A Variational Approach to Obstacle Problems for Shearable Nonlinearly Elastic Rods

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Abstract

We use variational methods to study obstacle problems for geometrically exact (Cosserat) theories for the planar deformation of nonlinearly elastic rods. These rods can suffer flexure, extension, and shear. There is a marked difference between the behavior of a shearable and an unshearable rod. The set of admissible deformations is not convex, because of the exact geometry used. We first investigate the fundamental question of describing contact forces, which we necessarily treat as vector-valued Borel measures. Moreover, we introduce techniques for describing point obstacles. Then we prove existence for a very large class of problems. Finally, using nonsmooth analysis for handling the obstacle, we show that the Euler-Lagrange equations are satisfied almost everywhere. These equations provide very detailed structural information about the contact forces.

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Introduction

Analyses of existence and regularity problems for elastic rods that are constrained by obstacles were carried out before, but to the best of my knowledge, only for planar deformations on the basis of the Euler elastica (restricted to small deflections) or its linearization, i.e., for models where the rod is idealized as a curve and where the obstacles can be described by a graph. In these cases the set of possible deformations is a closed convex subset in a suitable function space and the variational approach leads to a variational inequality as necessary solvability condition. The existence of a solution is in fact easy to show for this situation. Therefore, such authors as Bertocchi & Degiovanni [11], Degiovanni & Marino [16], Degiovanni & Lancelotti [15], Kučera [28, 29, 30], Link [32], Miersemann [33, 34], Miersemann & Mittelmann [35], Quittner [40, 41], Schuricht [43, 42], and Zeidler [45] were mainly interested in the investigation of multiple solutions, bifurcation problems and continuation. There are a few results about the regularity of such solutions for the linearized model (cf. Frehse [21], Gastaldi & Kinderlehrer [22], Kinderlehrer & Stampacchia [27], Lewy & Stampacchia [31]). These are not completely satisfactory, however, because one only knows that the reactions with the obstacle correspond to measures. But, roughly speaking, either there is contact on an interval and this fact itself gives all the information one can get (without special regularity investigations) or there is contact at some point at which a concentrated contact force acts. But, in general, the models used furnish no information about the actual direction of the contact forces. Moreover, effects caused by shearing, which are important for contact reactions, have been excluded. In short, the rod models previously used lack enough structure to give a detailed description of the effect of the obstacle.[1]

A more suitable model is the Cosserat theory, because it is geometrically exact and it allows not only flexure but also shear and extension and, furthermore, it involves very general constitutive relations which can describe a very large class of materials (cf. Antman [3]). Even for planar deformations it has enough structure to describe many aspects in a detailed and satisfactory way. On the other hand, it still leads to ordinary differential equations and is, therefore, not too complicated for the analysis. In this paper we study very general planar obstacle problems for

[1] After finishing this article I still found some unpublished existence and regularity results of Ball concerning obstacle problems for extensible rods (partially without restriction to small deformations). However, though the used models are less primitive than the Euler elastica, thickness and shear of the rod are still neglected and there are no regularity results for more realistic situations (cf. Ball [7]).

this model by variational methods. This theory is a consistent intrinsically one-dimensionally theory. For purposes of interpretation, however, we also regard it as being derived from a higher-dimensional theory. In contrast to the usual obstacle problems studied so far in elasticity, the set of admissible deformations for Cosserat rods is not convex. The nature of contact reactions can be quite subtle. To handle them in a physically and mathematically reasonable way, we have to investigate the foundations of the concept of force and moment with more than usual care. We verify the existence of a solution for a very large class of obstacle problems. Then we derive the Euler-Lagrange equations as a necessary condition. These equations are identical with the usual equilibrium conditions. They provide new and very detailed information about the structure of solutions of contact problems in elasticity. This information is obtained by a special description of obstacles and by computing a corresponding generalized gradient with methods of nonsmooth analysis. In our existence and regularity theory we follow the ideas which are presented in the book of Antman [3, Chapt. VII] and which are based on Antman [1, 2] and Ball [8].

In Section 1 we introduce the Cosserat theory for planar deformations, following Antman [3]. However, we have to extend some aspects so that we can handle obstacle problems. In continuum mechanics it is usually assumed that the forces are either body forces or surface tractions with integrable densities, where the stresses corresponding to surface tractions can be described by a constitutive rule. But this is too restrictive for contact problems, because there can actually arise concentrated contact forces without an integrable density. For this reason we must introduce forces and couples as vector-valued Borel measures and no longer suppose that all contact forces acting at a surface are absolutely continuous. This approach coincides with some general axiomatics introduced by Noll [39] and has the special consequence that we cannot apply Cauchy's fundamental theorem about the existence of a stress tensor in general. Nevertheless, we are able to handle the analysis for general obstacle problems by variational methods. Another special question, which becomes more interesting by this generalization and which will be discussed in this paper, is how we can determine the real contact reactions by means of the constitutive functions at points where concentrations occur. We cannot expect that at such points where the stress tensor does not exist the reactions can be determined by the strains at this point only.

Section 2 is devoted to the question of the formulation of obstacle problems. In contrast to previous investigations where the rod is idealized as a curve, we consider the rod as a two-dimensional body in the plane. First we discuss the normally used model where it is assumed that the elastic body can deform within the closure of the complement of the obstacle. Then we introduce a new and more sophisticated approach which can also describe point obstacles, e.g. To justify this formulation we must show that the deformations under consideration correspond to open mappings. That we can verify the existence of a solution, corresponding to a minimizer of the energy, with the Weierstrass Theorem, we need that the set of admissible deformations is weakly closed in a suitable function space. Usually one exploits the fact that a closed convex set is weakly closed. But this argument does not work in our situation, because the admissible set is not convex even if the rod can only deform within a convex set in the plane. However, the weak closedness can be verified relatively easy directly for the standard model. For the more general model we have to impose the natural additional assumption that the stored energy goes to infinity under total compression.

The minimization of the energy of the rod, subject to constraints describing obstacles, boundary conditions, and orientation preservation, is studied in Section 3. We impose usual growth conditions to the stored energy for coercivity and apply the Weierstrass Theorem. However, a sophisticated argument is necessary for the more general obstacle model. In this way we get the existence of minimizers for a very large class of external forces, boundary conditions and obstacles. Finally we point out that we lose some information about the growth of the stored energy if we describe the rod by coordinates with respect to axis which are fixed in the plane, i.e., we obtain existence of minimizers only in a larger space in general.

In Section 4 we formulate and discuss the Euler-Lagrange equations for a very general class of obstacle problems. In the case without obstacles it is standard to derive the Euler-Lagrange equations, which coincide with the equilibrium conditions and, hence, finally justify the variational approach. This step has frequently been omitted for obstacle problems, the analysis terminating with the derivation of a variational inequality and containing no further information on regularity. At first glance, our problem seems to be even worse than those for more primitive models, since even those obstacles that confine the rod to a convex set in Euclidean space do not correspond to a convex set of admissible deformations in a suitable function space. Therefore the formulation of a reasonable variational inequality is not possible in the usual way. However, we choose a different approach and describe the obstacle by an inequality side condition with a locally Lipschitz continuous functional which contains very detailed information about the obstacle. Then we apply the nonsmooth Lagrange Multiplier Rule of Clarke's calculus of generalized gradients and obtain, in a first step, a variational equation which is in fact equivalent to an abstract variational inequality. But we then determine the exact structure of the gradient corresponding to the obstacle and finally obtain the Euler-Lagrange equations by usual arguments. Since these equations coincide with the equilibrium conditions for the rod, we obtain that the minimizers of our variational problem are solutions of a corresponding obstacle problem. Moreover, we get a very obvious description of the contact forces which correspond to a vector-valued Borel measure supported on the contact set and directed, roughly speaking, normal to the boundary of the obstacle, i.e., we in fact get the usual condition for frictionless contact, that the tangential components of the traction vanish at contact points, as a regularity result. However, since we need some regularity for the side conditions in order to get normality in the Lagrange Multiplier Rule, we need some restrictions for the obstacle and the boundary conditions. We cannot handle the case where, e.g., the position of a material point is prescribed on the boundary of the obstacle, i.e., we encounter difficulties which also arise in the simple linearized model (cf. Kinderlehrer & Stampacchia [27, Chapt. IV,VII], Lewy & Stampacchia [31]). Furthermore, we have to restrict our attention to obstacles that can be described by the standard model (i.e., without point obstacles) and such that the reactions are not dominated by the constraints as, e.g., if the rod is rigidly clamped by the obstacle. A special difficulty is provided by the orientation-preserving condition: The minimizers of Section 3 are found within a class which allows total compression on a set with positive measure. The stored energy has bad differentiability properties on this class. By modifying the method used by Antman [3, Chapter VII] for problems without obstacles, we can overcome these difficulties and finally obtain that the solutions even satisfy the stronger orientation-preserving condition.

Section 5 presents a short introduction to Clarke's calculus of generalized gradients and provides the tools we need for our analysis. In particular, we show how to compute the generalized gradients for the functional describing the obstacle. Finally the proof that the Euler-Lagrange equations are fulfilled is given in Section 6 in several steps.

For the mathematical analysis of obstacle problems which are not restricted to rods we refer the reader to Hlaváček, Haslinger, Nečas & Lovíšek [26], Ciarlet & Nečas [12], Fichera [20], Do [18].

I am greatly indebted to Stuart Antman for encouraging my studies in obstacle problems for the Cosserat theory and for many interesting and helpful discussions, especially during my one-year stay in College Park.

Notation. We denote by cl $\mathcal{A}$, int $\mathcal{A}$, $\partial\mathcal{A}$ and $\mathcal{A}^c$ the closure, the interior, the boundary and the complement of the set $\mathcal{A}$. If X is a Banach space, then X^* stands for its dual space and $\langle \cdot, \cdot \rangle$ for the duality form on $X^* \times X$. The scalar product on $\mathbb{R}^n$ is expressed by $\mathbf{a} \cdot \mathbf{b}$.

1 Rod theory

In this section we formulate an extended version of the special Cosserat theory of rods which describes planar deformations of nonlinearly elastic rods which can bend, stretch and shear. To handle the general forces arising in obstacle problems, we have to extend some aspects of the model introduced by Antman [3], which was derived both as an intrinsic one-dimensional model, and exactly from three-dimensional elasticity by the imposition of a simple material constraint.

The main point is that in standard continuum mechanics the forces are supposed to be the sum of contact and body forces with integrable densities (cf. Antman [3], Gurtin [23], Truesdell [44], Zeidler [47]). This is however too restrictive for obstacle problems, because for them concentrated forces can occur. Thus we have to use a more general notion for forces. Such concentrated forces cause discontinuities in the surface traction and, consequently, Cauchy's fundamental theorem on the existence of a stress tensor cannot be applied. On the other hand, we cannot expect the existence of a stress tensor at points where the surface traction has a discontinuity. Respecting a usual constitutive rule, a discontinuity in the surface traction should correspond to a discontinuity in the strain. In this connection the next question is how we can determine the resultant reaction at a surface, where no stress tensor exists, by the strains through the constitutive functions. We see that obstacle problems raise many questions touching the foundations of classical continuum mechanics. However, we do not need to study these difficulties in all detail for our variational approach to obstacle problems for rods. We merely discuss those aspects that finally can be justified by our results.

To get an appropriate formulation for obstacle problems for rods, we have to focus on the geometry of the rod and the location where the reactions take place more carefully than usual. Therefore modelling questions have to be studied first in a very detailed way from a higher-dimensional point of view. For simplicity of representation we restrict such questions to a planar setting and give a short explanation how to extend them to the three-dimensional case.

In this way we present a consistent one-dimensional model which allows a geometrically exact interpretation in a two- or three-dimensional setting.

1.1 Geometry of deformation

Let $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be a fixed right-handed orthonormal basis in $\mathbb{R}^3$. We consider a slender three-dimensional body $\mathcal{B}$ that is symmetric with respect to the $\{\mathbf{i}, \mathbf{j}\}$-plane and we restrict our studies to deformations that preserve this symmetry. (Alternatively, we may regard $\mathcal{B}$ as a very long cylindrical body with generators parallel to $\mathbf{k}$. This is called the plain-strain problem.)

Let us identify the deformed body $\mathcal{B}$ with the region occupied by its intersection with the $\{\mathbf{i}, \mathbf{j}\}$-plane. We assume that the *position* $\mathbf{p}$ of the deformed material points can be given in the form

$$\mathbf{p}(s, \zeta) \;=\; \mathbf{r}(s) + \zeta \mathbf{b}(s) \quad \text{for } \; (s, \zeta) \in \Omega, \;\text{ where} \tag{1.1}$$

$$\Omega \;:=\; \{(s, \zeta) \in \mathbb{R}^2 : \; s \in [0, L], \; \zeta \in [h_1(s), h_2(s)]\}. \tag{1.2}$$

Here $\mathbf{r}(\cdot)$ describes the deformed configuration of some material curve in the body $\mathcal{B}$, the so-called *base curve* (e.g., the curve of centroids or a suitable boundary curve), and $\mathbf{b}(s)$ is a unit vector, called the *director* at s, describing the orientation of the cross-section at s. We interpret s as length parameter and ζ as thickness parameter and suppose h_1 and h_2 to be given bounded real functions on $[0, L]$ with

$$h_1(s) \le 0 \le h_2(s) \qquad \text{for all} \quad s \in [0, L]. \tag{1.3}$$

This condition ensures that the base curve belongs to the rod and excludes reversed orientation along this curve. Moreover, h_1 and $-h_2$ be lower-semicontinuous that Ω is closed.

The previous considerations show that we can describe a *planar configuration of a rod* by a pair of vector-valued functions

$$s \;\mapsto\; \mathbf{r}(s), \; \mathbf{b}(s) \;\in\; \operatorname{span}\{\mathbf{i}, \mathbf{j}\}, \quad s \in [0, L] \tag{1.4}$$

which we suppose to be absolutely continuous. The continuity ensures that the rod does not fracture. This setting gives us a one-dimensional model in the mathematical sense. In the following we sometimes argue, for the purpose of motivation and interpretation, from a higher-dimensional point of view. The reader should however observe, that we develop a consistent one-dimensional theory.

Let us set $\mathbf{a} := -\mathbf{k} \times \mathbf{b}$ and let us denote by θ the angle measured counter-clockwise from $\mathbf{i}$ to $\mathbf{a}$. Then

$$\mathbf{a} = \cos\theta\, \mathbf{i} + \sin\theta\, \mathbf{j}, \quad \mathbf{b} = -\sin\theta\, \mathbf{i} + \cos\theta\, \mathbf{j}. \tag{1.5}$$

Thus a configuration can be alternatively described by $\mathbf{r}$ and θ. Observe that the absolute continuity of $\mathbf{b}(\cdot)$ implies the absolute continuity of $\theta(\cdot)$ (with the obvious choice of branch of θ modulo 2π).

Since $\{\mathbf{a}, \mathbf{b}\}$ is a natural basis for the description of deformations of the rod, we decompose vector-valued functions with respect to it. We set

$$\mathbf{r}' = \nu \mathbf{a} + \eta \mathbf{b}, \quad \mu := \theta'. \tag{1.6}$$

By the absolute continuity of $\mathbf{r}$ and θ, the derivatives are defined almost everywhere on $[0, L]$. The question of further regularity will be a subject of our investigation.

We call the functions

$$\nu, \quad \eta, \quad \mu \tag{1.7}$$

the *strains* of the configuration (1.4). For the natural undeformed state of the rod, which we take to call the *reference configuration*, we identify all variables by a superposed circle and assume that

$$\mathring{\nu} = 1, \quad \mathring{\eta} = 0, \quad \text{i.e.,} \quad \mathring{\mathbf{r}}' = \mathring{\mathbf{a}} \,. \tag{1.8}$$

This means that the cross-sections are orthogonal to the base curve and that s is the arc length of the base curve in the reference configuration. An originally straight rod is obviously characterized by $\mathring{\mu} = 0$.

The requirement that the deformations be locally orientation-preserving can be expressed by the condition that

$$\det\left(\frac{\partial \mathbf{p}(s,\zeta)}{\partial(s,\zeta)}\right) \;=\; \nu(s) - \zeta\mu(s) \;>\; 0 \quad \text{for all} \quad (s,\zeta) \in \Omega \tag{1.9}$$

(cf. Antman [3, Chapter IV]). This can be written equivalently by the one-dimensional inequality

$$\nu(s) > V(\mu(s), s) \;\text{ for }\; s \in [0, L], \quad \text{where} \quad V(\mu, s) := \begin{cases} h_2(s)\mu & \text{for } \mu \geq 0, \\ h_1(s)\mu & \text{for } \mu \leq 0. \end{cases} \tag{1.10}$$

Observe that V is a convex function in μ.

Note. In analogy to our planar setting let us consider a three-dimensional rod whith a constrained position field $\mathbf{p}$ of the form

$$\mathbf{p}(s, \zeta^1, \zeta^2) \;=\; \mathbf{r}(s) + \zeta^1 \mathbf{b}(s) + \zeta^2 \mathbf{k} \quad \text{for} \;\; (s, \zeta^1, \zeta^2) \in \tilde{\Omega}, \tag{1.11}$$

where $\tilde{\Omega} := \{(s, \zeta^1, \zeta^2) : s \in [0, L], \;\; (\zeta^1, \zeta^2) \in \mathcal{A}(s)\}$ and $\mathcal{A}(s)$ are parameter sets for the cross-sections at s, which are symmetric with respect to $\zeta^2 = 0$. The corresponding condition for orientation preservation is equivalent to (1.10) if, e.g.,

$$(s, \zeta^1, \zeta^2) \in \tilde{\Omega} \quad \text{implies that} \quad (s, \zeta^1, 0) \in \tilde{\Omega} \tag{1.12}$$

(cf. Antman & Marlow [5, 6]). This means that our (one-dimensional) theory allows a geometrically exact three-dimensional interpretation in this case. We shall use (1.12) below again to justify our planar setting for obstacle problems.

For given integrable functions $\nu(\cdot), \eta(\cdot), \mu(\cdot)$, $\mathbf{r}_0 \in \mathbb{R}^2$, and $\theta_0 \in \mathbb{R}$, we can represent a configuration by

$$\begin{aligned} \mathbf{r}(s) &= \mathbf{r}_0 + \int_0^s (\nu \mathbf{a} + \eta \mathbf{b})\, dt, \\ &= \mathbf{r}_0 + \int_0^s \Big[(\nu\cos\theta - \eta\sin\theta)\mathbf{i} + (\nu\sin\theta + \eta\cos\theta)\mathbf{j}\Big]\, dt, && (1.13) \\ \theta(s) &= \theta_0 + \int_0^s \mu\, dt. && (1.14) \end{aligned}$$

We introduce cartesian coordinate functions x, y by

$$\mathbf{r} \;=\; x\,\mathbf{i} \;+\; y\,\mathbf{j}, \tag{1.15}$$

so that

$$\nu = \mathbf{r}' \cdot \mathbf{a} = x'\cos\theta + y'\sin\theta, \quad \eta = \mathbf{r}' \cdot \mathbf{b} = -x'\sin\theta + y'\cos\theta, \quad \mu = \theta'. \tag{1.16}$$

It is easy to see from these transformations that one can formulate a rod problem either in terms of the strains (ν, η, μ), which we call the *intrinsic formulation*, or by means of the functions (x, y, θ), which we call the *extrinsic formulation*. While the intrinsic formulation is natural for describing the mechanical response of the rod, the extrinsic formulation seems to be convenient for the description of fixed obstacles in the $\{\mathbf{i}, \mathbf{j}\}$-plane. We shall discuss the differences in the analysis for these two apparently equivalent formulations in this paper. As we shall see, the extrinsic formulation gives a less effective existence theory and is inapplicable for the regularity theory. Therefore we shall mainly work with the intrinsic formulation.

1.2 Forces and equilibrium conditions

As already mentioned, only body and contact forces with integrable densities are considered in usual continuum mechanics, since this restriction is essential for the developed analysis, in particular for the existence of a stress tensor (cf. Antman [3], Gurtin [23], Gurtin & Martins [24], Noll [37, 39], Truesdell [44], Zeidler [47], Ziemer [49]). Obstacle problems, however, do not fit in this setting, because concentrated forces can occur. Therefore we need a more general approach for such problems.

Since a force can only be observed by the interaction between bodies, there is no physical evidence for a force to have an integrable density. It is rather reasonable to describe a *force* by a vector-valued mapping

$$(\mathcal{P}_1, \mathcal{P}_2) \mapsto \mathfrak{f}(\mathcal{P}_1, \mathcal{P}_2) \tag{1.17}$$

which assigns the resultant force exerted from a body $\mathcal{P}_2$ to a body $\mathcal{P}_1$ to suitable pairs $(\mathcal{P}_1, \mathcal{P}_2)$. We impose the natural condition that $\mathfrak{f}$ is additive in each component corresponding to disjoint decompositions. From special interest are the values of $\mathfrak{f}$ for pairs $(\mathcal{P}, \mathcal{P}^c)$, i.e., the resultant force exerted to a body $\mathcal{P}$ by its environment $\mathcal{P}^c$. Without danger of confusion we set

$$\mathfrak{f}(\mathcal{P}) \;:=\; \mathfrak{f}(\mathcal{P}, \mathcal{P}^c). \tag{1.18}$$

From the mathematical point of view it is reasonable to assume that $\mathcal{P} \mapsto \mathfrak{f}(\mathcal{P})$ is a vector-valued measure on a suitable class of bodies and subbodies $\mathcal{P}$.

Analogously we introduce a *couple* (which is a pure torque) as a vector-valued measure

$$(\mathcal{P}_1, \mathcal{P}_2) \mapsto \mathfrak{l}(\mathcal{P}_1, \mathcal{P}_2). \tag{1.19}$$

The force $\mathfrak{f}$ and the couple $\mathfrak{l}$ are called balanced if

$$\mathfrak{f}(\mathcal{P}) \;=\; \mathbf{0} \quad \text{and} \quad \mathfrak{l}(\mathcal{P}) \;=\; \mathbf{0} \tag{1.20}$$

for all bodies and subbodies $\mathcal{P}$, i.e., the physical system is in *equilibrium*.

In continuum mechanics we decompose the reactions $\mathfrak{f}$ and $\mathfrak{l}$ into a *contact reaction*, which is exerted from the material of a body to the material of a neighbor body through a common surface, and other *external reactions*. In the standard theory it is assumed that all forces acting at a surface are contact forces with an integrable density and that all external forces are body forces with an integrable density. Furthermore, couples are excluded. But, this is too restrictive for obstacle problems. Here we have to handle surface forces which are external and can even have concentrations.

Note. The previous discussion is a little sloppy in some aspects, since we do not need these things in all detail for our analysis. Let us however give some further discussion for the interested reader.

A very sophisticated question is to define a reasonable class of bodies and subbodies. We can ask whether a body is an open or closed set. The fundamental difficulty here is to handle common surfaces of different bodies. From the mathematical point of view, we have to say to which part we take a common surface if we cut a body into two pieces. Noll proposed the class of bodies and subbodies to be a Boolean Algebra where the only known reasonable example for continuum mechanics is, roughly speaking, the class of regularly open sets, i.e., sets with $\operatorname{int}\operatorname{cl}\mathcal{P} = \mathcal{P}$ (cf. Noll [38, 39] and, for a more general approach using geometric measure theory, cf. Gurtin, Williams & Ziemer [25]). This problem of bodies and subbodies is also connected with the formulation of obstacle problems, because we have to say which points a body can occupy when its deformation is restricted by another body, e.g., by a rigid obstacle. We discuss this question a little more in Section 2.

Forces are introduced by Noll by some general axioms which roughly express the properties mentioned above (cf. Noll [37, 39]). Since the analysis in continuum mechanics is mainly based on the existence of a stress tensor, forces are usually restricted to body and contact forces with integrable densities such that Cauchy's famous theorem is applicable. It is still open to formulate a theory which can handle more general forces which really occur in contact problems and where concentrated forces can cause discontinuities in the traction and in the strains. In such cases also the determination of resultant forces exerted through a surface by means of a constitutive function must be investigate with more care.

We do not intend to adopt the full axiomatic treatment of Noll for subbodies and forces in our investigations and it is not necessary to have a rigorous three-dimensional theory allowing such general reactions for our analysis. However, we have to invoke some of these aspects in our rod model. Following the previous ideas and some argumentation in Antman & Lanza [4], we present an extended analysis for the Cosserat theory of nonlinearly elastic rods which can handle more general forces and couples than is usual. In contrast to standard continuum mechanics we do not distinguish a-priori between body and contact forces. We rather select, under all forces acting at some body, the surface traction exerted by the material of a neighbor body through a common surface as a special force. We, however, do not assume that this traction is identical with all forces acting at this surface. Analogously we proceed with couples. This way we are finally able to describe all effects arising in obstacle problems in a physically satisfactory sense. A special advantage of our setting is that a very large class of external forces like boundary forces, weight, surface traction and any concentrations can be handled by one simple expression during all computations. Some consequences of this more general approach related to the constitutive functions are discussed in Section 1.3. Let us finally mention that we present a mathematically exact approach sufficient for our purposes without emphasizing modelling questions too much. But our results can possibly give some hints for a general rigorous modelling.

In our investigations we identify a subbody of the rod with the corresponding subset of Ω. It is reasonable that we take at least all sets relatively open with respect to Ω as subbodies. Since we intend to define measures on the class of subbodies, we have to choose all Borel sets in Ω as subbodies. Of special interest for the rod model are subbodies of the form

$$\Omega_{\mathcal{I}} := \{(\tau,\zeta) \in \Omega : \tau \in \mathcal{I}\} \quad \text{for } \mathcal{I} \subset [0,L], \qquad \Omega_s := \Omega_{[s,L]} \tag{1.21}$$

where the $\mathcal{I}$'s as Borel sets.

For a given configuration, the material of $\Omega_{[s,L]}$ exerts across section s a *resultant force* $\mathbf{n}(s)$ and a *resultant couple* $\mathbf{m}(s)$ on the material of $\Omega_{[0,s)}$. Naturally we have

$$\mathbf{n}(0) = \mathbf{0} \quad \text{and} \quad \mathbf{m}(0) = \mathbf{0}. \tag{1.22}$$

Let us mention that $\mathbf{n}$ and $\mathbf{m}$ can also be introduced as measures over subbodies of the form (1.21) and that the mappings $s \mapsto \mathbf{n}(s), \mathbf{m}(s)$ then are distributions of these measures with respect to the arc length s.

Note. Let $\mathfrak{n}(\breve{\Omega})$ denote the resultant force exerted by the material of $\breve{\Omega}^c$ through the common surface on the material of $\breve{\Omega}$ where $\breve{\Omega} \subset \Omega$ are Borel sets. According to our previous discussion, $\mathfrak{n}$ is assumed to be a vector-valued Borel measure.

Since the position field of the rod has the form given in (1.1), it is sufficient to consider material reactions for subbodies of the form $\Omega_{\mathcal{I}}$. With $\mathcal{I} \mapsto \mathfrak{n}(\Omega_{\mathcal{I}})$ we have a Borel measure on $[0, L]$ which is uniquely determined by one of its distribution functions

$$s \mapsto \mathbf{n}^{<}(s) := \mathfrak{n}(\Omega_{[0,s)}) \quad \text{or} \quad s \mapsto \mathbf{n}^{\leq}(s) := \mathfrak{n}(\Omega_{[0,s]}) \tag{1.23}$$

combined with the natural condition $\mathbf{n}^{<}(0) = \mathbf{0}$ or $\mathbf{n}^{\leq}(L) = \mathbf{0}$, respectively. For our theory we have chosen $\mathbf{n}(s) := \mathbf{n}^{<}(s)$. Observe that $\mathbf{n}^{<}(s)$ and $\mathbf{n}^{\leq}(s)$ differ when a concentrated force acts at section s. Analogously we can introduce the resultant couple.

Let us mention that the equivalence between measure and distribution function is very useful for the experimental determination of $\mathfrak{n}$. While the practical determination of $\mathfrak{n}(\Omega_{\mathcal{I}})$ is impossible for complicated Borel sets $\mathcal{I}$, this can be done for subintervals $\mathcal{I} \subset [0, L]$ which is sufficient for the distribution function.

We suppose that all other forces acting at the body, i.e., forces other than $\mathbf{n}$, can be described by a finite vector-valued Borel measure

$$\mathcal{P} \mapsto \mathfrak{f}(\mathcal{P}) \tag{1.24}$$

assigning the resultant force to subbodies $\mathcal{P} \subset \Omega$. $\mathfrak{f}$ is said to be the *external force*. Let us recall that the components of $\mathfrak{f}$ are finite signed Borel measures (cf. Evans & Gariepy [19], Dinculeanu [17]).

The external force causes the *induced couple of* $\mathfrak{f}$

$$\mathfrak{l}_{\mathfrak{f}}(\mathcal{P}) := \int_{\mathcal{P}} \Big(\mathbf{p}(s,\zeta) - \mathbf{r}(s) \Big) \times d\mathfrak{f}(s,\zeta) = \int_{\mathcal{P}} \zeta \mathbf{b}(s) \times d\mathfrak{f}(s,\zeta) \tag{1.25}$$

(recall (1.1)). Observe that, in contrast to the force $\mathbf{n}$, it is really important to know the distribution of $\mathfrak{f}$ on the whole body and not only along the base curve, because it makes obviously a difference for $\mathfrak{l}_{\mathfrak{f}}$ whether, e.g., a force is acting at the top curve or at the bottom curve of the rod.

We now analogously suppose that all couples wich are different from $\mathbf{m}$ and $\mathfrak{l}_{\mathfrak{f}}$ can be given by a finite vector-valued Borel measure

$$\mathcal{P} \mapsto \mathfrak{l}(\mathcal{P}) \tag{1.26}$$

which we call *external couple*.

In general, $\mathfrak{f}$ and $\mathfrak{l}$ can still depend on the configuration of the rod, on the time, on the motion and also on the history of the deformation. We however restrict our attention to stationary problems in this paper. Moreover we assume for our planar model that all forces take values in the $\{\mathbf{i}, \mathbf{j}\}$-plane and that all torques and couples are orthogonal to this plane. Observe that this is fulfilled by the induced couple $\mathfrak{l}_{\mathfrak{f}}$ if the external force $\mathfrak{f}$ meets this condition.

Note. For a three-dimensional rod, which can be represented as in (1.11) and which should suffer only planar deformations, it is reasonable to assume that the external force $\mathfrak{f}$ takes values in the $\{\mathbf{i}, \mathbf{j}\}$-plane only and that $\mathfrak{f}$ is even with respect to σ. This in particular implies the orthogonality of $\mathfrak{l}_{\mathfrak{f}}$ to the $\{\mathbf{i}, \mathbf{j}\}$-plane and justifies our assumption.

According to (1.20), the rod is in *equilibrium* if the resultant force and the resultant torque about the origin vanish for each subbody. However, for our special model, we can restrict our attention to subbodies of the form $\Omega_{\mathcal{I}}$. Since a measure is uniquely determined by its distribution, it is sufficient to have the following balance of forces and moments, where we use the notation $\mathbf{z} = (\tau, \zeta)$:

$$\mathbf{n}(s) - \mathfrak{f}(\Omega_s) \; = \; \mathbf{n}(s) - \int_{\Omega_s} d\mathfrak{f}(\mathbf{z}) \; = \; \mathbf{0} \quad \text{for } s \in [0, L], \tag{1.27}$$

$$\mathbf{m}(s) + \mathbf{r}(s) \times \mathbf{n}(s) - \int_{\Omega_s} \mathbf{p}(\mathbf{z}) \times d\mathfrak{f}(\mathbf{z}) - \int_{\Omega_s} d\mathfrak{l}(\mathbf{z}) \; = \; \mathbf{0} \quad \text{for } s \in [0, L]. \tag{1.28}$$

Observe that (1.22) tells us that the resultant external force and the resultant moment of all external actions vanish for the whole body in equilibrium.

Let us still transform equation (1.28). It can be rewritten as

$$\mathbf{m}(s) + \mathbf{r}(s) \times \mathbf{n}(s) - \int_{\Omega_s} \Big(\mathbf{p}(\mathbf{z}) - \mathbf{r}(\tau) \Big) \times d\mathfrak{f}(\mathbf{z}) - \int_{\Omega_s} \mathbf{r}(\tau) \times d\mathfrak{f}(\mathbf{z}) - \int_{\Omega_s} d\mathfrak{l}(\mathbf{z}) = \mathbf{0}. \tag{1.29}$$

Using Fubini's Theorem and (1.27), we obtain a special form of partial integration for the fourth term:

$$\begin{aligned}
\int_{\Omega_s} \mathbf{r}(\tau) \times d\mathfrak{f}(\mathbf{z}) \; &= \; \int_{\Omega_s} \Big[\mathbf{r}(s) + \int_s^\tau \mathbf{r}'(\xi)\, d\xi \Big] \times d\mathfrak{f}(\mathbf{z}) \\
&= \; \mathbf{r}(s) \times \int_{\Omega_s} d\mathfrak{f}(\mathbf{z}) \; + \; \int_s^L \Big[\mathbf{r}'(\xi) \times \int_{\Omega_\xi} d\mathfrak{f}(\mathbf{z}) \Big]\, d\xi \\
&= \; \mathbf{r}(s) \times \mathbf{n}(s) \; + \; \int_s^L \mathbf{r}'(\xi) \times \mathbf{n}(\xi)\, d\xi.
\end{aligned} \tag{1.30}$$

Let us set

$$\mathbf{f}(s) \; := \; \int_{\Omega_s} d\mathfrak{f}(\mathbf{z}), \qquad \mathbf{l}(s) \; := \; \int_{\Omega_s} d\mathfrak{l}(\mathbf{z}), \tag{1.31}$$

$$\mathbf{l}_f(s) \; := \; \int_{\Omega_s} d\mathfrak{l}_{\mathfrak{f}}(\mathbf{z}) \; = \; \int_{\Omega_s} \zeta \mathbf{b}(\tau) \times d\mathfrak{f}(\tau, \zeta). \tag{1.32}$$

For the last relation recall (1.25). We can now reformulate (1.27) and (1.28) and obtain the *integral form of the equilibrium conditions*

$$\mathbf{n}(s) \; - \; \mathbf{f}(s) \; = \; \mathbf{0} \quad \text{for } s \in [0, L], \tag{1.33}$$

$$\mathbf{m}(s) \; - \; \int_s^L \mathbf{r}'(\tau) \times \mathbf{n}(\tau)\, d\tau \; - \; \mathbf{l}_f(s) \; - \; \mathbf{l}(s) \; = \; \mathbf{0} \quad \text{for } s \in [0, L]. \tag{1.34}$$

Let us mention that, from the point of view of a one-dimensional model, $\mathbf{f}$, $\mathbf{l}$ and $\mathbf{l}_f$ are given functions where the induced couple $\mathbf{l}_f$ can still depend on $\mathbf{b}(\cdot)$ or, alternatively, on $\theta(\cdot)$ (cf. Example 1.57). Inspired by the interpretation of $\mathbf{f}$, $\mathbf{l}$ and $\mathbf{l}_f$ as distributions of measures describing certain forces and couples it is reasonable to assume that $\mathbf{f}$, $\mathbf{l}$ and $\mathbf{l}_f$ are BV-functions (functions of bounded variation). Observe in this connection the continuity of $\mathbf{b}$ for absolutely continuous strains. Furthermore, we can suppose a linear dependence of $\mathbf{l}_f$ on $\mathbf{b}$.

If the measures $\mathfrak{f}$, $\mathfrak{l}_{\mathfrak{f}}$ and $\mathfrak{l}$ have integrable densities such that

$$\mathbf{f}(s) = \int_s^L \bar{\mathbf{f}}(\tau)\, d\tau, \quad \mathbf{l}_f(s) = \int_s^L \bar{\mathbf{l}}_f(\tau)\, d\tau, \quad \mathbf{l}(s) = \int_s^L \bar{\mathbf{l}}(\tau)\, d\tau, \tag{1.35}$$

then we get the *classical differential form of the equations of equilibrium*

$$\mathbf{n}' + \bar{\mathbf{f}} = \mathbf{0} \quad \text{a.e. on } [0,L], \tag{1.36}$$

$$\mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \bar{\mathbf{l}}_f + \bar{\mathbf{l}} = \mathbf{0} \quad \text{a.e. on } [0,L]. \tag{1.37}$$

We introduce the component functions $s \mapsto N(s)$, $H(s)$, $M(s)$, the so-called *stress resultants* of the problem, by

$$\mathbf{n} = N\,\mathbf{a} + H\,\mathbf{b}, \quad \mathbf{m} = M\,\mathbf{k}. \tag{1.38}$$

1.3 Constitutive functions

We call a material *elastic* if there are *constitutive functions* $\hat{N}$, $\hat{H}$, $\hat{M}$ such that the stress resultants are determined by the strains through

$$N = \hat{N}(\nu,\eta,\mu,s), \quad H = \hat{H}(\nu,\eta,\mu,s), \quad M = \hat{M}(\nu,\eta,\mu,s). \tag{1.39}$$

The domain of these functions is restricted obviously by (1.10). We assume that $\hat{N}$, $\hat{H}$, $\hat{M}$ are smooth with respect to (ν,η,μ) and that

$$\begin{pmatrix} \hat{N}_\nu & \hat{N}_\eta & \hat{N}_\mu \\ \hat{H}_\nu & \hat{H}_\eta & \hat{H}_\mu \\ \hat{M}_\nu & \hat{M}_\eta & \hat{M}_\mu \end{pmatrix} \quad \text{is positive-definite,} \tag{1.40}$$

$$\hat{N}(\nu,\eta,\mu,s) \to \begin{Bmatrix} +\infty \\ -\infty \end{Bmatrix} \quad \text{as} \quad \nu \to \begin{Bmatrix} +\infty \\ V(\mu,s) \end{Bmatrix}, \tag{1.41}$$

$$\hat{H}(\nu,\eta,\mu,s) \to \pm\infty \quad \text{as} \quad \eta \to \pm\infty, \tag{1.42}$$

$$\hat{M}(\nu,\eta,\mu,s) \to \pm\infty \quad \text{as} \quad \mu \text{ approaches its positive and negative extremes of the region (1.10).} \tag{1.43}$$

Furthermore, we require the mild symmetry condition

$$\hat{N}(\nu,\cdot,\mu,s), \ \hat{M}(\nu,\cdot,\mu,s) \quad \text{are even,} \quad \hat{H}(\nu,\cdot,\mu,s) \quad \text{is odd.} \tag{1.44}$$

Condition (1.40) is a rod-theoretic version of the Strong Ellipticity Condition and implies that $\hat{N}$ is an increasing function of ν, $\hat{H}$ is increasing with η, and $\hat{M}$ increases with μ.

In view of (1.13), (1.14), a configuration is determined to within a rigid displacement by a triple of integrable strains $\nu(\cdot), \eta(\cdot), \mu(\cdot)$. Using the constitutive relations (1.39) we can define integrable functions $s \mapsto \check{N}(s), \check{H}(s), \check{M}(s)$ by

$$\check{N}(s) := \hat{N}(\nu(s),\eta(s),\mu(s),s), \quad \text{etc.} \tag{1.45}$$

In this way we obtain the stress resultants and, simultaneously, the resultant reactions $\mathbf{n}(s)$, $\mathbf{m}(s)$ at the cross-section s. Clearly the shape of a configuration is not influenced by a change of the strains on a set of measure zero. However, the constitutive functions give different stress resultants by such a change according to formula (1.45). Therefore at best they can provide the correct resultant reactions at cross-section s almost everywhere. This aspect is interesting from the physical point of view an will be discussed a little more in the following note. But it does not influence our analysis.

Note. From the previous discussion, the question arises of how to get the real resultant reactions at every cross-section for an equilibrium configuration. In the case where the measures $\mathfrak{f}$, $\mathfrak{l}_\mathfrak{f}$ and $\mathfrak{l}$ have integrable densities, the equilibrium equations (1.36) and (1.37) imply that the real stress resultants must be continuous. Since in this case each triple of stress resultants $(\check{N}, \check{H}, \check{M})$ provided through the constitutive functions by the strains (ν, η, μ) as in (1.45) coincides with the real continuous stress resultants almost everywhere, the continuous representative can be selected by the following limit of the average

$$N(s) = \lim_{\varepsilon \to 0} \fint_{[s-\varepsilon, s+\varepsilon]} \check{N}(\tau) d\tau, \quad \text{etc.}, \tag{1.46}$$

because the average is not influenced by a change on a null set. In previous investigations of problems without obstacles the forces are usually assumed to have integrable densities and one in fact tacitly assumes that the right continuous representative for the stress resultants is chosen.

Let us now study the more interesting case of equilibria subject to general external forces $\mathfrak{f}$ and external couples $\mathfrak{l}$, which are measures. In this case the distribution functions $\mathbf{f}(\cdot), \mathbf{l}(\cdot), \mathbf{l}_f(\cdot)$ are BV-functions. Moreover, they are continuous from the left and the limit from the right exists at every point (cf. Benedetto [10, pp. 121,123,187]). Hence the equilibrium conditions (1.33) and (1.34) imply the same properties for the stress resultants $N(\cdot), H(\cdot), M(\cdot)$. We can now argue as above. Unfortunately the method from (1.46) does not work in this case. However, by the continuity from the left, the real stress resultants can be found by

$$N(s) = \lim_{\varepsilon \to 0} \fint_{[s-\varepsilon, s]} \check{N}(\tau) d\tau, \quad \text{etc.} \tag{1.47}$$

Alternatively, the limit of the average in (1.47) can be replaced by the approximate limit from the left (cf. Evans & Gariepy [19]). If we put the procedure from (1.47) into the definition of the constitutive functions, then they still respect the Principle of Determinism of Noll [36] (also called Principle of Local Action in Truesdell [44, p. 201]) and we avoid all difficulties. In the following we can tacitly assume that (1.47) is incorperated with (1.39).

For a large class of materials the matrix in (1.40) is symmetric (cf. Antman [3]). In this case the material is called *hyperelastic* and there exists a real-valued function W of (ν, η, μ, s), the so-called *stored energy function*, such that

$$\hat{N} = W_\nu, \quad \hat{H} = W_\eta, \quad \hat{M} = W_\mu. \tag{1.48}$$

Since we use variational methods, we restrict our attention to hyperelastic materials. The *total stored energy* of the rod is the functional

$$E_s(\nu, \eta, \mu) = \int_0^L W(\nu(s), \eta(s), \mu(s), s)\, ds. \tag{1.49}$$

Observe that E_s is not influenced by a change of the strains on a null-set, i.e., E_s is uniquely determined for a given configuration.

Our variational approach begins with the existence of the stored energy function W. It delivers Euler-Lagrange equations that are equivalent to the integral form of the equilibrium conditions (1.33) and (1.34) almost everywhere on $[0, L]$ by means of the constitutive equations (1.48). In this way we finally justify our rod theory.

In the following we suppose that there are no prescribed external couples $\mathfrak{l}$ and, for simplicity, that the prescribed external force $\mathfrak{f}$ only depends on the coordinates $\mathbf{z} = (s, \zeta)$ and not on the configuration $\mathbf{p}(\cdot)$. Then this force is conservative and has the *potential energy*

$$E_p(\mathbf{p}) := -\int_\Omega \mathbf{p}(\mathbf{z}) \cdot d\mathfrak{f}(\mathbf{z}) = -\int_\Omega \Big(\mathbf{r}(s) + \zeta \mathbf{b}(s) \Big) \cdot d\mathfrak{f}(\mathbf{z}). \tag{1.50}$$

Note. By our assumption that $\mathfrak{f}$ is perpendicular to $\mathbf{k}$ we get the same expression for the potential energy if we use the three-dimensional setting from (1.11).

Our analysis is applicable to more general conservative forces. The main difference would be to find suitable growth conditions for the existence result in Section 3 (cf. Antman [3, Chapter VII]). We omit such special technical questions, because they obscue our main goal of treating difficulties associated with contact.

1.4 Special forces

In order to demonstrate the response of different external forces, we compute $\mathbf{f}(s)$ and $\mathbf{l}_f(s)$ for some examples. Though the two-dimensional setting from (1.1) would be sufficient for our analysis, we argue from the three-dimensional point of view based on formula (1.11), because this is preferable in the light of applications. Recall that $\mathcal{A}(s)$ denotes the cross-section of the three-dimensional rod at s, and we define $\tilde{\Omega}_s$ in analogy to (1.21). Instead of (1.31) and (1.32) we then have

$$\mathbf{f}(s) := \int_{\tilde{\Omega}_s} d\mathfrak{f}(\tau,\zeta^1,\zeta^2), \quad \mathbf{l}_f(s) := \int_{\tilde{\Omega}_s} \Big(\zeta^1 \mathbf{b}(\tau) + \zeta^2 \mathbf{k} \Big) \times d\mathfrak{f}(\tau,\zeta^1,\zeta^2). \tag{1.51}$$

Let us mention that $\mathbf{f}(s)$ and $\mathbf{l}_f(s)$, determined in (1.51) by three-dimensional arguments, are appropriate one-dimensional entries for our theory.

Example 1.52 (Terminal loads) .

(a) First we assume that we have a terminal load caused by a uniformly distributed contact force $\tilde{\mathbf{f}}$ on the cross-section $\mathcal{A}(0)$. This situation can be met if the end of the rod is welded to a rigid body. We readily get

$$\mathbf{f}(s) = \begin{cases} m^0 \tilde{\mathbf{f}} & \text{for} \quad s = 0, \\ \mathbf{0} & \text{for} \quad s > 0, \end{cases} \qquad \text{where } m^0 := \int_{\mathcal{A}(0)} d\zeta^1\, d\zeta^2, \tag{1.53}$$

$$\mathbf{l}_f(s) = \begin{cases} m^1 \mathbf{b}(0) \times \tilde{\mathbf{f}} & \text{for} \quad s = 0, \\ \mathbf{0} & \text{for} \quad s > 0, \end{cases} \qquad \text{where } m^1 := \int_{\mathcal{A}(0)} \zeta^1\, d\zeta^1\, d\zeta^2 \,. \tag{1.54}$$

The concentrated couple at $s = 0$ vanishes if $m^1 = 0$, i.e., if $\mathbf{r}(0)$ is the centroid of the cross-section $\mathcal{A}(0)$.

(b) Let us now assume that there is a terminal force $\breve{\mathbf{f}}$ that only acts at the point $\mathbf{p}(0, h_0, 0)$. Thus

$$\mathbf{f}(s) = \begin{cases} \breve{\mathbf{f}} & \text{for} \quad s = 0, \\ \mathbf{0} & \text{for} \quad s > 0, \end{cases} \tag{1.55}$$

$$\mathbf{l}_f(s) = \begin{cases} h_0 \mathbf{b}(0) \times \breve{\mathbf{f}} & \text{for} \quad s = 0, \\ \mathbf{0} & \text{for} \quad s > 0, \end{cases} \tag{1.56}$$

Here the concentrated couple at $s = 0$ vanishes if the force $\breve{\mathbf{f}}$ acts at the base curve, i.e., at the point $\mathbf{r}(0)$.

Example 1.57 (Weight) We assume that the rod has an integrable mass densitiy $\varrho(\tau,\zeta^1,\zeta^2) > 0$ which is even in ζ^2. Let g be the acceleration of gravity and let $\mathbf{j}$ point upward. Then

$$\begin{aligned} \mathbf{f}(s) &= \int_{\tilde{\Omega}_s} -g\varrho(\mathbf{z})\,\mathbf{j}\, d\mathbf{z} = -g\,\mathbf{j} \int_s^L \int_{\mathcal{A}(\tau)} \varrho(\tau,\zeta^1,\zeta^2)\, d\zeta^1\, d\zeta^2\, d\tau \\ &= -\mathbf{j} \int_s^L \varrho^0(\tau)\, d\tau \qquad \text{with} \quad \varrho^0(\tau) := g \int_{\mathcal{A}(\tau)} \varrho(\tau,\zeta^1,\zeta^2)\, d\zeta^1\, d\zeta^2, \end{aligned} \tag{1.58}$$

$$\begin{aligned} \mathbf{l}_f(s) &= \int_{\tilde{\Omega}_s} \Big(\zeta^1 \mathbf{b}(\tau) + \zeta^2 \mathbf{k} \Big) \times \Big(-g\varrho(\mathbf{z})\,\mathbf{j} \Big)\, d\mathbf{z} \\ &= g\,\mathbf{j} \times \int_s^L \mathbf{b}(\tau) \Big[\int_{\mathcal{A}(\tau)} \zeta^1 \varrho(\tau,\zeta^1,\zeta^2)\, d\zeta^1\, d\zeta^2 \Big]\, d\tau \\ &= \mathbf{k} \int_s^L \varrho^1(\tau)\, \sin(\theta(\tau))\, d\tau \qquad \text{with} \quad \varrho^1(\tau) := g \int_{\mathcal{A}(\tau)} \zeta^1 \varrho(\tau,\zeta^1,\zeta^2)\, d\zeta^1\, d\zeta^2. \end{aligned} \tag{1.59}$$

Hence, the distributions $\mathbf{f}$, $\mathbf{l}_f$ have the integrable densities

$$\bar{\mathbf{f}}(\tau) = -\varrho^0(\tau)\,\mathbf{j}, \quad \bar{\mathbf{l}}_f(\tau) = \varrho^1(\tau)\sin(\theta(\tau))\,\mathbf{k}. \tag{1.60}$$

For the special case where $\varrho(\cdot)$ is constant and $\mathbf{r}(\cdot)$ is a curve of centroids, we get $\mathbf{l}_f = \mathbf{0}$.

Example 1.61 (Hydrostatic pressure) Though we do not study such cases, we finally give an example of a force depending on the configuration of the body (cf. Antman [3, Chapter VII]). For simplicity of representation we argue in a two-dimensional setting in this case. Assume that the hydrostatic pressure acts along the bottom curve of the rod which we also take as base curve, i.e., $h_1(s) = 0$ on $[0, L]$. If ϱ is the pressure, then we can describe the corresponding force by the line density $\bar{\mathbf{f}} := \varrho\mathbf{k} \times \mathbf{r}'$ and obtain

$$\mathbf{f}(s) = \varrho \int_s^L \mathbf{k} \times \mathbf{r}'(\tau)\, d\tau. \tag{1.62}$$

Since the force acts only at the base curve, the corresponding couple $\mathbf{l}_f(s)$ vanishes everywhere.

In addition to these examples, we study general contact forces in Section 4 below.

2 Obstacle problems

We now give a mathematical description of obstacles for the deformation of elastic rods. We study two slightly different methods. While the first follows the intuitively preferred standard procedure, the second seems to be a little artificial at the first sight. It is, however, a more sophisticated and powerful version. In contrast to the first method, the second can handle point obstacles.

To formulate obstacle or contact problems exact, we are faced with difficulties like those that arise in defining subbodies. The question which points in the space can be occupied by a body is a little sophisticated in the case where two bodies are in contact, because we cannot assign the common boundary points uniquely to a body. In the axiomatics for subbodies of Noll this problem plays an important role (cf. Noll [38, 39]). Since we do not claim to adopt these axiomatics in full generality, let us briefly discuss this point more intuitively, which is sufficient for our purposes. Observe that an obstacle can be considered as fixed rigid body.

We start with the nice case in which the bodies are closures of open sets, i.e., where we assume a natural "thickness" for bodies. Here we can demand that the interior of one body not be penetrated by another body. In this way we obtain a correct modelling if, roughly speaking, the topological structure of all bodies is preserved during deformations. For the formulation of an obstacle problem it is therefore sufficient in this case to restrict the deformable body to the complement of the interior of the obstacle or, equivalently, to the closure of the complement of the obstacle. This is the method used in the literature for contact problems. We shall use it for a first variant of obstacle problems.

We can also consider "thin" bodies such as points and curves. This provides serious difficulties for a reasonable mathematical formulation. The previous method from "thick" bodies can be transferred in an obvious way only to the case in which there is at most one "thin" obstacle. For the contact between two "thin" bodies it does not work. To overcome this difficulty partially, we could "fatten" such bodies at least in certain directions. On the other hand we could argue that bodies are always "thick" in the real world. But it seems convenient, especially for qualitative investigations, to idealize, e.g., very thin obstacles as lower-dimensional objects. We do not claim to study this modelling question in full generality. We shall, however, present a formulation for obstacle problems, corresponding to the Cosserat theory, which can handle any closed set as obstacle.

Since our investigations are restricted to planar deformations of rods, we restrict our attention to obstacle problems that are reasonable in such a setting.

Note. In the plain-strain problem, where we regard also the obstacle as an infinitely long cylindrical body with generators parallel to $\mathbf{k}$, our planar setting is very natural and geometrically exact. Obstacle problems for slender three-dimensional rods, whith a position field $\mathbf{p}$ constrained as in (1.11), can be formulated reasonably in a planar setting if, e.g., we assume that also the obstacle is symmetric about the $\{\mathbf{i}, \mathbf{j}\}$-plane, that the rod satisfies (1.12), and that the obstacle meets a condition analogous to (1.12).

Let $\mathbf{X}$ denote a space of vector-valued functions $s \mapsto \mathbf{u}(s)$ which uniquely determine configurations (e.g., $\mathbf{u} = (\mathbf{r}, \mathbf{b})$ or $\mathbf{u} = (\nu, \eta, \mu, \mathbf{r}_0, \theta_0)$ or $\mathbf{u} = (x, y, \theta)$). Hence we can express $\mathbf{p}(s, \zeta)$ in terms of $\mathbf{u}$, which we indicate by

$$\mathbf{p}(s, \zeta) = \mathbf{p}[\mathbf{u}](s, \zeta) \quad \text{for} \quad (s, \zeta) \in \Omega. \tag{2.1}$$

Later we specify the space $\mathbf{X}$.

For our purposes we define an *obstacle* $\mathcal{O}$ as a closed subset of $\mathbb{R}^2$ with $\mathcal{O}^c \neq \emptyset$. In a first variant we assume that the rod can occupy the points of $\mathrm{cl}(\mathcal{O}^c)$.

Variant 1. We define the set of *admissible configurations with respect to the obstacle* $\mathcal{O}$ by

$$\mathbf{A}_1 := \{\mathbf{u} \in \mathbf{X} : \mathbf{p}[\mathbf{u}](s, \zeta) \in \mathrm{cl}\, \mathcal{O}^c \ \text{ for all } \ (s, \zeta) \in \Omega\}. \tag{2.2}$$

It is easy to see that this variant only works if

$$\mathcal{O}^c = \mathrm{int}\,(\,\mathrm{cl}\, \mathcal{O}^c\,). \tag{2.3}$$

Obviously this relation is not fulfilled if the obstacle $\mathcal{O}$ contains isolated points or curves, i.e., if we have a "thin" obstacle. For this reason we consider a second variant which is only slightly changed compared with the previous one. However, it is more powerful, but it needs some deeper justification.

Variant 2. We define the set of *admissible configurations with respect to the obstacle* $\mathcal{O}$ by

$$\mathbf{A}_2 := \{\mathbf{u} \in \mathbf{X} : \mathbf{p}[\mathbf{u}](s, \zeta) \in \mathcal{O}^c \ \text{ for all } \ (s, \zeta) \in \mathrm{int}\, \Omega\}. \tag{2.4}$$

Following the ideas mentioned above, we demand in this second variant that inner points of the rod remain in the complement of the closed obstacle. That we do not get physical nonsense in this case, we have to ensure that $(s, \zeta) \mapsto \mathbf{p}[\mathbf{u}](s, \zeta)$ is an open mapping on $\mathrm{int}\, \Omega$ for all admissible deformations. We justify this version by

Lemma 2.5 *If* $\mathbf{u} \in \mathbf{X}$ *defines a configuration that satifies the orientation-preserving condition* (1.10)*, then* $(s, \zeta) \mapsto \mathbf{p}[\mathbf{u}](s, \zeta)$ *is an open mapping on* $\mathrm{int}\ \Omega$*, i.e., the images of open sets are open.*

The proof mainly uses (1.10) and is given at the end of this section. It is easy to see that we are able to handle point and curve obstacles in a physically reasonable sense in this model.

At this point it arises a new problem. To carry out the analysis of an existence theory by minimization of the energy, we need side conditions in our variational problem which represent weakly closed sets in the space $\mathbf{X}$. The condition (1.10) of orientation preservation, however, does not satisfy this condition. Therefore, for technical purposes, we have to enlarge the set of admissible deformations by the weaker condition

$$\nu(s) \geq V(\mu(s), s) \quad \text{a.e. on } [0, L]. \tag{2.6}$$

This means that we also take into account configurations with $\nu = 0$, $\mu = 0$, i.e., complete compression, on a whole interval for existence theory. Later we shall however show by regularity arguments that, under some mild additional conditions, solutions really satisfy the orientation-preserving condition in the strong variant.

Unfortunately, Lemma (2.5) does not hold under the weaker condition (2.6). But let us suppose the very natural condition that the stored energy W approaches infinity under complete compression, i.e.,

$$W(\nu, \eta, \mu, s) \to \infty \quad \text{as} \quad \nu - V(\mu, s) \to 0. \tag{2.7}$$

This condition will also play an important role for our regularity results in Section 4. Now we can give a more subtle variant of Lemma (2.5).

Lemma 2.8 *Let W fulfil (2.7). If $\mathbf{u} \in \mathbf{X}$ defines a configuration that satisfies (2.6) and which has a finite total stored energy $E_s(\mathbf{u}) = \int_0^L W[\mathbf{u}](\nu(s), \eta(s), \mu(s), s)\, ds < \infty$, then $(s, \zeta) \mapsto \mathbf{p}[\mathbf{u}](s, \zeta)$ is an open mapping on $\operatorname{int} \Omega$.*

The proof is also postponed to the end of this section. Let us now discuss the weak closedness of $\mathbf{A}_1$ or $\mathbf{A}_2$ in a suitable space $\mathbf{X}$. For $p_1, p_2, p_3 > 1$, we choose for the intrinsic formulation

$$\mathbf{X} := \mathcal{L}^{p_1} \times \mathcal{L}^{p_2} \times \mathcal{L}^{p_3} \times \mathbb{R}^2 \times \mathbb{R} \quad \text{with} \quad \mathbf{u} = (\nu, \eta, \mu, \mathbf{r}_0, \theta_0). \tag{2.9}$$

Here $\mathcal{L}^p$ denotes the usual Lebesgue space of p-integrable functions on $[0, L]$.

Proposition 2.10 .

1) $\mathbf{A}_1$ is weakly closed in $\mathbf{X}$.

2) Let $(\nu, \eta, \mu, s) \mapsto W(\nu, \eta, \mu, s)$ be measurable in s and twice continuously differentiable in the other arguments. Let (2.7) be hold and suppose that there is an integrable function $s \mapsto \gamma(s)$ such that

$$W(\nu, \eta, \mu, s) \geq \gamma(s) \quad \textit{for all } \nu, \eta, \mu, s. \tag{2.11}$$

Then, for a given constant $c \in \mathbb{R}$, the set

$$\mathbf{A}_3 := \{\mathbf{u} \in \mathbf{A}_2 : E_s(\mathbf{u}) \leq c, \ \ \mathbf{u} \textit{ satisfies } (2.6)\ \} \tag{2.12}$$

is weakly closed in $\mathbf{X}$.

Note. With respect to the extrinsic formulation, we can choose

$$\mathbf{X}_0 := \mathcal{W}^{1,p_1} \times \mathcal{W}^{1,p_2} \times \mathcal{W}^{1,p_3} \quad \text{with} \quad \mathbf{u} = (x, y, \theta) \tag{2.13}$$

for $p_1, p_2, p_3 > 1$. Here $\mathcal{W}^{1,p}$ denotes the usual Sobolev space of p-integrable functions with generalized p-integrable derivative. Using the compact embedding of $\mathcal{W}^{1,p}$ into the space $\mathcal{C}$ of continuous functions, we can argue as in the proof of Proposition (2.10) below and get an analogous assertion.

Let us now discuss some questions which are necessary for our regularity investigations in Section 4. In order to derive the Euler-Lagrange equations for the variational problem, we shall apply a general Lagrange Multiplier Rule. For this purpose, the restrictions caused by the obstacle must be described by an inequality side condition. Let $\text{dist}_{\mathcal{O}^c}(\mathbf{q})$ denote the distance of point $\mathbf{q}$ from the set $\mathcal{O}^c$, then this can be done by

$$\text{dist}_{\mathcal{O}^c}\, \mathbf{p}[\mathbf{u}](s,\zeta) \le 0 \quad \text{for all} \quad (s,\zeta) \in \Omega \tag{2.14}$$

or, equivalently,

$$\max_{(s,\zeta)\in\Omega} \text{dist}_{\mathcal{O}^c}\, \mathbf{p}[\mathbf{u}](s,\zeta) \le 0. \tag{2.15}$$

Since the functions $\text{dist}_{\mathcal{O}^c}(\cdot)$ and $\text{dist}_{\text{cl}\,\mathcal{O}^c}(\cdot)$ are identical, this method however only works in the case where Variant 1 is applicable (cf. (2.3)). Roughly speaking, the inequality condition does not recognize a lower-dimensional obstacle.

To get normality in the Lagrange Multiplier Rule, we need some regularity in condition (2.14), which is not fulfilled by the function $\text{dist}_{\mathcal{O}^c}(\cdot)$ appearing there, because the generalized gradient $\partial\text{dist}_{\mathcal{O}^c}(\mathbf{p})$ contains the zero vector for points $\mathbf{p}$ where equality holds in (2.14) (cf. Section 5). Moreover, condition (2.14) cannot indicate the points of the rod which are in contact with the obstacle, since $\text{dist}_{\mathcal{O}^c}(\mathbf{q}) = 0$ for all points $\mathbf{q}$ which can be occupied by the rod. It is rather desirable to choose a function which equals zero only for boundary points of the obstacle. Using such a function we are able to determine the contact points and, consequently, the points where contact forces can act. For this reason we use

$$d(\mathbf{q}) := \text{dist}_{\mathcal{O}^c}\, \mathbf{q} - \text{dist}_{\mathcal{O}}\, \mathbf{q}, \quad \mathbf{q} \in \mathbb{R}^2, \tag{2.16}$$

instead of $\text{dist}_{\mathcal{O}^c}(\cdot)$ in conditions (2.14), (2.15) and we restrict our attention, roughly speaking, to obstacles with the property that

$$0 \notin \partial d(\mathbf{q}) \quad \text{for} \ \ \mathbf{q} \in \mathbb{R}^2 \ \ \text{with} \ \ d(\mathbf{q}) = 0 \tag{2.17}$$

(cf. condition (4.8) below).

P r o o f of Proposition 2.10. (1) We assume that

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in} \ \ \mathbf{X}, \quad \text{where either all} \ \ \mathbf{u}_n \in \mathbf{A}_1 \ \text{or all} \ \ \mathbf{u}_n \in \mathbf{A}_3. \tag{2.18}$$

This means that

$$\nu_n \rightharpoonup \nu, \ \ \eta_n \rightharpoonup \eta, \ \ \mu_n \rightharpoonup \mu, \ \ \mathbf{r}_{0,n} \to \mathbf{r}_0, \ \ \theta_{0,n} \to \theta_0 \tag{2.19}$$

in the corresponding spaces. Using (1.13), (1.14), (1.5), we define $\mathbf{r}_n$, θ_n, $\mathbf{b}_n$, and $\mathbf{r}$, θ, $\mathbf{b}$ with respect to $\mathbf{u}_n$ and $\mathbf{u}$, respectively. Thus $\mathbf{r}_n$, θ_n, $\mathbf{b}_n$, $\mathbf{r}$, θ, $\mathbf{b}$ are absolutely continuous functions on $[0, L]$. By the compactness of integral operators, we obtain

$$\mathbf{r}_n \to \mathbf{r}, \ \ \theta_n \to \theta, \ \ \mathbf{b}_n \to \mathbf{b} \tag{2.20}$$

in corresponding spaces of continuous functions. Hence we have the pointwise convergence

$$\mathbf{p}[\mathbf{u}_n](s,\zeta) \to \mathbf{p}[\mathbf{u}](s,\zeta) \quad \text{for all} \quad (s,\zeta) \in \Omega. \tag{2.21}$$

This directly implies that $\mathbf{u} \in \mathbf{A}_1$, and we have verified the first case.

(2) Let us continue with the second case. By our assumptions W is a Carathéodory function and by (1.40), (1.48), $W(\cdot,\cdot,\cdot,s)$ is convex. The boundedness of W by (2.11) then implies that $E_s(\cdot)$ is weakly lower-semicontinuous (cf. Dacorogna [14, Theorem 3.4]). Using (1.3), (1.10), we get that (2.6) describes a convex set. Since a strongly convergent sequence in a space $\mathcal{L}^{p_i}$ has a subsequence that converges a.e., this set is also strongly closed. Thus the set given by (2.6) is weakly closed in $\mathbf{X}$. Hence, $\mathbf{u}$ satisfies (2.6) and $E_s(\mathbf{u}) \le c$. We now assume that

$$\mathbf{q}_0 := \mathbf{p}[\mathbf{u}](s_0,\zeta_0) \notin \mathcal{O}^c \quad \text{for some} \quad (s_0,\zeta_0) \in \operatorname{int}\Omega. \tag{2.22}$$

According to the properties of the set U_0 in the proof of Lemma (2.5) and (2.8) below, there exists a closed ball $B_0 \subset \operatorname{int}\Omega$ with center (s_0,ζ_0) and positive radius such that $\mathbf{p}[\mathbf{u}]$ is injective on B_0.

Let us define

$$\mathbf{q}_n := \mathbf{p}[\mathbf{u}_n](s_0,\zeta_0) \quad \text{and} \quad \mathcal{Q}_n := (\mathbf{q}_0,\mathbf{q}_n), \tag{2.23}$$

where the last expression denotes the open line segment between the points $\mathbf{q}_0$ and $\mathbf{q}_n$. We study the set $\mathcal{P}_n := \mathbf{p}[\mathbf{u}_n](B_0) \cap \mathcal{Q}_n$. Since $\mathcal{P}_n \subset \mathcal{O}^c$ and $\mathbf{q}_0 \notin \mathcal{O}^c$, there is a boundary point $\tilde{\mathbf{q}}_n$ of $\mathcal{P}_n$ with respect to $\mathcal{Q}_n$. By the continuity of $\mathbf{p}[\mathbf{u}_n]$ on Ω, the image $\mathbf{p}[\mathbf{u}_n](B_0)$ is closed and we can find $(s_n,\zeta_n) \in B_0$ such that $\mathbf{p}[\mathbf{u}_n](s_n,\zeta_n) = \tilde{\mathbf{q}}_n$. Lemma 2.8 tells us that $\mathbf{p}[\mathbf{u}_n]$ is also an open mapping on $\operatorname{int}\Omega$. Hence we even have $(s_n,\zeta_n) \in \partial B_0$. Thus there is a subsequence, denoted the same way, for which $(s_n,\zeta_n) \to (\tilde{s},\tilde{\zeta}) \in \partial B_0$. By (2.21),

$$\mathbf{q}_n = \mathbf{p}[\mathbf{u}_n](s_0,\zeta_0) \;\to\; \mathbf{p}[\mathbf{u}](s_0,\zeta_0) = \mathbf{q}_0. \tag{2.24}$$

Since $\tilde{\mathbf{q}}_n \in \mathcal{Q}_n = (\mathbf{q}_0,\mathbf{q}_n)$, this implies that

$$\tilde{\mathbf{q}}_n \;\to\; \mathbf{q}_0. \tag{2.25}$$

On the other hand, the equicontinuity of $\mathbf{p}[\mathbf{u}_n](\cdot,\cdot)$, which follows because of the definition of $\mathbf{r}_n$, $\mathbf{b}_n$ as integral operators, implies that

$$\tilde{\mathbf{q}}_n = \mathbf{p}[\mathbf{u}_n](s_n,\zeta_n) \to \mathbf{p}[\mathbf{u}](\tilde{s},\tilde{\zeta}). \tag{2.26}$$

This yields $\mathbf{p}[\mathbf{u}](s_0,\zeta_0) = \mathbf{p}[\mathbf{u}](\tilde{s},\tilde{\zeta})$, which contradicts the injectivity of $\mathbf{p}[\mathbf{u}]$ on B_0, i.e., (2.22) cannot be true. Thus $\mathbf{p}[\mathbf{u}](s,\zeta) \in \mathcal{O}^c$ for all $(s,\zeta) \in \operatorname{int}\Omega$, and this implies the assertion for the second case. $\diamondsuit$

P r o o f of Lemma 2.5. Let us fix any $\mathbf{u} \in \mathbf{X}$ that satisfies (1.10). First we show that $\mathbf{p}[\mathbf{u}](\cdot,\cdot)$ is locally injective on $\operatorname{int}\Omega$. For this purpose we choose any $(s_0,\zeta_0) \in \operatorname{int}\Omega$. Then we can find $\varepsilon > 0$ and $\delta > 0$ such that

$$\operatorname{cl} U_0 \subset \operatorname{int}\Omega, \quad \text{where} \quad U_0 := \{(s,\zeta) \in \Omega : s \in (s_0-\varepsilon, s_0+\varepsilon),\ \zeta \in (\zeta_0-\delta,\zeta_0+\delta)\}. \tag{2.27}$$

Define

$$\tilde{\zeta} := \inf_{s\in[s_0-\varepsilon,s_0+\varepsilon]} \{h_2(s) - (\zeta_0+\delta), (\zeta_0-\delta) - h_1(s)\}. \tag{2.28}$$

Then

$$h_2(s) \geq \zeta + \tilde{\zeta}, \quad h_1(s) \leq \zeta - \tilde{\zeta} \quad \text{for all} \quad (s,\zeta) \in U_0. \tag{2.29}$$

Using (1.10), we obtain

$$\nu(s) - \zeta\mu(s) > \tilde{\zeta}|\mu| \quad \text{on } U_0. \tag{2.30}$$

Since $(\bar{s}_1, \bar{s}_2) \mapsto \mathbf{a}(\theta(\bar{s}_1)) \cdot \mathbf{a}(\theta(\bar{s}_2))$ is continuous and $\|\mathbf{a}\| = 1$, we can choose ε so small that

$$\tfrac{1}{2} < \mathbf{a}(\theta(\bar{s}_1)) \cdot \mathbf{a}(\theta(\bar{s}_2)) < \tfrac{3}{2} \quad \text{for} \quad \bar{s}_1, \bar{s}_2 \in (s_0-\varepsilon, s_0+\varepsilon). \tag{2.31}$$

Observing that $\tilde{\zeta} > 0$, we can assume ε to be so small that

$$\int_{s_0-\varepsilon}^{s_0+\varepsilon} |\eta| \, d\tau < \tfrac{1}{3}\tilde{\zeta}. \tag{2.32}$$

If $\mathbf{p}[\mathbf{u}](\cdot,\cdot)$ is not injective on U_0 for given $\mathbf{u} \in \mathbf{X}$, then there are points $(s_1,\zeta_1), (s_2,\zeta_2) \in U_0$, $s_1 < s_2$ such that

$$\mathbf{p}[\mathbf{u}](s_1,\zeta_1) = \mathbf{p}[\mathbf{u}](s_2,\zeta_2). \tag{2.33}$$

Using $\tilde{\mathbf{p}}(s) := \mathbf{p}[\mathbf{u}](s,\zeta_2)$, we study

$$\begin{aligned} \Delta\mathbf{p} &:= \tilde{\mathbf{p}}(s_2) - \tilde{\mathbf{p}}(s_1) \\ &= (s_2-s_1)\int_0^1 \tilde{\mathbf{p}}'(s_1+t(s_2-s_1))\,dt \\ &= (s_2-s_1)\int_0^1 \mathbf{r}'(s_1+t(s_2-s_1)) - \zeta_2\mu(s_1+t(s_2-s_1))\mathbf{a}(\theta(s_1+t(s_2-s_1)))\,dt \\ &= (s_2-s_1)\int_0^1 \tilde{\nu}(t)\mathbf{a}(\tilde{\theta}(t)) + \tilde{\eta}(t)\mathbf{b}(\tilde{\theta}(t)) - \zeta_2\tilde{\mu}(t)\mathbf{a}(\tilde{\theta}(t))\,dt \qquad (2.34) \\ &\quad \text{where} \quad \tilde{\nu}(t) := \nu(s_1+t(s_2-s_1)), \quad \text{etc.} \end{aligned}$$

Setting $\mathbf{a}_1 := \mathbf{a}(\theta(s_1))$, we get

$$\Delta\mathbf{p}\cdot\mathbf{a}_1 = (s_2-s_1)\int_0^1 \Big(\tilde{\nu}(t) - \zeta_2\tilde{\mu}(t)\Big)\mathbf{a}(\tilde{\theta}(t))\cdot\mathbf{a}_1 + \tilde{\eta}(t)\mathbf{b}(\tilde{\theta}(t))\cdot\mathbf{a}_1\,dt. \tag{2.35}$$

Let us show that this expression is positive. We have

$$\begin{aligned} \mathbf{b}(\tilde{\theta}(t)) &= \mathbf{b}(\theta(s_1+t(s_2-s_1))) \\ &= \mathbf{b}(\theta(s_1)) - (s_2-s_1)\int_0^t \mu(s_1+\tau(s_2-s_1))\mathbf{a}(\theta(s_1+\tau(s_2-s_1)))\,d\tau. \qquad (2.36) \end{aligned}$$

Since $\mathbf{a}(\theta(s_1)) \cdot \mathbf{b}(\theta(s_1)) = 0$,

$$\mathbf{b}(\tilde{\theta}(t))\cdot\mathbf{a}_1 = -(s_2-s_1)\int_0^t \tilde{\mu}(\tau)\mathbf{a}(\tilde{\theta}(\tau))\cdot\mathbf{a}_1\,d\tau. \tag{2.37}$$

Observing (2.31), (2.32), we can estimate

$$\begin{aligned} |\mathbf{b}(\tilde\theta(t))\cdot\mathbf{a}_1| &\le (s_2-s_1)\int_0^t |\tilde\mu(\tau)||\mathbf{a}(\tilde\theta(\tau))\cdot\mathbf{a}_1|\,d\tau \\ &\le \tfrac{3}{2}(s_2-s_1)\int_0^1 |\tilde\mu(\tau)|\,d\tau, \end{aligned} \tag{2.38}$$

$$\begin{aligned} \int_0^1 |\tilde\eta(t)|\,dt &= \int_0^1 |\eta(s_1+t(s_2-s_1))|\,dt = \frac{1}{s_2-s_1}\int_{s_1}^{s_2} |\eta(\sigma)|\,d\sigma \\ &< \frac{\tilde\zeta}{3(s_2-s_1)}\ . \end{aligned} \tag{2.39}$$

In view of (2.30), (2.31), (2.38), equation (2.35) implies

$$\Delta\mathbf{p}\cdot\mathbf{a}_1 \ \ge\ \frac{(s_2-s_1)}{2}\tilde\zeta\int_0^1 |\tilde\mu(t)|\,dt \ -\ \tfrac{3}{2}(s_2-s_1)^2\int_0^1 |\tilde\eta(t)|\,dt\int_0^1 |\tilde\mu(\tau)|\,d\tau. \tag{2.40}$$

Inequalities (2.39) and (2.40) yield that

$$\Delta\mathbf{p}\cdot\mathbf{a}_1 > 0 \quad \text{if} \quad \int_0^1 |\tilde\mu(t)|\,dt > 0. \tag{2.41}$$

Otherwise, $\mu = 0$ a.e. on $[s_1, s_2]$. This implies that $\theta(s) = \theta(s_1)$ and, therefore, $\mathbf{a}(\theta(s))\cdot\mathbf{a}_1 = 1$, $\mathbf{b}(\theta(s))\cdot\mathbf{a}_1 = 0$ on $[s_1, s_2]$ (recall (1.14), (1.5)). Since $\nu > 0$ by (1.3) and (1.10), equation (2.35) gives

$$\Delta\mathbf{p}\cdot\mathbf{a}_1 \ =\ \int_0^1 \tilde\nu(t)\,dt \ >\ 0 \quad \text{if} \quad \int_0^1 |\tilde\mu(t)|\,dt = 0. \tag{2.42}$$

Recalling (1.1), (2.33), (2.41), (2.42), we get the contradiction

$$0 \ =\ \Big(\mathbf{p}(s_2,\zeta_2)-\mathbf{p}(s_1,\zeta_1)\Big)\cdot\mathbf{a}_1 \ =\ \Big(\mathbf{p}(s_2,\zeta_2)-\mathbf{p}(s_1,\zeta_2)\Big)\cdot\mathbf{a}_1 \ =\ \Delta\mathbf{p}\cdot\mathbf{a}_1 \ >\ 0. \tag{2.43}$$

Consequently (2.33) cannot be true and $\mathbf{p}[\mathbf{u}](\cdot,\cdot)$ must be injective on U_0. Since $(s_0,\zeta_0)\in \operatorname{int}\Omega$ is arbitrary, we have the local injectivity on $\operatorname{int}\Omega$. The continuity of $\mathbf{p}[\mathbf{u}](\cdot,\cdot)$ then implies that $\mathbf{p}[\mathbf{u}](\cdot,\cdot)$ maps open sets onto open sets (cf. Zeidler [46, Theorem 16.C]). $\diamondsuit$

Observe that $\mathbf{p}(\cdot,\cdot)$ is not continuously differentiable in general and, therefore, we cannot conclude in the usual way that the invertibility of the matrixes $\partial\mathbf{p}/\partial(s,\zeta)$, which follows from (1.10), is sufficient for the injectivity of $\mathbf{p}$ (cf. Zeidler [46, Theorem 4F]). Moreover, the Jacobian of $\mathbf{p}$ need not be invertible in the case of Lemma 2.8.

P r o o f of Lemma 2.8. As in the proof of Lemma 2.5, we get (2.41). For the proof of (2.42), we use (2.7) and $E(\mathbf{u}) < \infty$, which implies that $\nu > 0$ a.e. on $[0, L]$ and therefore that $\int_0^1 \tilde\nu(t)\,dt > 0$. Then we can continue as in the previous proof. $\diamondsuit$

3 Existence of minimizers

In this section we verify the existence of solutions for very general obstacle problems. If we assume that a given external force $\mathfrak{f}$ is applied to the rod, then we have the total energy

$$E(\mathbf{u}) := E_s(\mathbf{u}) + E_p(\mathbf{u}) = \int_0^L W[\mathbf{u}](s)\,ds \ -\ \int_\Omega \mathbf{p}[\mathbf{u}](\mathbf{z})\cdot d\mathfrak{f}(\mathbf{z}). \tag{3.1}$$

As in Section 2 for the intrinsic formulation, we take

$$(\nu, \eta, \mu, \mathbf{r}_0, \theta_0) = \mathbf{u} \in \mathbf{X} = \mathcal{L}^{p_1} \times \mathcal{L}^{p_2} \times \mathcal{L}^{p_3} \times \mathbb{R}^2 \times \mathbb{R}, \quad p_1, p_2, p_3 > 1, \tag{3.2}$$

$$\|\mathbf{u}\| := \|\nu\|_{p_1} + \|\eta\|_{p_2} + \|\mu\|_{p_3} + \|\mathbf{r}_0\| + |\theta_0| \tag{3.3}$$

where $\|\cdot\|_p$ denotes the usual $\mathcal{L}^p$-norm. Recall that $\mathbf{u}$ defines a configuration by (1.13), (1.14).

Now we study the following variational problem:

$$E(\mathbf{u}) \to \text{Min!}\,, \quad \mathbf{u} \in \mathbf{X}, \tag{3.4}$$

$$\mathbf{u} \;\in\; \mathbf{A}, \tag{3.5}$$

$$\mathbf{F}(\mathbf{u}) \;=\; \mathbf{0}, \tag{3.6}$$

$$\nu(s) - V(\mu(s), s) \;\geq\; 0 \quad \text{a.e. on } [0, L]. \tag{3.7}$$

Here $\mathbf{A}$ stands for either $\mathbf{A}_1$ or $\mathbf{A}_2$ and $\mathbf{F}$ is a map from $\mathbf{X}$ into a Banach space $\mathbf{Y}$, which we suppose to be weakly continuous, i.e., $\mathbf{u}_n \rightharpoonup \mathbf{u}$ implies that $\mathbf{F}(\mathbf{u}_n) \rightharpoonup \mathbf{F}(\mathbf{u})$. By (3.6) we can express, e.g., very general boundary conditions like confining $\mathbf{r}_0$ to a bounded set. As we already mentioned, we have to work with the weaker orientation-preserving condition (3.7) for existence theory, since the stronger variant (1.10) does not define a weakly closed set in $\mathbf{X}$. We later however show that solutions really satisfy the stronger condition under some mild additional assumptions.

In view of the translation invariance of the stored energy, we need some kind of boundedness for the admissible configurations in our problem, so that we can derive something like a generalized Poincaré inequality as basis for coercivity. In this connection we introduce the following notion. We say that the constraints (3.5) and (3.6) *imply pseudo-bounded configurations* if there exist linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^2$, numbers $s_1, s_2 \in [0, L]$ and $K \in \mathbb{R}$ such that

$$|\mathbf{e}_1 \cdot \mathbf{r}[\mathbf{u}](s_1)| < K, \;\; |\mathbf{e}_2 \cdot \mathbf{r}[\mathbf{u}](s_2)| < K \quad \text{for all} \quad \mathbf{u} \in \mathbf{X} \;\text{ satisfying (3.5), (3.6).} \tag{3.8}$$

This condition is fulfilled, e.g., if some point $\mathbf{r}(s_0)$ is confined to a bounded set or if $\mathbf{r}(s_1)$ and $\mathbf{r}(s_2)$ are enforced to move on nonparallel lines.

Recalling (1.1), (1.5), (1.13), (1.14), we observe a 2π-periodicity of E and $\mathbf{A}$ with respect to θ_0. Since we can suppose the same periodicity in condition (3.6) for a physically reasonable problem, we have a multiplicity in our problem which prevents coercivity for E. However, from the geometrical point of view, this multiplicity is artificial. Therefore we can demand without any loss of generality that

$$\theta_0 \in [0, 2\pi]. \tag{3.9}$$

Here we choose the closed interval, because we need compactness. Thus we again have a multiplicity. But this does not bother our analysis.

A key role for coercivity is played by the growth condition

$$W(\nu, \eta, \mu, s) \geq c(|\nu|^{p_1} + |\eta|^{p_2} + |\mu|^{p_3}) + \gamma(s) \tag{3.10}$$

where $c > 0$ is a constant and γ an integrable function.

Theorem 3.11 (Existence) . *Let $(\nu, \eta, \mu, s) \mapsto W(\nu, \eta, \mu, s)$ be measurable in s and twice continuously differentiable in the other arguments and let the growth condition* (3.10) *be satisfied. Let $\mathbf{F}$ be weakly continuous and $\mathfrak{f}$ be a given vector-valued Borel measure on Ω. Let* (3.5), (3.6) *imply pseudo-bounded configurations and let* (2.7) *hold when $\mathbf{A} = \mathbf{A}_2$. If there exists at least one admissible configuration with finite energy E, then there is a minimizer $\mathbf{u} \in \mathbf{X}$ of the variational problem* (3.4) – (3.7) *combined with* (3.9).

P r o o f. The weak convergence $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in $\mathbf{X}$ implies that

$$\nu_n \rightharpoonup \nu, \;\; \eta_n \rightharpoonup \eta, \;\; \mu_n \rightharpoonup \mu, \;\; \mathbf{r}_{0,n} \to \mathbf{r}_0, \;\; \theta_{0,n} \to \theta_0 \tag{3.12}$$

in the corresponding spaces. If $\mathbf{u}_n \to \mathbf{u}$, then all the convergences in (3.12) are strong.

Condition (3.6) defines a weakly closed subset of $\mathbf{X}$ by the weak continuity of $\mathbf{F}$. As shown in the proof of Proposition 2.10, the set determined by (3.7) is weakly closed. Obviously, restriction (3.9) also forms a weakly closed set. Condition (3.5) will be studied later.

By the Riesz Representation Theorem, there is a unique correspondence between the linear continuous functionals on the space $\mathcal{C}(\Omega, \mathbb{R}^2)$ of continuous functions and the vector-valued measures on Ω (cf. Bauer [9], Benedetto [10], Zeidler [48, Appendix (87)]). This implies that

$$\mathbf{q} \mapsto \int_\Omega \mathbf{q}(\mathbf{z}) \cdot d\mathfrak{f}(\mathbf{z}) \tag{3.13}$$

defines a linear continuous functional on $\mathcal{C}(\Omega, \mathbb{R}^2)$ satisfying

$$\Big| \int_\Omega \mathbf{q}(\mathbf{z}) \cdot d\mathfrak{f}(\mathbf{z}) \Big| \le |\mathfrak{f}| \, \|\mathbf{q}\|_{\mathcal{C}} \tag{3.14}$$

where $\| \cdot \|_{\mathcal{C}}$ denotes the supremum norm and

$$|\mathfrak{f}| := \sup_{\mathbf{e}(\cdot) \in \mathcal{C}(\Omega), \, \|\mathbf{e}\|_{\mathcal{C}} \le 1} \int_\Omega \mathbf{e}(\mathbf{z}) \cdot d\mathfrak{f}(\mathbf{z}) \tag{3.15}$$

is the total variation of the measure $\mathfrak{f}$.

Recalling (2.20), we obtain that the weak convergence of a sequence $\{\mathbf{u}_n\}$ in $\mathbf{X}$ implies the strong convergence of $\mathbf{p}[\mathbf{u}_n]$ in the space of continuous functions. Consequently (3.14) implies the weak continuity of the potential energy $\mathbf{u} \mapsto E_p(\mathbf{u})$ in $\mathbf{X}$. Since the functions h_1, h_2 are bounded on $[0, L]$, say by a constant $\tilde{c}_1 > 0$, and using (1.1), (3.14), we can find a constant $d_0 > 0$ such that

$$|E_p(\mathbf{u})| \;\le\; |\mathfrak{f}| \Big(\| \mathbf{r}[\mathbf{u}] \|_{\mathcal{C}} + 2\tilde{c}_1 \Big) \;\le\; d_0 \Big(\| \mathbf{r}[\mathbf{u}] \|_{\mathcal{C}} + 1 \Big). \tag{3.16}$$

By the continuity properties of W, the growth condition (3.10) and the convexity of W caused by (1.40), we get the weak lower-semicontinuity of $\mathbf{u} \mapsto E_s(\mathbf{u})$ by standard arguments (see also the proof of Proposition 2.10). Thus the total energy $E(\cdot)$ is weakly lower-semicontinuous.

In order to study the coercivity of E, we first need some preliminary investigations. Using (1.6), (3.8) and the Hölder inequality, we can find a constant $d_1 > 0$ such that

$$\begin{aligned} |\mathbf{r}(s) \cdot \mathbf{e}_i| &= \Big| \mathbf{r}(s_i) \cdot \mathbf{e}_i + \int_{s_i}^s \mathbf{r}' \cdot \mathbf{e}_i \, d\tau \Big| \\ &\le |\mathbf{r}(s_i) \cdot \mathbf{e}_i| + \int_{s_i}^s |\, \nu \, \mathbf{a} \cdot \mathbf{e}_i \,| d\tau + \int_{s_i}^s |\, \eta \, \mathbf{b} \cdot \mathbf{e}_i \,| d\tau \\ &\le K + d_1(\|\nu\|_{p_1} + \|\eta\|_{p_2}) \quad \text{for all } \; s \in [0, L], \;\; i = 1, 2, \end{aligned} \tag{3.17}$$

if $\mathbf{r}(\cdot)$ corresponds to any $\mathbf{u}$ that satisfies the side conditions (3.5) and (3.6). By the equivalence of norms in $\mathbb{R}^2$, this implies that for suitable constants $d_2, d_3 > 0$,

$$\begin{aligned} \|\mathbf{r}(s)\| &\leq d_2(\,|\mathbf{r}(s)\cdot\mathbf{e}_1| + |\mathbf{r}(s)\cdot\mathbf{e}_2|\,) \\ &\leq d_3(\,\|\nu\|_{p_1} + \|\eta\|_{p_2} + 1\,) \quad \text{for all } s \in [0, L]. \end{aligned} \tag{3.18}$$

Consequently, if $\mathbf{u}$ respects the side conditions (3.5) and (3.6), then

$$\|\mathbf{r}_0\| \leq d_3(\,\|\nu\|_{p_1} + \|\eta\|_{p_2} + 1\,), \quad \|\,\mathbf{r}[\mathbf{u}]\,\|_{\mathcal{C}} \leq d_3(\,\|\nu\|_{p_1} + \|\eta\|_{p_2} + 1\,). \tag{3.19}$$

This is something like a generalized Poincaré inequality and is essentially based on the boundedness condition (3.8).

Since θ_0 is bounded by (3.9) and since $\|\mathbf{r}_0\|$ can be estimated by (3.19), for an admissible $\mathbf{u}$ we have that

$$\|\mathbf{u}\| \to \infty \quad \text{implies} \quad \|\nu\|_{p_1} + \|\eta\|_{p_2} + \|\mu\|_{p_3} \to \infty. \tag{3.20}$$

From the growth condition (3.10), we can derive

$$\begin{aligned} E_s(\mathbf{u}) &\geq \int_0^L \Big[\, c\Big(\,|\nu(s)|^{p_1} + |\eta(s)|^{p_2} + |\mu(s)|^{p_3}\,\Big) + \gamma(s)\,\Big]\, ds \\ &\geq c(\,\|\nu\|_{p_1}^{p_1} + \|\eta\|_{p_2}^{p_2} + \|\mu\|_{p_3}^{p_3}\,) + d_4, \quad d_4 > 0. \end{aligned} \tag{3.21}$$

Using the estimates (3.16) and (3.19), we get

$$|E_p(\mathbf{u})| \leq d_5(\,\|\nu\|_{p_1} + \|\eta\|_{p_2} + 1\,), \quad d_5 > 0. \tag{3.22}$$

Therefore,

$$\begin{aligned} E(\mathbf{u}) &\geq c(\,\|\nu\|_{p_1}^{p_1} + \|\eta\|_{p_2}^{p_2} + \|\mu\|_{p_3}^{p_3}\,) + d_4 - d_5(\,\|\nu\|_{p_1} + \|\eta\|_{p_2} + 1\,) \\ &= \|\nu\|_{p_1}(\,c\|\nu\|_{p_1}^{p_1-1} - d_5\,) + \|\eta\|_{p_2}(\,c\|\eta\|_{p_2}^{p_2-1} - d_5\,) + c\|\mu\|_{p_3}^{p_3} + d_4 - d_5. \end{aligned} \tag{3.23}$$

Recalling (3.20), we obtain the coercivity of E for admissible configurations, i.e.,

$$E(\mathbf{u}) \to \infty \quad \text{as} \quad \|\mathbf{u}\| \to \infty \tag{3.24}$$

as long as $\mathbf{u}$ respects the side conditions (3.5), (3.6), (3.9).

Since there exists an admissible configuration $\tilde{\mathbf{u}} \in \mathbf{X}$ with finite energy E, to minimize E we can restrict our attention to the set

$$\tilde{\mathbf{A}}_0 := \{\mathbf{u} \in \mathbf{X} :\ E(\mathbf{u}) \leq E(\tilde{\mathbf{u}}),\ \ \mathbf{u} \text{ satisfies (3.5), (3.6), (3.7), (3.9)}\}. \tag{3.25}$$

The coercivity (3.24) gives that $\tilde{\mathbf{A}}_0$ is bounded.

By (3.22), $E_p(\cdot)$ is bounded on the bounded set $\tilde{\mathbf{A}}_0$. Since $E(\cdot)$ is bounded from above on $\tilde{\mathbf{A}}_0$ by definition, $E_s(\cdot)$ must also be bounded from above on $\tilde{\mathbf{A}}_0$, say by a constant $\tilde{c} > 0$. Therefore, we can replace $\mathbf{A}$ in (3.5) with the subset

$$\breve{\mathbf{A}} := \{\mathbf{u} \in \mathbf{A} :\ E_s(\mathbf{u}) \leq \tilde{c}\}, \tag{3.26}$$

and we can restrict the minimization to the set

$$\tilde{\mathbf{A}} := \{\mathbf{u} \in \mathbf{X} : \; E(\mathbf{u}) \leq E(\tilde{\mathbf{u}}), \;\; \mathbf{u} \in \breve{\mathbf{A}}, \;\; \mathbf{u} \text{ satisfies (3.6), (3.7), (3.9)}\}. \tag{3.27}$$

$\tilde{\mathbf{A}}$ is bounded as subset of $\tilde{\mathbf{A}}_0$. As we have already shown, the sets given by (3.6), (3.7), (3.9) are weakly closed and E and E_s are weakly lower-semicontinuous. Proposition 2.10 tells us that $\mathbf{A}_1$ is weakly closed and, for the case $\mathbf{A} = \mathbf{A}_2$, that all $\mathbf{u} \in \breve{\mathbf{A}}$ satisfying (3.7) form a weakly closed set. Thus $\tilde{\mathbf{A}}$ is weakly compact. By the Theorem of Weierstrass, the weakly lower-semicontinuous funtional E admits a minimum on the weakly compact set $\tilde{\mathbf{A}}$ which satisfies our variational problem. $\diamondsuit$

Let us now discuss the existence with respect to the extrinsic formulation. For this purpose, we mainly have to study the properties of the stored energy W under transformation (1.16). The transformation of the side conditions does not provide serious problems in this connection. The stored energy has the form

$$(x', y', \theta', \theta, s) \mapsto \tilde{W}(x', y', \theta', \theta, s) \tag{3.28}$$

The convexity and the coercivity of $\tilde{W}$ in the highest derivatives are important for the existence problem. As we can see in the following lemma, the convexity is not influenced by the transformation.

Lemma 3.29 (Convexity) *If $W(\cdot,\cdot,\cdot,s)$ is twice continuously differentiable and convex, then the transformed function $\tilde{W}(\cdot,\cdot,\cdot,\theta,s)$ is also twice continuously differentiable and convex.*

Before we give the proof, let us still discuss the coercivity. As shown by Antman [3], the stored energy must be a function of (ν, η, μ) which are the natural strains for the rod model. Here every strain has a different material response and, therefore, (3.10) is a natural growth condition for the stored energy. However, if we intend to adopt the ideas of the proof of Theorem 3.11 for the extrinsic formulation, then we need an equivalent growth condition for $\tilde{W}$ with respect to (x', y', θ').

Lemma 3.30 (Coercivity) *The growth condition* (3.10) *implies that*

$$\tilde{W}(x', y', \theta', \theta, s) \geq \tilde{c}(\, |x'|^{\tilde{p}} + |y'|^{\tilde{p}} + |\theta'|^{p_3} \,) + \tilde{\gamma}(s) \tag{3.31}$$

where $\tilde{p} = \min\{p_1, p_2\}$, $\tilde{c} > 0$ is a constant and $\tilde{\gamma}$ is an integrable function. Moreover, the exponent $\tilde{p}$ in (3.31) *is optimal and the growth conditions* (3.10) *and* (3.31) *are equivalent in the case $p_1 = p_2$ only.*

We see that the desired equivalence is not fulfilled. In fact this equivalence cannot be expected, because x' and y' are independent of a special material direction. In paticular if $\mathbf{a} = \mathbf{i}$, then x' behaves like ν, and if $\mathbf{b} = \mathbf{i}$, then x' behaves like η. This means that the behavior of x' depends on the direction θ, and one always has the worst exponent $\tilde{p}$ for a special θ. The same is valid for y'. We can conclude that we lose some information about the growth of W if we use coordinates which are independent of special material directions. Possibly one could bypass this problem by working in a suitable Orlicz space. However, this would provide other technical difficulties, since one has to imitate in fact the natural situation from (3.10) with unsuitable coordinates.

In summary, we can formulate an existence result analogous to Theorem 3.11 for the extrinsic formulation, but only with less efficient growth condition for the stored energy.

P r o o f of Lemma 3.29. Recall the transformation (1.16). The differentiability of $\tilde{W}$ is a simple consequence of the chain rule. We set

$$\mathbf{v} := (\nu, \eta, \mu), \quad \mathbf{z} := (x', y', \theta'). \tag{3.32}$$

Suppressing the dependence on s, θ, we consider W and $\tilde{W}$ as functions of $\mathbf{v}$ and $\mathbf{z}$, respectively. By the chain rule we get

$$\frac{\partial}{\partial \mathbf{z}} \tilde{W}(\mathbf{z}) = \frac{\partial}{\partial \mathbf{z}} W(\mathbf{v}(\mathbf{z})) = \frac{\partial W}{\partial \mathbf{v}}(\mathbf{v}(\mathbf{z})) \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{z}}(\mathbf{z}) = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{z}}\right)^{*}(\mathbf{z}) \cdot \frac{\partial W}{\partial \mathbf{v}}(\mathbf{v}(\mathbf{z})) \tag{3.33}$$

(here * denotes the adjoint matrix),

$$\frac{\partial^2}{\partial \mathbf{z}^2}\tilde{W}(\mathbf{z}) = \frac{\partial}{\partial \mathbf{z}}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{z}}\right)^*(\mathbf{z})\cdot\frac{\partial W}{\partial \mathbf{v}}(\mathbf{v}(\mathbf{z})) \;+\; \left(\frac{\partial \mathbf{v}}{\partial \mathbf{z}}\right)^*(\mathbf{z})\cdot\frac{\partial^2 W}{\partial \mathbf{v}^2}(\mathbf{v}(\mathbf{z}))\cdot\frac{\partial \mathbf{v}}{\partial \mathbf{z}}(\mathbf{z}). \tag{3.34}$$

Observing (1.16), we find

$$\frac{\partial \mathbf{v}}{\partial \mathbf{z}}(\mathbf{z}) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and, thus,} \quad \frac{\partial}{\partial \mathbf{z}}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{z}}\right)^*(\mathbf{z}) = 0 \tag{3.35}$$

for all $\mathbf{z}$. Consequently,

$$\frac{\partial^2}{\partial \mathbf{z}^2}\tilde{W} \;=\; \left(\frac{\partial \mathbf{v}}{\partial \mathbf{z}}\right)^*\cdot\frac{\partial^2 W}{\partial \mathbf{v}^2}\cdot\frac{\partial \mathbf{v}}{\partial \mathbf{z}}. \tag{3.36}$$

Since $\partial^2 W/\partial \mathbf{v}^2$ is a positive-definite matrix by the convexity of W and since $\partial \mathbf{v}/\partial \mathbf{z}$ is a regular matrix, $\partial^2\tilde{W}/\partial \mathbf{z}^2$ is also positive-definite, i.e., $\tilde{W}$ is convex in $\mathbf{z}$. ◇

P r o o f of Lemma 3.30. Obviously,

$$|\nu|^{p_1} \ge |\nu|^{\tilde{p}} - 1 \quad \text{and} \quad |\eta|^{p_2} \ge |\eta|^{\tilde{p}} - 1 \quad \text{for} \quad \nu, \eta \in \mathbb{R}. \tag{3.37}$$

Therefore, (3.10) implies that

$$W(\nu,\eta,\mu,s) \ge c(\,|\nu|^{\tilde{p}} + |\eta|^{\tilde{p}} + |\mu|^{p_3}\,) + \gamma(s) - 2c. \tag{3.38}$$

If we exploit the equivalence of norms in $\mathbb{R}^2$ and use the transformation formulas (1.16), then we get

$$\begin{aligned} |\nu|^{\tilde{p}} + |\eta|^{\tilde{p}} \;&\ge\; d_1(\nu^2+\eta^2)^{\tilde{p}/2} \\ &=\; d_1\Big(\,(x'\cos\theta + y'\sin\theta)^2 + (-x'\sin\theta + y'\cos\theta)^2\Big)^{\tilde{p}/2} \\ &=\; d_1(|x'|^2+|y'|^2)^{\tilde{p}/2} \;\ge\; d_2(|x'|^{\tilde{p}} + |y'|^{\tilde{p}}), \quad d_1, d_2 > 0. \end{aligned} \tag{3.39}$$

Inequalities (3.38), (3.39) yield condition (3.31). The optimality of the exponent $\tilde{p}$ can be seen from (3.10) if we choose $\theta = 0$ and $\theta = \frac{\pi}{2}$ in (1.16). This also means that we cannot have equivalence between (3.10) and (3.31) in the case of $p_1 \ne p_2$. If $p_1 = p_2 = p$, then this equivalence follows from the estimates

$$\begin{aligned} |x'|^p + |y'|^p \;&\ge\; d_3(|x'|^2+|y'|^2)^{p/2} = d_3(|\nu|^2+|\eta|^2)^{p/2} \\ &\ge\; d_4(|\nu|^p + |\eta|^p), \qquad d_3, d_4 > 0. \end{aligned} \tag{3.40}$$

Use (1.13) for the inverse transformation of (1.16). ◇

4 Euler-Lagrange equations

We now study the regularity of solutions of the variational problem. Under some mild additional assumptions we derive the Euler-Lagrange equations, which are equivalent to the equilibrium conditions introduced in Section 1, as necessary condition. It follows that the orientation-preserving condition must be fulfilled in the strong variant (1.10). Furthermore, we obtain very detailed information about the contact forces, i.e., they correspond to a vector-valued Borel measure that is supported on the contact set and directed, roughly speaking, normal to the obstacle. However, the case in which the position of a point of the rod is confined to lie on the boundary of the obstacle cannot be handled completely, because the side conditions are not independent in this case and, consequently, one cannot verify the normality for the Lagrange Multiplier Rule in general. As far as I know, such cases are always excluded in regularity investigations in the literature (see, e.g., Kinderlehrer & Stampacchia [27]).

Clarke's calculus of generalized gradients $\partial g(\cdot)$ for locally Lipschitz continuous functionals is a fundamental tool for our analysis. We actually use this concept in the main result in this section, though it is formally introduced only at the beginning of the next section. Readers who are not familiar with this concept could replace it for a moment with the subdifferential for convex functionals or the derivative for continuously differentiable funtionals.

For our regularity investigations it is reasonable to describe the obstacle by an inequality side condition. We accordingly choose the method introduced in the discussion centered on (2.14) and, roughly speaking, restrict our attention to obstacles satisfying (2.17). Furthermore, we specialize the general side condition (3.6) to prescribe restrictions for $\mathbf{p}(\cdot,\cdot)$, $\theta(\cdot)$ at a finite number of points in Ω. Thus we consider solutions of the variational problem:

$$E(\mathbf{u}) \to \text{Min!}, \quad \mathbf{u} \in \mathbf{X}, \tag{4.1}$$

$$g_0(\mathbf{u}) := \max_{(s,\zeta)\in\Omega} d(\mathbf{p}[\mathbf{u}](s,\zeta)) \leq 0, \tag{4.2}$$

$$g_i(\mathbf{u}) := \tilde{g}_i(\mathbf{p}[\mathbf{u}](s_1,\zeta_1), \theta[\mathbf{u}](s_1), \dots, \mathbf{p}[\mathbf{u}](s_m,\zeta_m), \theta[\mathbf{u}](s_m)) = 0, \quad i = 1,\dots,n, \tag{4.3}$$

$$\nu(s) - V(\mu(s), s) \geq 0 \quad \text{a.e. on } [0,L]. \tag{4.4}$$

Here $\mathbf{X}$ is the same Banach space as used in the previous section (cf. (3.2)). $d : \mathbb{R}^2 \mapsto \mathbb{R}$ is the function given by

$$d(\mathbf{q}) := \text{dist}_{\mathcal{O}^c}\mathbf{q} - \text{dist}_{\mathcal{O}}\mathbf{q}, \quad \mathbf{q} \in \mathbb{R}^2, \tag{4.5}$$

where $\mathcal{O}$ denotes the obstacle (cf. Section 2), $(s_j, \zeta_j) \in \Omega$ are prescribed points for $j = 1,\dots,m$ and

$$\tilde{g}_i : (\mathbb{R}^2 \times \mathbb{R})^m \mapsto \mathbb{R}, \quad i = 1,\dots,n, \tag{4.6}$$

are given functions. While (4.2) describes the obstacle, (4.3) can impose boundary conditions or more general "concentrated" side conditions. We assume that the functions $\tilde{g}_i$ are continuously differentiable and that the matrix

$$\frac{\partial(\tilde{g}_1,\dots,\tilde{g}_n)}{\partial(\mathbf{p}_1,\theta_1,\dots,\mathbf{p}_m,\theta_m)} \quad \text{is regular if} \quad \tilde{g}_i(\mathbf{p}_1,\theta_1,\dots,\mathbf{p}_m,\theta_m) = 0 \ \text{ for all } \ i = 1,\dots,n. \tag{4.7}$$

Here regularity means that the matrix maps onto $\mathbb{R}^n$.

We call $s_c \in [0,L]$ a *contact parameter* of a configuration $\mathbf{u}$ if $d(\mathbf{p}[\mathbf{u}](s_c,\zeta)) = 0$ for some $\zeta \in [h_1(s_c), h_2(s_c)]$. A configuration $\mathbf{u}$ is said to have *regular contact* if for every contact parameter $s_c \in [0,L]$ there is an open neighborhood $\mathcal{I}(s_c) \subset \mathbb{R}$ such that

$$\mathbf{0} \notin \overline{\text{co}}\, \{\partial d(\mathbf{q}) |\, d(\mathbf{q}) = 0, \ \mathbf{q} = \mathbf{p}[\mathbf{u}](s,\zeta) \ \text{for some} \ (s,\zeta) \in \Omega_{\mathcal{I}(s_c)}\}, \tag{4.8}$$

where $\Omega_{\mathcal{I}(s_c)} := \{(s,\zeta) \in \Omega |\, s \in \mathcal{I}(s_c)\}$. This is in fact a condition for both the configuration and the obstacle. It is used to prevent the reactions in the rod from being enforced by the contact reactions only. For a large class of obstacles, (4.8) is fulfilled by all configurations. Roughly speaking, if the complement of an obstacle with piecewise smooth boundary and only Lipschitz corners is so "thick" that the rod can touch the obstacle near a contact parameter s_c only along either the bottom or the top curve (i.e. $\zeta = h_1(s)$ or $\zeta = h_2(s)$), then the previous condition is always satisfied (cf. also (2.17)).

A main difficulty for regularity investigations is to handle the singularity caused by (1.41) – (1.43) on the boundary of the set described by $\nu - V(\mu, s) > 0$ (observe (1.48)), because this singularity yields bad differentiability properties for the total stored energy E_s, and the standard technique with growth restrictions for the derivatives W_ν, W_η, W_μ cannot be employed. For this reason, we impose the natural condition that a total compression implies an infinite stored energy W:

$$W(\nu, \eta, \mu, s) \to \infty \quad \text{as} \quad \nu - V(\mu, s) \to 0. \tag{4.9}$$

Otherwise, we would get rods with finite total stored energy E_s which are compressed to length zero on a whole interval. (Observe that we already used this condition in Section 2.) Assuming (4.9) instead of the usual growth restrictions, we can adopt a modified version of a method used by Antman for problems without obstacles to bypass the difficulties with the singularity (cf. Antman [3, Chapter VII.5]).

Theorem 4.10 (Euler-Lagrange equations) *Let W be measurable in s, continuously differentiable in the other arguments and satisfy* (4.9)*. Let the functions $\tilde{g}_i$ be continuously differentiable with* (4.7)*. Let $\mathbf{u} = (\nu, \eta, \mu, \mathbf{r}_0, \theta_0)$ be a solution of the variational problem* (4.1) – (4.4) *with regular contact, such that all points s_j, $j = 1, \ldots, m$ are not contact parameters. Define*

$$\begin{aligned} \mathbf{n}[\mathbf{u}](s) &:= W_\nu(\nu(s), \eta(s), \mu(s), s)\mathbf{a}(\theta[\theta_0, \mu](s)) + W_\eta(\nu(s), \eta(s), \mu(s), s)\mathbf{b}(\theta[\theta_0, \mu](s)), && (4.11)\\ \mathbf{m}[\mathbf{u}](s) &:= W_\mu(\nu(s), \eta(s), \mu(s), s)\,\mathbf{k}. && (4.12) \end{aligned}$$

Then there exist a vector-valued Borel measure $\mathfrak{f}_c$ on Ω and vector-valued step functions $\mathbf{f}_s$, $\mathbf{l}_{f_s}$, $\mathbf{l}_s$ with jumps at most at $s = s_j$ such that the following Euler-Lagrange equations are valid a.e. on $[0, L]$:

$$\begin{aligned} \mathbf{0} &= \mathbf{n}[\mathbf{u}](s) - \mathbf{f}(s) - \mathbf{f}_c(s) - \mathbf{f}_s(s), && (4.13)\\ \mathbf{0} &= \mathbf{m}[\mathbf{u}](s) - \int_s^L \mathbf{r}'(\tau) \times \mathbf{n}[\mathbf{u}](\tau)\, d\tau - \mathbf{l}_f(s) - \mathbf{l}_{f_c}(s) - \mathbf{l}_{f_s}(s) - \mathbf{l}_s(s), && (4.14)\\ \mathbf{0} &= -\,\mathbf{f}(0) - \mathbf{f}_c(0) - \mathbf{f}_s(0), && (4.15)\\ \mathbf{0} &= -\int_0^L \mathbf{r}'(\tau) \times \mathbf{n}[\mathbf{u}](\tau)\, d\tau - \mathbf{l}_f(0) - \mathbf{l}_{f_c}(0) - \mathbf{l}_{f_s}(0) - \mathbf{l}_s(0), && (4.16) \end{aligned}$$

where

$$\mathbf{f}_c(s) := \int_{\Omega_s} d\mathfrak{f}_c(\tau, \zeta), \quad \mathbf{l}_{f_c}(s) := \int_{\Omega_s} \zeta \mathbf{b}[\mathbf{u}](\tau) \times d\mathfrak{f}_c(\tau, \zeta). \tag{4.17}$$

Moreover,

$$\operatorname{ess\,inf}_{s \in [0,L]} \Big(\nu(s) - V(\mu(s), s)\Big) \;>\; 0, \tag{4.18}$$

$$\mathbf{n}[\mathbf{u}](s),\ \mathbf{m}[\mathbf{u}](s) \quad \textit{are essentially bounded on } [0, L]. \tag{4.19}$$

The measure $\mathfrak{f}_c$ describes the contact force exerted by the obstacle on the rod and is supported on the contact set

$$\Omega_{\mathbf{u}} := \{(s, \zeta) \in \Omega :\ d(\mathbf{p}[\mathbf{u}](s, \zeta)) = 0\}. \tag{4.20}$$

More precisely, there exist a real non-negative Borel measure ρ on Ω supported on $\Omega_{\mathbf{u}}$ and a ρ-integrable mapping $(s, \zeta) \mapsto \mathbf{d}^*(s, \zeta) \in \partial d(\mathbf{p}[\mathbf{u}](s, \zeta))$ (∂ with respect to $d(\cdot)$) such that

$$\mathfrak{f}_c(\breve{\Omega}) \;=\; -\int_{\breve{\Omega}} \mathbf{d}^*(s, \zeta)\, d\rho(s, \zeta) \quad \textit{for all Borel sets } \breve{\Omega} \subset \Omega. \tag{4.21}$$

$\mathbf{l}_{f_c}$ *is the distribution of the induced couple of* $\mathfrak{f}_c$.

The step functions $\mathbf{f}_s$, $\mathbf{l}_{f_s}$ *and* $\mathbf{l}_s$ *are the distributions of concentrated forces, the corresponding induced couples, and other concentrated couples, respectively, caused by the side conditions* (4.3). *In particular, there are numbers* $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$, *such that*

$$\mathbf{f}_s(s) \;=\; \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} \breve{\chi}_j(s)\mathbf{p}_{ij}^*, \quad \mathbf{l}_s(s) \;=\; \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} \breve{\chi}_j(s)\theta_{ij}^*, \tag{4.22}$$

$$\mathbf{l}_{f_s}(s) \;=\; \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} \breve{\chi}_j(s)\zeta_j \mathbf{b}(\theta(s_j)) \times \mathbf{p}_{ij}^*, \tag{4.23}$$

where

$$\mathbf{p}_{ij}^* \;:=\; \frac{\partial \tilde{g}_i}{\partial \mathbf{p}_j}(\mathbf{p}[\mathbf{u}](s_1,\zeta_1), \theta[\mathbf{u}](s_1), \ldots, \mathbf{p}[\mathbf{u}](s_m,\zeta_m), \theta[\mathbf{u}](s_m)), \tag{4.24}$$

$$\theta_{ij}^* \;:=\; \frac{\partial \tilde{g}_i}{\partial \theta_j}(\mathbf{p}[\mathbf{u}](s_1,\zeta_1), \theta[\mathbf{u}](s_1), \ldots, \mathbf{p}[\mathbf{u}](s_m,\zeta_m), \theta[\mathbf{u}](s_m)), \tag{4.25}$$

$$\breve{\chi}_j(s) = \begin{cases} 1 & \text{for} \quad s \in [0, s_j], \\ 0 & \text{for} \quad s \in (s_j, L]. \end{cases} \tag{4.26}$$

Corollary 4.27 *Theorem* 4.10 *holds without the assumption that* s_{j_0} *not be contact parameter under the following condition:*

If s_{j_0} *is a contact parameter of* $\mathbf{u}$, *then the sets*

$$\Lambda_{j_0} \;:=\; \text{span}\,\{\frac{\partial}{\partial \mathbf{r}_{j_0}}\tilde{g}_1[\mathbf{u}], \ldots, \frac{\partial}{\partial \mathbf{r}_{j_0}}\tilde{g}_n[\mathbf{u}]\}, \tag{4.28}$$

$$\Sigma_{j_0} \;:=\; \overline{\text{co}}\,\{\partial d(\mathbf{q}) |\, d(\mathbf{q}) = \mathbf{0},\ \mathbf{q} = \mathbf{p}[\mathbf{u}](s_{j_0}, \zeta) \ \textit{for some}\ \zeta \in [h_1(s_{j_0}), h_2(s_{j_0})]\} \tag{4.29}$$

are disjoint.

Remark 4.30 .

1) Recalling Lemma 2.8 we can see that $\Omega_{\mathbf{u}} \subset \partial\Omega$ under some very natural additional conditions which we already assumed for the existence in Theorem 3.11.

2) The condition $\mathbf{d}^*(s,\zeta) \in \partial d(\mathbf{p}[\mathbf{u}](s,\zeta))$ in fact means that, for parameters (s,ζ) with $d(\mathbf{p}[\mathbf{u}](s,\zeta)) = 0$, the vector $\mathbf{d}^*(s,\zeta)$ belongs to the normal cone of $\mathcal{O}^c$ at the point $\mathbf{p}[\mathbf{u}](s,\zeta)$. Equation (4.21) then tells us that the contact forces are directed normal to the obstacle. In this way we obtain the usually prescribed contact condition that the tangential components of the traction vanish in the case of contact without friction. The result that the contact force is normal to the obstacle can be sharpened in the case where the rod touches the obstacle at a corner.

3) Let us denote by $\mathfrak{f}_s$ the measure on Ω that is given by the concentrations $\sum_{i=1}^n \lambda_i \mathbf{p}_{ij}^*$ at the points (s_j, ζ_j), $j = 1, \ldots, m$. Then $\mathbf{f}_s$ is just the corresponding distribution according to formula (1.31) and $\mathbf{l}_{f_s}$ is the distribution of the induced couple $\mathfrak{l}_{\mathfrak{f}_s}$ according to (1.32). $\mathbf{l}_s$ is the distribution of pure couples caused by constraints for the angles $\theta(s_j)$.

4) The Euler-Lagrange equations (4.13) – (4.16) coincide with the equilibrium equations (1.33), (1.34) combined with (1.22) and provide very detailed information about the structure of the contact forces.

5) Observe that we get the Euler-Lagrange equations in natural way only almost everywhere on $[0, L]$. Thus we are led to the discussion in Section 1.3 again. Since all functions except $\mathbf{n}[\mathbf{u}](\cdot)$ and $\mathbf{m}[\mathbf{u}](\cdot)$ in (4.13) and (4.14) are BV-functions, $\mathbf{n}[\mathbf{u}](\cdot)$ and $\mathbf{m}[\mathbf{u}](\cdot)$ also have to be BV-functions. In this way we can once more justify the necessity and sufficiency of a setting like (1.47).

6) The condition in Corollary 4.27 roughly means that the equality side conditions and the obstacle cannot cause concentrated forces at $s = s_{j_0}$ in the opposite direction and, consequently, the side conditions are linearly independent.

Obviously the sets Λ_{j_0} and Σ_{j_0} lie in $\mathbb{R}^2$, because we study planar configurations. Since Λ_{j_0} is a linear space, it must be a line or only the origin. Otherwise Λ_{j_0} and Σ_{j_0} cannot be disjoint. If $\Lambda_{j_0} = \{\mathbf{0}\}$, then the condition of the Corollary is always fulfilled by (4.8). This situation is met if all functions $\tilde{g}_i$ are independent of $\mathbf{r}_{j_0}$. If Λ_{j_0} is a line, then both sets are disjoint, roughly speaking, in the case where the point $\mathbf{r}(s_{j_0})$ is restricted by (4.3) to move along a curve which intersects the boundary of the obstacle transversally. The condition of the Corollary is not fulfilled if (4.3) prescribes the position of $\mathbf{r}(s_{j_0})$ on the boundary of the obstacle.

7) The conditions for the functions $\tilde{g}_i$ can be weakened; e.g., that they need only be locally Lipschitz continuous. For this purpose one can work with partial generalized gradients (cf. Clarke [13]). However, we do not intend to complicate our analysis with such technicalities.

8) One could try to prove a similar result in terms of the extrinsic formulation in the space $\mathbf{X}_0$ (cf. (2.13)). In this case the stored energy depends on $(x', y', \theta', \theta, s)$ (cf. (1.16)). However, the singularity of W prevents the imposition of the usual growth condition on the derivatives of W. To handle the bad differentiability properties of E_s in the proof of Theorem 4.10, we employ a subtle method using variations only on some measurable subsets of $[0, L]$ for the intrinsic formulation. With respect to the extrinsic formulation this causes very serious difficulties in getting suitable variations for θ and is probably not applicable. This observation completely justifies the use of the intrinsic formulation.

Theorem 4.31 *Let the assumptions of Theorem* 4.10 *be fulfilled, let* $W(\cdot,\cdot,\cdot,s)$ *be strictly convex for all* $s \in [0, L]$*, and let* W *be continuous in* s*. If* $\mathbf{u} = (\nu, \eta, \mu, \mathbf{r}_0, \theta_0)$ *is a solution of the variational problem, then* ν, η, μ *belong to* $\mathcal{L}^\infty$ *and, therefore,* $\mathbf{r}$ *and* θ *are Lipschitz continuous.*

If, in addition, W *is independent of* s *and twice continuously differentiable in the other arguments, then* ν, η, μ *are actually* BV*-functions.*

5 Generalized gradients of Clarke

Let us now give a short introduction to Clarke's generalized gradients for locally Lipschitz continuous functionals. This calculus is a fundamental tool for handling nonsmooth problems. A comprehensive exposition can be found in Clarke [13]. We also provide some special results which we need for our regularity investigations, i.e., we prepare the computation of the generalized gradient $\partial g_0(\mathbf{u})$ (cf. (4.2)).

Let X be a Banach space and $f : X \mapsto \mathbb{R}$ a locally Lipschitz continuous functional. The *generalized directional derivative* $f^0(u;h)$ of f at u in the direction h is given by

$$f^0(u;h) := \limsup_{v\in X,\, v\to u,\, t\to +0} \frac{f(v+th)-f(v)}{t}\,. \tag{5.1}$$

Then we define the *generalized gradient* $\partial f(u)$ of f at u as the set

$$\partial f(u) := \{f^* \in X^* : \langle f^*, h\rangle \le f^0(u;h) \text{ for all } h \in X\}. \tag{5.2}$$

$\partial f(u)$ is a nonempty, bounded, convex and weak*-compact subset of X^*. If f is continuously differentiable, then $\partial f(u)$ is the singleton $\{f'(u)\}$. For convex functionals, $f^0(u;h)$ is the usual one-sided directional derivative and $\partial f(u)$ is the subdifferential of convex analysis.

Let us summerize some additional properties of the generalized gradient for our analysis (cf. Clarke [13]).

Proposition 5.3 *Let f be Lipschitz continuous near $u \in X$ and let l_0 be its Lipschitz constant near u.*

1) $\|f^*\| \le l_0$ for all $f^* \in \partial f(u)$.

2.1) $\partial(\alpha f)(u) = \alpha \partial f(u)$ for all $\alpha \in \mathbb{R}$.

2.2) $\partial \sum_{i=1}^n f_i(u) \subset \sum_{i=1}^n \partial f_i(u)$ for locally Lipschitz continuous functionals f_i.

3) If $\{u_i\} \subset X$ and $\{f_i^*\} \subset X^*$ are sequences with $f_i^* \in \partial f(u_i)$, $u_i \to u$ and $f_i^* \overset{*}{\rightharpoonup} f^*$ for some $f^* \in X^*$, then $f^* \in \partial f(u)$.

4) (Chain Rule). Let Y be a Banach space, $F : X \mapsto Y$ continuously differentiable at $u \in X$ and $d : Y \mapsto \mathbb{R}$ Lipschitz continuous near $F(u)$. Then $g := d \circ F$ is Lipschitz continuous near u and

$$\partial g(u) \subset \partial d(F(u)) \circ F'(u), \tag{5.4}$$

i.e., for $g^* \in \partial g(u)$ there exists $d^* \in \partial d(F(u))$ such that

$$\langle g^*, w\rangle = \langle d^* \circ F'(u), w\rangle = \langle d^*, F'(u)\, w\rangle_{Y^*\times Y} \quad \text{for all} \quad w \in X. \tag{5.5}$$

5.1) (Minimum). If f attains a local minimum (or maximum) at u, then $0 \in \partial f(u)$.

5.2) (Lagrange Multiplier Rule). Assume that $g_0, g_1, \ldots, g_n : X \mapsto \mathbb{R}$ are locally Lipschitz continuous. If u is a local minimizer of f subject to the restrictions $g_0(v) \le 0$ and $g_i(v) = 0$, $i = 1, \ldots, n$, then there exist constants $\lambda_f, \lambda_0 \ge 0$, and $\lambda_i \in \mathbb{R}$, not all zero, such that

$$0 \;\in\; \lambda_f \partial f(u) + \lambda_0 \partial g_0(u) + \sum_{i=1}^n \lambda_i \partial g_i(u) \tag{5.6}$$

and $\lambda_0 g_0(u) = 0$.

To handle the inequality side condition (4.2), we study the generalized gradient of functionals of the following type

$$g(v) := \max_{\xi\in\Omega} d(p(v,\xi))\,. \tag{5.7}$$

We assume that

(i) X, Y are Banach spaces where Y is supposed to be reflexive and Ω is a metrizable compact topological space,

(ii) $p : X \times \Omega \mapsto Y$ is continuous and $v \mapsto p(v, \xi)$ is differentiable for all $\xi \in \Omega$ such that the derivative $p_v(\cdot, \cdot)$ is continuous on $X \times \Omega$,

(iii) $d : Y \mapsto \mathbb{R}$ is Lipschitz continuous.

Since Ω is compact, g is well defined and

$$\Omega(v) := \{\xi \in \Omega : \, g(v) = d(p(v, \xi))\} \tag{5.8}$$

is a nonempty closed subset of Ω. At the end of this section, we prove

Lemma 5.9 *g is locally Lipschitz continuous on X.*

We can describe the generalized gradient of g as a composition of $\partial d(\cdot)$ and $p_v(\cdot, \cdot)$. Let us denote the set of all regular probability Borel measures on Ω supported on $\breve{\Omega} \subset \Omega$ by $R[\breve{\Omega}]$.

Proposition 5.10 *Suppose that* (i) – (iii) *hold. Then*

$$\partial g(v) \subset \{ \int_\Omega \partial d(p(v, \xi)) \circ p_v(v, \xi) \, d\rho(\xi) : \, \rho \in R[\Omega(v)] \} \quad \textit{for } v \in X, \tag{5.11}$$

where the term on the right hand side describes the subset of X^* with the property that every element g^* of this set corresponds to a mapping $d^* : \Omega \mapsto Y^*$ with $d^*(\xi) \in \partial d(p(v, \xi))$ (∂ with respect to $d(\cdot)$) and to a measure $\rho \in R[\Omega(v)]$ such that

$$\xi \mapsto \langle d^*(\xi) \circ p_v(v, \xi), \, w \rangle = \langle d^*(\xi), p_v(v, \xi) w \rangle_{Y^* \times Y} \tag{5.12}$$

is ρ-integrable for all $w \in X$ and that

$$\langle g^*, w \rangle = \int_\Omega \langle d^*(\xi), p_v(v, \xi) w \rangle \, d\rho(\xi) \quad \textit{for all } w \in X. \tag{5.13}$$

P r o o f. As in Clarke [13, p. 85] we define a different kind of generalized gradients for the parameter-dependent functionals

$$f_{[\xi]} : X \mapsto \mathbb{R} \quad \text{with} \quad f_{[\xi]}(v) := d(p(v, \xi)), \tag{5.14}$$

which take into account variations of the parameter, by

$$\partial_{[\Omega]} f_{[\xi]}(v) := \overline{\text{co}}^* \Big\{ f^* \in X^* : \, f_i^* \overset{*}{\rightharpoonup} f^*, \, f_i^* \in \partial f_{[\xi_i]}(v_i), \, v_i \to v, \, \xi_i \to \xi, \, \xi_i \in \Omega \Big\} \tag{5.15}$$

where $\overline{\text{co}}^*$ denotes the weak*-closed convex hull. Since $f_{[\xi]}$ is locally Lipschitz continuous on X for all $\xi \in \Omega$, this gradient is well defined and, obviously, $\partial f_{[\xi]}(v) \subset \partial_{[\Omega]} f_{[\xi]}(v)$. By Clarke [13, Theorem 2.8.2] we obtain

$$\partial g(v) \subset \Big\{ \int_\Omega \partial_{[\Omega]} f_{[\xi]}(v) \, d\rho(\xi) : \, \rho \in R[\Omega(v)] \Big\} =: \partial_0 g(v) \subset X^* \quad \text{for } v \in X. \tag{5.16}$$

The interpretation of $\partial_0 g(v)$ is analogous to that of the right-hand side in (5.11), which we denote by $\partial_1 g(v)$. Thus we have to verify that $\partial_0 g(v) \subset \partial_1 g(v)$.

By the Chain Rule (cf. Proposition 5.3), for $\xi \in \Omega$ we get

$$\partial f_{[\xi]}(v) \subset \partial d(p(v,\xi)) \circ p_v(v,\xi), \quad v \in X. \tag{5.17}$$

This means that every element $\tilde{f}^* \in \partial f_{[\xi]}(v)$ corresponds to $d^* \in \partial d(p(v,\xi))$ such that

$$\langle \tilde{f}^*, w\rangle = \langle d^*, p_v(v,\xi)w\rangle_{Y^*\times Y} \quad \text{for all } \ w \in X. \tag{5.18}$$

We now assume that

$$v_k \to v, \quad \xi_k \to \xi, \quad \xi_k \in \Omega, \quad f_k^* \in \partial f_{[\xi_k]}(v_k), \quad f_k^* \stackrel{*}{\rightharpoonup} f^*. \tag{5.19}$$

By definition (5.15), $f^* \in \partial_{[\Omega]} f_{[\xi]}(v)$. We show later in this proof that

$$f^* \in \partial d(p(v,\xi)) \circ p_v(v,\xi). \tag{5.20}$$

As a generalized gradient, $\partial d(p(v,\xi))$ is a convex and weak*-compact subset of Y^*. Hence the set $\partial d(p(v,\xi)) \circ p_v(v,\xi) \subset X^*$ is convex and weak*-closed. Observing (5.15), (5.19) and (5.20), we get

$$\partial_{[\Omega]} f_{[\xi]}(v) \subset \partial d(p(v,\xi)) \circ p_v(v,\xi). \tag{5.21}$$

But this means that $\partial_0 g(v) \subset \partial_1(v)$ and thus gives the assertion.

It remains to prove (5.20). Let f^* be given as in (5.19). By (5.17) there are elements

$$d_k^* \in \partial d(p(v_k,\xi_k)) \quad \text{with} \quad f_k^* = d_k^* \circ p_v(v_k,\xi_k). \tag{5.22}$$

Since $d(\cdot)$ is Lipschitz continuous, the sets $\partial d(q) \subset Y^*$, $q \in Y$, are uniformly bounded (cf. Proposition 5.3). Using the reflexivity of Y, we can therefore assume, possibly for a subsequence, that $d_k^* \rightharpoonup d^* \in Y^*$. Thus, by the continuity of $p_v(\cdot,\cdot)$,

$$\begin{aligned}\langle f^*, w\rangle &= \lim_{k\to\infty} \langle f_k^*, w\rangle = \lim_{k\to\infty} \langle d_k^*, p_v(v_k,\xi_k)w\rangle_{Y^*\times Y} \\ &= \langle d^*, p_v(v,\xi)w\rangle_{Y^*\times Y} = \langle d^* \circ p_v(v,\xi), w\rangle \quad \text{for all } \ w \in X,\end{aligned} \tag{5.23}$$

i.e.,

$$f^* = d^* \circ p_v(v,\xi). \tag{5.24}$$

Applying Proposition 5.3.3 and the continuity of $p(\cdot,\cdot)$, we obtain

$$d^* \in \partial d(p(v,\xi)). \tag{5.25}$$

Conditions (5.24), (5.25) now imply (5.20) and the proof is complete. ♢

P r o o f of Lemma 5.9. We study g near $v_0 \in X$. First we show that there exists a Lipschitz constant independent of $\xi \in \Omega$ for $v \mapsto p(v,\xi)$ near v_0.

Since $p_v(\cdot,\cdot)$ is continuous and Ω is compact, we can choose a neighborhood U_0 of v_0 such that

$$\|p_v(v,\xi)\| \le l_1 := \max_{\xi\in\Omega} \|p_v(v_0,\xi)\| \ + 1 \quad \text{for all} \quad v \in U_0, \ \xi \in \Omega. \tag{5.26}$$

We can suppose that U_0 is convex and, thus,

$$\begin{aligned}\|p(v_1,\xi)-p(v_2,\xi)\| &= \|\int_0^1 p_v(tv_1+(1-t)v_2,\xi)(v_1-v_2)\,dt\| \\ &\le l_1\|v_1-v_2\| \quad \text{for all} \quad v_1,v_2\in U_0,\ \ \xi\in\Omega. \end{aligned} \tag{5.27}$$

If l_2 denotes the Lipschitz constant of the functional d, for $v_1, v_2 \in U_0$ we finally can estimate

$$\begin{aligned}|g(v_1)-g(v_2)| &= |\max_{\xi\in\Omega} d(p(v_1,\xi)) - \max_{\xi\in\Omega} d(p(v_2,\xi)\,| \\ &\le \max_{\xi\in\Omega}|d(p(v_1,\xi))-d(p(v_2,\xi))| \\ &= |d(p(v_1,\xi_0))-d(p(v_2,\xi_0))| \qquad \text{for some}\ \ \xi_0\in\Omega \\ &\le l_2\|p(v_1,\xi_0)-p(v_2,\xi_0)\| \ \le\ l_1 l_2\|v_1-v_2\|. \end{aligned} \tag{5.28}$$

This verifies the assertion. ♢

6 Proof that the Euler-Lagrange equations are satisfied

We prove Theorem 4.10, Corollary 4.27 and Theorem 4.31 in this section. Some serious difficulties arise from the fact that E_s is only Gâteaux differentiable in special directions in $\mathbf{X}$. Since we also have to handle generalized gradients, the situation is even worse than usual, because the generalized gradient coincides with the derivative only in the case of stronger differentiability (e.g., for a continuous derivative). Therefore we cannot employ the usual arguments to get Lagrange multipliers independent of a sufficiently large class of variations by means of the Gâteaux derivative (cf. Antman [3, Chapter VII]). However, we bypass these difficulties by considering a corresponding modified variational problem in a different space where the functionals have nice differentiability properties on suitable subspaces. To get normality in the Lagrange Multiplier Rule, we need the regularity of the functionals $g_0, g_1, \ldots, g_n$ and, moreover, the linear independence of the corresponding generalized gradients on such subspaces of variations. We derive this from conditions (4.7) and (4.8). However, we have to ensure that these subspaces are large enough to get regularity for g_0, since (4.8) expresses a local property both of the solution $\mathbf{u}$ and of the obstacle. Thus we finally get the Euler-Lagrange equations.

6.1 Modified problem

Let us introduce the following space of variations

$$(\overset{\triangle}{\nu}, \overset{\triangle}{\eta}, \overset{\triangle}{\mu}, \overset{\triangle}{\mathbf{r}}_0, \overset{\triangle}{\theta}_0) \ = \overset{\triangle}{\mathbf{u}} \in \ \mathcal{V} \ := \ \mathcal{L}^\infty\times\mathcal{L}^\infty\times\mathcal{L}^\infty\times\mathbb{R}^2\times\mathbb{R}\,. \tag{6.1}$$

$\mathcal{V}$ is a Banach space with the norm

$$\|\overset{\triangle}{\mathbf{u}}\|_\infty := \|\nu\|_\infty + \|\eta\|_\infty + \|\mu\|_\infty + \|\overset{\triangle}{\mathbf{r}}_0\| + |\theta_0|. \tag{6.2}$$

We define the functionals $\breve{E}, \breve{E}_s, \breve{E}_p, \breve{g}_0, \breve{g}_1, \ldots, \breve{g}_n$ on $\mathcal{V}$ by

$$\breve{E}(\overset{\triangle}{\mathbf{u}}) := E(\mathbf{u}+\overset{\triangle}{\mathbf{u}}), \quad etc., \tag{6.3}$$

where $\mathbf{u}$ is a solution of the problem (4.1) – (4.4). We can now consider the following modified variational problem

$$\breve{E}(\overset{\triangle}{\mathbf{u}}) \to \text{Min!}, \quad \overset{\triangle}{\mathbf{u}} \in \mathcal{V}, \tag{6.4}$$

$$\breve{g}_0(\overset{\triangle}{\mathbf{u}}) \leq 0, \quad \breve{g}_i(\overset{\triangle}{\mathbf{u}}) = 0, \quad i = 1, \dots, n, \tag{6.5}$$

$$\nu(s) + \overset{\triangle}{\nu}(s) - V(\mu(s) + \overset{\triangle}{\mu}(s), s) \geq 0 \quad \text{a.e. on } [0, L]. \tag{6.6}$$

Since $\mathcal{V} \subset \mathbf{X}$, this problem obviously has a solution at $\overset{\triangle}{\mathbf{u}} = \mathbf{0}$. We now study this modified problem on suitable subspaces of $\mathcal{V}$. Thus all differentiability properties which we consider are taken with respect to $\mathcal{V}$ or to certain of it subspaces.

We provide some more notation. For $k \in \mathbb{N}$, set

$$U_k(s) := \{\xi := (\bar{\nu}, \bar{\eta}, \bar{\mu}) \in \mathbb{R}^3 : |\bar{\nu} - \nu(s)| + |\bar{\eta} - \eta(s)| + |\bar{\mu} - \mu(s)| < 3/k\}, \tag{6.7}$$

$$\mathcal{I}_k := \{s \in [0, L] : \sup_{\xi \in U_k(s)} \Big(|W_\nu(\xi, s)| + |W_\eta(\xi, s)| + |W_\mu(\xi, s)| \Big) \leq k\}, \tag{6.8}$$

$$\chi_k(s) := \begin{cases} 1 & \text{for } s \in \mathcal{I}_k, \\ 0 & \text{for } s \notin \mathcal{I}_k. \end{cases} \tag{6.9}$$

Our assumptions on W ensure that the sets $\mathcal{I}_k$ are measurable and, obviously, $\mathcal{I}_k \subset \mathcal{I}_{k+1}$. By (1.41), (1.42), (1.43), (4.9), the gradient of $W(\cdot, \cdot, \cdot, s)$ is unbounded exactly where W is unbounded. Hence, $[0, L] \setminus \bigcup_{k=1}^{\infty} \mathcal{I}_k$ has measure zero. Otherwise $E_s(\mathbf{u})$ would not be finite.

We shall study the modified variational problem on the following subspaces of $\mathcal{V}$

$$\mathcal{V}_k := \{\overset{\triangle}{\mathbf{u}} \in \mathcal{V} : \overset{\triangle}{\nu}(s) = \overset{\triangle}{\eta}(s) = \overset{\triangle}{\mu}(s) = 0 \text{ for } s \notin \mathcal{I}_k\}. \tag{6.10}$$

Within these subspaces we consider the special neighborhoods of the origin

$$\mathcal{V}_k^0 := \{\overset{\triangle}{\mathbf{u}} \in \mathcal{V}_k : \|\overset{\triangle}{\nu}\|_\infty, \|\overset{\triangle}{\eta}\|_\infty, \|\overset{\triangle}{\mu}\|_\infty < 1/k\}. \tag{6.11}$$

Clearly,

$$\Big(\nu(s) + \overset{\triangle}{\nu}(s), \eta(s) + \overset{\triangle}{\eta}(s), \mu(s) + \overset{\triangle}{\mu}(s)\Big) \in U_k(s) \quad \text{a.e. on } [0, L] \text{ for } \overset{\triangle}{\mathbf{u}} \in \mathcal{V}_k^0. \tag{6.12}$$

Observing (4.9), (6.8), (6.10), we readily see that $\overset{\triangle}{\mathbf{u}}$ respects the orientation-preserving condition (6.6) for all $\overset{\triangle}{\mathbf{u}} \in \mathcal{V}_k^0$, $k \in \mathbb{N}$, i.e., this condition is fulfilled automatically in a neighborhood of the origin in the subspace $\mathcal{V}_k$. Thus we can drop this condition as long as we study the modified problem in such a subspace.

Since $\mathbf{u}$ has regular contact, (4.8) yields a covering of the set $\mathcal{I}_c \subset [0, L]$ of all contact parameters s_c of $\mathbf{u}$ with open sets $\mathcal{I}(s_c)$ which we can assume to be open intervals. Obviously $\mathcal{I}_c$ is compact and we can select a finite open covering $\{\bar{\mathcal{I}}_i\}_{i=1}^l$. For a sufficiently small number $\varepsilon_0 > 0$, we get a new open covering $\{\tilde{\mathcal{I}}_i\}_{i=1}^l$ of $\mathcal{I}_c$ if we cut off at the ends of each $\bar{\mathcal{I}}_i$ an interval of length ε_0. We use these coverings for regularity arguments below.

6.2 Differentiability of the stored energy

Let us now investigate the differentiability of the modified energy $\breve{E}_s$ in the spaces $\mathcal{V}_k$.

For $\overset{\vartriangle}{\mathbf{v}}\in \mathcal{V}_k^0$ and $\overset{\vartriangle}{\mathbf{u}}\in \mathcal{V}_k$, we get that $\overset{\vartriangle}{\mathbf{v}} +t \overset{\vartriangle}{\mathbf{u}}\in \mathcal{V}_k^0$ for $t \in \mathbb{R}$ sufficiently small and, with the notation $\operatorname{grad} W := (W_\nu, W_\eta, W_\mu)$,

$$\frac{\breve{E}_s(\overset{\vartriangle}{\mathbf{v}} +t \overset{\vartriangle}{\mathbf{u}}) - \breve{E}_s(\overset{\vartriangle}{\mathbf{v}})}{t} = \int_0^L \left[\int_0^1 \operatorname{grad}\breve{W}[\overset{\vartriangle}{\mathbf{v}} +\tau t \overset{\vartriangle}{\mathbf{u}}](s)\, d\tau\right] \cdot \Big(\overset{\vartriangle}{\nu}(s), \overset{\vartriangle}{\eta}(s), \overset{\vartriangle}{\mu}(s)\Big)^T ds. \tag{6.13}$$

Observing (6.8), (6.10), (6.11), we can apply the Lebesgue Dominated Convergence Theorem and compute (cf. also Zeidler [48, p. 1018]),

$$\begin{aligned}
\langle \breve{E}_s'(\overset{\vartriangle}{\mathbf{v}}), \overset{\vartriangle}{\mathbf{u}}\rangle &= \frac{d}{dt}\breve{E}_s(\overset{\vartriangle}{\mathbf{v}} +t \overset{\vartriangle}{\mathbf{u}})\Big|_{t=0} \\
&= \int_0^L \frac{d}{dt}\breve{W}[\overset{\vartriangle}{\mathbf{v}} +t \overset{\vartriangle}{\mathbf{u}}](s)\Big|_{t=0} ds \\
&= \int_{\mathcal{I}_k} \breve{W}_\nu[\overset{\vartriangle}{\mathbf{v}}](s)\, \overset{\vartriangle}{\nu}(s) + \breve{W}_\eta[\overset{\vartriangle}{\mathbf{v}}](s)\, \overset{\vartriangle}{\eta}(s) + \breve{W}_\mu[\overset{\vartriangle}{\mathbf{v}}](s)\, \overset{\vartriangle}{\mu}(s)\, ds.
\end{aligned} \tag{6.14}$$

We readily verify that the convergence of the derivative as $t \to 0$ is uniform with respect to $\| \overset{\vartriangle}{\mathbf{u}} \|_\infty$ and, consequently, $\breve{E}_s'(\cdot)$ exists as a Fréchet derivative on a neighborhood of the origin in the space $\mathcal{V}_k$. Let us now choose a sequence $\overset{\vartriangle}{\mathbf{v}}_i\mapsto\overset{\vartriangle}{\mathbf{v}}$ in $\mathcal{V}_k^0$. Then

$$\begin{aligned}
\|\breve{E}_s'(\overset{\vartriangle}{\mathbf{v}}_i) - \breve{E}_s'(\overset{\vartriangle}{\mathbf{v}})\|_k &= \max_{\overset{\vartriangle}{\mathbf{u}}\in\mathcal{V}_k, \|\overset{\vartriangle}{\mathbf{u}}\|_\infty\le 1} |\langle \breve{E}_s'(\overset{\vartriangle}{\mathbf{v}}_i) - \breve{E}_s'(\overset{\vartriangle}{\mathbf{v}}), \overset{\vartriangle}{\mathbf{u}}\rangle| \\
&\le \int_{\mathcal{I}_k} \Big|\breve{W}_\nu[\overset{\vartriangle}{\mathbf{v}}_i](s) - \breve{W}_\nu[\overset{\vartriangle}{\mathbf{v}}](s)\Big| + \Big|\breve{W}_\eta[\overset{\vartriangle}{\mathbf{v}}_i](s) - \breve{W}_\eta[\overset{\vartriangle}{\mathbf{v}}](s)\Big| + \\
&\qquad + \Big|\breve{W}_\mu[\overset{\vartriangle}{\mathbf{v}}_i](s) - \breve{W}_\mu[\overset{\vartriangle}{\mathbf{v}}](s)\Big|\, ds.
\end{aligned} \tag{6.15}$$

Since the strong convergence in $\mathcal{L}^\infty$ implies convergence a.e. and by the smoothness hypothesis for W, we can again use the Dominated Convergence Theorem to see the continuity of the derivative $\breve{E}_s'(\cdot)$ near the origin in $\mathcal{V}_k$. Clearly, for $\overset{\vartriangle}{\mathbf{u}}\in \mathcal{V}_k$

$$\langle \breve{E}_s'(\mathbf{0}), \overset{\vartriangle}{\mathbf{u}}\rangle = \int_{\mathcal{I}_k} W_\nu[\mathbf{u}](s)\, \overset{\vartriangle}{\nu}(s) + W_\eta[\mathbf{u}](s)\, \overset{\vartriangle}{\eta}(s) + W_\mu[\mathbf{u}](s)\, \overset{\vartriangle}{\mu}(s)\, ds. \tag{6.16}$$

6.3 Differentiability of the potential energy

We proceed with the differentiability of the modified potential energy on $\mathcal{V}$. With $\mathbf{v} = \mathbf{u}+ \overset{\vartriangle}{\mathbf{v}}$ we have

$$\breve{E}_p(\overset{\vartriangle}{\mathbf{v}}) = -\int_\Omega \mathbf{p}[\mathbf{v}](\mathbf{z}) \cdot d\mathfrak{f}(\mathbf{z}) = -\int_\Omega \Big(\mathbf{r}[\mathbf{v}](s) + \zeta \mathbf{b}[\mathbf{v}](s)\Big) \cdot d\mathfrak{f}(s,\zeta) \tag{6.17}$$

for $\overset{\vartriangle}{\mathbf{v}}\in \mathcal{V}$ (recall (1.50)).

Obviously, $\breve{E}_p(\cdot)$ is the sum of a linear continuous function in $\mathbf{r}$ and a linear continuous function in $\mathbf{b}$ and one easily verifies that $\mathbf{r}[\cdot]$, $\mathbf{b}[\cdot]$ are continuously differentiable by (1.13), (1.14), (1.5). Therefore, $\breve{E}_p(\cdot)$ is continuously Fréchet differentiable on $\mathcal{V}$ and, consequently, also on the subspaces $\mathcal{V}_k$.

Let us now compute this derivative at $\mathbf{0} \in \mathcal{V}$. Using (1.13), (1.14), for $\overset{\triangle}{\mathbf{u}} \in \mathcal{V}$ and $t \in \mathbb{R}$ we get

$$\begin{aligned}
\breve{E}_p(t\,\overset{\triangle}{\mathbf{u}}) &= -\int_\Omega \mathbf{r}[\mathbf{u}+t\,\overset{\triangle}{\mathbf{u}}](s)\cdot d\mathfrak{f}(s,\zeta) - \int_\Omega \zeta\mathbf{b}[\mathbf{u}+t\,\overset{\triangle}{\mathbf{u}}](s)\cdot d\mathfrak{f}(s,\zeta) \\
&= -\int_\Omega \Big[\ \mathbf{r}_0 + t\,\overset{\triangle}{\mathbf{r}}_0\ +\ \int_0^s \Big(\nu(\tau)+t\,\overset{\triangle}{\nu}(\tau)\Big)\mathbf{a}\Big(\theta[\mu+t\,\overset{\triangle}{\mu},\theta_0+t\,\overset{\triangle}{\theta}_0](\tau)\Big) \\
&\qquad +\Big(\eta(\tau)+t\,\overset{\triangle}{\eta}(\tau)\Big)\mathbf{b}\Big(\theta[\mu+t\,\overset{\triangle}{\mu},\theta_0+t\,\overset{\triangle}{\theta}_0](\tau)\Big)\,d\tau\ \Big]\cdot d\mathfrak{f}(s,\zeta) \\
&\quad -\int_\Omega \zeta\mathbf{b}\Big(\theta[\mu+t\,\overset{\triangle}{\mu},\theta_0+t\,\overset{\triangle}{\theta}_0](s)\Big)\cdot d\mathfrak{f}(s,\zeta) \qquad (6.18)
\end{aligned}$$

where

$$\theta[\mu+t\,\overset{\triangle}{\mu},\theta_0+t\,\overset{\triangle}{\theta}_0](s) = \theta_0 + t\,\overset{\triangle}{\theta}_0 + \int_0^s (\,\mu(\tau)+t\,\overset{\triangle}{\mu}(\tau)\,)\,d\tau. \qquad (6.19)$$

Equation (1.5) tells us that $\partial\mathbf{a}/\partial\theta = \mathbf{b}$ and $\partial\mathbf{b}/\partial\theta = -\mathbf{a}$. Consequently

$$\begin{aligned}
&\langle \breve{E}_p'(\mathbf{0}),\overset{\triangle}{\mathbf{u}}\rangle \\
&= -\int_\Omega \overset{\triangle}{\mathbf{r}}_0\cdot d\mathfrak{f}(s,\zeta)\ -\int_\Omega\int_0^s \Big(\overset{\triangle}{\nu}(\tau)\mathbf{a}(\theta[\mu,\theta_0](\tau)) + \overset{\triangle}{\eta}(\tau)\mathbf{b}(\theta[\mu,\theta_0](\tau))\Big)\,d\tau\cdot d\mathfrak{f}(s,\zeta) \\
&\quad -\int_\Omega\int_0^s \Big[\Big(\nu(\tau)\mathbf{b}(\theta[\mu,\theta_0](\tau)) - \eta(\tau)\mathbf{a}(\theta[\mu,\theta_0](\tau))\Big)\Big(\overset{\triangle}{\theta}_0 + \int_0^\tau \overset{\triangle}{\mu}(\omega)d\omega\Big)\Big]\,d\tau\cdot d\mathfrak{f}(s,\zeta) \\
&\quad +\int_\Omega \zeta\mathbf{a}(\theta[\mu,\theta_0](s))\,\Big(\overset{\triangle}{\theta}_0 + \int_0^s \overset{\triangle}{\mu}(\omega)\,d\omega\Big)\cdot d\mathfrak{f}(s,\zeta). \qquad (6.20)
\end{aligned}$$

Since all terms are integrable, we can apply Fubini's Theorem. Writing $\theta(\cdot)$ instead of $\theta[\mu,\theta_0](\cdot)$, we obtain

$$\begin{aligned}
\langle \breve{E}_p'(\mathbf{0}),\overset{\triangle}{\mathbf{u}}\rangle &= -\,\overset{\triangle}{\mathbf{r}}_0\cdot\int_\Omega d\mathfrak{f}(s,\zeta) \\
&\quad -\int_0^L \Big[\Big(\overset{\triangle}{\nu}(\tau)\mathbf{a}(\theta(\tau)) + \overset{\triangle}{\eta}(\tau)\mathbf{b}(\theta(\tau))\Big)\cdot\int_{\Omega_\tau} d\mathfrak{f}(s,\zeta)\Big]\,d\tau \\
&\quad -\,\overset{\triangle}{\theta}_0\int_0^L \Big[\Big(\nu(\tau)\mathbf{b}(\theta(\tau)) - \eta(\tau)\mathbf{a}(\theta(\tau))\Big)\cdot\int_{\Omega_\tau} d\mathfrak{f}(s,\zeta)\Big]\,d\tau \\
&\quad -\int_0^L \overset{\triangle}{\mu}(\omega)\Big[\int_\omega^L \Big(\nu(\tau)\mathbf{b}(\theta(\tau)) - \eta(\tau)\mathbf{a}(\theta(\tau))\Big)\cdot\int_{\Omega_\tau} d\mathfrak{f}(s,\zeta)\,d\tau\Big]\,d\omega \\
&\quad +\,\overset{\triangle}{\theta}_0\int_\Omega \zeta\mathbf{a}(\theta(s))\cdot d\mathfrak{f}(s,\zeta) \\
&\quad +\int_0^L \overset{\triangle}{\mu}(\omega)\Big[\int_{\Omega_\omega} \zeta\mathbf{a}(\theta(s))\cdot d\mathfrak{f}(s,\zeta)\Big]\,d\omega. \qquad (6.21)
\end{aligned}$$

Recalling (1.31), (1.32), the orthogonality of $\mathbf{a},\mathbf{b}$ and changing variables, we obtain

$$\begin{aligned}
\langle \breve{E}_p'(\mathbf{0}),\overset{\triangle}{\mathbf{u}}\rangle &= -\,\overset{\triangle}{\mathbf{r}}_0\cdot\mathbf{f}(0)\ -\ \int_0^L \Big(\overset{\triangle}{\nu}(s)\mathbf{a}(\theta(s)) + \overset{\triangle}{\eta}(s)\mathbf{b}(\theta(s))\Big)\cdot\mathbf{f}(s)\,ds \\
&\quad -\int_0^L \overset{\triangle}{\mu}(s)\Big[\int_s^L (\,\mathbf{r}'(\tau)\times\mathbf{f}(\tau)\,)\cdot\mathbf{k}\,d\tau\ +\ \mathbf{l}_f(s)\cdot\mathbf{k}\Big]\,ds \\
&\quad -\,\overset{\triangle}{\theta}_0\ \Big[\int_0^L (\,\mathbf{r}'(\tau)\times\mathbf{f}(\tau)\,)\cdot\mathbf{k}\,d\tau\ +\ \mathbf{l}_f(0)\cdot\mathbf{k}\Big]. \qquad (6.22)
\end{aligned}$$

6.4 Generalized gradient of the obstacle condition

We now determine the generalized gradient of the functional $\breve{g}_0$ on the space $\mathcal{V}$ by applying Proposition 5.10. With $\mathbf{v} = (\breve{\nu}, \breve{\eta}, \breve{\mu}, \breve{\mathbf{r}}_0, \breve{\theta}_0)$, we have by (1.1), (1.5), (1.13), (1.14) that

$$\mathbf{p}[\mathbf{v}](s,\zeta) = \breve{\mathbf{r}}_0 + \int_0^s \Big(\breve{\nu}(\tau)\mathbf{a}(\theta[\breve{\mu},\breve{\theta}_0](\tau)) + \breve{\eta}(\tau)\mathbf{b}(\theta[\breve{\mu},\breve{\theta}_0](\tau)) \Big)\, d\tau \;+\; \zeta\mathbf{b}(\theta[\breve{\mu},\breve{\theta}_0](s)). \tag{6.23}$$

$\mathbf{p}[\cdot](\cdot)$ is continuous on $(\mathbf{u}+\mathcal{V})\times\Omega$ and the mapping $\mathbf{v}\mapsto\mathbf{p}[\mathbf{v}](s,\zeta)$ is continuously differentiable on $\mathbf{u}+\mathcal{V}$ for all $(s,\zeta)\in\Omega$. Consequently the mapping $(\overset{\triangle}{\mathbf{v}},(s,\zeta))\mapsto\breve{\mathbf{p}}[\overset{\triangle}{\mathbf{v}}](s,\zeta) := \mathbf{p}[\mathbf{u}+\overset{\triangle}{\mathbf{v}}](s,\zeta)$ is continuous on $\mathcal{V}\times\Omega$ and $\overset{\triangle}{\mathbf{v}}\mapsto\breve{\mathbf{p}}[\overset{\triangle}{\mathbf{v}}](s,\zeta)$ has a derivative $\breve{\mathbf{p}}_v[\overset{\triangle}{\mathbf{v}}](s,\zeta)$ on $\mathcal{V}$ for all $(s,\zeta)\in\Omega$. A straightforward computation yields the continuity of $\breve{\mathbf{p}}_v[\cdot](\cdot)$ on $\mathcal{V}\times\Omega$ where

$$\begin{aligned}
\breve{\mathbf{p}}_v[\mathbf{0}](s,\zeta)\,\overset{\triangle}{\mathbf{u}} \;&=\; \overset{\triangle}{\mathbf{r}}_0 + \int_0^s \Big[\overset{\triangle}{\nu}(\tau)\,\mathbf{a}(\theta(\tau)) + \overset{\triangle}{\eta}(\tau)\,\mathbf{b}(\theta(\tau)) + \mathbf{r}'^{\perp}(\tau)\Big(\overset{\triangle}{\theta}_0 + \int_0^\tau \overset{\triangle}{\mu}(\omega)d\omega\Big)\Big]d\tau \\
&\quad -\; \zeta\mathbf{a}(\theta(s))\Big(\overset{\triangle}{\theta}_0 + \int_0^s \overset{\triangle}{\mu}(\tau)d\tau\Big) \quad \text{for all } \overset{\triangle}{\mathbf{u}}\in\mathcal{V}.
\end{aligned} \tag{6.24}$$

Here again $\theta(\cdot)$ stands for $\theta[\mu,\theta_0](\cdot)$ and $\mathbf{r}'^{\perp}(\cdot) := \nu(\cdot)\mathbf{b}(\theta(\cdot)) - \eta(\cdot)\mathbf{a}(\theta(\cdot))$.

Clearly, d defined in (4.5) is Lipschitz continuous with constant 1. Therefore $\breve{g}_0$ is locally Lipschitz continuous on $\mathcal{V}$ by Lemma 5.9. Hence the generalized gradient $\partial\breve{g}_0(\cdot)$ exists on $\mathcal{V}$ and can be characterized by Proposition 5.10. If $g^*\in\partial\breve{g}_0(\mathbf{0})$, then there exist a mapping $(s,\zeta)\mapsto\mathbf{d}^*(s,\zeta)\in\partial d(\mathbf{p}[\mathbf{u}](s,\zeta))$ and a probability Borel measure ρ supported on the set $\breve{\Omega}_{g_0} := \{(s,\zeta)\in\Omega:\ \breve{g}_0(\mathbf{0}) = d(\breve{\mathbf{p}}[\mathbf{0}](s,\zeta))\}$ such that $\mathbf{d}^*(s,\zeta)\cdot\breve{\mathbf{p}}_v[\mathbf{0}](s,\zeta)\,\overset{\triangle}{\mathbf{u}}$ is ρ-integrable and

$$\langle g^*,\overset{\triangle}{\mathbf{u}}\rangle = \int_\Omega \mathbf{d}^*(s,\zeta)\cdot\breve{\mathbf{p}}_v[\mathbf{0}](s,\zeta)\,\overset{\triangle}{\mathbf{u}}\,d\rho(s,\zeta). \tag{6.25}$$

Observe that $\mathbf{d}^*,\rho$ depend on g^*, but let us suppress this dependence. Moreover, $\breve{\Omega}_{g_0}$ is equal to the contact set $\Omega_{\mathbf{u}}$ if $\breve{g}_0(\mathbf{0}) = 0$ (cf. (4.20)) and obviously $\|\mathbf{d}^*(s,\zeta)\|\le 1$ on Ω. Substituting (6.24) into (6.25), we get

$$\begin{aligned}
\langle g^*,\overset{\triangle}{\mathbf{u}}\rangle \;&=\; \int_\Omega \mathbf{d}^*(s,\zeta)\cdot\Big[\overset{\triangle}{\mathbf{r}}_0 \;+\; \int_0^s \Big(\overset{\triangle}{\nu}(\tau)\mathbf{a}(\theta(\tau)) + \overset{\triangle}{\eta}(\tau)\mathbf{b}(\theta(\tau))\Big)\,d\tau\,\Big]\,d\rho(s,\zeta) \\
&\quad +\; \int_\Omega \mathbf{d}^*(s,\zeta)\cdot\Big[\int_0^s \mathbf{r}'^{\perp}(\tau)\Big(\overset{\triangle}{\theta}_0 + \int_0^\tau \overset{\triangle}{\mu}(\omega)d\omega\Big)\,d\tau\,\Big]\,d\rho(s,\zeta) \\
&\quad -\; \int_\Omega \mathbf{d}^*(s,\zeta)\cdot\mathbf{a}(\theta(s))\zeta\Big(\overset{\triangle}{\theta}_0 + \int_0^s \overset{\triangle}{\mu}(\tau)d\tau\Big)\,d\rho(s,\zeta).
\end{aligned} \tag{6.26}$$

Let us choose $\overset{\triangle}{\mathbf{u}}\in\mathcal{V}$ such that $\overset{\triangle}{\nu}(s) = \overset{\triangle}{\eta}(s) = \overset{\triangle}{\mu}(s) = 0$ on $[0,L]$ and $\overset{\triangle}{\theta}_0 = 0$. Then we get that $\mathbf{d}^*(s,\zeta)\cdot\,\overset{\triangle}{\mathbf{r}}_0$ is ρ-integrable for all $\overset{\triangle}{\mathbf{r}}_0\in\operatorname{span}\{\mathbf{i},\mathbf{j}\}$. But this implies that $\mathbf{d}^*(s,\zeta)$ itself is ρ-integrable. Since all other terms in the integrand in (6.26) are also integrable, we can invoke Fubini's Theorem. By straightforward computations, we obtain

$$\begin{aligned}
\langle g^*,\overset{\triangle}{\mathbf{u}}\rangle \;&=\; \overset{\triangle}{\mathbf{r}}_0\cdot\int_\Omega \mathbf{d}^*(\tau,\zeta)d\rho(\tau,\zeta) \\
&\quad + \int_0^L \Big[\Big(\overset{\triangle}{\nu}(s)\mathbf{a}(\theta(s)) + \overset{\triangle}{\eta}(s)\mathbf{b}(\theta(s))\Big)\cdot\int_{\Omega_s}\mathbf{d}^*(\tau,\zeta)\,d\rho(\tau,\zeta)\,\Big]\,ds \\
&\quad + \int_0^L \overset{\triangle}{\mu}(s)\Big[\int_s^L\Big(\mathbf{r}'^{\perp}(\tau)\cdot\int_{\Omega_\tau}\mathbf{d}^*(\omega,\zeta)\,d\rho(\omega,\zeta)\Big)d\tau - \int_{\Omega_s}\zeta\mathbf{d}^*(\tau,\zeta)\cdot\mathbf{a}(\theta(\tau))\,d\rho(\tau,\zeta)\Big]ds
\end{aligned}$$

$$+\stackrel{\triangle}{\theta}_0 \int_0^L \Big(\mathbf{r}'^{\perp}(\tau)\cdot \int_{\Omega_\tau} \mathbf{d}^*(\omega,\zeta)\, d\rho(\omega,\zeta) \Big)\, d\tau$$
$$-\stackrel{\triangle}{\theta}_0 \int_\Omega \zeta \mathbf{d}^*(\tau,\zeta)\cdot \mathbf{a}(\theta(\tau))\, d\rho(\tau,\zeta) \tag{6.27}$$

where $\Omega_s := \{(\tau,\zeta)\in\Omega:\ \tau\in[s,L]\}$. For Borel sets $\breve{\Omega}\subset\Omega$,

$$\breve{\Omega}\mapsto \tilde{\mathfrak{f}}_c(\breve{\Omega}) := \int_{\breve{\Omega}} \mathbf{d}^*(\tau,\zeta)\, d\rho(\tau,\zeta) \tag{6.28}$$

defines a vector-valued Borel measure on Ω supported on the set $\breve{\Omega}_{g_0}$ (cf. Benedetto [10, p. 171]). As we shall see, for some $g^*\in\partial\breve{g}_0(\mathbf{0})$ and a suitable real number $\lambda_0\ge 0$, the measure $-\lambda_0\tilde{\mathfrak{f}}_c$ describes the contact force exerted by the obstacle. In analogy with (1.31), (1.32), set

$$\tilde{\mathbf{f}}_c(s) := \int_{\Omega_s} d\tilde{\mathfrak{f}}_c(\tau,\zeta) = \int_{\Omega_s} \mathbf{d}^*(\tau,\zeta)\, d\rho(\tau,\zeta), \tag{6.29}$$
$$\tilde{\mathbf{l}}_{f_c}(s) := \int_{\Omega_s} \zeta\mathbf{b}(\tau)\times d\tilde{\mathfrak{f}}_c(\tau,\zeta) = \int_{\Omega_s} \zeta\mathbf{b}(\tau)\times\mathbf{d}^*(\tau,\zeta)\, d\rho(\tau,\zeta). \tag{6.30}$$

The identities in (6.29), (6.30) are simple consequences of measure theory (cf. Bauer [9, p. 110] for (6.30)). Recalling the orthogonality $\mathbf{a}\perp\mathbf{b}$ and $\mathbf{r}'\perp\mathbf{r}'^{\perp}$, we finally get

$$\begin{aligned}\langle g^*,\stackrel{\triangle}{\mathbf{u}}\rangle &= \stackrel{\triangle}{\mathbf{r}}_0\cdot\tilde{\mathbf{f}}_c(0) + \int_0^L \Big(\stackrel{\triangle}{\nu}(s)\mathbf{a}(\theta(s))+\stackrel{\triangle}{\eta}(s)\mathbf{b}(\theta(s))\Big)\cdot\tilde{\mathbf{f}}_c(s)\, ds\\ &+ \int_0^L \stackrel{\triangle}{\mu}(s)\Big[\int_s^L (\mathbf{r}'(\tau)\times\tilde{\mathbf{f}}_c(\tau))\cdot\mathbf{k}\, d\tau + \tilde{\mathbf{l}}_{f_c}(s)\cdot\mathbf{k}\Big]\, ds\\ &+ \stackrel{\triangle}{\theta}_0\Big[\int_0^L (\mathbf{r}'(\tau)\times\tilde{\mathbf{f}}_c(\tau))\cdot\mathbf{k}\, d\tau + \tilde{\mathbf{l}}_{f_c}(0)\cdot\mathbf{k}\Big].\end{aligned} \tag{6.31}$$

Note that this differential has the same structure as (6.22).

We now consider the regularity of $\breve{g}_0$ in the subspaces $\mathcal{V}_k$ for the case that $\breve{g}_0(\mathbf{0})=0$ and $k\in\mathbb{N}$ is sufficiently large. More precisely, we suppose that the Lebesgue measure of $[0,L]\setminus\mathcal{I}_k$ is smaller than ε_0. The support of the measure ρ belongs to the set $\breve{\Omega}_{g_0}$ which is equal to $\Omega_{\mathbf{u}}$ in our case (cf. (4.20)). Therefore the set of all $s\in[0,L]$ with the property that (s,ζ) lies in the support of ρ for some ζ, which we call s-support of ρ, is contained in the set $\mathcal{I}_c$ of all contact parameters of $\mathbf{u}$. Since ρ is a probability measure, the support and the s-support must be nonempty. Thus there exists a largest point $\tilde{s}$ in the s-support and, clearly, $\tilde{s}\in\mathcal{I}_c$. From the covering $\{\tilde{\mathcal{I}}_i\}_{i=1}^l$ of $\mathcal{I}_c$ we can choose some $\tilde{\mathcal{I}}_i$ containing $\tilde{s}$. Now observe that $\tilde{\mathcal{I}}_i\subset\bar{\mathcal{I}}_i$ and that (4.8) is fulfilled with $\bar{\mathcal{I}}_i$ instead of $\mathcal{I}(s_c)$.

Let us first assume that $0\notin\bar{\mathcal{I}}_i$. Then we can find an interval $\tilde{\mathcal{I}}\subset\bar{\mathcal{I}}_i$ of length ε_0 having $\tilde{s}$ as the right boundary point. Using the properties of the integral defining $\tilde{\mathbf{f}}_c$ we obtain that $\tilde{\mathbf{f}}_c(s)=\mathbf{0}$ for $s>\tilde{s}$ and from $\mathbf{d}^*(s,\zeta)\in\partial d(\mathbf{p}[\mathbf{u}](s,\zeta))$, we obtain

$$\tilde{\mathbf{f}}_c(s)\in\rho([s,\tilde{s}])\,\overline{\mathrm{co}}\,\{\partial d(\mathbf{q})\,|\,d(\mathbf{q})=0,\ \ \mathbf{q}=\mathbf{p}[\mathbf{u}](\tau,\zeta)\ \text{ for some }\ (\tau,\zeta)\in\Omega_{[s,\tilde{s}]}\}\quad\text{for }\ s\le\tilde{s}. \tag{6.32}$$

Then (4.8) implies

$$\tilde{\mathbf{f}}_c(s)\neq\mathbf{0}\quad\text{on }\tilde{\mathcal{I}}. \tag{6.33}$$

Since we have chosen $k \in \mathbb{N}$ large, the Lebesgue measure of $\tilde{\mathcal{I}} \cap \mathcal{I}_k$ is positive. We take $\overset{\triangle}{\mathbf{u}} \in \mathcal{V}_k$ with $\overset{\triangle}{\mathbf{r}}_0 = \mathbf{0}$, $\overset{\triangle}{\theta}_0 = 0$, $\overset{\triangle}{\mu}(s) = 0$ on $\mathcal{I}_k$, with

$$\overset{\triangle}{\nu}(s) = \mathbf{a}(\theta(s)) \cdot \tilde{\mathbf{f}}_c(s), \quad \overset{\triangle}{\eta}(s) = \mathbf{b}(\theta(s)) \cdot \tilde{\mathbf{f}}_c(s) \quad \text{for } s \in \mathcal{I}_k \cap \tilde{\mathcal{I}}, \tag{6.34}$$

and with $\overset{\triangle}{\nu}(s) = \overset{\triangle}{\eta}(s) = 0$ otherwise. Hence (recall (6.9))

$$\langle g^*, \overset{\triangle}{\mathbf{u}} \rangle \;=\; \int_{\tilde{\mathcal{I}}} \chi_k(s)\, \|\tilde{\mathbf{f}}_c(s)\|^2\, ds \;>\; 0\,. \tag{6.35}$$

We now suppose that $0 \in \bar{\mathcal{I}}_i$. By similar arguments we get that $\tilde{\mathbf{f}}_c(0) \neq \mathbf{0}$. Taking $\overset{\triangle}{\mathbf{u}} \in \mathcal{V}_k$ with $\overset{\triangle}{\nu}, \overset{\triangle}{\eta}, \overset{\triangle}{\mu}, \overset{\triangle}{\theta}_0$ equal to zero and $\overset{\triangle}{\mathbf{r}}_0 = \tilde{\mathbf{f}}_c(0)$, we have

$$\langle g^*, \overset{\triangle}{\mathbf{u}} \rangle \;=\; \|\tilde{\mathbf{f}}_c(0)\|^2 \;>\; 0. \tag{6.36}$$

Condition (6.35) and (6.36) imply that $g^* \neq \mathbf{0}$ and, consequently,

$$\mathbf{0} \notin \partial \breve{g}_0(\mathbf{0}) \quad \text{if} \quad \breve{g}_0(\mathbf{0}) = 0 \quad \text{in the space } \mathcal{V}_k \text{ for } k \text{ large.} \tag{6.37}$$

6.5 Differentiability of the concentrated side condition

By arguments analogous to (6.23), (6.24), we obtain the continuous differentiability of

$$\overset{\triangle}{\mathbf{v}} \mapsto \breve{\mathbf{p}}[\overset{\triangle}{\mathbf{v}}](s_j, \zeta_j) := \mathbf{p}[\mathbf{u} + \overset{\triangle}{\mathbf{v}}](s_j, \zeta_j) \quad \text{and} \quad \overset{\triangle}{\mathbf{v}} \mapsto \breve{\theta}[\overset{\triangle}{\mathbf{v}}](s_j) := \theta[\mathbf{u} + \overset{\triangle}{\mathbf{v}}](s_j) \tag{6.38}$$

on $\mathcal{V}$ for $j = 1, \ldots, n$. Using Fubini's Theorem and the notation $\breve{\chi}_j(s)$ defined in (4.26), we get the differentials

$$\begin{aligned}
\breve{\mathbf{p}}_v[\mathbf{0}](s_j, \zeta_j)\, \overset{\triangle}{\mathbf{u}} \;&=\; \overset{\triangle}{\mathbf{r}}_0 + \int_0^{s_j} \Big[\overset{\triangle}{\nu}(\tau)\mathbf{a}(\theta(\tau)) + \overset{\triangle}{\eta}(\tau)\mathbf{b}(\theta(\tau)) + \mathbf{r}'^{\perp}(\tau) \Big(\overset{\triangle}{\theta}_0 + \int_0^{\tau} \overset{\triangle}{\mu}(\omega) d\omega \Big) \Big] d\tau \\
&\quad -\; \zeta_j \mathbf{a}(\theta(s_j)) \Big(\overset{\triangle}{\theta}_0 + \int_0^{s_j} \overset{\triangle}{\mu}(\tau) d\tau \Big) \\
&=\; \overset{\triangle}{\mathbf{r}}_0 + \int_0^L \breve{\chi}_j(s) \Big(\overset{\triangle}{\nu}(s)\mathbf{a}(\theta(s)) + \overset{\triangle}{\eta}(s)\mathbf{b}(\theta(s)) \Big) ds \\
&\quad + \int_0^L \overset{\triangle}{\mu}(s) \Big[\int_s^L \breve{\chi}_j(\tau)\mathbf{r}'^{\perp}(\tau)\, d\tau \;-\; \breve{\chi}_j(s)\zeta_j \mathbf{a}(\theta(s_j)) \Big] ds \\
&\quad + \overset{\triangle}{\theta}_0 \Big[\int_0^L \breve{\chi}_j(s)\mathbf{r}'^{\perp}(s)\, ds \;-\; \zeta_j \mathbf{a}(\theta(s_j)) \Big],
\end{aligned} \tag{6.39}$$

$$\breve{\theta}_v[\mathbf{0}](s_j)\, \overset{\triangle}{\mathbf{u}} \;=\; \overset{\triangle}{\theta}_0 + \int_0^L \breve{\chi}_j(s)\, \overset{\triangle}{\mu}(s)\, ds. \tag{6.40}$$

Now it is easy to see that $\breve{g}_i(\cdot)$ is continuously differentiable on $\mathcal{V}$. In the notation of (4.24), (4.25), we obtain

$$\begin{aligned}
\langle \breve{g}_i'(\mathbf{0}), \overset{\triangle}{\mathbf{u}} \rangle \;&=\; \sum_{j=1}^m \mathbf{p}_{ij}^* \cdot \overset{\triangle}{\mathbf{r}}_0 \;+\; \sum_{j=1}^m \mathbf{p}_{ij}^* \cdot \int_0^L \breve{\chi}_j(s) \Big(\overset{\triangle}{\nu}(s)\mathbf{a}(\theta(s)) + \overset{\triangle}{\eta}(s)\mathbf{b}(\theta(s)) \Big) ds \\
&\quad + \sum_{j=1}^m \mathbf{p}_{ij}^* \cdot \Big[\int_0^L \overset{\triangle}{\mu}(s) \Big(\int_s^L \breve{\chi}_j(\tau)\, \mathbf{r}'^{\perp}(\tau)\, d\tau \;-\; \breve{\chi}_j(s)\zeta_j \mathbf{a}(\theta(s_j)) \Big) ds \Big]
\end{aligned}$$

$$+ \overset{\triangle}{\theta}_0 \sum_{j=1}^{m} \mathbf{p}_{ij}^* \cdot \Big[\int_0^L \breve{\chi}_j(s)\, \mathbf{r}'^{\perp}(s)\, ds \;-\; \zeta_j \mathbf{a}(\theta(s_j)) \Big]$$

$$+ \overset{\triangle}{\theta}_0 \sum_{j=1}^{m} \theta_{ij}^* \;+\; \sum_{j=1}^{m} \theta_{ij}^* \int_0^L \breve{\chi}_j(s) \overset{\triangle}{\mu}(s)\, ds. \tag{6.41}$$

We define

$$\tilde{\mathbf{f}}_i(s) := \sum_{j=1}^{m} \breve{\chi}_j(s) \mathbf{p}_{ij}^*, \quad \tilde{\mathbf{l}}_{\tilde{f}_i}(s) := \sum_{j=1}^{m} \breve{\chi}_j(s) \zeta_j \mathbf{b}(\theta(s_j)) \times \mathbf{p}_{ij}^*, \quad \tilde{\mathbf{l}}_i(s) := \mathbf{k} \sum_{j=1}^{m} \breve{\chi}_j(s) \theta_{ij}^* \tag{6.42}$$

for $s \in [0, L]$, $i = 1, \ldots, n$. Obviously $\tilde{\mathbf{f}}_i$, $\tilde{\mathbf{l}}_{\tilde{f}_i}$, $\tilde{\mathbf{l}}_i$ are step functions with jumps (at most) at $s = s_j$. Observing that $\mathbf{a}, \mathbf{b}$ are orthogonal, for $\overset{\triangle}{\mathbf{u}} \in \mathcal{V}$ we get

$$\begin{aligned} \langle \breve{g}_i'(\mathbf{0}), \overset{\triangle}{\mathbf{u}} \rangle \;&=\; \overset{\triangle}{\mathbf{r}}_0 \cdot \tilde{\mathbf{f}}_i(0) \;+\; \int_0^L \Big(\overset{\triangle}{\nu}(s) \mathbf{a}(\theta(s)) + \overset{\triangle}{\eta}(s) \mathbf{b}(\theta(s)) \Big) \cdot \tilde{\mathbf{f}}_i(s)\, ds \\ &+ \int_0^L \overset{\triangle}{\mu}(s) \Big[\int_s^L (\, \mathbf{r}'(\tau) \times \tilde{\mathbf{f}}_i(\tau)\,) \cdot \mathbf{k}\, d\tau \;+\; \tilde{\mathbf{l}}_{\tilde{f}_i}(s) \cdot \mathbf{k} \;+\; \tilde{\mathbf{l}}_i(s) \cdot \mathbf{k} \Big] ds \\ &+ \overset{\triangle}{\theta}_0 \Big[\int_0^L (\, \mathbf{r}'(\tau) \times \tilde{\mathbf{f}}_i(\tau)\,) \cdot \mathbf{k}\, d\tau \;+\; \tilde{\mathbf{l}}_{\tilde{f}_i}(0) \cdot \mathbf{k} \;+\; \tilde{\mathbf{l}}_i(0) \cdot \mathbf{k} \Big]. \end{aligned} \tag{6.43}$$

Let us investigate the linear independence of the derivatives $\breve{g}_1'(\mathbf{0}), \ldots, \breve{g}_n'(\mathbf{0})$ with respect to the subspaces $\mathcal{V}_k$. For this purpose we assume that there are real numbers $\bar{\lambda}_i$, $i = 1, \ldots, n$ with

$$0 \;=\; \sum_{i=1}^{n} \bar{\lambda}_i \langle \breve{g}_i'(\mathbf{0}), \overset{\triangle}{\mathbf{u}} \rangle \quad \text{for all} \quad \overset{\triangle}{\mathbf{u}} \in \mathcal{V}_k. \tag{6.44}$$

If we choose special variations with $\overset{\triangle}{\mu} = 0$, $\overset{\triangle}{\theta} = 0$, then (6.43) implies that

$$\mathbf{0} \;=\; \sum_{i=1}^{n} \bar{\lambda}_i \tilde{\mathbf{f}}_i(s) \quad \text{a.e. on } \mathcal{I}_k \quad \text{and} \quad \mathbf{0} \;=\; \sum_{i=1}^{n} \bar{\lambda}_i \tilde{\mathbf{f}}_i(0). \tag{6.45}$$

Since $\tilde{\mathbf{f}}_i$ are step functions, (6.45) must be valid for all $s \in [0, L]$ if $k \in \mathbb{N}$ is large. Observing (6.42) and (4.26), we then get

$$\sum_{i=1}^{n} \bar{\lambda}_i \mathbf{p}_{ij}^* = \mathbf{0}, \;\; j = 1, \ldots, m, \quad \text{and, thus,} \quad \sum_{i=1}^{n} \bar{\lambda}_i \tilde{\mathbf{l}}_{\tilde{f}_i}(s) = \mathbf{0} \quad \text{for } \; s \in [0, L]. \tag{6.46}$$

We now choose variations with $\overset{\triangle}{\mathbf{r}}_0 = \mathbf{0}$, $\overset{\triangle}{\nu} = \overset{\triangle}{\eta} = 0$. From (6.43), (6.44) with the same arguments as above, we obtain

$$0 \;=\; \sum_{i=1}^{n} \bar{\lambda}_i \tilde{\mathbf{l}}_i(s) \cdot \mathbf{k} \quad \text{for all} \;\; s \in [0, L]. \tag{6.47}$$

Recalling (6.42) and (6.46) we have

$$\mathbf{0} \;=\; \sum_{i=1}^{n} \bar{\lambda}_i \mathbf{p}_{ij}^*, \quad 0 \;=\; \sum_{i=1}^{n} \bar{\lambda}_i \theta_{ij}^* \quad \text{for all} \;\; j = 1, \ldots, m. \tag{6.48}$$

By assumption (4.7) and Banach's Closed Range Theorem all $\bar{\lambda}_i$ must be zero (cf. Zeidler [46, p. 777]). This means that, with respect to the space $\mathcal{V}_k$, the gradients

$$\breve{g}_i'(\mathbf{0}), \;\; i = 1, \ldots, n, \;\; \text{are linearly independent for large} \;\; k \in \mathbb{N}. \tag{6.49}$$

6.6 Euler-Lagrange equations

Let us now study the modified problem (6.4) – (6.6) in some space $\mathcal{V}_k$ instead of $\mathcal{V}$. As we have seen, we can drop condition (6.6) in this case, $\mathbf{0} \in \mathcal{V}_k$ is a solution, and all functionals occurring in our analysis are Lipschitz continuous or even continuously differentiable near this solution with respect to the space $\mathcal{V}_k$ for $k \in \mathbb{N}$ large. Thus we can apply the Lagrange Multiplier Rule from Proposition 5.3, i.e., there exist real numbers $\lambda_E^k \geq 0$, $\lambda_0^k \geq 0$, $\lambda_1^k, \ldots, \lambda_n^k$, not all zero, with the property that

$$\mathbf{0} \;\in\; \lambda_E^k \breve{E}'(\mathbf{0}) \;+\; \lambda_0^k \partial \breve{g}_0(\mathbf{0}) \;+\; \sum_{i=1}^n \lambda_i^k \breve{g}_i'(\mathbf{0}) \tag{6.50}$$

where $\lambda_0^k = 0$ if $\breve{g}_0(\mathbf{0}) < 0$. This yields the existence of a gradient $g_0^{*k} \in \partial \breve{g}_0(\mathbf{0})$ such that

$$0 \;=\; \lambda_E^k \langle \breve{E}'(\mathbf{0}), \overset{\triangle}{\mathbf{u}} \rangle \;+\; \lambda_0^k \langle g_0^{*k}, \overset{\triangle}{\mathbf{u}} \rangle \;+\; \sum_{i=1}^n \lambda_i^k \langle \breve{g}_i'(\mathbf{0}), \overset{\triangle}{\mathbf{u}} \rangle \qquad \text{for all } \overset{\triangle}{\mathbf{u}} \in \mathcal{V}_k. \tag{6.51}$$

The gradient g_0^{*k} corresponds to a probability measure ρ^k and a ρ-measurable function $\mathbf{d}^{*k}$ on Ω with $\|\mathbf{d}^{*k}(\cdot,\cdot)\| \leq 1$. According to (6.28), (6.29), (6.30), we define $\tilde{\mathbf{f}}_c^k$, $\tilde{\mathbf{f}}_c^k$ and $\tilde{\mathbf{l}}_{f_c}^k$.

In view of (6.16), (6.22), (6.31), (6.43), equation (6.51) yields the following weak form of the Euler-Lagrange equations

$$\begin{aligned}
0 \;=\; & \lambda_E^k \int_{\mathcal{I}_k} \breve{W}_\nu[\mathbf{0}](s) \overset{\triangle}{\nu}(s) + \breve{W}_\eta[\mathbf{0}](s) \overset{\triangle}{\eta}(s) + \breve{W}_\mu[\mathbf{0}](s) \overset{\triangle}{\mu}(s)\, ds \\
& - \lambda_E^k \overset{\triangle}{\mathbf{r}}_0 \cdot \mathbf{f}(0) \;-\; \lambda_E^k \int_{\mathcal{I}_k} \Big(\overset{\triangle}{\nu}(s) \mathbf{a}(\theta(s)) + \overset{\triangle}{\eta}(s) \mathbf{b}(\theta(s)) \Big) \cdot \mathbf{f}(s)\, ds \\
& - \lambda_E^k \int_{\mathcal{I}_k} \overset{\triangle}{\mu}(s) \Big[\int_s^L (\mathbf{r}'(\tau) \times \mathbf{f}(\tau)) \cdot \mathbf{k}\, d\tau \;+\; \mathbf{l}_f(s) \cdot \mathbf{k} \Big] ds \\
& - \lambda_E^k \overset{\triangle}{\theta}_0 \Big[\int_0^L (\mathbf{r}'(\tau) \times \mathbf{f}(\tau)) \cdot \mathbf{k}\, d\tau \;+\; \mathbf{l}_f(0) \cdot \mathbf{k} \Big] \\
& + \lambda_0^k \overset{\triangle}{\mathbf{r}}_0 \cdot \tilde{\mathbf{f}}_c^k(0) \;+\; \lambda_0^k \int_{\mathcal{I}_k} \Big(\overset{\triangle}{\nu}(s) \mathbf{a}(\theta(s)) + \overset{\triangle}{\eta}(s) \mathbf{b}(\theta(s)) \Big) \cdot \tilde{\mathbf{f}}_c^k(s)\, ds \\
& + \lambda_0^k \int_{\mathcal{I}_k} \overset{\triangle}{\mu}(s) \Big[\int_s^L (\mathbf{r}'(\tau) \times \tilde{\mathbf{f}}_c^k(\tau)) \cdot \mathbf{k}\, d\tau \;+\; \tilde{\mathbf{l}}_{f_c}^k(s) \cdot \mathbf{k} \Big] ds \\
& + \lambda_0^k \overset{\triangle}{\theta}_0 \Big[\int_0^L (\mathbf{r}'(\tau) \times \tilde{\mathbf{f}}_c^k(\tau)) \cdot \mathbf{k}\, d\tau \;+\; \tilde{\mathbf{l}}_{f_c}^k(0) \cdot \mathbf{k} \Big] \\
& + \overset{\triangle}{\mathbf{r}}_0 \cdot \sum_{i=1}^n \lambda_i^k \tilde{\mathbf{f}}_i(0) \;+\; \int_{\mathcal{I}_k} \Big(\overset{\triangle}{\nu}(s) \mathbf{a}(\theta(s)) + \overset{\triangle}{\eta}(s) \mathbf{b}(\theta(s)) \Big) \cdot \Big(\sum_{i=1}^n \lambda_i^k \tilde{\mathbf{f}}_i(s) \Big) ds \\
& + \int_{\mathcal{I}_k} \overset{\triangle}{\mu}(s) \sum_{i=1}^n \lambda_i^k \Big[\int_s^L (\mathbf{r}'(\tau) \times \tilde{\mathbf{f}}_i(\tau)) \cdot \mathbf{k}\, d\tau \;+\; \tilde{\mathbf{l}}_{\tilde{f}_i}(s) \cdot \mathbf{k} \;+\; \tilde{\mathbf{l}}_i(s) \cdot \mathbf{k} \Big] ds \\
& + \overset{\triangle}{\theta}_0 \sum_{i=1}^n \lambda_i^k \Big[\int_0^L (\mathbf{r}'(\tau) \times \tilde{\mathbf{f}}_i(\tau)) \cdot \mathbf{k}\, d\tau \;+\; \tilde{\mathbf{l}}_{\tilde{f}_i}(0) \cdot \mathbf{k} \;+\; \tilde{\mathbf{l}}_i(0) \cdot \mathbf{k} \Big] \quad \text{for all } \overset{\triangle}{\mathbf{u}} \in \mathcal{V}_k
\end{aligned} \tag{6.52}$$

Set

$$\tilde{\mathbf{f}}_s^k(s) \;:=\; \sum_{i=1}^n \lambda_i^k \tilde{\mathbf{f}}_i(s), \quad \tilde{\mathbf{l}}_{\tilde{f}_s}^k \;:=\; \sum_{i=1}^n \lambda_i^k \tilde{\mathbf{l}}_{\tilde{f}_i}, \quad \tilde{\mathbf{l}}_s^k(s) \;:=\; \sum_{i=1}^n \lambda_i^k \tilde{\mathbf{l}}_i(s) \quad \text{for } s \in [0, L]. \tag{6.53}$$

We remark that $\tilde{\mathbf{f}}_s^k$, $\tilde{\mathbf{l}}_{\tilde{f}_s}^k$, $\tilde{\mathbf{l}}_s^k$ are again step functions with jumps at most at $s = s_j$, $j = 1, \ldots, m$. Remembering that $\breve{W}_\nu[\mathbf{0}](\cdot) = W_\nu[\mathbf{u}](\cdot)$, etc., we obtain the following Euler-Lagrange equations a.e. on $\mathcal{I}_k$

$$0 = \lambda_E^k W_\nu[\mathbf{u}](s) - \lambda_E^k \mathbf{a}(\theta(s)) \cdot \mathbf{f}(s) + \lambda_0^k \mathbf{a}(\theta(s)) \cdot \tilde{\mathbf{f}}_c^k(s) + \mathbf{a}(\theta(s)) \cdot \tilde{\mathbf{f}}_s^k(s), \tag{6.54}$$

$$0 = \lambda_E^k W_\eta[\mathbf{u}](s) - \lambda_E^k \mathbf{b}(\theta(s)) \cdot \mathbf{f}(s) + \lambda_0^k \mathbf{b}(\theta(s)) \cdot \tilde{\mathbf{f}}_c^k(s) + \mathbf{b}(\theta(s)) \cdot \tilde{\mathbf{f}}_s^k(s), \tag{6.55}$$

$$\begin{aligned} 0 = \; & \lambda_E^k W_\mu[\mathbf{u}](s) - \lambda_E^k \int_s^L (\, \mathbf{r}'(\tau) \times \mathbf{f}(\tau)\,) \cdot \mathbf{k}\, d\tau - \lambda_E^k \mathbf{l}_f(s) \cdot \mathbf{k} \\ & + \lambda_0^k \int_s^L (\, \mathbf{r}'(\tau) \times \tilde{\mathbf{f}}_c^k(\tau)\,) \cdot \mathbf{k}\, d\tau + \lambda_0^k \tilde{\mathbf{l}}_{f_c}^k(s) \cdot \mathbf{k} \\ & + \int_s^L (\, \mathbf{r}'(\tau) \times \tilde{\mathbf{f}}_s^k(\tau)\,) \cdot \mathbf{k}\, d\tau + \tilde{\mathbf{l}}_{\tilde{f}_s}^k(s) \cdot \mathbf{k} + \tilde{\mathbf{l}}_s^k(s) \cdot \mathbf{k}, \end{aligned} \tag{6.56}$$

$$\mathbf{0} = - \lambda_E^k \mathbf{f}(0) + \lambda_0^k \tilde{\mathbf{f}}_c^k(0) + \tilde{\mathbf{f}}_s^k(0), \tag{6.57}$$

$$\begin{aligned} 0 = \; & - \lambda_E^k \int_0^L (\, \mathbf{r}'(\tau) \times \mathbf{f}(\tau)\,) \cdot \mathbf{k}\, d\tau - \lambda_E^k \mathbf{l}_f(0) \cdot \mathbf{k} + \lambda_0^k \int_0^L (\, \mathbf{r}'(\tau) \times \tilde{\mathbf{f}}_c^k(\tau)\,) \cdot \mathbf{k}\, d\tau \\ & + \lambda_0^k \tilde{\mathbf{l}}_{f_c}^k(0) \cdot \mathbf{k} + \int_0^L (\, \mathbf{r}'(\tau) \times \tilde{\mathbf{f}}_s^k(\tau)\,) \cdot \mathbf{k}\, d\tau + \tilde{\mathbf{l}}_{\tilde{f}_s}^k(0) \cdot \mathbf{k} + \tilde{\mathbf{l}}_s^k(0) \cdot \mathbf{k}. \end{aligned} \tag{6.58}$$

Using the notation from (4.11) we can reformulate (6.54), (6.55) as

$$\mathbf{0} = \lambda_E^k \mathbf{n}[\mathbf{u}](s) - \lambda_E^k \mathbf{f}(s) + \lambda_0^k \tilde{\mathbf{f}}_c^k(s) + \tilde{\mathbf{f}}_s^k(s) \quad \text{a.e. on } \mathcal{I}_k. \tag{6.59}$$

Suppose that $\lambda_E^k = 0$. By (6.50), the case that $\lambda_i^k = 0$ for all $i = 1, \ldots, n$ can be excluded, because $\lambda_0^k = 0$ if $\breve{g}_0(\mathbf{0}) < 0$ and $0 \notin \partial \breve{g}_0(\mathbf{0})$ if $\breve{g}_0(\mathbf{0}) = 0$ (cf. (6.37)). If not all λ_i^k are zero, then (6.49) and the discussion surrounding it imply that either $\tilde{\mathbf{f}}_s^k(s)$ or $\tilde{\mathbf{l}}_s^k(s)$ is not identical zero. Since $\tilde{\mathbf{f}}_s^k$ and $\tilde{\mathbf{l}}_s^k$ are step functions, one of them must have a non-zero jump at some s_j. Equation (6.59) tells us that

$$\mathbf{0} = \lambda_0^k \tilde{\mathbf{f}}_c^k(s) + \tilde{\mathbf{f}}_s^k(s) \quad \text{a.e. on } \mathcal{I}_k \tag{6.60}$$

in this case. If $\tilde{\mathbf{f}}_s^k$ is not identical zero, then $\lambda_0^k \tilde{\mathbf{f}}_c^k$ must have a non-zero jump at some s_j. This is a contradiction, since s_j is not a contact parameter and the support of ρ^k only contains points (s, ζ) where s is a contact parameter. If $\tilde{\mathbf{f}}_s^k$ is identical zero, then $\lambda_0^k = 0$ by (6.60), because $\tilde{\mathbf{f}}_c^k(s)$ cannot be identically zero on $\mathcal{I}_k$ by (4.8) and the fact that ρ^k is a probability measure (cf. (6.29)). Furthermore $\tilde{\mathbf{l}}_{\tilde{f}_s}^k$ is identical zero in this case (cf. (6.45), (6.46)). Equations (6.56) and (6.58) with $\lambda_E^k = 0$ then give

$$\mathbf{0} = \tilde{\mathbf{l}}_s^k(s) \quad \text{for all } s \in [0, L], \tag{6.61}$$

i.e., $\tilde{\mathbf{f}}_s^k$ and $\tilde{\mathbf{l}}_s^k$ are both identically zero. As we have seen, this is impossible by our regularity assumptions. Consequently we must have $\lambda_E^k > 0$ and, without loss of generality, we can take $\lambda_E^k = 1$.

We now show that the Euler-Lagrange equations hold a.e. on $[0, L]$ instead of merely on $\mathcal{I}_k$. If there are no obstacles or if $g_0(\mathbf{u}) < 0$, then $\lambda_0^k = 0$ and the step functions $\tilde{\mathbf{f}}_s^k$, $\tilde{\mathbf{l}}_{\tilde{f}_s}^k$, $\tilde{\mathbf{l}}_s^k$ are independent of k for large $k \in \mathbb{N}$, because $\mathcal{I}_k \subset \mathcal{I}_{k+1}$ (cf. Antman [3, Chapter VII.5]). By the linear independence of the gradients $\breve{g}_i'(\mathbf{0})$, arguments similar to those just given imply that the multipliers λ_i^k, $i = 1, \ldots, n$ are also independent of k for large $k \in \mathbb{N}$. This extends the previous Euler-Lagrange equations a.e. on $[0, L]$ if $g_0(\mathbf{u}) < 0$.

Let us now consider the case that $g_0(\mathbf{u}) = 0$, which needs some more effort. We study the question of whether the Lagrange multipliers λ_i^k, $i = 0, 1, \ldots, n$, are bounded for $k \to \infty$. We first assume for contradiction that, at least for a subsequence,

$$\lambda_0^k \to \infty \quad \text{as} \quad k \to \infty. \tag{6.62}$$

Recall that $\{\tilde{\mathcal{I}}_i\}_{i=1}^l$ is a finite covering of the set $\mathcal{I}_c$ of contact parameters with open intervals and that (4.8) holds with $\bar{\mathcal{I}}_i \supset \tilde{\mathcal{I}}_i$ instead of $\mathcal{I}(s_c)$. Without loss of generality we can suppose that no $\bar{\mathcal{I}}_i$ contains any point s_j. Set $\Omega_i := \{(s, \zeta) \in \Omega |\ s \in \tilde{\mathcal{I}}_i\}$. For each $k \in \mathbb{N}$ there is an $i_k \in \{1, \ldots, l\}$ with $\rho^k(\Omega_{i_k}) \geq 1/l$, since ρ^k is a probability measure with support in $\Omega_{\mathcal{I}_c}$. By the finiteness of the number of indices we can assume, at least for a subsequence, that all indices i_k are equal, e.g., to $i_0 \in \{1, \ldots, l\}$. We agree that the open interval $\tilde{\mathcal{I}}_{i_0}$ coincides with $(\check{s}_1, \check{s}_2)$.

Let us start with the case that $\check{s}_1, \check{s}_2 \in (0, L)$. By $\mathcal{I}_k^e$ we denote the subset of $\mathcal{I}_k$ where equality holds in (6.59). We can fix some $k_0 \in \mathbb{N}$ sufficiently large such that the Lebesgue measure of the sets $(0, \check{s}_1) \cap \bar{\mathcal{I}}_{i_0} \cap \mathcal{I}_{k_0}^e$ and $(\check{s}_2, L) \cap \bar{\mathcal{I}}_{i_0} \cap \mathcal{I}_{k_0}^e$ is positive. Since $\mathcal{I}_{k_0} \subset \mathcal{I}_k$ for all $k > k_0$, we can find

$$\check{s}_a^k,\ \check{s}_b^k \ \in\ \bar{\mathcal{I}}_{i_0} \cap \mathcal{I}_{k_0}^e \cap \mathcal{I}_k^e \quad \text{with} \quad \check{s}_a^k \in (0, \check{s}_1),\ \ \check{s}_b^k \in (\check{s}_2, L) \quad \text{for } k > k_0. \tag{6.63}$$

Obviously the functions $\mathbf{f}$, $\tilde{\mathbf{f}}_c^{k_0}$ and $\tilde{\mathbf{f}}_s^{k_0}$ are bounded on $[0, L]$. In view of (6.59), there exist constants $c_f > 0$, $c_{k_0} > 0$ with

$$\|\mathbf{f}(s)\| \leq c_f \quad \text{on } [0, L], \qquad \|\mathbf{n}[\mathbf{u}](s)\| \ \leq\ c_{k_0} \quad \text{on } \mathcal{I}_{k_0}^e. \tag{6.64}$$

Since $\bar{\mathcal{I}}_{i_0}$ does not contain any s_j, it follows that $\tilde{\mathbf{f}}_s$ is constant on $\bar{\mathcal{I}}_{i_0}$ and (6.59) gives (observe that $\lambda_E^k = 1$)

$$\mathbf{0} \ =\ \mathbf{n}[\mathbf{u}](\check{s}_b^k) - \mathbf{n}[\mathbf{u}](\check{s}_a^k) - (\,\mathbf{f}(\check{s}_b^k) - \mathbf{f}(\check{s}_a^k)\,) + \lambda_0^k(\,\tilde{\mathbf{f}}_c^k(\check{s}_b^k) - \tilde{\mathbf{f}}_c^k(\check{s}_a^k)\,) \tag{6.65}$$

and, consequently,

$$\lambda_0^k \,\|\, \tilde{\mathbf{f}}_c^k(\check{s}_b^k) - \tilde{\mathbf{f}}_c^k(\check{s}_a^k) \,\| \ \leq\ 2c_{k_0} + 2c_f \quad \text{for all } k > k_0. \tag{6.66}$$

Recall the definition of $\tilde{\mathbf{f}}_c^k$ according to (6.29). With the notation $\Omega^k := \{(s, \zeta) \in \Omega | \check{s}_a^k \leq s < \check{s}_b^k\}$,

$$\mathbf{q}_c^k \ :=\ \tilde{\mathbf{f}}_c^k(\check{s}_a^k) - \tilde{\mathbf{f}}_c^k(\check{s}_b^k) \ =\ \int_{\Omega^k} \mathbf{d}^{*k}(\tau, \zeta)\, d\rho^k(\tau, \zeta). \tag{6.67}$$

We define

$$\Sigma \ :=\ \overline{\mathrm{co}}\, \{\partial d(\mathbf{q}) |\ d(\mathbf{q}) = 0,\ \ \mathbf{q} = \mathbf{p}[\mathbf{u}](\tau, \zeta) \ \text{ for some } \ (\tau, \zeta) \in \Omega_{\bar{\mathcal{I}}_{i_0}}\}, \tag{6.68}$$

Condition (4.8) implies that $\mathbf{0} \notin \Sigma$. By the compactness of Σ and a simple separation argument, there exist a vector $\check{\mathbf{q}} \in \mathbb{R}^2$ and a constant $c_q > 0$ with

$$\check{\mathbf{q}} \cdot \mathbf{q} \ >\ c_q \quad \text{for all } \mathbf{q} \in \Sigma. \tag{6.69}$$

By (6.67) and the definition of the integral, $\mathbf{q}_c^k$ belongs to a set Σ^k which can be recovered from Σ by scaling with a number $c^k \geq \rho^k(\Omega_{i_0}) \geq 1/l > 0$. Observing (6.69) we obtain for $k \to \infty$, that

$$\lambda_0^k \to \infty \quad \text{implies} \quad \lambda_0^k \check{\mathbf{q}} \cdot \mathbf{q}_c^k \to \infty. \tag{6.70}$$

This, however, contradicts (6.66), and only the cases $\check{s}_1 \leq 0$ or $\check{s}_2 \geq L$ remain.

If $\check{s}_1 \leq 0$ and $\check{s}_2 \in (0, L)$, then we argue as in the previous case with the special choice that $\check{s}_a^k := 0$, and, instead of (6.65), we derive a formula using (6.57) instead of (6.59) at $\check{s}_a^k$. This also leads to a contradiction.

If $\check{s}_2 \geq L$, then we choose $\check{s}_a^k$ as in the previous cases and replace (6.65) with either (6.59) at $\check{s}_a^k$ or with (6.57). Observe that $\tilde{\mathbf{f}}_s(\check{s}_a^k) = \mathbf{0}$ in these cases. Then we can argue as before and again obtain a contradiction. Thus we conclude that

$$\lambda_0^k \quad \text{is bounded for} \quad k \in \mathbb{N}. \tag{6.71}$$

Now let $\check{\mathcal{I}}$ be an open interval which contains exactly one of the points s_j. We first assume that this s_j lies in $(0, L)$. Since s_j is not a contact parameter, we can suppose that there are no contact parameters in $\check{\mathcal{I}}$ and, hence, $\tilde{\mathbf{f}}_c^k$ is constant on $\check{\mathcal{I}}$. As in (6.63) we can fix k_0 so large that there exist

$$\check{s}_a^k,\ \check{s}_b^k \ \in\ \check{\mathcal{I}} \cap \mathcal{I}_{k_0}^e \cap \mathcal{I}_k^e \quad \text{with} \quad \check{s}_a^k < s_j < \check{s}_b^k \quad \text{for } \ k > k_0. \tag{6.72}$$

Equation (6.59) then tells us that

$$\mathbf{0} \ = \ \mathbf{n}[\mathbf{u}](\check{s}_b^k) - \mathbf{n}[\mathbf{u}](\check{s}_a^k) - (\ \mathbf{f}(\check{s}_b^k) - \mathbf{f}(\check{s}_a^k)\ ) + (\ \tilde{\mathbf{f}}_s^k(\check{s}_b^k) - \tilde{\mathbf{f}}_s^k(\check{s}_a^k)\ ). \tag{6.73}$$

As in (6.66), we obtain

$$\|\ \tilde{\mathbf{f}}_s^k(\check{s}_b^k) - \tilde{\mathbf{f}}_s^k(\check{s}_a^k)\ \| \ \leq \ 2c_{k_0} + 2c_f \quad \text{for all } \ k > k_0. \tag{6.74}$$

Since $\tilde{\mathbf{f}}_s^k$ is a step function with only one jump in $\check{\mathcal{I}}$ at $s = s_j$, (6.74) implies the boundedness of this jump for all large k. If $s_j = 0$ or $s_j = L$, then we adopt the same ideas as in the proof of the boundedness of λ_0^k, i.e., we employ (6.57) in the case of $s_j = 0$ and, for $s_j = L$, we use (6.59) only at $s = \check{s}_a^k$. Thus we finally obtain that

$$\tilde{\mathbf{f}}_s^k(s) \quad \text{is uniformly bounded for all } \ s \in [0, L], \ \ k \in \mathbb{N}. \tag{6.75}$$

Recalling the definition of $\tilde{\mathbf{f}}_s^k$ we see that

$$\sum_{i=1}^{n} \lambda_i^k \mathbf{p}_{ij}^* \quad \text{is uniformly bounded for all } \ j = 1, \ldots, m, \ \ k \in \mathbb{N}. \tag{6.76}$$

Thus

$$\tilde{\mathbf{l}}_{\tilde{f}_s}^k \quad \text{is uniformly bounded for all } \ s \in [0, L], \ \ k \in \mathbb{N}. \tag{6.77}$$

Using (6.71), (6.75) and analogous arguments as before, we also get that

$$\tilde{\mathbf{l}}_s^k(s) \quad \text{is uniformly bounded for all } \ s \in [0, L], \ \ k \in \mathbb{N}. \tag{6.78}$$

Obviously (6.71), (6.75), (6.77), and (6.78) are also valid if $g_0(\mathbf{u}) < 0$. Since ρ^k is a probability measure and $\|\mathbf{d}^{*k}(\cdot,\cdot)\| \le 1$,

$$\|\tilde{\mathbf{f}}_c^k(s)\| \;\le\; 1 \quad \text{for } s \in [0,1],\ k \in \mathbb{N}. \tag{6.79}$$

Observing (6.71), (6.75), (6.77), (6.78), (6.79), we can derive from the Euler-Lagrange equations (6.56) and (6.59) that $\mathbf{n}[\mathbf{u}](\cdot)$ and $W_\mu[\mathbf{u}](\cdot)$ are essentially bounded on $[0,L]$, i.e., there is a constant $\kappa > 0$ such that

$$\|\mathbf{n}[\mathbf{u}](s)\| \;\le\; \kappa \quad \text{and} \quad |W_\mu[\mathbf{u}](s)| \;\le\; \kappa \quad \text{a.e. on } [0,L]. \tag{6.80}$$

This, however, implies that $\mathcal{I}_k = [0,L]$ for all $k \in \mathbb{N}$ sufficiently large and, consequently, (6.56) – (6.59) hold a.e. on $[0,L]$ for large k. This means that there exist Lagrange multipliers $\lambda_0 \ge 0$, $\tilde{\lambda}_i \in \mathbb{R}$, a probability measure $\tilde{\rho}$ and a measurable function $\tilde{\mathbf{d}}^*$ such that, with the notation of (4.12) and with $\tilde{\mathfrak{f}}_c$, $\tilde{\mathbf{f}}_c$, $\tilde{\mathbf{l}}_{f_c}$, $\tilde{\mathbf{f}}_s$, $\tilde{\mathbf{l}}_{\tilde{f}_s}$, $\tilde{\mathbf{l}}_s$ according to (6.28), (6.29), (6.30), (6.42), (6.53),

$$\mathbf{0} \;=\; \mathbf{n}[\mathbf{u}](s) \;-\; \mathbf{f}(s) \;+\; \lambda_0\tilde{\mathbf{f}}_c(s) \;+\; \tilde{\mathbf{f}}_s(s) \quad \text{a.e. on } [0,L], \tag{6.81}$$

$$\begin{aligned}\mathbf{0} \;=\;& \mathbf{m}[\mathbf{u}](s) \;-\; \int_s^L \mathbf{r}'(\tau)\times\mathbf{f}(\tau)\,d\tau \;-\; \mathbf{l}_f(s) \;+\; \lambda_0 \int_s^L \mathbf{r}'(\tau)\times\tilde{\mathbf{f}}_c(\tau)\,d\tau \\ &+\; \lambda_0\tilde{\mathbf{l}}_{f_c}(s) \;+\; \int_s^L \mathbf{r}'(\tau)\times\tilde{\mathbf{f}}_s(\tau)\,d\tau \;+\; \tilde{\mathbf{l}}_{\tilde{f}_s}(s) \;+\; \tilde{\mathbf{l}}_s(s) \quad \text{a.e. on } [0,L],\end{aligned} \tag{6.82}$$

$$\mathbf{0} \;=\; -\;\mathbf{f}(0) \;+\; \lambda_0\tilde{\mathbf{f}}_c(0) \;+\; \tilde{\mathbf{f}}_s(0), \tag{6.83}$$

$$\begin{aligned}\mathbf{0} \;=\;& -\int_0^L \mathbf{r}'(\tau)\times\mathbf{f}(\tau)\,d\tau \;-\; \mathbf{l}_f(0) \;+\; \lambda_0\int_0^L \mathbf{r}'(\tau)\times\tilde{\mathbf{f}}_c(\tau)\,d\tau \;+\; \lambda_0\tilde{\mathbf{l}}_{f_c}(0) \\ &+\; \int_0^L \mathbf{r}'(\tau)\times\tilde{\mathbf{f}}_s(\tau)\,d\tau \;+\; \tilde{\mathbf{l}}_{\tilde{f}_s}(0) \;+\; \tilde{\mathbf{l}}_s(0).\end{aligned} \tag{6.84}$$

Using (6.81) we can reformulate (6.82) and (6.84) as

$$\begin{aligned}\mathbf{0} \;=\;& \mathbf{m}[\mathbf{u}](s) \;-\; \int_s^L \mathbf{r}'(\tau)\times\mathbf{n}[\mathbf{u}](\tau)\,d\tau \\ &-\; \mathbf{l}_f(s) \;+\; \lambda_0\tilde{\mathbf{l}}_{f_c}(s) \;+\; \tilde{\mathbf{l}}_{\tilde{f}_s}(s) \;+\; \tilde{\mathbf{l}}_s(s) \qquad \text{a.e. on } [0,L],\end{aligned} \tag{6.85}$$

$$\mathbf{0} \;=\; -\int_0^L \mathbf{r}'(\tau)\times\mathbf{n}[\mathbf{u}](\tau)\,d\tau \;-\; \mathbf{l}_f(0) \;+\; \lambda_0\tilde{\mathbf{l}}_{f_c}(0) \;+\; \tilde{\mathbf{l}}_{\tilde{f}_s}(0) \;+\; \tilde{\mathbf{l}}_s(0). \tag{6.86}$$

With

$$\rho := \lambda_0\tilde{\rho},\;\; \mathbf{d}^* := \tilde{\mathbf{d}}^*,\;\; \lambda_i = -\tilde{\lambda}_i,\;\; i = 1,\dots,n, \tag{6.87}$$

we introduce $\mathfrak{f}_c$, $\mathbf{f}_c$, $\mathbf{l}_{f_c}$, $\mathbf{f}_s$, $\mathbf{l}_{f_s}$, $\mathbf{l}_s$ as in (4.17), (4.21), (4.22), (4.23). Obviously

$$\mathbf{f}_c = -\lambda_0\tilde{\mathbf{f}}_c,\;\; \mathbf{l}_{f_c} = -\lambda_0\tilde{\mathbf{l}}_{f_c},\;\; \mathbf{f}_s = -\tilde{\mathbf{f}}_s,\;\; \mathbf{l}_{f_s} = -\tilde{\mathbf{l}}_{\tilde{f}_s},\;\; \mathbf{l}_s = -\tilde{\mathbf{l}}_s. \tag{6.88}$$

This way we verify the Euler-Lagrange equations in Theorem 4.10 and the corresponding structure assertions. As a special consequence of (6.80) we get

$$\operatorname{ess\,inf}_{s\in[0,L]} \Big(\nu(s) - V(\mu(s),s)\Big) \;>\; 0 \tag{6.89}$$

and the theorem is proved. $\diamondsuit$

P r o o f of Corollary 4.27. We can argue as in the proof of the Theorem until formula (6.59). In order to get normality, i.e., $\lambda_E^k \neq 0$, under the assumption of the corollary, we use the fact that $\tilde{\mathbf{f}}_s^k$ can only have a jump at s_{j_0} belonging to Λ_{j_0} and, on the other hand, a jump of $\tilde{\mathbf{f}}_c^k$ at s_{j_0} must lie in a set $c_0\Sigma_{j_0}$ for some $c_0 > 0$. Since $\Lambda_{j_0} \cap c_0\Sigma_{j_0} = \emptyset$, we can derive that $\lambda_E^k = 1$ as before.

By definition, Σ_{j_0} is closed and, by the Lipschitz continuity of $d(\cdot)$, Σ_{j_0} is bounded (cf. Proposition 5.3.1), i.e., Σ_{j_0} is compact. Thus we can find a neighborhood U_0 of Σ_{j_0} such that

$$\Lambda_{j_0} \cap U_0 \;=\; \emptyset. \tag{6.90}$$

By Proposition 5.3.3, there must be an open interval $\check{I}_{j_0} := (s_{j_0} - \varepsilon, s_{j_0} + \varepsilon)$, $\varepsilon > 0$, with the property that

$$\Sigma_0 := \overline{\text{co}}\{\partial d(\mathbf{q}) |\, d(\mathbf{q} = 0,\ \mathbf{q} = \mathbf{p}[\mathbf{u}](s,\zeta) \text{ for some } (s,\zeta) \in \Omega_{\check{I}_{j_0}}\} \;\subset\; U_0. \tag{6.91}$$

Otherwise, we would have sequences $\{(\check{s}_i, \check{\zeta}_i)\}_{i=1}^{\infty} \subset \Omega$ and $\{\mathbf{d}_i^*\}_{i=1}^{\infty} \subset \mathbb{R}^2$ with

$$\check{s}_i \to s_{j_0}, \quad d(\mathbf{q}_i) = 0 \text{ for } \mathbf{q}_i := \mathbf{p}[\mathbf{u}](\check{s}_i, \check{\zeta}_i), \tag{6.92}$$

$$\mathbf{d}_i^* \in \partial d(\mathbf{q}_i) \quad \text{and} \quad \mathbf{d}_i^* \notin U_0. \tag{6.93}$$

At least for a subsequence we can assume that

$$(\check{s}_i, \check{\zeta}_i) \to (s_{j_0}, \zeta_{j_0}) \in \Omega, \quad \mathbf{d}_i^* \to \mathbf{d}_{j_0}^* \in \mathbb{R}^2. \tag{6.94}$$

Consequently,

$$\mathbf{q}_i \to \mathbf{q}_{j_0} := \mathbf{p}[\mathbf{u}](s_{j_0}, \zeta_{j_0}), \quad d(\mathbf{q}_{j_0}) = 0. \tag{6.95}$$

By Proposition 5.3.3, $\mathbf{d}_{j_0}^* \in \Sigma_{j_0}$, and this contradicts $\mathbf{d}_i^* \notin U_0$.

Without loss of generality we can assume for the covering that $\{\bar{\mathcal{I}}_i\}_{i=1}^l$ of $\mathcal{I}_c$,

$$s_{j_0} \in \bar{\mathcal{I}}_i \quad \text{implies} \quad \bar{\mathcal{I}}_i \subset \check{\mathcal{I}}_{j_0} \tag{6.96}$$

and that each $\bar{\mathcal{I}}_i$ contains at most one of the s_j. In order to derive the boundedness of the multipliers λ_0^k and of the step functions $\tilde{\mathbf{f}}_s^k$, $\tilde{\mathbf{l}}_s^k$, we can argue as in the previous proof. However, we have to include the case that

$$s_{j_0} \;\in\; \tilde{\mathcal{I}}_{i_0} = (\check{s}_1, \check{s}_2). \tag{6.97}$$

We again start with $\check{s}_1, \check{s}_2 \in (0, L)$ and we choose $\check{s}_a^k, \check{s}_b^k$ as in (6.63). Instead of (6.65) we get

$$\mathbf{0} = \mathbf{n}[\mathbf{u}](\check{s}_b^k) - \mathbf{n}[\mathbf{u}](\check{s}_a^k) - (\, \mathbf{f}(\check{s}_b^k) - \mathbf{f}(\check{s}_a^k)\,) + \lambda_0^k(\, \tilde{\mathbf{f}}_c^k(\check{s}_b^k) - \tilde{\mathbf{f}}_c^k(\check{s}_a^k)\,) + (\, \tilde{\mathbf{f}}_s^k(\check{s}_b^k) - \tilde{\mathbf{f}}_s^k(\check{s}_a^k)\,) \tag{6.98}$$

and, consequently,

$$\|\lambda_0^k(\, \tilde{\mathbf{f}}_c^k(\check{s}_b^k) - \tilde{\mathbf{f}}_c^k(\check{s}_a^k)\,) + (\, \tilde{\mathbf{f}}_s^k(\check{s}_b^k) - \tilde{\mathbf{f}}_s^k(\check{s}_a^k)\,)\| \;\leq\; 2c_{k_0} + 2c_f \quad \text{for all } k > k_0. \tag{6.99}$$

Define $\mathbf{q}_c^k$ as in (6.67) and

$$\mathbf{q}_s^k := \tilde{\mathbf{f}}_s^k(\check{s}_a^k) - \tilde{\mathbf{f}}_s^k(\check{s}_b^k), \quad \mathbf{q}^k := \lambda_0^k \mathbf{q}_c^k + \mathbf{q}_s^k. \tag{6.100}$$

By (6.90), (6.91), (6.96) and with Σ from (6.68), we have

$$\Sigma \subset \Sigma_0 \subset U_0 \quad \text{and} \quad \Lambda_{j_0} \cap \Sigma \; = \; \emptyset. \tag{6.101}$$

Since Λ_{j_0} is a linear subspace, this implies

$$\mathbf{0} \;\; \notin \;\; \Lambda_{j_0} + \Sigma := \{\mathbf{q} \in \mathbb{R}^2 |\; \mathbf{q} = \mathbf{q}_\Lambda + \mathbf{q}_\Sigma,\;\; \mathbf{q}_\Lambda \in \Lambda_{j_0},\;\; \mathbf{q}_\Sigma \in \Sigma\}. \tag{6.102}$$

By the closedness of Λ_{j_0} and the compactness of Σ, we obtain the closedness of $\Lambda_{j_0} + \Sigma$. Hence we can separate $\mathbf{0}$ and $\Lambda_{j_0} + \Sigma$, i.e., there are a vector $\check{\mathbf{q}}$ and a constant $c_q > 0$ such that

$$\check{\mathbf{q}} \cdot \mathbf{q} \;\; > \;\; c_q \quad \text{for all } \; \mathbf{q} \in \Lambda_{j_0} + \Sigma \tag{6.103}$$

and, therefore,

$$\check{\mathbf{q}} \cdot \mathbf{q}_\Lambda \;\; > \;\; c_q - \check{\mathbf{q}} \cdot \mathbf{q}_\Sigma \quad \text{for all } \; \mathbf{q}_\Lambda \in \Lambda_{j_0} \; \text{ and some fixed } \; \mathbf{q}_\Sigma \in \Sigma. \tag{6.104}$$

The linearity of the space Λ_{j_0} implies that

$$\check{\mathbf{q}} \cdot \mathbf{q}_\Lambda = 0 \quad \text{for all } \; \mathbf{q}_\Lambda \in \Lambda_{j_0}. \tag{6.105}$$

We again have $\mathbf{q}_c^k \in c^k\Sigma$ for some $c^k \geq 1/l > 0$ and, clearly, $\mathbf{q}_s^k \in \Lambda_{j_0}$ (cf. (6.42) and (6.53)). Thus

$$\check{\mathbf{q}} \cdot \mathbf{q}^k = \check{\mathbf{q}} \cdot (\; \lambda_0^k \mathbf{q}_c^k + \mathbf{q}_s^k \;) = \lambda_0^k c^k \check{\mathbf{q}} \cdot \Big(\frac{1}{c^k}\mathbf{q}_c^k\Big) > \lambda_0^k c^k c_q, \tag{6.106}$$

i.e.,

$$\lambda_0^k \to \infty \quad \text{implies} \quad \check{\mathbf{q}} \cdot \mathbf{q}^k \to \infty. \tag{6.107}$$

But this contradicts (6.99). The remaining cases $\check{s}_1 \leq 0$ and $\check{s}_2 \geq L$ can be handled as in the proof of the theorem by combining with these arguments. This finally gives the boundedness of λ_0^k for $k \in \mathbb{N}$. In view of (6.79), condition (6.99) also implies the uniform boundedness of $\tilde{\mathbf{f}}_s^k$ as in (6.75). The uniform boundedness of $\tilde{\mathbf{l}}_s^k$ as in (6.78) can be obtained again by similar arguments.

At this point we can proceed as in the proof of the theorem, and the corollary is established. $\diamondsuit$

P r o o f of Theorem 4.31. We introduce the functions

$$N(s) := W_\nu(\nu(s), \eta(s), \mu(s), s), \;\; H(s) := W_\eta(\ldots), \;\; M(s) := W_\mu(\ldots). \tag{6.108}$$

By the strict convexity of W we can find continuous functions $\hat{\nu}, \hat{\eta}, \hat{\mu}$ such that a.e. on $[0, L]$

$$\nu(s) = \hat{\nu}(N(s), H(s), M(s), s), \;\; \eta(s) = \hat{\eta}(\ldots), \;\; \mu(s) = \hat{\mu}(\ldots). \tag{6.109}$$

Since the continuous functions $\hat{\nu}, \hat{\eta}, \hat{\mu}$ are bounded on compact sets, the first assertion is a consequence of (4.19).

Under the additional assumptions of Theorem 4.31 we have that the functions $\hat{\nu}, \hat{\eta}, \hat{\mu}$ are independent of s and continuously differentiable as functions of $(\bar{N}, \bar{H}, \bar{M}) \in \mathbb{R}^3$. By (4.19), $N(\cdot), H(\cdot), M(\cdot)$ are essentially bounded by a constant κ. Obviously we can suppose that all partial derivatives $\hat{\nu}_N$, $\hat{\nu}_H$, $\hat{\nu}_M$, $\hat{\eta}_N, \ldots$ are uniformly bounded by a constant κ_1 on the compact subset of $\mathbb{R}^3$ where $|\bar{N}|, |\bar{H}|, |\bar{M}| \leq \kappa$. Using the Mean Value Theorem we can hence estimate for a.e. $s^1, s^2 \in [0, L]$

$$\begin{aligned} &|\hat{\nu}(N(s^2), H(s^2), M(s^2)) - \hat{\nu}(N(s^1), H(s^1), M(s^1))| \\ &\quad\leq \; |\hat{\nu}_N(\tilde{N}, \tilde{H}, \tilde{M})(N(s^2) - N(s^1))| \; + \; |\hat{\nu}_H(\ldots)(H(s^2) - H(s^1))| \\ &\qquad + \; |\hat{\nu}_M(\ldots)(M(s^2) - M(s^1))| \\ &\quad\leq \; \kappa_1 \Big(|N(s^2) - N(s^1)| + |H(s^2) - H(s^1)| + |M(s^2) - M(s^1)| \Big) \end{aligned} \tag{6.110}$$

and analogously for $\hat{\eta}, \hat{\mu}$. In Remark 4.30 we have seen that the real functions $N(\cdot), H(\cdot), M(\cdot)$ have bounded variation. Using the above estimates and the identities (6.109), we easily verify that the bounded variation of N, H, M implies that of ν, η, μ (cf. Benedetto [10], Evans & Gariepy [19]). ♢

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