# SUMMER SCHOOL ON HODGE THEORY, DRESDEN
# EXERCISE SHEET

## 1. Lecture one

**Exercise 1.1.** Consider the Hopf surface obtained as the quotient $X_{\lambda\mu}$ of the action

$$\mathbf{Z} \curvearrowright \mathbf{C}^2 \backslash \{0\} \qquad n \cdot (z_1, z_2) = (\lambda^n z_1, \mu^n z_2),$$

where $\lambda$ and $\mu$ are complex numbers satisfying $0 < |\lambda| \leq |\mu| < 1$. Show that

(1) $X_{\lambda\mu}$ is a compact complex surface;
(2) $X_{\lambda\mu}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^3$.

**Exercise 1.2.** Let $A$ be an Abelian surface; prove that $A$ has exactly 16 two-torsion points. *Hint: write it as a complex torus* $\mathbf{C}^2/\Gamma$ *with lattice* $\Gamma \simeq \mathbf{Z}^4$.

**Exercise 1.3.** Let $Y$ be a smooth complete intersection of degrees $\underline{d} = (d_1, \ldots d_n)$ into $\mathbf{P}^{n+2}$. Up to replacing the ambient projective space with one of smaller dimension, we can assume that each $d_i$ is strictly bigger than 1. What are all the possible values of $\underline{d}$ such that $Y$ is a $K3$ surface?
*Hint: use the adjunction formula.*

## 2. Lecture two

**Exercise 2.1.** Recall that a Kummer surface $S$ is obtained by blowing up the 16 singular points of the quotient $Y := A/\tau$ of an Abelian surface $A$ by the involution map $\tau \colon a \mapsto -a$. Using this definition, give a proof that $S$ is indeed a $K3$ surface.

**Exercise 2.2.** Let $Z$ be a smooth subvariety and $[Z] \in H^{2p}(X, \mathbf{Z})$ be its image under the cycle map. Show that the class $[Z]$, seen as an element of $H^{2p}(X, \mathbf{C})$, is of type $(p, p)$.

**Exercise 2.3.** Show that for a $K3$ surface $S$, the following equality holds:

$$\mathrm{Pic}(S) = H^2(S, \mathbf{Z}) \cap H^{1,1}(S).$$

In other words, the Picard group of $S$ coincides with its *Néron-Severi group*.

**Exercise 2.4.** Use the Gauß–Bonnet formula ($e(X) = (-1)^n \int_X c_n(X)$ for $n = \dim X$) together with the Hirzebruch–Riemann–Roch formula to show that the topological Euler characteristic of a K3 surface is 24.

*Date*: September 9, 2024.

**Exercise 2.5.** A pure *real* Hodge structure of weight $k$ is given by a finite dimensional $\mathbf{R}$-vector space $H$, together with a decomposition

$$H_{\mathbf{C}} = H \otimes \mathbf{C} = \bigoplus_{p+q=k} H^{p,q}, \quad \text{such that } \overline{H^{p,q}} = H^{q,p}.$$

Prove that any real Hodge structure can be written as the direct sum of indecomposable real Hodge structures of dimension 1 or 2.

**Exercise 2.6.** Prove that the two definitions of (integral, rational, real) Hodge structure, namely the one given in terms of the direct sum decomposition and the one given in terms of the filtration, are indeed equivalent.
*Hint: use an inductive argument on the dimension of the underlying complex vector space.*

## 3. Lecture three

**Exercise 3.1.** The following is called the *Lefschetz hyperplane Theorem*: given a smooth projective hypersurface $Y \subset \mathbf{P}^{n+1}$ of degree $d$, then the natural restriction map

$$H^i(\mathbf{P}^{n+1}, \mathbf{Z}) \longrightarrow H^i(Y, \mathbf{Z})$$

is an isomorphism for $i < n$, and it is injective for $i = n$.
Using this result, show that the upper part of Hodge diamond of a *smooth cubic threefold* looks as follows.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 5 & & 5 & & 0 \end{array}$$

**Exercise 3.2.** Let $X$ be a compact Kähler manifold and let $\mathcal{P}^{p,k}\colon B \to G(b^{p,k}, H^k(X, \mathbb{C}))$ be the period map defined in Lecture 3, which sends $b$ to $F^pH^k(X)$ (which is of dimension $b^{p,k}$). Is this map ever surjective? If so, when?

**Exercise 3.3.** Let $X$ be a K3 surface with holomorphic symplectic 2-form $\sigma$ and fix a marking $\phi\colon H^2(X, \mathbb{C}) \to \Lambda$. Show that the class $[\phi(\sigma)] \in \mathbb{P}(\Lambda \otimes \mathbb{C})$ lies in

$$\mathcal{D}_\Lambda = \{[x] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | (x, x) = 0, (x, \overline{x}) > 0\}.$$

## 4. Lecture four

**Exercise 4.1.** Consider a lattice $\Lambda$, that is, a $\mathbf{Z}$-module of finite rank together with a non-degenerate symmetric bilinear form. Let $G_\Lambda$ denote the Grassmannian of positive, oriented planes in $\Lambda_{\mathbf{R}} = \Lambda \otimes \mathbf{R}$. A point in $G_\Lambda$ is thus a pair $(P, \omega)$ consisting of a 2-dimensional subspace $P \subset \Lambda_{\mathbf{R}}$ on which the restriction of the bilinear form is positive definite and an orientation $\omega$ on $P$. Assume that $\Lambda$ is now the K3 lattice and let $D_\Lambda \subset \mathbf{P}(\Lambda_{\mathbf{C}})$ be the corresponding period domain. Show that the map

$$\delta : D_\Lambda \to G_\Lambda, \quad \ell = [x + iy] \mapsto (\operatorname{span}\{x, y\}, [x \wedge y])$$

is a diffeomorphism. Here, $x, y \in \Lambda_{\mathbf{R}}$ are the real and imaginary part of a generating vector for the line $\ell$.

**Exercise 4.2.** With the notation of the previous exercise, show that the group $\mathrm{SO}(\Lambda_{\mathbf{R}})$ acts transitively on $G_\Lambda$. What is the stabilizer of a plane $P \in G_\Lambda$?

**Exercise 4.3.** Show that the K3 period domain $D_\Lambda$ is connected and simply connected.

**Exercise 4.4.** Let $A$ be an ample divisor on a K3 surface. Prove that the linear system $|A|$ has no divisorial fixed locus.
*Hint: write $|A| = |M| + F$ for a movable divisor $M$ and a fixed part $F$. Show that every $0 \le F' \le F$ has to have negative square. Then compare $\chi(M+F)$ and $\chi(M)$ using Riemann–Roch.*