Real algebraic geometry and Hodge theory
Dresden, September 2024

Define $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$. This group is generated by the complex conjugation $\sigma \in G$.

Exercise 1. Let M be a complex manifold of (complex) dimension n. Let $\sigma : M \to M$ be an antiholomorphic involution. Show that the fixed locus M^σ of σ is a $\mathcal{C}^\infty$ submanifold of M of (real) dimension n.

Exercise 2. Let $\sigma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be an antiholomorphic involution.

1. Show that in an appropriate system of coordinates of $\mathbb{P}^1(\mathbb{C})$, one has $\sigma([1:0]) = [0:1]$).
2. Show that in an appropriate system of coordinates of $\mathbb{P}^1(\mathbb{C})$, one has either $\sigma([x:y]) = [\bar{x}:\bar{y}]$ or $\sigma([x:y]) = [-\bar{y}:\bar{x}]$.
3. In each of these two cases, exhibit a real algebraic variety X such that $X(\mathbb{R})$ is G-equivariantly isomorphic to $(\mathbb{P}^1(\mathbb{C}), \sigma)$.
4. Show that these two real algebraic varieties are not isomorphic.

Exercise 3. Let M be a connected compact Riemann surface of genus g. Let $\sigma : M \to M$ be an antiholomorphic involution with fixed locus M^σ.

1. Let $N \subset M^\sigma$ be a connected component. Show that there exists a neighborhood U of N in M such that $U \setminus N$ has two connected components exchanged by σ.
2. Shows that $M \setminus M^\sigma$ has at most two connected components.
3. Show that M^σ has at most $g+1$ connected components.
4. Deduce Harnack's theorem: if $C \subset \mathbb{P}^2_{\mathbb{R}}$ is a smooth real plane curve of degree d, then $C(\mathbb{R})$ has at most $\frac{d^2-3d+4}{2}$ connected components.

Exercise 4. Consider Motzkin's polynomial $f(x,y) := 1 + x^2y^4 + x^4y^2 - 3x^2y^2$.

1. Show that $f \geq 0$ on $\mathbb{R}^2$.
2. Suppose that $f = \sum_{i=1}^k P_i^2$ with $P_i \in \mathbb{R}[x,y]$. Show that, for any $d > 0$, the monomial x^d does not appear in any of the P_i.
3. Reach a contradiction and deduce that f is not a sum of squares in $\mathbb{R}[x,y]$. (Hint: look at the low degree terms of f.)

Exercise 5.

1. Show that $H^k_G(\mathrm{pt}, \mathbb{Z}/2) = \mathbb{Z}/2$ for $k \geq 0$. (Hint: one can use the real-complex exact sequence.)
2. Compute $H^k_G(\mathrm{pt}, \mathbb{Z}(j))$ for $k \geq 0$ and $j \in \mathbb{Z}$.

Exercise 6. Let S be a (reasonable) topological space endowed with a continuous action of G.

1. Recall that $H^1(S, \mathbb{Z})$ is in bijection with the set of Galois topological coverings $T \to S$ with Galois group $\mathbb{Z}$.
2. Give an analogous description of $H^1_G(S, \mathbb{Z})$ in terms of G-equivariant coverings. (Hint: reduce to the case where $S^G = \varnothing$).

3. Show that $H^1_G(S,\mathbb{Z}) \to H^1(S,\mathbb{Z})^G$ is injective.
4. Show that if $S = S^G$, then $H^1_G(S,\mathbb{Z}) = H^1(S,\mathbb{Z})^G = H^1(S,\mathbb{Z})$.
5. Give an example in which $H^1_G(S,\mathbb{Z}) \to H^1(S,\mathbb{Z})^G$ is not surjective.

Exercise 7. Let S be a (reasonable) topological space endowed with a continuous action of G. Let $\mathcal{F}$ be a G-equivariant sheaf on S (to simplify, one may assume that $\mathcal{F} = \mathbb{Z}$).

1. Show that the kernel of $H^k_G(S,\mathcal{F}) \to H^k(S,\mathcal{F})^G$ is a 2-torsion abelian group.
2. Show that the cokernel of $H^k_G(S,\mathcal{F}) \to H^k(S,\mathcal{F})^G$ is a 2-torsion abelian group.

Exercise 8. Let X be a smooth variety over $\mathbb{R}$. Let $\mathcal{L}$ be a line bundle on X. Consider the composition

$$\psi : H^2_G(X(\mathbb{C}),\mathbb{Z}(1)) \to H^2_G(X(\mathbb{C}),\mathbb{Z}/2) \to H^2_G(X(\mathbb{R}),\mathbb{Z}/2) \to H^2(X(\mathbb{R}),\mathbb{Z}/2)$$

of the reduction modulo 2, of the restriction to $X(\mathbb{R})$ and of the morphism forgetting the G-action.

1. Show that $\psi(\mathrm{cl}(\mathcal{L})) = \mathrm{cl}_{\mathbb{R}}(\mathcal{L})^2$ in the particular case where $X = \mathbb{P}^2_{\mathbb{R}}$ and $\mathcal{L} = \mathcal{O}(1)$.
2. Show that $\psi(\mathrm{cl}(\mathcal{L})) = \mathrm{cl}_{\mathbb{R}}(\mathcal{L})^2$ in general.

Exercise 9. Let X be a smooth projective real algebraic variety such that $X_{\mathbb{C}}$ and $\mathbb{P}^N_{\mathbb{C}}$ are isomorphic as complex algebraic varieties.

1. Show that there exists an algebraic line bundle $\mathcal{L}$ on X with $\mathcal{L}_{\mathbb{C}} \simeq \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(N+1)$.
2. Show that there exists an algebraic line bundle $\mathcal{L}'$ on X with $\mathcal{L}'_{\mathbb{C}} \simeq \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(2)$. (Hint: a possibility is to apply the real Lefschetz $(1,1)$ theorem.)
3. If N is even, show that X is isomorphic to $\mathbb{P}^N_{\mathbb{R}}$ as a real algebraic variety.

Exercise 10. Let X be a smooth projective real algebraic curve of genus g with $X(\mathbb{R}) = \varnothing$.

1. Show that -1 is a sum of 2 squares in $\mathbb{R}(X)$ (this is a theorem of Witt).
2. Let $\Gamma := \{x_0^2 + x_1^2 + x_2^2 = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$ be the real conic with no real points. Show that there exists a morphism of real algebraic varieties $f : X \to \Gamma$.
3. Show that $\deg(f) \equiv g + 1 \pmod 2$. (Hint: apply Exercise 9.3 to an appropriate linear system.)

Exercise 11. Let C be a smooth projective curve over $\mathbb{R}$ with $C(\mathbb{C})$ connected and $C(\mathbb{R})$ not connected.

1. Give an example of such a C.
2. Show that $H^0(C(\mathbb{R}),\mathbb{Z}/2)_{\mathrm{alg}} \neq H^0(C(\mathbb{R}),\mathbb{Z}/2)$.
3. Show that $X := C \times \mathbb{P}^1_{\mathbb{R}}$ satisfies $H^2(X,\mathcal{O}_X) = 0$ but $H^1(X(\mathbb{R}),\mathbb{Z}/2)_{\mathrm{alg}} \neq H^1(X(\mathbb{R}),\mathbb{Z}/2)$.
4. Construct a smooth projective surface X over $\mathbb{R}$ with $X(\mathbb{R})$ connected and $H^2(X,\mathcal{O}_X) = 0$ but such that $H^1(X(\mathbb{R}),\mathbb{Z}/2)_{\mathrm{alg}} \neq H^1(X(\mathbb{R}),\mathbb{Z}/2)$. (Hint: choose S to be a conic bundle over C.)

Exercise 12.

1. Let $X \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth hypersurface of degree d. Compute $h^2(X,\mathcal{O}_X)$.
2. Let $F \in \mathbb{C}[x_0,x_1,x_2]$ be a homogeneous polynomial of degree $2d$ with smooth zero locus in $\mathbb{P}^2_{\mathbb{C}}$. Let X be the double cover of $\mathbb{P}^2_{\mathbb{C}}$ with equation $\{z^2 + F(x_0,x_1,x_2) = 0\}$. Compute $h^2(X,\mathcal{O}_X)$.

Exercise 13.

1. Let $X \subset \mathbb{P}^3_{\mathbb{R}}$ be a smooth hypersurface of degree $d \leq 3$. Show that $H^1(X(\mathbb{R}),\mathbb{Z}/2)_{\mathrm{alg}} = H^1(X(\mathbb{R}),\mathbb{Z}/2)$.
2. For $d \geq 2$, construct a smooth hypersurface $X \subset \mathbb{P}^3_{\mathbb{R}}$ of degree d with $h^1(X(\mathbb{R}),\mathbb{Z}/2) \geq 2$.
3. For $d \geq 4$, show the existence of a smooth hypersurface $X \subset \mathbb{P}^3_{\mathbb{R}}$ of degree d with $H^1(X(\mathbb{R}),\mathbb{Z}/2)_{\mathrm{alg}} \neq H^1(X(\mathbb{R}),\mathbb{Z}/2)$.