quad S \mapsto \begin{cases} 0, & \text{if } S = \emptyset \\ (w_r)_S, & \text{else.} \end{cases}$$

By construction, we have

$$r(T) = (v_r)_T = (\mathfrak{A}_n\, w_r)_T = \sum_{T \cap S \neq \emptyset} r^*(S), \text{ for all } T \subseteq [n].$$

If we have another map $t : 2^{[n]} \to \mathbb{Z}$ with $t(\emptyset) = 0$, then it holds that $(r + t)^* = r^* + t^*$.

2.4.2 Proposition. *Let* $r : 2^{[n]} \to \mathbb{Z}$ *be a map with* $r(\emptyset) = 0$. *If* $r^*(S) \geq 0$ *for all* $S \subseteq [n]$, *then* r *is a polymatroid which is representable over every field.*

Proof. Let $N = r([n])$, let K be an arbitrary field and let $(V_1, \dots, V_n)$ be the subspace arrangement in K^N defined by

$$V_k = \mathrm{Span}_K\{v_{S,j} : S \in \mathrm{J}_n,\ k \in S,\ 1 \leq j \leq r^*(S)\},$$

where the $v_{S,j} \in K^N$ are $N = \sum_{S\in \mathrm{J}_n} r^*(S)$ linearly independent vectors. Then we have for $T \subseteq [n]$

$$\dim(\sum_{k\in T} V_k) = \sum_{T\cap S\neq\emptyset} r^*(S) = r(T).$$

□

2.4.3 Definition. Let $r : 2^{[n]} \to \mathbb{Z}$ be a map with $r(\emptyset) = 0$. The *support* of r is the set

$$\mathrm{supp}(r) = \{S \subseteq [n] :\ r(S) = r([n])\}.$$

The *deficiency set* of r is the set

$$\mathrm{def}(r) = \{S \subseteq [n] :\ r^*(S) < 0\}.$$

2.4.4 Proposition. *Let $r : 2^{[n]} \to \mathbb{Z}$ be a function which obeys heredity (i.e. $A \subseteq B$ implies $r(A) \leq r(B)$) and $r(\emptyset) = 0$. Let $S \subseteq [n]$, then the following are equivalent:*

(i) $S \in \mathrm{supp}(r)$.

(ii) $\sum_{T\cap S=\emptyset} r^*(T) = 0$.

(iii) $r^*(T) = 0$ *for all* $T \subseteq [n]$ *with* $S \cap T = \emptyset$.

Proof.

$(iii) \Rightarrow (ii) :$ true.

$(i) \Leftrightarrow (ii) :\ S \in \mathrm{supp}(r) \Leftrightarrow 0 = r([n]) - r(S) = \sum_{T\cap S=\emptyset} r^*(T)$.

$(i), (ii) \Rightarrow (iii) :$ Let $S \in \mathrm{supp}(r)$ and let $T \subseteq [n]$ with $S \cap T = \emptyset$. Consider the set $R = [n] \setminus T$. It is $S \subseteq R$, so we have $R \in \mathrm{supp}(r)$. From this it follows that $\sum_{A\subseteq T} r^*(A) = \sum_{A\cap R=\emptyset} r^*(A) = 0$. We will argue by induction with respect to $k = |T|$. If $k = 1$, we are done. If $k > 1$, we can apply the induction hypothesis on every strict subset of T:

$$0 = \sum_{M\subseteq T} r^*(M) = r^*(T).$$

□

2.4.5 Corollary. *Let $r : 2^{[n]} \to \mathbb{Z}$ be a polymatroid, let $S \in \mathrm{supp}(r)$ and let $T \in \mathrm{def}(r)$. Then $S \cap T \neq \emptyset$ is true.*

Proof. Let $S \in \mathrm{supp}(r)$ and suppose $S\cap T = \emptyset$. Then the preceding Proposition implies that $r^*(T) = 0$, thus $T \notin \mathrm{def}(r)$. □

The next Theorem ensures that none of the methods explained in the introduction of this section can be used to disprove the Generalized Lax Conjecture.

2.4.6 Theorem. *Let $r : 2^{[n]} \to \mathbb{Z}$ be a polymatroid. Then there is a polymatroid $t : 2^{[n]} \to \mathbb{Z}$ which is representable over every field, so that*

(i) *$r + t$ is representable over every field.*

(ii) $\mathrm{supp}(r) \subseteq \mathrm{supp}(t)$.

Proof. Let $t : 2^{[n]} \to \mathbb{Z}$ be the unique map defined by $t(\emptyset) = 0$ and

$$t^*(S) = \begin{cases} -r^*(S), & \text{if } S \in \mathrm{def}(r) \\ 0, & \text{else.} \end{cases}$$

Proposition 2.4.2 implies that t and $r + t$ are both polymatroids and representable over every field. Let $S \in \mathrm{supp}(r)$. It follows from Corollary 2.4.5, that $S \cap T \neq \emptyset$ for all $T \in \mathrm{def}(r)$. So we have $t^*(R) = 0$ for all $R \subseteq [n]$ with $R \cap S = \emptyset$. Proposition 2.4.4 implies $S \in \mathrm{supp}(t)$ and thus implies the claim. □

Finally, we give a formulation of Theorem 2.4.6 that does not use the language of polymatroids.

2.4.7 Theorem. *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be a homogeneous, stable polynomial. Then there are homogeneous, stable polynomials $f, g \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ that have a definite determinantal representation, such that*

$$\mathrm{supp}(g \cdot h) = \mathrm{supp}(f)$$

and every $a \in (\mathbb{R}_{\geq 0})^n$ that lies in the hyperbolicity cone of h also lies in the hyperbolicity cone of g.

Proof. The set $J = \mathrm{supp}(h)$ is a jump system by Theorem 2.3.3. By Proposition 2.2.8 the rank function r of J is a polymatroid. Thus, we can find by Theorem 2.4.6 a polymatroid $t : 2^{[n]} \to \mathbb{Z}$ which is representable over $\mathbb{R}$, such that $r + t$ is representable over $\mathbb{R}$ and $\mathrm{supp}(r) \subseteq \mathrm{supp}(t)$. By Theorem 2.3.8 there are homogeneous, stable polynomials $f, g \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ that have a definite determinantal representation, such that

$$t(S) = \deg(g(e + \mathrm{t} \sum_{i \in S} \delta_i)) \text{ and } (r + t)(S) = \deg(f(e + \mathrm{t} \sum_{i \in S} \delta_i))$$

for all $S \subseteq [n]$. Therefore, by construction, $\mathrm{supp}(g \cdot h)$ and $\mathrm{supp}(f)$ are two constant sum jump systems, whose rank functions coincide. By Theorem 2.2.19 we have $\mathrm{supp}(g \cdot h) = \mathrm{supp}(f)$. Now let $a \in (\mathbb{R}_{\geq 0})^n$ and consider the set $T_a = \{i \in [n] : \ a_i > 0\}$. If we have in mind, that h has only non-negative or non-positive coefficients, we see that $h(a) \neq 0$ if and only if $T_a \in \mathrm{supp}(r)$. If a lies in the hyperbolicity cone of h, we have by Theorem 1.3.9 $h(a) \neq 0$ and therefore we have $T_a \in \mathrm{supp}(r) \subseteq \mathrm{supp}(t)$. Thus, we have $g(a) \neq 0$. Because $(\mathbb{R}_{\geq 0})^n$ is contained in the closure of the hyperbolicity cone of g, it follows, that a lies in the hyperbolicity cone of g, see Corollary 1.3.10. □

3 Multiaffine Polynomials

In this chapter we will restrict to the multiaffine case, i.e. we will study polynomials, where every variable occurs at most of degree one. We will see that this is not a big restriction with regard to the Generalized Lax Conjecture 1.4.4, since it would suffice to prove it in the case of multiaffine polynomials, cf. Remark 3.2.15. In the first two sections we work towards a result of Brändén, which characterizes the multiaffine, real polynomials, that are stable, (Theorem 3.2.7). Then we present some of our own results, concerning homogeneous multiaffine polynomials that have a definite determinantal representation. In particular, we acquire a necessary and sufficient condition for a homogeneous, multiaffine and stable polynomial to have a definite determinantal representation, which reminds of Brändén's result, cf. Theorem 3.4.7.

3.1 Univariate Stable Polynomials

First of all, we need to recall some classic facts about stable polynomials in one variable. In the main, we will stick to [19, Section 6.3], see also [16]. We borrowed the notation in this and the following section from [6].

3.1.1 Definition. Let $f, g \in \mathbb{R}[t]$ be two non-zero univariate polynomials. We say that f and g *interlace*, if the following holds:

(i) $|\deg(f) - \deg(g)| \leq 1$.

(ii) All zeros of f and g are real.

(iii) Let $\alpha_1 \leq \cdots \leq \alpha_d$ be the zeros of f and let $\beta_1 \leq \cdots \leq \beta_e$ be the zeros of g. It holds
$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \ldots,$$
or
$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \ldots.$$

For technical reasons we also say that the polynomial 0 interlaces any (non-zero) real-rooted polynomial f.

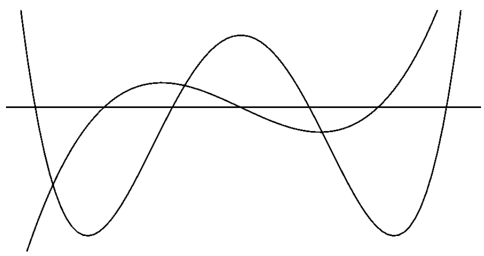

Figure 6: A quartic polynomial interlaced by a cubic polynomial.

3.1.2 Remark.

1. If f, g are non-zero and $\deg(f), \deg(g) \leq 1$, then f and g interlace.

2. If f and g interlace and if $h \in \mathbb{R}[\mathrm{t}]$ has only real roots, then fh and gh interlace too.

3.1.3 Example. Let $f \in \mathbb{R}[\mathrm{t}]$ be a non-constant polynomial with only real roots. The most natural example of an interlacing polynomial is the derivative: It is an immediate consequence of the mean value theorem, that f and f' interlace.

3.1.4 Proposition (cf. [24], Section 2.3)**.** *Let $f, g \in \mathbb{R}[\mathrm{t}]$ such that f and g interlace. Then the polynomial $f'g - fg'$ is non-negative or non-positive on $\mathbb{R}$.*

Proof. Without loss of generality, we can assume $\deg(f) \leq \deg(g) = n$. For a start, suppose that f and g have no common roots. Because f and g interlace, it follows that f and g have only simple roots (every multiple root of f would be also a root of g and vice versa). Let $\alpha_1 < \ldots < \alpha_n$ be the roots of g and let $g_i = \frac{g}{\mathrm{t} - \alpha_i}$ for each $i \in [n]$. The polynomials $g, g_1, \ldots, g_n$ are linearly independent: Let $a_0, \ldots, a_n \in \mathbb{R}$ such that

$$a_0 g + a_1 g_1 + \ldots + a_n g_n = 0.$$

It follows immediately that $a_0 = 0$, because g has bigger degree than the g_i. Evaluating at α_i yields $a_i g_i(\alpha_i) = 0$, thus $a_i = 0$ for each $i \in [n]$. This implies that $g, g_1, \ldots, g_n$ are linearly independent and they therefore form a basis of the vector space

$$\{p \in \mathbb{R}[\mathrm{t}] : \deg(p) \leq n\}.$$

Thus, there are unique $c_0, \ldots, c_n \in \mathbb{R}$ such that

$$f = c_0 g + c_1 g_1 + \ldots + c_n g_n.$$

Moreover, we notice that $g' = g_1 + \ldots + g_n$. Since f has exactly one root between two consecutive roots of g, the sequence $f(\alpha_1), \ldots, f(\alpha_n)$ alternates strictly in sign. It holds $g_i(\alpha_i) = g'(\alpha_i)$ and g and g' interlace, thus, by the same argument, the sequence $g_1(\alpha_1), \ldots, g_n(\alpha_n)$ alternates strictly in sign. Since $c_i g_i(\alpha_i) = f(\alpha_i) \neq 0$ for each $i \in [n]$, we get that all of $c_1, \ldots, c_n$ have the same sign. It holds

$$\frac{f}{g} = c_0 + \frac{c_1}{\mathrm{t} - \alpha_1} + \ldots + \frac{c_n}{\mathrm{t} - \alpha_n}.$$

Looking at the derivative of this yields

$$\frac{f'g - fg'}{g^2} = \left(\frac{f}{g}\right)' = \frac{-c_1}{(\mathrm{t} - \alpha_1)^2} + \ldots + \frac{-c_n}{(\mathrm{t} - \alpha_n)^2}.$$

This implies that $f'g - fg'$ is non-negative or non-positive on $\mathbb{R}$.

Now let $f, g \in \mathbb{R}[\mathrm{t}]$ be an arbitrary pair of interlacing polynomials. Since we can approximate f, g arbitrary closely by such a pair without common zeros, this implies the claim. □

3.1.5 Definition. Let $f, g \in \mathbb{R}[\mathrm{t}]$ be two polynomials. We write $f \ll g$, if f and g interlace and if the *Wronskian*

$$\mathrm{W}(f, g) = f' \cdot g - f \cdot g'$$

is non-positive on $\mathbb{R}$. For technical reasons, we write $0 \ll f$ and $f \ll 0$, if f has only real roots.

3.1.6 Example. Let $f \in \mathbb{R}[\mathrm{t}]$ be a polynomial that has only real roots. Then f and f' interlace. We have $\mathrm{W}(f', f) = ff'' - (f')^2$. If $\alpha \in \mathbb{R}$ is a simple root of f, then $W(f', f)(\alpha) = -f'(\alpha)^2 < 0$, therefore $\mathrm{W}(f', f) \leq 0$ on $\mathbb{R}$ by Proposition 3.1.4. Therefore we have in this case $f' \ll f$. Since we can approximate polynomials that have only real roots arbitrary closely by such polynomials without multiple zeros, $f' \ll f$ also holds true if f has no simple roots.

3.1.7 Remark. Let $f, g \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be two polynomials that do not vanish identically. Then $\mathrm{W}(f, g) = 0$ implies $\left(\frac{f}{g}\right)' = 0$, thus $f = c \cdot g$ for some $c \in \mathbb{R}$. Therefore, if $\frac{f}{g} \notin \mathbb{R}$, the Wronskian $\mathrm{W}(f, g)$ does not vanish identically.

3.1.8 Lemma. *Let $g, h \in \mathbb{C}[\mathrm{t}]$. Then we have for all $x \in \mathbb{C}$*

$$\lim_{\epsilon \to 0} \frac{h(x-\epsilon)g(x+\epsilon) - h(x+\epsilon)g(x-\epsilon)}{2\epsilon} = h(x)g'(x) - g(x)h'(x).$$

Proof. For all $x \in \mathbb{C}$ holds

$$\begin{aligned}
& \lim_{\epsilon \to 0} \frac{h(x-\epsilon)g(x+\epsilon) - h(x+\epsilon)g(x-\epsilon)}{2\epsilon} \\
= \; & \frac{1}{2} \lim_{\epsilon \to 0} h(x-\epsilon) \cdot \frac{g(x+\epsilon) - g(x)}{\epsilon} + \frac{1}{2} \lim_{\epsilon \to 0} h(x-\epsilon) \cdot \frac{g(x) - g(x-\epsilon)}{\epsilon} \\
& - \frac{1}{2} \lim_{\epsilon \to 0} g(x-\epsilon) \cdot \frac{h(x) - h(x-\epsilon)}{\epsilon} - \frac{1}{2} \lim_{\epsilon \to 0} g(x-\epsilon) \cdot \frac{h(x+\epsilon) - h(x)}{\epsilon} \\
= \; & h(x)g'(x) - g(x)h'(x).
\end{aligned}$$

□

3.1.9 Lemma. *Let $f \in \mathbb{C}[\mathrm{t}]$ be a stable polynomial, that has no real zeros. Then we have $|f(\mu)| > |f(\overline{\mu})|$ for all $\mu \in \mathbb{C}$ with positive imaginary part.*

Proof. Since f is stable without real zeros, we can write $f = \gamma \cdot \prod_{i=1}^{n} (\mathrm{t} + \alpha_i)$ with $\alpha_i, \gamma \in \mathbb{C} \smallsetminus \{0\}$, where $\operatorname{Im} \alpha_i > 0$ for $i \in [n]$. Let $\mu \in \mathbb{C}$ such that $\operatorname{Im} \mu > 0$. We clearly have $|\mu + \alpha_i| > |\overline{\mu} + \alpha_i|$ for all $i \in [n]$. This implies the claim. □

3.1.10 Lemma (cf. [19], Theorem 6.3.4)**.** *Let $f = h + \mathrm{i}\, g \in \mathbb{C}[\mathrm{t}]$ with real polynomials $h, g \in \mathbb{R}[\mathrm{t}]$. Let f have no real zeros and let f be stable. Then we have $g \ll h$.*

Proof. Let $H \subseteq \mathbb{C}$ be the upper open half-plane. By Lemma 3.1.9 we obtain $|f(\mu)| > |f(\overline{\mu})|$ for all $\mu \in H$. This implies

$$\begin{aligned}
& 2\,\mathrm{i} \cdot (h(\overline{\mu}) \cdot g(\mu) - h(\mu) \cdot g(\overline{\mu})) \\
= \; & h(\mu)h(\overline{\mu}) - \mathrm{i}\, h(\mu)g(\overline{\mu}) + \mathrm{i}\, h(\overline{\mu})g(\mu) + g(\mu)g(\overline{\mu}) \\
& -(h(\mu)h(\overline{\mu}) - \mathrm{i}\, h(\overline{\mu})g(\mu) + \mathrm{i}\, h(\mu)g(\overline{\mu}) + g(\mu)g(\overline{\mu})) \\
= \; & (h(\mu) + \mathrm{i}\, g(\mu)) \cdot (h(\overline{\mu}) - \mathrm{i}\, g(\overline{\mu})) - (h(\overline{\mu}) + \mathrm{i}\, g(\overline{\mu})) \cdot (h(\mu) - \mathrm{i}\, g(\mu)) \\
= \; & f(\mu) \cdot \overline{f(\mu)} - f(\overline{\mu}) \cdot \overline{f(\overline{\mu})} \\
= \; & |f(\mu)|^2 - |f(\overline{\mu})|^2 \\
> \; & 0
\end{aligned}$$

for all $\mu \in H$. Because non-real zeros of g and h would appear in pairs of conjugates, this implies that g and h have only real zeros. Since f does not have real zeros, g and h have no common zero. The expression $4\operatorname{Im}\mu = -2\,\mathrm{i}\cdot(\mu - \overline{\mu})$ is positive for all $\mu \in H$, thus

$$\frac{h(\overline{\mu}) \cdot g(\mu) - h(\mu) \cdot g(\overline{\mu})}{\mu - \overline{\mu}} < 0$$

for all $\mu \in H$. Letting μ approach the real line, we find by Lemma 3.1.8 that

$$\mathrm{W}(g, h) = g' \cdot h - g \cdot h' \leq 0$$

on $\mathbb{R}$. Therefore $r = \frac{g}{h}$ has non-positive derivative at all real points except for its poles. This implies that between any two consecutive poles, r decreases monotonically from ∞ to $-\infty$. Thus, between two consecutive zeros of h, there is exactly one zero of g. Considering $\frac{h}{g}$ instead, we see by the same argument that g and h interlace, thus $g \ll h$. □

3.1.11 Lemma (cf. [19], Theorem 6.3.4)**.** *Let $f = h + \mathrm{i}\,g \in \mathbb{C}[\mathrm{t}]$, with real polynomials $h, g \in \mathbb{R}[\mathrm{t}]$. Suppose that g and h have no common zero and $g \ll h$. Then f is stable.*

Proof. Let $H \subseteq \mathbb{C}$ be the upper open half-plane and let $H' \subseteq \mathbb{C}$ be the lower open half-plane. First note, that f has no real zeros $x \in \mathbb{R}$, because the identity $f(x) = h(x) + \mathrm{i}\,g(x) = 0$ would imply $h(x) = g(x) = 0$, but then x would be a common zero of g and h.

First suppose that all zeros of f lie in H. Then $h - \mathrm{i}\,g$ is stable and Lemma 3.1.10 implies $\mathrm{W}(g, h) = -\mathrm{W}(-g, h) \geq 0$, but this contradicts $g \ll h$. Thus f has at least one zero $\mu \in H'$.

Now suppose that there is a $\lambda \in H$ with $f(\lambda) = 0$. Define the rational function $r = \frac{h}{g}$. We have $r(\lambda) = \frac{f(\lambda) - \mathrm{i}\,g(\lambda)}{g(\lambda)} = -\mathrm{i}$ and similar $r(\overline{\mu}) = \frac{\overline{f(\mu)} + \mathrm{i}\,g(\overline{\mu})}{g(\overline{\mu})} = \mathrm{i}$. The function

$$\varphi(t) = \operatorname{Im} r(\lambda + t(\overline{\mu} - \lambda))$$

is continuous for $t \in [0, 1]$, since $\lambda + t(\overline{\mu} - \lambda) \in H$ for $t \in [0, 1]$ and since g has only real roots. Moreover it is $\varphi(0) = -1$ and $\varphi(1) = 1$. Thus by the intermediate value theorem there is some $\xi \in H$ such that $r(\xi) \in \mathbb{R}$ and $r(\xi) \neq 0$. Now consider the real polynomial

$$d = h - \frac{h(\xi)}{g(\xi)} g = h - r(\xi) g \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]\,.$$

It is $d(\xi) = d(\overline{\xi}) = 0$. Let $\beta_1 < \ldots < \beta_m$ denote the zeros of g (note that by assumption g has only simple zeros: every multiple root of g would be a common root of g and h). Then $d(\beta_i) = h(\beta_i)$ for all $i \in [m]$. Since h and g interlace, we have

$$d(\beta_i) d(\beta_{i+1}) = h(\beta_i) h(\beta_{i+1}) < 0$$

for all $i \in [m-1]$. Thus d changes sign between two consecutive zeros of g and therefore has a real zero between to consecutive zeros of g. But taking into account the pair of non-real zeros ξ and $\overline{\xi}$, we find that d has more zeros than g. If $\deg(g) \geq \deg(h)$, this implies that d has to be the zero-polynomial. It follows $h = r(\xi) g$, therefore h and g are constant, since they have no common zero, and thus f is also constant. If $\deg(h) > \deg(g)$, we can argue analogously with the polynomial $d' = g - \frac{g(\xi)}{h(\xi)} h$. □

3.1.12 Theorem (Hermite-Biehler, cf. [19], Theorem 6.3.4)**.** *Consider the univariate polynomial $f = h + \mathrm{i}\, g \in \mathbb{C}[\mathrm{t}]$, where $h, g \in \mathbb{R}[\mathrm{t}]$. Then f is stable if and only if $g \ll h$.*

Proof. Let f be stable. Let $f = p \cdot \tilde{f}$ where $p \in \mathbb{R}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ has only real zeros and $\tilde{f} \in \mathbb{C}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ is stable and has no real zeros. Every zero of p is a zero of h and g. We conclude that p is a common factor of h and g. Let $g = p \cdot \tilde{g}$ and $h = p \cdot \tilde{h}$. Then we have $\tilde{f} = \tilde{h} + \mathrm{i}\,\tilde{g}$. By Lemma 3.1.10 we have $\tilde{g} \ll \tilde{h}$. Thus g and h interlace and

$$\begin{aligned} \mathrm{W}(g, h) &= (p\tilde{g})' \cdot (p\tilde{h}) - (p\tilde{g}) \cdot (p\tilde{h})' \\ &= (p'\tilde{g} + p\tilde{g}') \cdot (p\tilde{h}) - (p\tilde{g}) \cdot (p'\tilde{h} + p\tilde{h}') \\ &= p^2 \cdot \mathrm{W}(\tilde{g}, \tilde{h}) \\ &\leq 0. \end{aligned}$$

Conversely, let $g \ll h$. Let $p \in \mathbb{R}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ be the greatest common factor of h and g and let $g = p \cdot \tilde{g}$ and $h = p \cdot \tilde{h}$. Then $\tilde{g}$ and $\tilde{h}$ interlace and as seen above $\mathrm{W}(\tilde{g}, \tilde{h}) \leq 0$. Thus $\tilde{g} \ll \tilde{h}$ and $\tilde{h} + \mathrm{i}\,\tilde{g}$ is stable by Lemma 3.1.11. Thus $f = p \cdot (\tilde{h} + \mathrm{i}\,\tilde{g})$ is also stable. □

3.1.13 Lemma. *Let $f, g \in \mathbb{R}[\mathrm{t}]$. Let $x_0 \in \mathbb{R}$ be a zero of f with even multiplicity and let $g(x_0) \neq 0$. Then there is some $\lambda \in \mathbb{R}$, such that $f + \lambda g$ has a non-real zero.*

Proof. We can find a small $\epsilon > 0$, such that g does not have a zero in the open interval $U = (x_0 - \epsilon, x_0 + \epsilon)$ and such that f has no change of sign in U. We first consider the case where $g > 0$ and $f \geq 0$ on U. Because the zeros of f depend continuously on its coefficients, we can choose some small $\lambda > 0$, such that $f + \lambda g$ has a zero $\alpha \in \mathbb{C}$ with $|\alpha - x_0| < \epsilon$. But since $f + \lambda g$ is strictly positive on U, α cannot be real. The remaining cases are proved analogously, choosing $\lambda < 0$ if necessary. □

3.1.14 Lemma (cf. [19], Theorem 6.3.8)**.** *Let $f, g \in \mathbb{R}[\mathrm{t}]$ be polynomials that have no common zero. For all $\lambda, \mu \in \mathbb{R}$, not both zero, let $\lambda f + \mu g$ have only real roots. Then f and g interlace.*

Proof. For a start, we show, that if $\left(\frac{f}{g}\right)'$ has a change of sign in $x_0 \in \mathbb{R}$, then $g(x_0) = 0$. Suppose $g(x_0) \neq 0$ and let $y_0 = \frac{f(x_0)}{g(x_0)}$. Consider the polynomial $p = f - y_0 g$. We have $p(x_0) = 0$ and $\left(\frac{p}{g}\right)' = \left(\frac{f}{g}\right)'$ has a change of sign in $x_0 \in \mathbb{R}$. Therefore $\frac{p}{g}$ has no change of sign in x_0 and thus p does not have one either. But then x_0 is a zero of p with even multiplicity and we can apply the preceding Lemma. Thus there exists some $\lambda \in \mathbb{R}$ such that $p + \lambda g = f + (\lambda - y_0) g$ has a non-real zero, which contradicts the assumption.

Therefore, between two consecutive poles $\frac{f}{g}$ is monotonic. This implies that between two consecutive zeros of g, there is exactly one zero of f. Changing roles of f and g yields the claim. □

3.1.15 Theorem (Hermite-Kakeya-Obreschkoff, cf. [19], Theorem 6.3.8)**.** *Let $g, h \in \mathbb{R}[\mathrm{t}]$. Then are equivalent:*

(i) $h \ll g$ *or* $g \ll h$ *or* $h = g = 0$.

(ii) For all $\alpha, \beta \in \mathbb{R}$ the polynomial $\alpha g + \beta h$ is either zero or has only real roots.

Proof. First note that the equivalence is clear, if g or h is zero. Therefore let $gh \neq 0$. It suffices to show the implication $(i) \Rightarrow (ii)$ in the case $g \ll h$. The Hermite-Biehler Theorem implies that $h + \mathrm{i}\, g$ is stable. Thus, for all $\alpha, \beta \in \mathbb{R}$, the polynomial

$$(\alpha - \mathrm{i}\,\beta) \cdot (h + \mathrm{i}\, g) = (\alpha h + \beta g) + \mathrm{i}(\alpha g - \beta h)$$

is also stable. Thus $\alpha h + \beta g$ is either zero or has only real roots, again by the Hermite-Biehler Theorem.

In order to show $(ii) \Rightarrow (i)$, let $h = fp$ and $g = fq$ with $f, p, q \in \mathbb{R}[\mathrm{t}] \setminus \{0\}$, such that p and q do not have a common zero. By assumption, the polynomial $f \cdot (\alpha p + \beta q) = \alpha h + \beta g$ is either zero or has only real roots for all $\alpha, \beta \in \mathbb{R}$. By the preceding Lemma, this implies that p and q interlace. Hence g and h interlace. Proposition 3.1.4 then implies the claim. □

3.2 A Stability Criterion for Multiaffine Real Polynomials

In this section, we present a convenient stability criterion for multiaffine real polynomials, discovered by Brändén. The entire section is based on [6, Section 5].

3.2.1 Definition. A polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ is called *affine* in x_i, if the degree of h in x_i is at most 1. If h is affine in $\mathrm{x}_1, \ldots, \mathrm{x}_n$, we call h *multiaffine*.

3.2.2 Remark. Let $i \in [n]$ and let $f, g \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ such that $f \cdot g$ is affine in x_i. Then f and g are both affine in x_i and x_i does not occur in both f and g. Thus, if a multiaffine polynomial is not irreducible, then every factor depends on separate variables. Moreover, every factor is multiaffine itself.

3.2.3 Example. Let $X = (\mathrm{x}_{ij})_{ij}$ be the $n \times n$ square matrix with distinct variables x_{ij} as entries. The determinant $h = \det(X) \in \mathbb{R}[\mathrm{x}_{ij}]$ is multiaffine. We will show that h is irreducible. Let $h = f \cdot g$ where $f, g \in \mathbb{R}[\mathrm{x}_{ij}]$ with $h = f \cdot g$ and such that the degree of f in x_{11} is 1. Let $1 < j \leq n$ and suppose that g depends on x_{1j}, say $f = a\,\mathrm{x}_{11} + b$ and $g = c\,\mathrm{x}_{1j} + d$ where a, b, c, d are multiaffine polynomials that do not depend on x_{11} or x_{1j} and $ac \neq 0$. It follows

$$h = f \cdot g = ac\,\mathrm{x}_{11}\,\mathrm{x}_{1j} + ad\,\mathrm{x}_{11} + bc\,\mathrm{x}_{1j} + bd.$$

But since x_{11} and x_{1j} do not both occur in some monomial of h, this is a contradiction. Therefore g does not depend on x_{1j}, but f does. Analogously, for fixed $j \in [n]$, the variables x_{1j} and x_{ij} for $i \in [n]$, $i \neq j$, do not both occur in some monomial of h. Thus, in the same way, we can follow that every x_{ij} appears in f, thus g is a constant.

3.2.4 Definition. Let $h \in \mathbb{C}\,[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be a polynomial and let $i, j \in [n]$. The *Rayleigh difference of* (i, j) *in* h is defined as

$$\Delta_{ij} h = \frac{\partial h}{\partial\,\mathrm{x}_i} \cdot \frac{\partial h}{\partial\,\mathrm{x}_j} - h \cdot \frac{\partial^2 h}{\partial\,\mathrm{x}_i\,\partial\,\mathrm{x}_j}.$$

3.2.5 Lemma. *Let $i, j \in [n]$. Then the Rayleigh difference has the following properties:*

(i) Let $h = f \cdot g$ with $f, g \in \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$. Then we have

$$\Delta_{ij} h = f^2 \Delta_{ij} g + g^2 \Delta_{ij} f.$$

(ii) Let $h = g^N$ for $N \in \mathbb{Z}_{\geq 0}$ with $g \in \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$. Then we have

$$\Delta_{ij} h = N g^{2(N-1)} \Delta_{ij} g.$$

(iii) Let $h \in \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be affine in x_i, x_j. Then $\Delta_{ij} h$ does not depend on x_i and x_j.

Proof. Let $h = f \cdot g$ with $f, g \in \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$. It holds

$$\begin{aligned}
\Delta_{ij} h &= \frac{\partial f \cdot g}{\partial \mathrm{x}_i} \cdot \frac{\partial f \cdot g}{\partial \mathrm{x}_j} - f \cdot g \cdot \frac{\partial^2 f \cdot g}{\partial \mathrm{x}_i \, \partial \mathrm{x}_j} \\
&= (f \cdot \frac{\partial g}{\partial \mathrm{x}_i} + g \cdot \frac{\partial f}{\partial \mathrm{x}_i}) \cdot (f \cdot \frac{\partial g}{\partial \mathrm{x}_j} + g \cdot \frac{\partial f}{\partial \mathrm{x}_j}) \\
&\quad - f \cdot g \cdot (f \cdot \frac{\partial^2 g}{\partial \mathrm{x}_i \, \partial \mathrm{x}_j} + \frac{\partial f}{\partial \mathrm{x}_i} \cdot \frac{\partial g}{\partial \mathrm{x}_j} + \frac{\partial g}{\partial \mathrm{x}_i} \cdot \frac{\partial f}{\partial \mathrm{x}_j} + g \cdot \frac{\partial^2 f}{\partial \mathrm{x}_i \, \partial \mathrm{x}_j}) \\
&= f^2 \cdot (\frac{\partial g}{\partial \mathrm{x}_i} \cdot \frac{\partial g}{\partial \mathrm{x}_j} - g \cdot \frac{\partial^2 g}{\partial \mathrm{x}_i \, \partial \mathrm{x}_j}) + g^2 \cdot (\frac{\partial f}{\partial \mathrm{x}_i} \cdot \frac{\partial f}{\partial \mathrm{x}_j} - f \cdot \frac{\partial^2 f}{\partial \mathrm{x}_i \, \partial \mathrm{x}_j}) \\
&= f^2 \Delta_{ij} g + g^2 \Delta_{ij} f.
\end{aligned}$$

Thus we have (i). Now let $h = g^N$ for $N \in \mathbb{Z}_{\geq 0}$ with $g \in \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$. The case $N = 1$ of (ii) is clear. We get the general case by induction on N from (i):

$$\begin{aligned}
\Delta_{ij} h &= \Delta_{ij}(g \cdot g^{N-1}) \\
&= g^2 \Delta_{ij} g^{N-1} + g^{2(N-1)} \Delta_{ij} g \\
&= g^2 \cdot ((N-1) g^{2(N-2)} \Delta_{ij} g) + g^{2(N-1)} \Delta_{ij} g \\
&= N g^{2(N-1)} \Delta_{ij} g.
\end{aligned}$$

In order to show (iii) let $h \in \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be affine in x_i, x_j. Then we have

$$\frac{\partial(\Delta_{ij} h)}{\partial \mathrm{x}_i} = \underbrace{\frac{\partial^2 h}{\partial \mathrm{x}_i^2}}_{=0} \cdot \frac{\partial h}{\partial \mathrm{x}_j} + \frac{\partial h}{\partial \mathrm{x}_i} \cdot \frac{\partial^2 h}{\partial \mathrm{x}_i \, \partial \mathrm{x}_j} - \frac{\partial h}{\partial \mathrm{x}_i} \cdot \frac{\partial^2 h}{\partial \mathrm{x}_i \, \partial \mathrm{x}_j} - h \cdot \underbrace{\frac{\partial^3 h}{\partial \mathrm{x}_i^2 \, \partial \mathrm{x}_j}}_{=0} = 0,$$

thus $\Delta_{ij} h$ does not depend on x_i. Analogously we see that $\Delta_{ij} h$ does not depend on x_j. ☐

3.2.6 Lemma ([6], Corollary 5.5)**.** *Let $h = f + \mathrm{i}\, g \neq 0$, where $f, g \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ and let x_{n+1} be a new variable. Then the following are equivalent:*

(i) $f + \mathrm{x}_{n+1}\, g$ is stable.

(ii) $h = f + \mathrm{i}\, g$ is stable.

(iii) For all $\alpha, \beta \in \mathbb{R}$ the polynomial $\alpha g + \beta f$ is stable, and

$$\frac{\partial f}{\partial \mathrm{x}_j} g - f \frac{\partial g}{\partial \mathrm{x}_j} \geq 0$$

on $\mathbb{R}^{\mathrm{n}}$ for all $j \in [n]$.

Proof. The implication $(i) \Rightarrow (ii)$ is clear, since i lies in the upper open half-plane. In order to show $(ii) \Rightarrow (iii)$, let $v \in \mathbb{R}^{\mathrm{n}}$ and $e \in (\mathbb{R}_{>0})^n$. Further let $\alpha, \beta \in \mathbb{R}$ such that $\alpha g + \beta f \neq 0$. By the characterization of stability in Lemma 1.5.5 we have to show that

$$\alpha \cdot g(v + \mathrm{t}\, e) + \beta \cdot f(v + \mathrm{t}\, e)$$

is stable. Since

$$f(v + \mathrm{t}\, e) + \mathrm{i}\, g(v + \mathrm{t}\, e)$$

is stable, we know by the Hermite-Biehler theorem that $g(v + \mathrm{t}\, e) \ll f(v + \mathrm{t}\, e)$. By the Hermite-Kakeya Theorem we obtain, that

$$\alpha \cdot g(v + \mathrm{t}\, e) + \beta \cdot f(v + \mathrm{t}\, e)$$

has only real roots and it thus is stable. It follows analogously from the Hermite-Biehler Theorem that for all $\epsilon > 0$ and $j \in [n]$

$$\mathrm{W}(g(v + \mathrm{t}(\delta_j + \epsilon e)), f(v + \mathrm{t}(\delta_j + \epsilon e))) \leq 0.$$

Letting $\epsilon \to 0$, we obtain

$$\frac{\partial f}{\partial \mathrm{x}_j}(v) g(v) - f(v) \frac{\partial g}{\partial \mathrm{x}_j}(v) = -\mathrm{W}(g(v + \mathrm{t}\, \delta_j), f(v + \mathrm{t}\, \delta_j))|_{t=0} \geq 0.$$

It remains the implication $(iii) \Rightarrow (i)$. Let $a, b \in \mathbb{R}$ and $b > 0$. We have to show that $f + (a + \mathrm{i}\, b) g$ is stable. Let $v \in \mathbb{R}^{\mathrm{n}}$ and $e \in (\mathbb{R}_{>0})^n$ and consider the univariate polynomials $\tilde{f} = f(v + \mathrm{t}\, e)$ and $\tilde{g} = g(v + \mathrm{t}\, e)$. Like above, we have to show that

$$\tilde{f} + (a + \mathrm{i}\, b)\tilde{g} = (\tilde{f} + a\tilde{g}) + \mathrm{i}\, b\tilde{g}$$

is stable. For all $\alpha, \beta \in \mathbb{R}$ we have by assumption that if the polynomial

$$\alpha(\tilde{f} + a\tilde{g}) + \beta b\tilde{g} = \alpha\tilde{f} + (\alpha a + \beta b)\tilde{g}$$

is not zero, it is stable and has thus only real roots by Example 1.5.4. Therefore we have by the Hermite-Kakeya Theorem $\tilde{f} + a\tilde{g} \ll b\tilde{g}$ or $b\tilde{g} \ll \tilde{f} + a\tilde{g}$. It holds

$$\begin{aligned}
\mathrm{W}(b\tilde{g}, \tilde{f} + a\tilde{g}) &= b\mathrm{W}(g(v + \mathrm{t}\, e), f(v + \mathrm{t}\, e)) \\
&= -b \sum_{j=1}^{n} e_j \left(\frac{\partial f}{\partial \mathrm{x}_j} g - f \frac{\partial g}{\partial \mathrm{x}_j}\right)(v + \mathrm{t}\, e) \\
&\leq 0.
\end{aligned}$$

Thus we have $b\tilde{g} \ll \tilde{f} + a\tilde{g}$ and therefore by Hermite-Biehler we get the claim. $\square$

3.2.7 Theorem ([6], Theorem 5.6)**.** *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be multiaffine. Then are equivalent:*

(i) h is stable.

(ii) $\Delta_{ij}h$ is non-negative on $\mathbb{R}^{\mathrm{n}}$ for all $i, j \in [n]$.

Proof. In order to show the implication $(i) \Rightarrow (ii)$, let $i, j \in [n]$, $g = \frac{\partial h}{\partial \mathrm{x}_i}$ and $f = h|_{\mathrm{x}_i=0}$. We have $h = \mathrm{x}_i\, g + f$, so we can calculate the Rayleigh difference:

$$\begin{aligned}\Delta_{ij}h &= \frac{\partial(\mathrm{x}_i\, g + f)}{\partial \mathrm{x}_i} \cdot \frac{\partial(\mathrm{x}_i\, g + f)}{\partial \mathrm{x}_j} - (\mathrm{x}_i\, g + f) \cdot \frac{\partial^2(\mathrm{x}_i\, g + f)}{\partial \mathrm{x}_i\, \partial \mathrm{x}_j} \\ &= g \cdot (\mathrm{x}_i \frac{\partial g}{\partial \mathrm{x}_j} + \frac{\partial f}{\partial \mathrm{x}_j}) - (\mathrm{x}_i\, g + f) \cdot \frac{\partial g}{\partial \mathrm{x}_j} \\ &= g \cdot \frac{\partial f}{\partial \mathrm{x}_j} - f \cdot \frac{\partial g}{\partial \mathrm{x}_j}.\end{aligned}$$

Now Lemma 3.2.6 $(i) \Rightarrow (iii)$ implies $\Delta_{ij}h \geq 0$.

We will proof the direction $(ii) \Rightarrow (i)$ by induction on n. The case $n = 0$ is clear. Let $h = \mathrm{x}_{n+1}\, g + f$ with $f, g \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ such that $\Delta_{ij}h$ is non-negative for all $i, j \in [n+1]$. We want to show that f and g satisfy condition (iii) in Lemma 3.2.6. Let $v \in \mathbb{R}^{\mathrm{n}}$ and $\lambda \in \mathbb{R}$. Then we have for all $i, j \in [n]$:

$$(\Delta_{ij}(h|_{\mathrm{x}_{n+1}=\lambda}))(v) = (\Delta_{ij}h)(v, \lambda) \geq 0.$$

Thus, by our induction hypothesis, $f + \lambda g$ is stable. It follows, that $\alpha f + \beta g$ is stable for all $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$. But since the roots of every polynomial depend continuously on the coefficients of that polynomial, we get that g is stable too. Furthermore we have for all $j \in [n]$:

$$\frac{\partial f}{\partial \mathrm{x}_j} g - f \frac{\partial g}{\partial \mathrm{x}_j} = \Delta_{j,n+1}h \geq 0.$$

Thus, by Lemma 3.2.6, the Theorem is proved. □

3.2.8 Remark. The original proof, that the bases generated polynomial of the Vámos cube is stable, cf. Lemma 2.3.14, was carried out by Wagner and Wei [23] using an improved version of Theorem 3.2.7.

3.2.9 Example. As we will see in Proposition 3.2.14, the direction $(i) \Rightarrow (ii)$ still holds true, if we drop the condition that h is multiaffine. But $(ii) \Rightarrow (i)$ is not true without this requirement: Let $h = \mathrm{x}_1^2 + \mathrm{x}_2^2$. We have for instance $h(1 + \mathrm{i}, -1 + \mathrm{i}) = 0$, thus h is not stable. Now let $q = \mathrm{x}_1 + \mathrm{x}_2$ and $N \in \mathbb{Z}_{\geq 0}$. Clearly $q^N h$ is not stable as well, but for all $i, j \in [2]$ we have

$$\begin{aligned}\Delta_{ij}(q^N h) &= q^{2N}\Delta_{ij}h + Nq^{2N-2}h^2\Delta_{ij}q \\ &= q^{2N-2}(q^2\Delta_{ij}h + Nh^2).\end{aligned}$$

Now let $z \in \mathbb{R}$ be the minimal value that $q^2\Delta_{ij}h$ takes on the unit sphere and let $N > |z|$. Then, since $\Delta_{ij}(q^N h)$ is homogeneous, we get that $\Delta_{ij}(q^N h)$ is non-negative on $\mathbb{R}^2$. Because $\Delta_{ij}(q^N h)$ is a homogeneous polynomial in two variables, it is even a sum of squares.

3.2.10 Example. Consider the multiaffine polynomial

$$h = a_{12}\,\mathrm{x}_1\,\mathrm{x}_2 + a_{13}\,\mathrm{x}_1\,\mathrm{x}_3 + a_{14}\,\mathrm{x}_1\,\mathrm{x}_4 + a_{23}\,\mathrm{x}_2\,\mathrm{x}_3 + a_{24}\,\mathrm{x}_2\,\mathrm{x}_4 + a_{34}\,\mathrm{x}_3\,\mathrm{x}_4$$

where $a_{ij} \in \mathbb{R}$. A short calculation verifies that

$$\Delta_{12}h = a_{1,4}a_{2,4}\,\mathrm{x}_4^2 - (a_{1,2}a_{3,4} - a_{1,3}a_{2,4} - a_{1,4}a_{2,3})\,\mathrm{x}_3\,\mathrm{x}_4 + a_{1,3}a_{2,3}\,\mathrm{x}_3^2\,.$$

By permuting indices, Theorem 3.2.7 implies that h is stable if and only if $a_{kl}a_{rs} \geq 0$ for all $k, l, r, s \in [4]$ and $D \geq 0$, where D is the common discriminant of all the $\Delta_{ij}h$:

$$\begin{aligned} D &= -2a_{12}a_{34}a_{13}a_{24} - 2a_{12}a_{34}a_{14}a_{23} - 2a_{13}a_{24}a_{14}a_{23} \\ &\quad + a_{12}^2a_{34}^2 + a_{14}^2a_{23}^2 + a_{13}^2a_{24}^2. \end{aligned}$$

The next Theorem, better known as the Theorem of Grace-Walsh-Szegö, will enable us to apply our results concerning multiaffine polynomials to arbitrary polynomials. We will not give a proof here.

3.2.11 Theorem ([7], Theorem 2.12)**.** *Let $f \in \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x_n}]$ be a symmetric multiaffine polynomial and let $H \subseteq \mathbb{C}$ be an open half-plane. Let $\xi_1, \ldots, \xi_n \in H$, then there is a point $\xi \in H$, such that $f(\xi_1, \ldots, \xi_n) = f(\xi, \ldots, \xi)$.*

3.2.12 Corollary. *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x_n}]$ be stable and let $d = \deg(h)$. Then there exists a stable multiaffine polynomial $h_0 \in \mathbb{R}[\mathrm{x}_{ij} : i \in [n], j \in [d]]$, such that*

$$h = h_0(\underbrace{\mathrm{x}_1, \ldots, \mathrm{x}_1}_{d\ times}, \ldots, \underbrace{\mathrm{x}_n, \ldots, \mathrm{x}_n}_{d\ times}).$$

Proof. Let $1 \leq k \leq d$. Then let $p_k = \binom{d}{k}^{-1} e_k$, where $e_k = \sum_{S \in \binom{[d]}{k}} \prod_{i \in S} \mathrm{x}_i$ is the elementary symmetric polynomial in d variables of degree k. Let $i \in [n]$. Clearly, $p_k(x_{i1}, \ldots, x_{id})$ is symmetric, multiaffine and $p_k(\mathrm{x}_i, \ldots, \mathrm{x}_i) = \mathrm{x}_i^k$. Now let

$$h = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_\alpha\, \mathrm{x}_1^{\alpha_1} \cdots \mathrm{x}_n^{\alpha_n},$$

for appropriate $c_\alpha \in \mathbb{R}$. Then the polynomial

$$h_0 = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_\alpha p_{\alpha_1}(\mathrm{x}_{11}, \ldots, \mathrm{x}_{1d}) \cdots p_{\alpha_n}(\mathrm{x}_{n1}, \ldots, \mathrm{x}_{nd})$$

is multiaffine and satisfies

$$h = h_0(\underbrace{\mathrm{x}_1, \ldots, \mathrm{x}_1}_{d \text{ times}}, \ldots, \underbrace{\mathrm{x}_n, \ldots, \mathrm{x}_n}_{d \text{ times}}).$$

Thus, it remains to show that h_0 is stable. Let H be the upper open half-plane and let $\xi_{ij} \in H$ for all $i \in [n]$ and $j \in [d]$. Suppose that $h_0(\xi_{11}, \ldots, \xi_{nd}) = 0$. The polynomial $h_0(\mathrm{x}_{11}, \ldots, \mathrm{x}_{1d}, \xi_{21}, \ldots, \xi_{nd})$ is symmetric and multiaffine, thus by the Grace-Walsh-Szegö Theorem, there exists a $\xi_1 \in H$ such that

$$h_0(\underbrace{\xi_1, \ldots, \xi_1}_{d \text{ times}}, \xi_{21} \ldots, \xi_{nd}) = 0.$$

Iterating this for the other variables shows, that there are also $\xi_2, \ldots, \xi_n \in H$ such that

$$h(\xi_1, \ldots, \xi_n) = h_0(\underbrace{\xi_1, \ldots, \xi_1}_{d \text{ times}}, \ldots, \underbrace{\xi_n, \ldots, \xi_n}_{d \text{ times}}) = 0.$$

This is a contradiction to the assumption that h is stable. $\square$

3.2.13 Remark. Let $h \in \mathbb{R}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ and h_0 as in the preceding Corollary. If h_0 is stable, it is clear that h is also stable. Thus, if we want to check, whether h is stable, we just have to apply Theorem 3.2.7 to h_0.

3.2.14 Proposition (cf. [6], Theorem 5.6)**.** *Let $h \in \mathbb{R}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ be stable of degree d. Then $\Delta_{ij}h$ is non-negative on $\mathbb{R}^{\mathrm{n}}$ for all $i, j \in [n]$.*

Proof. Let $x = (\mathrm{x}_1, \dots, \mathrm{x}_1, \dots, \mathrm{x}_n, \dots, \mathrm{x}_n)$ and let $h_0 \in \mathbb{R}[\mathrm{x}_{ij}]$ as in the preceding Corollary. Then we have by the chain rule

$$\begin{aligned}
\Delta_{ij}h &= \frac{\partial h}{\partial \mathrm{x}_i} \cdot \frac{\partial h}{\partial \mathrm{x}_j} - h \cdot \frac{\partial^2 h}{\partial \mathrm{x}_i \, \partial \mathrm{x}_j} \\
&= \left(\sum_{k=1}^{d} \frac{\partial h_0}{\partial \mathrm{x}_{ik}}(x) \right) \left(\sum_{l=1}^{d} \frac{\partial h_0}{\partial \mathrm{x}_{jl}}(x) \right) - h_0(x) \left(\sum_{l=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 h_0}{\partial \mathrm{x}_{ik} \, \partial \mathrm{x}_{jl}}(x) \right) \\
&= \sum_{l=1}^{d} \sum_{k=1}^{d} \left(\frac{\partial h_0}{\partial \mathrm{x}_{ik}}(x) \cdot \frac{\partial h_0}{\partial \mathrm{x}_{jl}}(x) - h_0(x) \cdot \frac{\partial^2 h_0}{\partial \mathrm{x}_{ik} \, \partial \mathrm{x}_{jl}}(x) \right).
\end{aligned}$$

Every summand is non-negative by Theorem 3.2.7. □

3.2.15 Remark. Another consequence of Corollary 3.2.12 is the fact, that it would suffice to proof the Generalized Lax Conjecture 1.4.4 in the case of homogeneous, irreducible, multiaffine and stable polynomials: Suppose that the closure of the hyperbolicity cone of every homogeneous, irreducible, multiaffine and stable polynomial is a spectrahedral cone and let $h \in \mathbb{R}[\mathrm{x_1}, \dots, \mathrm{x_n}]$ be a hyperbolic polynomial. By Remark 1.4.7, we can assume that h is irreducible. Since the property of being a spectrahedral cone is preserved under a linear change of variables, we can assume that h is stable. Let $\deg(h) = d$ and let $h_0 \in \mathbb{R}[\mathrm{x}_{ij} : i \in [n], j \in [d]]$ as in Corollary 3.2.12. Clearly h_0 is also irreducible. The closure of the hyperbolicity cone $\overline{\mathrm{C}_{h_0}}$ of h_0 is a spectrahedral cone, thus by Lemma 1.4.6, there is a homogeneous polynomial $q_0 \in \mathbb{R}[\mathrm{x}_{ij} : i \in [n], j \in [d]]$, which is stable and such that the hyperbolicity cone C_{q_0} of q_0 contains C_{h_0} and such that $q_0 h_0$ has a definite determinantal representation, i.e.

$$q_0 h_0 = \det\Big(\sum_{i \in [n], j \in [d]} \mathrm{x}_{ij} \, A_{ij} \Big),$$

where the A_{ij} are symmetric, positive semi-definite $d \times d$ matrices. Let

$$q = q_0(\underbrace{\mathrm{x}_1, \dots, \mathrm{x}_1}_{d \text{ times}}, \dots, \underbrace{\mathrm{x}_n, \dots, \mathrm{x}_n}_{d \text{ times}}) \in \mathbb{R}[\mathrm{x_1}, \dots, \mathrm{x_n}]$$

and $A_i = A_{i1} + \ldots + A_{id}$ for $i \in [n]$. The matrices $A_1, \dots, A_n$ are positive semi-definite. Clearly, q is stable and it holds

$$qh = \det(\mathrm{x}_1 \, A_1 + \ldots + \mathrm{x}_n \, A_n).$$

Thus, by Lemma 1.4.6, it remains to show that the hyperbolicity C_q of q contains the hyperbolicity cone C_h of h. In order to show this, let $e = (1, \dots, 1) \in \mathbb{R}^{\mathrm{n}}$ and let $v \in \mathrm{C}_h$, i.e. the polynomial $h(\mathrm{t}\, e + v)$ has only negative roots. Let $e_0 = (1, \dots, 1) \in \mathbb{R}^{nd}$ and let $v_0 = (\underbrace{v_1, \dots, v_1}_{d \text{ times}}, \dots, \underbrace{v_n, \dots, v_n}_{d \text{ times}}) \in \mathbb{R}^{nd}$. Then $h_0(\mathrm{t}\, e_0 + v_0) = h(\mathrm{t}\, e + v)$ has only negative roots, thus $v_0 \in \mathrm{C}_{h_0} \subseteq \mathrm{C}_{q_0}$. It follows that $q(\mathrm{t}\, e + v) = q_0(\mathrm{t}\, e_0 + v_0)$ has only negative zeros, and therefore $v \in \mathrm{C}_q$. This proofs that $\overline{\mathrm{C}_h}$ is a spectrahedral cone.

3.3 Multiaffine Polynomials with a Definite Determinantal Representation and the Grassmannian

At this time we want to remind of some basic facts of the Grassmannian $\mathbb{G}(d,n)$, see for example [11, Lecture 6] for proofs or further information. The Grassmannian $\mathbb{G} = \mathbb{G}(d,n)$ is the set of all $d-$dimensional subspaces of $\mathbb{C}^n$. Let $U \in \mathbb{G}$ be spanned by the vectors $u_1, \ldots, u_d$ and let

$$A = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_d \\ | & & | \end{pmatrix}$$

be the matrix, that has $u_1, \ldots, u_d$ as columns. Then we refer to the $d \times d$ minors of A as *Plücker coordinates* of U. Up to a scalar, the Plücker coordinates do not depend on the choice of the vectors $u_1, \ldots, u_d$. We thus have a well-defined map of sets $\mathbb{G} \to \mathbb{P}^N$ where $N = \binom{n}{d} - 1$, which, in fact, is an inclusion. The image of this map is closed with respect to the Zariski topology. Thus we can consider $\mathbb{G}$ as a projective variety. The variety $\mathbb{G}$ is irreducible, smooth and has dimension $d \cdot (n-d)$. The real points $\mathbb{G}(\mathbb{R})$ correspond to the $d-$dimensional subspaces of $\mathbb{C}^n$, that can be spanned by vectors with only real coordinates. By Proposition 1.2.1 $\mathbb{G}(\mathbb{R})$ lies Zariski-dense in $\mathbb{G}$. Further, $\mathbb{G}(\mathbb{R})$ is a connected set.

Consider the set of all homogeneous, multiaffine and stable polynomials in n variables of degree d, that have a definite determinantal representation. In the following we denote this set as a subset $\mathfrak{D}_{aff}(d,n) \subseteq \mathbb{P}^N$ of the projective space $\mathbb{P}^N$ where $N = \binom{n}{d} - 1$. The coordinates of some $h \in \mathfrak{D}_{aff}(d,n)$ under this identification are the coefficients of h. Note that this is possible, since for all $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$, that have these properties, λh with $\lambda \in \mathbb{R} \setminus \{0\}$ has by definition these properties too.

Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ have a definite determinantal representation. There are positive semi-definite matrices $A_1, \ldots, A_n$ of size $d \times d$ such that

$$h = \lambda \det(\mathrm{x}_1 A_1 + \ldots + \mathrm{x}_n A_n)$$

for some $\lambda \in \mathbb{R}$. Since h is multiaffine, we know by Theorem 2.3.8 that the A_i have rank at most one. There thus are vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ such that $A_i = v_i \cdot v_i^{\mathrm{T}}$. Therefore we have

$$\mathrm{x}_1 A_1 + \ldots + \mathrm{x}_n A_n = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} \mathrm{x}_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{x}_n \end{pmatrix} \begin{pmatrix} - & v_1^{\mathrm{T}} & - \\ & \vdots & \\ - & v_n^{\mathrm{T}} & - \end{pmatrix}.$$

Letting $V = \begin{pmatrix} - & v_1^{\mathrm{T}} & - \\ & \vdots & \\ - & v_n^{\mathrm{T}} & - \end{pmatrix}$ and $X = \begin{pmatrix} \mathrm{x}_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{x}_n \end{pmatrix}$, we obtain by the Cauchy-Binet Theorem:

$$h = \lambda \det(V^{\mathrm{T}} X V) = \lambda \sum_{S \in \binom{[n]}{d}} V(S)^2 \prod_{i \in S} \mathrm{x}_i \,.$$

The coefficients of h thus are exactly the squared Plücker coordinates of the vector space spanned by the columns of V. Therefore the following Proposition holds. Note that a remark in [7, Section 13.5] indicates, that the authors were aware of this connection between polynomials with a definite determinantal representation and the Grassmannian, but they did not amplify this.

3.3.1 Proposition. *Let $N = \binom{n}{d} - 1$ and let $\mathbb{G} = \mathbb{G}(d, n) \subseteq \mathbb{P}^N$ be the Grassmannian of all $d-$dimensional subspaces of $\mathbb{C}^n$. Further, let*

$$\Phi : \mathbb{P}^N \to \mathbb{P}^N, \ \ (x_0 : \ldots : x_N) \mapsto (x_0^2 : \ldots : x_N^2).$$

Let $\mathbb{G}^2 = \Phi(\mathbb{G})$, then it holds

$$\begin{aligned} \mathfrak{D}_{aff}(d, n) = & \ \Phi(\mathbb{G}(\mathbb{R})) \\ = & \ \{(y_0 : \ldots : y_N) \in \mathbb{G}^2(\mathbb{R}) : \ y_0, \ldots, y_N \in \mathbb{R} \ \textit{have the same sign}\}. \end{aligned}$$

Proof. By the consideration above, we have $\mathfrak{D}_{aff}(d, n) = \Phi(\mathbb{G}(\mathbb{R})) \subseteq \mathbb{G}^2(\mathbb{R})$. For every $y \in \Phi(\mathbb{G}(\mathbb{R}))$ it is clear that the y_i have the same sign. Conversely let $y \in \mathbb{G}^2(\mathbb{R})$ and $y_i \geq 0$ $(i = 0, \ldots, N)$, then there is a $(x_0 : \ldots : x_N) \in \mathbb{G}$ such that $x_i^2 = \lambda y_i$ $(\lambda \in \mathbb{C}^*)$. Let $\mu \in \mathbb{C}^*$ with $\mu^2 = \lambda$, then we have $(\mu^{-1} x_i)^2 = y_i \geq 0$, thus $\mu^{-1} x_i \in \mathbb{R}$. Therefore $y \in \Phi(\mathbb{G}(\mathbb{R}))$. □

3.3.2 Corollary. *The set $\mathfrak{D}_{aff}(d, n)$ is semi-algebraic, closed, connected and has dimension $d \cdot (n - d)$.*

Proof. Clearly $\mathfrak{D}_{aff}(d, n)$ is semi-algebraic and closed. Because $\mathbb{G}(\mathbb{R})$ is Zariski-dense in $\mathbb{G}$, the set $\mathfrak{D}_{aff}(d, n) = \Phi(\mathbb{G}(\mathbb{R}))$ lies Zariski-dense in $\mathbb{G}^2 = \Phi(\mathbb{G})$ and since Φ is a finite morphism, we have

$$\dim \mathfrak{D}_{aff}(d, n) = \dim \mathbb{G}^2 = \dim \mathbb{G} = d \cdot (n - d).$$

Finally, $\mathfrak{D}_{aff}(d, n)$ is connected, because $\mathbb{G}(\mathbb{R})$ is connected. □

3.3.3 Remark. Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be multiaffine and homogeneous of degree d. If h is assumed to be stable, all coefficients of h have the same sign by Corollary 1.5.10. Therefore, the condition, that h has a definite determinantal representation, is by Proposition 3.3.1 only an algebraic condition, i.e. there are polynomials $f_1, \ldots, f_r$, such that h has a definite determinantal representation if and only if all f_i vanish in the coefficients of h.

3.3.4 Example. Let $d = 1$ or $d = n - 1$. Then $\mathbb{G}(d, n) = \mathbb{P}^n$. Therefore, every multiaffine homogeneous polynomial of degree d in n variables with non-negative or non-positive coefficients has a definite determinantal representation. Thus, by Corollary 1.5.10 this is exactly the set of homogeneous, multiaffine, stable polynomials of degree d in n variables.

3.3.5 Example. The first non-trivial example is $d = 2$ and $n = 4$. It is

$$\mathbb{G}(2, 4) = \mathcal{V}_+(\mathrm{x}_{12}\,\mathrm{x}_{34} - \mathrm{x}_{13}\,\mathrm{x}_{24} + \mathrm{x}_{14}\,\mathrm{x}_{23}) \subseteq \mathbb{P}^5.$$

Letting $\mathrm{y}_{ij} = \mathrm{x}_{ij}^2$,

$$\mathrm{x}_{12}\,\mathrm{x}_{34} - \mathrm{x}_{13}\,\mathrm{x}_{24} + \mathrm{x}_{14}\,\mathrm{x}_{23} = 0$$

implies the equation

$$\mathrm{y}_{12}^2\,\mathrm{y}_{34}^2+\mathrm{y}_{14}^2\,\mathrm{y}_{23}^2+\mathrm{y}_{13}^2\,\mathrm{y}_{24}^2=2(\mathrm{y}_{12}\,\mathrm{y}_{34}\,\mathrm{y}_{13}\,\mathrm{y}_{24}+\mathrm{y}_{12}\,\mathrm{y}_{34}\,\mathrm{y}_{14}\,\mathrm{y}_{23}+\mathrm{y}_{13}\,\mathrm{y}_{24}\,\mathrm{y}_{14}\,\mathrm{y}_{23}).$$

It is not hard to see that the polynomial

$$\begin{aligned} f &= -2\,\mathrm{y}_{12}\,\mathrm{y}_{34}\,\mathrm{y}_{13}\,\mathrm{y}_{24}-2\,\mathrm{y}_{12}\,\mathrm{y}_{34}\,\mathrm{y}_{14}\,\mathrm{y}_{23}-2\,\mathrm{y}_{13}\,\mathrm{y}_{24}\,\mathrm{y}_{14}\,\mathrm{y}_{23}\\ &\quad +\mathrm{y}_{12}^2\,\mathrm{y}_{34}^2+\mathrm{y}_{14}^2\,\mathrm{y}_{23}^2+\mathrm{y}_{13}^2\,\mathrm{y}_{24}^2 \end{aligned}$$

is irreducible, thus $\mathbb{G}^2=\mathcal{V}_+(f)$. A polynomial

$$h=a_{12}\,\mathrm{x}_1\,\mathrm{x}_2+a_{13}\,\mathrm{x}_1\,\mathrm{x}_3+a_{14}\,\mathrm{x}_1\,\mathrm{x}_4+a_{23}\,\mathrm{x}_2\,\mathrm{x}_3+a_{24}\,\mathrm{x}_2\,\mathrm{x}_4+a_{34}\,\mathrm{x}_3\,\mathrm{x}_4$$

has therefore a definite determinantal representation if and only if $a_{ij}a_{kl}\geq 0$ and $f(a_{12},a_{13},a_{14},a_{23},a_{24},a_{34})=0$. As we have seen in Example 3.2.10, h is stable if and only if $a_{ij}a_{kl}\geq 0$ and $f(a_{12},a_{13},a_{14},a_{23},a_{24},a_{34})\geq 0$.

In [7] the authors introduced a notion of duality for multiaffine, stable polynomials. We will extend this on multiaffine polynomials with a definite determinantal representation.

3.3.6 Definition. Given a multiaffine polynomial $h\in\mathbb{R}[\mathrm{x}_1,\ldots,\mathrm{x}_\mathrm{n}]$ we define the *dual* polynomial h^* by

$$h^*(\mathrm{x}_1,\ldots,\mathrm{x}_n)=\mathrm{x}_1\cdots\mathrm{x}_n\cdot h(\mathrm{x}_1^{-1},\ldots,\mathrm{x}_n^{-1}).$$

3.3.7 Remark.

1. For every multiaffine h we have $h^{**}=h$.
2. Consider the multiaffine homogeneous polynomial

$$h=\sum_{S\in\binom{[n]}{d}} a_S\prod_{i\in S}\mathrm{x}_i,$$

   where $a_S\in\mathbb{R}$. Then its dual polynomial has the form

$$h^*=\sum_{S\in\binom{[n]}{d}} a_S\prod_{i\in([n]\setminus S)}\mathrm{x}_i\,.$$

At this place, we want to remind of the coordinate-free description of the Grassmannian. Let $V=\mathbb{C}^n$ and let $\bigwedge^d(V)$ be the dth exterior power of V. Then

$$\mathbb{G}(d,n)=\{[v_1\wedge\cdots\wedge v_d]:\ v_1,\ldots,v_d\in V\text{ linearly independent}\}\subseteq\mathbb{P}(\bigwedge^d(V))$$

is the set of equivalence classes of *totally decomposable* vectors. Let

$$I=\{i\in\mathbb{Z}^d:\ 1\leq i_1<\ldots<i_d\leq n\}$$

and for $i\in I$ let $i'=(i'_1,\ldots,i'_{n-d})$, such that

$$\{i_1,\ldots,i_d\}\cup\{i'_1,\ldots,i'_{n-d}\}=[n]$$

and $i'_1 < \ldots < i'_{n-d}$. The Plücker coordinates of $v_1 \wedge \cdots \wedge v_d$ are the coordinates with respect to the basis $\{\delta_{i_1} \wedge \cdots \wedge \delta_{i_d} \ : \ i \in I\}$ of $\bigwedge^d(V)$. There is an isomorphism

$$\varphi : \bigwedge^d(V) \to \bigwedge^{n-d}(V^*),$$

that maps totally decomposable vectors to totally decomposable vectors. If

$$\omega = \sum_{i \in I} a_i \cdot \delta_{i_1} \wedge \cdots \wedge \delta_{i_d} \in \bigwedge^d(V),$$

there are $\epsilon_i \in \{1, -1\}$, such that

$$\varphi(\omega) = \sum_{i \in I} \epsilon_i \cdot a_i \cdot \delta^*_{i'_1} \wedge \cdots \wedge \delta^*_{i'_{n-d}},$$

where $\delta^*_1, \ldots, \delta^*_n \in V^*$ is the dual basis of the standard basis $\delta_1, \ldots, \delta_n$. We get the following Proposition about determinantal representation of dual polynomials. The first part of this Proposition concerning stable polynomials is [7, Proposition 4.2]. The second part is our own result.

3.3.8 Proposition. *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be a homogeneous, stable and multiaffine polynomial. Then h^* is stable too. If h has a definite determinantal representation, then h^* also has one.*

Proof. If $\mu \in \mathbb{C}^n$ with $\operatorname{Re} \mu_i > 0$ for all $i = 1, \ldots n$, then $\operatorname{Re} \mu_i^{-1} > 0$. Thus

$$h^*(\mu) = \mu_1 \cdots \mu_n \cdot h(\mu_1^{-1}, \ldots, \mu_n^{-1}) \neq 0.$$

Now let h have a definite determinantal representation and let $\deg h = d$. Then there is a totally decomposable vector

$$\omega = \sum_{i \in I} a_i \cdot \delta_{i_1} \wedge \cdots \wedge \delta_{i_d},$$

with $a_i \in \mathbb{R}$, such that

$$h = \lambda \sum_{i \in I} a_i^2 \cdot \mathrm{x}_{i_1} \cdots \mathrm{x}_{i_d},$$

for appropriate $\lambda \in \mathbb{R}$. But then $\varphi(\omega)$ is totally decomposable too and there are $\epsilon_i \in \{1, -1\}$, such that

$$\varphi(\omega) = \sum_{i \in I} \epsilon_i \cdot a_i \cdot \delta^*_{i'_1} \wedge \cdots \wedge \delta^*_{i'_{n-d}}.$$

Therefore

$$h^* = \lambda \sum_{i \in I} (\epsilon_i \cdot a_i)^2 \cdot \mathrm{x}_{i'_1} \cdots \mathrm{x}_{i'_{n-d}}$$

has a definite determinantal representation. □

3.4 A Characterization of Multiaffine Stable Polynomials with a Determinantal Representation

Similarly to Theorem 3.2.7, the Rayleigh differences can be used to determine whether or not a homogeneous multiaffine stable polynomial has a definite determinantal representation (Theorem 3.4.7). We have published this result in a common paper with Daniel Plaumann and Cynthia Vinzant [15]. This main result of this section needs some preparation.

3.4.1 Lemma. *Let $v, w \in \mathbb{R}^d$. Consider the $d \times d$ matrix*

$$M = \mathrm{t}(vv^{\mathrm{T}}) + M_0,$$

where M_0 is a $d \times d$ matrix that does not depend on t. *For the partial derivative of the polynomial $h = \det(M)$ holds $\frac{\partial h}{\partial \mathrm{t}} = \det \begin{pmatrix} M & v \\ v^{\mathrm{T}} & 0 \end{pmatrix}$.*

Proof. By Corollary 1.1.3 we obtain by applying the chain rule

$$\begin{aligned} \frac{\partial h}{\partial \mathrm{t}} &= \sum_{i,j=1}^{d} (-1)^{i+j} v_i v_j M_{ij} \\ &= \sum_{i=1}^{d} (-1)^{i+d+1} v_i \sum_{j=1}^{d} (-1)^{j+d+1} v_j M_{ij} \\ &= \det \begin{pmatrix} M & v \\ v^{\mathrm{T}} & 0 \end{pmatrix}. \end{aligned}$$

□

3.4.2 Lemma ([18], Proposition 3.2)**.** *Let M be a $d \times d$ square matrix whose entries are polynomials in $\mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$. Let $v, w \in \mathbb{R}^d$. Then the polynomial*

$$\det \begin{pmatrix} M & v \\ v^{\mathrm{T}} & 0 \end{pmatrix} \det \begin{pmatrix} M & w \\ w^{\mathrm{T}} & 0 \end{pmatrix} - \det \begin{pmatrix} M & v \\ w^{\mathrm{T}} & 0 \end{pmatrix} \det \begin{pmatrix} M & w \\ v^{\mathrm{T}} & 0 \end{pmatrix}$$

is divisible by $\det(M)$.

Proof. Let $X = (\mathrm{x}_{ij})_{ij}$ be the $d \times d$ square matrix with distinct variables x_{ij} as entries and let $f = \det(X)$. Consider the variety $V = \mathcal{V}_{\mathbb{C}}(f)$. By Lemma 1.1.5 the matrix $\mathrm{adj}(X)$ has rank at most one on V. Therefore the determinant of the 2×2 matrix

$$\begin{pmatrix} v & w \end{pmatrix}^{\mathrm{T}} \mathrm{adj}(X) \begin{pmatrix} v & w \end{pmatrix}$$

vanishes on V. Since f is irreducible, see Example 3.2.3, f divides the polynomial

$$\begin{aligned} & \det \left(\begin{pmatrix} v & w \end{pmatrix}^{\mathrm{T}} \mathrm{adj}(X) \begin{pmatrix} v & w \end{pmatrix} \right) \\ = \; & \det \begin{pmatrix} v^{\mathrm{T}} \mathrm{adj}(X) v & v^{\mathrm{T}} \mathrm{adj}(X) w \\ w^{\mathrm{T}} \mathrm{adj}(X) v & w^{\mathrm{T}} \mathrm{adj}(X) w \end{pmatrix} \\ = \; & \left(v^{\mathrm{T}} \mathrm{adj}(X) v\right) \left(w^{\mathrm{T}} \mathrm{adj}(X) w\right) - \left(w^{\mathrm{T}} \mathrm{adj}(X) v\right) \left(v^{\mathrm{T}} \mathrm{adj}(X) w\right) \\ = \; & \det \begin{pmatrix} X & v \\ v^{\mathrm{T}} & 0 \end{pmatrix} \det \begin{pmatrix} X & w \\ w^{\mathrm{T}} & 0 \end{pmatrix} - \det \begin{pmatrix} X & v \\ w^{\mathrm{T}} & 0 \end{pmatrix} \det \begin{pmatrix} X & w \\ v^{\mathrm{T}} & 0 \end{pmatrix}. \end{aligned}$$

Now replacing the variables x_{ij} by the entries m_{ij} of M, we get the claim. □

3.4.3 Proposition. *Let $i, j \in [n]$. Let $f \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be affine in x_i and x_j. If f is stable with a definite determinantal representation, then $\frac{\partial f}{\partial \mathrm{x}_i} \cdot \frac{\partial f}{\partial \mathrm{x}_j}$ is a square modulo f.*

Proof. By Lemma 2.3.6 there are positive semi-definite matrices $A_1, \ldots, A_n$ and $\lambda \in \mathbb{R}$ such that

$$f = \lambda \det(M), \quad M = \mathrm{x}_1 A_1 + \ldots + \mathrm{x}_n A_n.$$

By Theorem 2.3.8, there are $v, w \in \mathbb{R}^d$ such that $A_i = vv^{\mathrm{T}}$ and $A_j = ww^{\mathrm{T}}$. Then the claim is a direct consequence of the preceding Lemmata:

$$\begin{aligned} \frac{\partial f}{\partial \mathrm{x}_i} \cdot \frac{\partial f}{\partial \mathrm{x}_j} &= \lambda^2 \det \begin{pmatrix} M & v \\ v^{\mathrm{T}} & 0 \end{pmatrix} \det \begin{pmatrix} M & w \\ w^{\mathrm{T}} & 0 \end{pmatrix} && \mod (f) \\ &= \lambda^2 \det \begin{pmatrix} M & w \\ v^{\mathrm{T}} & 0 \end{pmatrix}^2 && \mod (f). \end{aligned}$$

For the second equality, have in mind that M is a symmetric matrix. □

The next Lemma is a well-known fact from linear algebra, see for example [8, Aufgaben zu 3.3].

3.4.4 Lemma. *Let M be a $m \times m$ matrix with entries in some arbitrary field and let $\det(M) \neq 0$. Then holds:*

(i) $\det(\operatorname{adj}(M)) = \det(M)^{m-1}$.

(ii) $\operatorname{adj}(\operatorname{adj}(M)) = \det(M)^{m-2} M$.

Proof. We have by Proposition 1.1.5:

$$\begin{aligned} & \operatorname{adj}(M)M &=& \det(M) I_m \\ \Rightarrow & \operatorname{adj}(M) &=& \det(M) M^{-1} \\ \Rightarrow & \det(\operatorname{adj}(M)) &=& \det(M)^{m-1}. \end{aligned}$$

If we replace in M by M^{-1} in the second line above, we obtain:

$$\operatorname{adj}(M^{-1}) = \det(M^{-1}) M.$$

On the other side, we have

$$\begin{aligned} & M^{-1} &=& \operatorname{adj}(M) \det(M^{-1}) \\ \Rightarrow & \operatorname{adj}(M^{-1}) &=& \operatorname{adj}(\operatorname{adj}(M)) \det(M)^{-(m-1)}. \end{aligned}$$

Combining these, we get

$$\operatorname{adj}(\operatorname{adj}(M)) = \det(M)^{m-2} M.$$

□

The next Lemma provides a convenient method to construct a determinantal representation of a polynomial in some special cases.

3.4.5 Lemma (cf. [18], Lemma 4.7)**.** *Let $m > 2$ and $d \geq 2$. Let M be a $m \times m$ matrix whose entries are polynomials of degree $d-1$ with real coefficients. Let $h \in \mathbb{R}$ be irreducible of degree d. If h^{m-2} divides every $(m-1) \times (m-1)$ minor of M, then h^{m-1} divides the polynomial* $\det(M)$.

Proof. Without loss of generality we may assume $\det(M) \neq 0$. Consider the matrix $N = \frac{1}{h^{m-2}} \operatorname{adj}(M)$. The entries of N are polynomials and it holds

$$\begin{aligned} \operatorname{adj}(N) &= \operatorname{adj}(\frac{1}{h^{m-2}} \operatorname{adj}(M)) = h^{-(m-1)(m-2)} \operatorname{adj}(\operatorname{adj}(M)) \\ &= h^{-(m-1)(m-2)} \det(M)^{m-2} M. \end{aligned}$$

Let $M = (m_{ij})_{ij}$ where $m_{ij} \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$. Because the entries of $\operatorname{adj}(N)$ are polynomials, $h^{(m-1)(m-2)}$ divides $m_{ij} \det(M)^{m-2}$. But h and m_{ij} are coprime, because h is irreducible and $0 < \deg(m_{ij}) < \deg(h)$. Thus, it follows that $h^{(m-1)(m-2)}$ divides $\det(M)^{m-2}$ and thus h^{m-1} divides $\det(M)$. $\square$

3.4.6 Lemma. *Let $f \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be affine in x_i and x_j for some $i, j \in [n]$. If $f = g \cdot h$ with $g, h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$, then $\Delta_{ij} f$ is a square if and only if $\Delta_{ij} g$ and $\Delta_{ij} h$ are squares.*

Proof. Let $\Delta_{ij} f$ be a square. Since f is affine in $\mathrm{x}_i, \mathrm{x}_j$, both g and h are affine in $\mathrm{x}_i, \mathrm{x}_j$ and either $\frac{\partial g}{\partial \mathrm{x}_i} = 0$ or $\frac{\partial h}{\partial \mathrm{x}_i} = 0$. It follows that either $\Delta_{ij} g = 0$ or $\Delta_{ij} h = 0$. Using the identity $\Delta_{ij} f = g^2 \Delta_{ij} h + h^2 \Delta_{ij} g$ from Lemma 3.2.5, we see that either $\Delta_{ij} g = 0$ or $\Delta_{ij} g = \frac{\Delta_{ij} f}{h^2}$. In both cases $\Delta_{ij} g$ is a square. The same holds true for $\Delta_{ij} h$. Now let $\Delta_{ij} g$ and $\Delta_{ij} h$ be squares. As we have seen above, one of them is zero. Thus $\Delta_{ij} f = h^2 \Delta_{ij} g$ or $\Delta_{ij} f = g^2 \Delta_{ij} h$. $\square$

After this series of preparation lemmata, we are able to characterize all homogeneous multiaffine stable polynomials that have a definite determinantal representation. This is our main result of this section and it can also be found in [15, Theorem 5.5].

3.4.7 Theorem. *Let $f \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be homogeneous of degree d and stable. Suppose f is affine in the variables $\mathrm{x}_1, \ldots, \mathrm{x}_d$ and the coefficient of $\mathrm{x}_1 \cdots \mathrm{x}_d$ in f is non-zero. Then the following are equivalent:*

(i) f has a definite determinantal representation.

(ii) $\frac{\partial f}{\partial \mathrm{x}_i} \cdot \frac{\partial f}{\partial \mathrm{x}_j}$ is a square in $\mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}] / (f)$ for all $1 \leq i, j \leq d$.

(iii) $\Delta_{ij} f$ is a square in $\mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ for all $1 \leq i, j \leq d$.

Proof. The direction $(i) \Rightarrow (ii)$ is just Proposition 3.4.3.

In order to show $(ii) \Rightarrow (iii)$, let $\frac{\partial f}{\partial \mathrm{x}_1} \cdot \frac{\partial f}{\partial \mathrm{x}_2}$ be a square modulo (f), then $\Delta_{12} f$ also is. We can assume, that there is an irreducible factor g of f, such that $\frac{\partial g}{\partial \mathrm{x}_1} \neq 0$, because otherwise we would have $\Delta_{12} f = 0$. Then $\Delta_{12} f$ is a square modulo (g). Let $A = \mathbb{R}[\mathrm{x}_2, \ldots, \mathrm{x}_n]$. We can write $g = a \, \mathrm{x}_1 + b$ with $a, b \in A$ and $a \neq 0$. A is integrally closed and $\Delta_{12} f \in A$, by Lemma 3.2.5(iii). Thus, since $\Delta_{12} f$ is a square in

$$\mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}] / (g) = A[\mathrm{x}_1] / (a \, \mathrm{x}_1 + b) = A[\frac{b}{a}] \subseteq \operatorname{Quot}(A),$$

$\Delta_{12}f$ is already a square in A.

Finally, we proof the implication $(iii) \Rightarrow (i)$. For a start, suppose that f is irreducible. For every $i \leq j$, the polynomial $\Delta_{ij}f$ is a square, say a_{ij}^2 with $a_{ij} \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$. In the case $i = j$ we choose $a_{ii} = \frac{\partial f}{\partial \mathrm{x}_i}$ (note that $\Delta_{ii}f = (\frac{\partial f}{\partial \mathrm{x}_i})^2$). Then the product $\frac{\partial f}{\partial \mathrm{x}_i} \cdot \frac{\partial f}{\partial \mathrm{x}_j}$ is equivalent to a_{ij}^2 modulo the ideal (f). Therefore $a_{11}a_{ii}$ is equal to $a_{1i}^2 \bmod (f)$. Further, for every $2 \leq i < j \leq d$, the polynomials $(a_{11}a_{ij})^2$ and $(a_{1i}a_{1j})^2$ are equivalent modulo f. After changing the sign of a_{ij} if necessary, we see that $a_{11}a_{ij}$ equals $a_{1i}a_{1j} \bmod (f)$. It follows that f divides every 2×2 minors of the symmetric matrix $A = (a_{ij})_{ij}$. Then a_{ij} is affine in $\mathrm{x}_1, \ldots, \mathrm{x}_d$ and the variables x_i and x_j do not occur in a_{ij}, since they do not occur in $\Delta_{ij}f$ by Lemma 3.2.5(*iii*). All a_{ij} are homogeneous of degree $d-1$ and since f is irreducible we have $a_{ij} \neq 0$.

As f divides all the 2×2 minors of A, f^{d-2} divides all of the $(d-1) \times (d-1)$ minors of A and thus all of the entries of the adjugate matrix $\mathrm{adj}(A)$ by iterated application of Lemma 3.4.5. We can then consider the matrix $M = f^{2-d}\,\mathrm{adj}(A)$ with polynomial entries. Again by Lemma 3.4.5, f^{d-1} divides $\det(A)$. Because these both have degree $d(d-1)$, we conclude that $\det(A) = \lambda f^{d-1}$ for some $\lambda \in \mathbb{R}$. Putting all of this together, we find that

$$\det(M) \quad = \quad \frac{1}{f^{d(d-2)}} \cdot \det(\mathrm{adj}(A)) \quad = \quad \frac{1}{f^{d(d-2)}} \det(A)^{d-1} \quad = \quad \lambda^{d-1} f.$$

We now just need to argue that M is definite at some point. Let $u = \sum_{k=1}^{d} \delta_k$ and let $c = f(u)$. Since f is affine in $\mathrm{x}_1, \ldots, \mathrm{x}_d$, the only monomial of f not vanishing in u is $\mathrm{x}_1 \cdots \mathrm{x}_d$, thus c is the coefficient of this monomial and by assumption non-zero. By the same argument, we have $a_{ii}(u) = \frac{\partial f}{\partial \mathrm{x}_i}(u) = c$. If $i \neq j$, every monomial of a_{ij} vanishes in u, since a_{ij} has degree $d-1$ but at most $d-2$ of the first d variables occur. Thus, we have $A(u) = c \cdot I$. Then $M(u) = f(u)^{2-d}\,\mathrm{adj}(c \cdot I) = c \cdot I$. Therefore $\det(M)$ is a definite determinantal representation of f.

Now let f be reducible and let g be an irreducible factor of f. By Lemma 3.4.6, $\Delta_{ij}g$ is a square. Therefore every irreducible factor of f has a definite determinantal representation and so has f. □

3.4.8 Corollary. *Let $f \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be homogeneous, stable and multiaffine. Then the following are equivalent:*

- *(i) $\Delta_{ij}f$ is a square for all $1 \leq i, j \leq n$.*
- *(ii) f has a definite determinantal representation.*

Proof. This is an immediate consequence of the preceding theorem. □

3.4.9 Remark. Note that the preceding Corollary settles [7, Problem 13.14] in the case of real matrices.

We end this section with some immediate corollaries and examples.

3.4.10 Corollary. *Let $1 \leq k \leq n$ and let $f \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be a multiaffine stable polynomial. If f has a definite determinantal representation, then $\frac{\partial f}{\partial \mathrm{x}_k}$ and $f|_{\mathrm{x}_k=0}$ have also a definite determinantal representation.*

Proof. Let $i, j \in [n] \smallsetminus \{k\}$, $g = \frac{\partial f}{\partial \mathrm{x}_k}$ and $h = f|_{\mathrm{x}_k=0}$. Then it holds

$$\begin{aligned}
\Delta_{ij} f &= \frac{\partial}{\partial \mathrm{x}_i}(\mathrm{x}_k\, g + h) \cdot \frac{\partial}{\partial \mathrm{x}_j}(\mathrm{x}_k\, g + h) - (\mathrm{x}_k\, g + h) \cdot \frac{\partial^2}{\partial \mathrm{x}_i\, \partial \mathrm{x}_j}(\mathrm{x}_k\, g + h) \\
&= (\mathrm{x}_k \frac{\partial g}{\partial \mathrm{x}_i} + \frac{\partial h}{\partial \mathrm{x}_i})(\mathrm{x}_k \frac{\partial g}{\partial \mathrm{x}_j} + \frac{\partial h}{\partial \mathrm{x}_j}) - \\
&\quad (\mathrm{x}_k\, g + h)(\mathrm{x}_k \frac{\partial^2 g}{\partial \mathrm{x}_i\, \partial \mathrm{x}_j} + \frac{\partial^2 h}{\partial \mathrm{x}_i\, \partial \mathrm{x}_j}) \\
&= \mathrm{x}_k^2\, \Delta_{ij} g + \Delta_{ij} h \\
&\quad + \mathrm{x}_k (\frac{\partial g}{\partial \mathrm{x}_i} \frac{\partial h}{\partial \mathrm{x}_j} + \frac{\partial h}{\partial \mathrm{x}_i} \frac{\partial g}{\partial \mathrm{x}_j} - g \frac{\partial^2 h}{\partial \mathrm{x}_i\, \partial \mathrm{x}_j} - h \frac{\partial^2 g}{\partial \mathrm{x}_i\, \partial \mathrm{x}_j}).
\end{aligned}$$

Since $\Delta_{ij} f$ is a square, $\Delta_{ij} g$ and $\Delta_{ij} h$ are thus squares as well. Therefore g and h have a definite determinantal representation. □

3.4.11 Corollary. *Let $f = g \cdot h$, where $f, g, h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ are multiaffine stable polynomials. Then f has a definite determinantal representation if and only if g and h have one.*

Proof. This follows directly from Lemma 3.4.6 and Theorem 3.4.7. □

3.4.12 Example. Corollary 3.4.11 is a speciality of the multiaffine case, as the following example shows. Let

$$g = \mathrm{x}_1\,\mathrm{x}_2 + \mathrm{x}_1\,\mathrm{x}_3 + \mathrm{x}_1\,\mathrm{x}_4 + \mathrm{x}_2\,\mathrm{x}_3 + \mathrm{x}_2\,\mathrm{x}_4 + 2\,\mathrm{x}_3\,\mathrm{x}_4\,.$$

Consider the non-multiaffine polynomial $f = g^2$. Although f has a definite determinantal representation

$$f = \det \begin{pmatrix} \mathrm{x}_1 + \mathrm{x}_2 + 2\,\mathrm{x}_4 & \mathrm{x}_4 & 0 & -\mathrm{x}_2 - \mathrm{x}_4 \\ \mathrm{x}_4 & \mathrm{x}_2 + \mathrm{x}_3 + \mathrm{x}_4 & \mathrm{x}_2 + \mathrm{x}_4 & 0 \\ 0 & \mathrm{x}_2 + \mathrm{x}_4 & \mathrm{x}_1 + \mathrm{x}_2 + 2\,\mathrm{x}_4 & \mathrm{x}_4 \\ -\mathrm{x}_2 - \mathrm{x}_4 & 0 & \mathrm{x}_4 & \mathrm{x}_2 + \mathrm{x}_3 + \mathrm{x}_4 \end{pmatrix},$$

its factor g does not have one, since $\Delta_{12} g = \mathrm{x}_3^2 + \mathrm{x}_4^2$ is not a square.

3.4.13 Example. Consider the elementary symmetric polynomial of degree d in n variables $e_d \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$. Clearly $e_1 = \mathrm{x}_1 + \ldots + \mathrm{x}_n$ and $e_n = \mathrm{x}_1 \cdots \mathrm{x}_n$ have a definite determinantal representation. Since e_{n-1} is the dual polynomial of e_1, it has by Proposition 3.3.8 also a definite determinantal representation. It is a classic result, that these are the only cases where e_d has a definite determinantal representation, cf. [21, Theorem 1.3]. Indeed, for $n \geq 4$ and $2 \leq d \leq n-2$ the coefficients of the monomials $(\mathrm{x}_3\,\mathrm{x}_5 \cdots \mathrm{x}_{d+2})^2$, $(\mathrm{x}_4\,\mathrm{x}_5 \cdots \mathrm{x}_{d+2})^2$ and $\mathrm{x}_3\,\mathrm{x}_4(\mathrm{x}_5 \cdots \mathrm{x}_{d+2})^2$ in $\Delta_{12} e_d$ are all 1. Specializing to $\mathrm{x}_j = 1$ for $j \geq 5$ then shows that $\Delta_{12} f$ is not a square.

3.4.14 Remark. Let $f \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ be a multiaffine stable polynomial that has a definite determinantal representation. As Corollary 3.4.10 states, all derivatives of f in coordinate directions have a definite determinantal representations as well. This does not hold true for arbitrary derivatives: Let $n \geq 4$ and let $e_{n-1} \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ be the elementary symmetric polynomial of degree

$n-1$. As we have seen in Example 3.4.13, e_{n-1} has a definite determinantal representation, but

$$2e_{n-2} = \frac{\partial f}{\partial \mathrm{x}_1} + \ldots + \frac{\partial f}{\partial \mathrm{x}_n}$$

has not.

3.5 Some Extensions to the Non-Multiaffine Case

In this last section, we try to extend some of the results of the preceding section to the non-multiaffine case. We have published this together with Daniel Plaumann and Cynthia Vinzant in [15] too. The next Lemma is an analogy to Corollary 3.2.12 for polynomials with a definite determinantal representation.

3.5.1 Lemma. *Let $h \in \mathbb{R}[\mathrm{x_1}, \ldots, \mathrm{x_n}]$ be stable and homogeneous of degree d. If h has a definite determinantal representation, then there exists a multiaffine polynomial $h_0 \in \mathbb{R}[\mathrm{x}_{ij} : i \in [n], j \in [d]]$ with a definite determinantal representation, such that*

$$h = h_0(\underbrace{\mathrm{x}_1, \ldots, \mathrm{x}_1}_{d \text{ times}}, \ldots, \underbrace{\mathrm{x}_n, \ldots, \mathrm{x}_n}_{d \text{ times}}).$$

Proof. Let $h = \det(\mathrm{x}_1 A_1 + \ldots + \mathrm{x}_n A_n)$ with positive semi-definite $d \times d$ matrices $A_1, \ldots, A_n \in \mathrm{Sym}_d(\mathbb{R})$. Let $A_i = a_{i1}a_{i1}^{\mathrm{T}} + \ldots + a_{id}a_{id}^{\mathrm{T}}$ with suitable $a_{ij} \in \mathbb{R}^d$. Then we can choose $h_0 = \det(\sum_{i,j} \mathrm{x}_{ij}(a_{ij}a_{ij}^{\mathrm{T}}))$. Clearly h_0 is multiaffine, cf. Theorem 2.3.8. $\square$

3.5.2 Proposition. *Let $h \in \mathbb{R}[\mathrm{x_1}, \ldots, \mathrm{x_n}]$ be stable and homogeneous of degree d with a definite determinantal representation. Then $\Delta_{ij}h$ is a sum of squares for all $i, j \in [n]$.*

Proof. Let $x = (\mathrm{x}_1, \ldots, \mathrm{x}_1, \ldots, \mathrm{x}_n, \ldots, \mathrm{x}_n)$ and let $h_0 \in \mathbb{R}[\mathrm{x}_{ij}]$ as in the preceding Lemma. Then we have by the chain rule

$$\begin{aligned}
\Delta_{ij}h &= \frac{\partial h}{\partial \mathrm{x}_i} \cdot \frac{\partial h}{\partial \mathrm{x}_j} - h \cdot \frac{\partial^2 h}{\partial \mathrm{x}_i \, \partial \mathrm{x}_j} \\
&= \left(\sum_{k=1}^{d} \frac{\partial h_0}{\partial \mathrm{x}_{ik}}(x) \right) \left(\sum_{l=1}^{d} \frac{\partial h_0}{\partial \mathrm{x}_{jl}}(x) \right) - h_0(x) \left(\sum_{l=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 h_0}{\partial \mathrm{x}_{ik} \, \partial \mathrm{x}_{jl}}(x) \right) \\
&= \sum_{l=1}^{d} \sum_{k=1}^{d} \left(\frac{\partial h_0}{\partial \mathrm{x}_{ik}}(x) \cdot \frac{\partial h_0}{\partial \mathrm{x}_{jl}}(x) - h_0(x) \cdot \frac{\partial^2 h_0}{\partial \mathrm{x}_{ik} \, \partial \mathrm{x}_{jl}}(x) \right).
\end{aligned}$$

Every summand is a square by Theorem 3.4.7. $\square$

3.5.3 Definition. Let $h \in \mathbb{R}[\mathrm{x_1}, \ldots, \mathrm{x_n}]$. For $v \in \mathbb{R}^{\mathrm{n}}$ let $\mathrm{D}_v h$ be the directional derivative of h along v. We extend our notion of the Rayleigh difference in the following way. Let $a, b \in \mathbb{R}^{\mathrm{n}}$, then let

$$\Delta_{ab}h = \mathrm{D}_a h \cdot \mathrm{D}_b h - h \cdot \mathrm{D}_a \mathrm{D}_b h.$$

3.5.4 Corollary. *Let $h \in \mathbb{R}[\mathrm{x_1}, \ldots, \mathrm{x_n}]$ be hyperbolic with respect to $e \in \mathbb{R}^{\mathrm{n}}$ and let $a, b \in \overline{\mathrm{C}_h(e)}$. If h has a definite determinantal representation, then $\Delta_{ab}h$ is a sum of squares.*

Proof. By considering a linear change of variables we may assume that h is stable and $a = \delta_i$ and $b = \delta_j$ for some $i, j \in [n]$. Then the claim follows from Proposition 3.5.2. □

3.5.5 Proposition. *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be hyperbolic with respect to $e \in \mathbb{R}^\mathrm{n}$ and let $a, b \in \overline{\mathrm{C}_h(e)}$. If h^N has a definite determinantal representation for some $N \in \mathbb{Z}_{>0}$, then $\Delta_{ab}h$ is a sum of squares.*

Proof. Let $g = h^N$ have a definite determinantal representation. By Lemma 3.2.5 we have $\Delta_{ab}g = Nh^{2(N-1)}\Delta_{ab}h$. The left hand side is a sum of squares, for example

$$f_1^2 + \ldots + f_r^2 = Nh^{2(N-1)}\Delta_{ab}h$$

for some $f_i \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$. Let p be an irreducible factor of $h^{2(N-1)}$. Then p is hyperbolic and the right hand side vanishes on $V = \mathcal{V}_\mathbb{C}(p)$. Therefore, every f_i vanishes on $V(\mathbb{R})$ and thus, by the Lemma 1.3.12, on V. Thus we can divide the f_i by p. Iterating this, we get the claim. □

The next Theorem and the subsequent Example summarize our results of this section about polynomials with a definite determinantal representation and their relationship to arbitrary hyperbolic polynomials.

3.5.6 Theorem. *Let $h \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ be hyperbolic with respect to $e \in \mathbb{R}^\mathrm{n}$ and let $a, b \in \overline{\mathrm{C}_h(e)}$. Then $\Delta_{ab}h$ is non-negative on $\mathbb{R}^\mathrm{n}$. If some power of h has a definite determinantal representation, then $\Delta_{ab}h$ is actually a sum of squares.*

Proof. The non-negativity follows by a linear change of variables from Proposition 3.2.14. The statement about sum of squares is the preceding Proposition. □

3.5.7 Example. Let V_8 be the Vámos cube as in Example 2.1.22 and let

$$h_{V_8} = \sum_{B \in \mathfrak{B}(V_8)} \prod_{j \in B} \mathrm{x}_j$$

be the bases generated polynomial of this matroid. Wagner and Wei showed in [23] that h_{V_8} is stable. This also follows from Lemma 2.3.14 and the Grace-Walsh-Szegö Theorem 3.2.11, respectively Corollary 3.2.12. Thus $\Delta_{ij}h_{V_8}$ is non-negative for all $i, j \in [8]$. But it follows from Proposition 2.3.15 that no power of h_{V_8} has a definite determinantal representation. Thus we cannot expect that all $\Delta_{ij}h_{V_8}$ are even sum of squares. Indeed, we will show that $\Delta_{78}h_{V_8}$ is not a sum of squares. It suffices to restrict to the subspace

$$\{x = x_1 = x_2,\ y = x_3 = x_4,\ z = x_5 = x_6,\ w = x_7 = x_8\}$$

and show that the resulting polynomial $p = \frac{1}{4}\Delta_{78}h_{V_8}(x, x, y, y, z, z, w, w)$ is not a sum of squares. This restriction is given by

$$\begin{aligned} p \ = \ & x^4y^2 + 2x^3y^3 + x^2y^4 + x^4yz + 5x^3y^2z + 6x^2y^3z + 2xy^4z \\ & + x^4z^2 + 5x^3yz^2 + 10x^2y^2z^2 + 6xy^3z^2 + y^4z^2 + 2x^3z^3 \\ & + 6x^2yz^3 + 6xy^2z^3 + 2y^3z^3 + x^2z^4 + 2xyz^4 + y^2z^4. \end{aligned}$$

This polynomial vanishes in the following projective points:

$$(1:0:0),\ (0:1:0),\ (0:0:1),\ (1:-1:0),\ (1:0:-1),\ \text{and}\ (0:1:-1).$$

Thus if p is written as a sum of squares $\sum_k f_k^2$, then each f_k must vanish at each of these six points. The subspace of $\mathbb{R}[x,y,z]_3$ of cubics vanishing on these six points is spanned by the polynomials x^2y+xy^2, x^2z+xz^2, y^2z+yz^2 and xyz. Then p is a sum of squares if and only if there exists of positive semi-definite 4×4 matrix $G=(g_{ij})_{i,j\in[4]}$ so that $p=v^{\mathrm{T}}Gv$, with

$$v=\begin{pmatrix} x^2y+xy^2 \\ x^2z+xz^2 \\ y^2z+yz^2 \\ xyz \end{pmatrix}.$$

However, solving the resulting linear equations in the variables g_{ij}, result in a unique matrix G:

$$G=\begin{pmatrix} 1 & 1/2 & 1 & 2 \\ 1/2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 5 \end{pmatrix}.$$

One can see that G is not positive semi-definite from its determinant, which is $-\frac{1}{4}$. Thus p cannot be written as a sum of squares. Note that this provides another proof that no power of the polynomial h_{V_8} has a definite determinantal representation.

Zusammenfassung auf Deutsch

Ein homogenes Polynom $h \in \mathbb{R}[\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}]$ heißt hyperbolisch bezüglich $e \in \mathbb{R}^\mathrm{n}$, falls h nicht in e verschwindet und falls für jedes $v \in \mathbb{R}^\mathrm{n}$ das univariate Polynom $h(\mathrm{t}\, e + v) \in \mathbb{R}[\mathrm{t}]$ nur reelle Nullstellen hat. Der Hyperbolizitätskegel von h an e ist die Menge aller $v \in \mathbb{R}^\mathrm{n}$, für die alle Nullstellen von $h(\mathrm{t}\, e + v)$ negativ sind. Hyperbolizitätskegel sind semialgebraische konvexe Kegel. Ein spektraedrischer Kegel ist eine Menge, die durch homogene, lineare Matrixungleichungen definiert ist. Man interessiert sich für spektraedrische Kegel, da sie zulässige Bereiche der semidefiniten Optimierung sind, einer effizienten Verallgemeinerung der linearen Optimierung. Man prüft leicht nach, dass jeder spektraedrische Kegel der Hyperbolizitätskegel eines geeigneten hyperbolischen Polynoms ist. Für den Fall $n = 3$ ist auch die Umkehrung wahr: Jeder dreidimensionale Hyperbolizitätskegel ist ein spektraedrischer Kegel. Helton und Vinnikov konnten dies beweisen, indem sie zeigten, dass jedes hyperbolische Polynom $h \in \mathbb{R}[\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]$ eine definite Determinantendarstellung hat; das heißt es gibt symmetrische Matrizen A_1, A_2, A_3 mit reellen Einträgen, so dass $h = \det(\mathrm{x}_1\, A_1 + \mathrm{x}_2\, A_2 + \mathrm{x}_3\, A_3)$ gilt, wobei $v_1 A_1 + v_2 A_2 + v_3 A_3$ positiv definit ist für ein gewisses $v \in \mathbb{R}^3$ (beachte, dass jedes Polynom mit definiter Determinantendarstellung hyperbolisch ist). Dieses Ergebnis gab Anlass zu einer Reihe von Vermutungen, zum Beispiel:

Vermutung. *Sei $h \in \mathbb{R}[\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}]$ ein hyperbolisches Polynom. Dann gibt es eine natürliche Zahl N, so dass h^N eine definite Determinantendarstellung hat.*

Diese Vermutung stellte sich als falsch heraus: Brändén entdeckte ein Gegenbeispiel, das aus dem Gebiet der Matroidtheorie stammt. Nach ein paar vorbereitenden Abschnitten wird es ein erstes Ziel dieser Arbeit sein, darzustellen, wie dieses Gegenbeispiel konstruiert wurde und allgemein den Zusammenhang zwischen hyperbolischen Polynomen und Matroidtheorie zu diskutieren (Abschnitt 2.3, insbesondere Proposition 2.3.15). Es ist bisher ungeklärt, ob die nächste, abgeschwächte Vermutung stimmt:

Vermutung. *Jeder Hyperbolizitätskegel ist ein spektraedrischer Kegel.*

Diese Vermutung wird üblicherweise als verallgemeinerte Lax-Vermutung bezeichnet. Wir werden zeigen, dass man die verallgemeinerte Lax-Vermutung nicht auf analoge Weise wie die vorherige Vermutung widerlegen kann, indem wir eine Art diskrete Version der verallgemeinerten Lax-Vermutung beweisen (Abschnitt 2.4, insbesondere Theorem 2.4.6). Wir werden oft von stabilen Polynomen anstatt von hyperbolischen Polynomen sprechen. Ein homogenes Polynom $h \in \mathbb{R}[\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}]$ ist genau dann stabil, wenn es hyperbolisch bezüglich jedes Vektors des positiven Orthants ist. Da - nach einem linearen Koordinatenwechsel - jedes hyperbolische Polynom stabil ist, handelt es sich hierbei eher um eine Normierung als um eine Einschränkung. Im letzten Drittel werden wir uns multiaffinen Polynomen widmen, das sind Polynome, in denen jede Variable höchstens zum Grad eins vorkommt. Wir werden sehen, dass es reichen würde, die verallgemeinerte Lax-Vermutung für multiaffine Polynome zu zeigen (Remark 3.2.15). Nachdem wir eine Charakterisierung der multiaffinen stabilen Polynome von Brändén vorstellen (Theorem 3.2.7), werden wir ein sehr praktisches Kriterium dafür angeben, zu entscheiden, ob ein multiaffines stabiles Polynom eine definite Determinantendarstellung hat (Theorem 3.4.7). Indem wir diese beiden Resultate teilweise auf den nicht multiaffinen Fall ausdehnen, werden wir im letzten Abschnitt einen Zusammenhang zwischen Hyperbolizität und Nichtnegativität auf der einen Seite, und Darstellbarkeit durch Determinanten und Quadratsummen auf der anderen Seite andeuten können (Theorem 3.5.6).

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Erklärung

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   Real Polynomials with Definite Determinantal Representation

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Konstanz, 25. März 2013

Mario Kummer