Regularity for shearable nonlinearly elastic rods in obstacle problems

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Introduction

In this paper we are interested in regularity results to obstacle problems for shearable nonlinearly elastic rods. We work with the geometrically exact Cosserat theory for planar deformations which describes rods that can suffer not only flexure but also extension and shear and it involves general nonlinear constitutive relations. This is a consistent intrinsically one-dimensional theory which, however, allows a geometrically exact interpretation in a two- or three-dimensional setting.

The most obstacle problems studied in the literature are carried out for much simpler models neglecting shear, extension, and thickness and are restricted to small deformations. By these simplifications the set of admissible deformations is usually convex and the problem leads to a variational inequality. This is, meanwhile, a widely investigated subject where the results essentially rest on monotonicity and convexity arguments (cf., e.g., Fichera [6], Frehse [7], Hlaváček, Haslinger, Nečas & Lovíšek [8], Lewy & Stampacchia [11], Kikuchi & Oden [9], Kinderlehrer & Stampacchia [10], Rodrigues [12] and references therein). For more realistic models, however, a simple observations shows that even “nice” obstacles where the elastic body can move within a convex set do not correspond to a convex set of admissible deformations in a suitable function space. Thus we have to recognize that the theory of variational inequalities is unsuitable for that purpose.

We readily see that an obstacle brings a nonsmooth nonlinearity in the problem. But we cannot expect that the classical smooth analysis combined with the roughest nonsmooth tool, namely the variational inequality, is able to describe subtle nonsmooth effects. In particular the structure of the most interesting term describing the contact reactions, which fills the gap between inequality and equality, is not considered in the variational inequality. Therefore it seems to be necessary and natural to study obstacle problems by refined nonsmooth methods. In Schuricht [13] for the very large class of obstacles having Lipschitz boundary a more general nonsmooth

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variational approach to obstacle problems for Cosserat rods is given and, using Clarke's generalized gradients, the Euler-Lagrange equations can be derived. This provides more structure and shows as a regularity result that the reactions in frictionless contact are normal, in a generalized sense, to the boundary of the obstacle. This is usually a hypotheses in contact problems.

Motivated by these general variational results, in this paper we derive regularity results which are valid not only in the variational case. As we will see we obtain interesting qualitative differences between shearable and unshearable materials. This in particular allows a natural description of contact effects observed in experiments with soft shearable materials which was not possible in the unshearable context.

An introduction to the Cosserat theory for planar deformations is given in Section 1. In Section 2 we first formulate a general contact condition for frictionless contact inspired by previous variational results. Then we derive a refined contact condition which roughly expresses that the contact reactions are both normal to the boundary of the obstacle and to the boundary of the deformed rod. This refined condition seems reasonable also from the mechanical point of view. Both contact conditions are formulated by means of generalized gradients of certain distance functions and provide a powerful tool for a more detailed investigation of special situations in the rest of the paper. The case of an isolated active contact point where a concentrated force occurs is considered in Section 3. Here we obtain a discontinuity in the strains. In particular a rod with an originally smooth boundary will have a corner in the deformed shape which is impossible in the unshearable case. We furthermore discuss the question why it is reasonable to consider contact with sharp corners, though the direction of the contact force is not uniquely determined by the geometry in that case, instead of using a smooth approximation of the obstacle. Section 4 is devoted to obstacles having a $\mathcal{C}^1$-boundary. We show that the reactions are continuous at contact points for shearable materials, i.e., even at isolated contact points concentrations cannot occur. This is, however, wrong in the unshearable case. Obstacles with $\mathcal{C}^2$-boundary are studied in Section 5. Here we obtain that the contact reactions even have a continuous line density along the contact curve. In Section 6 we illuminate some qualitative differences in regularity between shearable and unshearable materials which show that shear causes new effects that cannot be neglected for soft materials. In the appendix we give a short introduction to Clarke's generalized gradients sufficient for the understanding of this paper.

Let us finally mention that the results of Schuricht [13] or Degiovanni & Schuricht [5] and of this paper provide a rigorous approach to a very large class of obstacle problems, i.e., we start with general existence results and without further hypotheses about the nature of contact reactions and the smoothness of the solutions we obtain general regularity results.

Notation. We denote by $\mathrm{cl}\,\mathcal{A}$, $\mathrm{int}\,\mathcal{A}$, $\mathcal{A}^c$, and $\partial\mathcal{A}$ the closure, the interior, the complement, and the boundary of the set $\mathcal{A}$. $\overline{\mathrm{cone}}\,\mathcal{A} := \mathrm{cl}\,\{tu|\, u \in \mathcal{A}, t \geq 0\}$ is the closed cone hull of $\mathcal{A}$. The function $\mathrm{dist}_{\mathcal{A}}(\cdot)$ assigns the distance to the set $\mathcal{A}$ to each point. For a locally Lipschitz continuous function $f : X \to \mathbb{R}$ Clarke's generalized gradient at u is denoted by $\partial f(u)$ and the generalized directional derivative by $f^0(u; v)$. For a function f defined on the real line $f(s\pm)$ stands for the limit from the right and from the left, respectively, at s. $\mathcal{L}^p$ denotes the Lebesgue space of $p-$integrable functions and $\mathcal{BV}$ the space of functions of bounded variation on a corresponding set. The scalar product on $\mathbb{R}^n$ is expressed by $\mathbf{a}\cdot\mathbf{b}$.

1 Rod theory

In this section we formulate the Cosserat or director theory describing planar deformations of nonlinearly elastic rods which can bend, stretch and shear. The presented version involves general forces and exact geometry as needed in obstacle problems. Though we sometimes argue from a higher dimensional point of view it is a consistent one-dimensional theory in the mathematical sense. However, it allows an exact two- or three-dimensional interpretation. For a more comprehensive presentation the reader is referred to Antman [1] and Schuricht [13].

Geometry of deformation. Let $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ be a fixed right-handed orthonormal basis in $\mathbb{R}^3$. We consider a slender three-dimensional body $\mathcal{B}$ that is symmetric with respect to the $\{\mathbf{i},\mathbf{j}\}$-plane and we restrict our studies to deformations that preserve this symmetry. Let us identify the deformed body $\mathcal{B}$ with the region occupied by its intersection with the $\{\mathbf{i},\mathbf{j}\}$-plane. We assume that the *position* $\mathbf{p}$ of the deformed material points can be given in the form

$$\mathbf{p}(s,\zeta) \;=\; \mathbf{r}(s)+\zeta\mathbf{b}(s) \quad \text{for } \ (s,\zeta)\in\Omega, \ \text{ where} \tag{1.1}$$

$$\Omega \;:=\; \{(s,\zeta)\in\mathbb{R}^2 : \, s\in[0,L],\ \zeta\in[h_1(s),h_2(s)]\}.$$

Here $\mathbf{r}(\cdot)$, $\mathbf{b}(\cdot)$ are vector-valued mappings lying in the $\{\mathbf{i},\mathbf{j}\}$-plane. $\mathbf{r}(\cdot)$ can be interpreted as the deformed configuration of some material curve in the body $\mathcal{B}$, the so-called *base curve* (e.g., the curve of centroids or a suitable boundary curve), and $\mathbf{b}(s)$ is a unit vector, called the *director* at s, describing the orientation of the cross-section at s. We understand s as length parameter and ζ as thickness parameter. h_1, h_2 are given real functions on $[0,L]$ which we assume to be continuous and

$$h_1(s)<h_2(s) \qquad \text{for all} \quad s\in[0,L].$$

Usually it is even supposed that $h_1(s)\le 0\le h_2(s)$ on $[0,L]$ which ensures that the base curve belongs to the rod and excludes reversed orientation along this curve under condition (1.2) below. But this is not necessary in general.

Thus we can describe a *planar configuration of a rod* by a pair $(\mathbf{r}(\cdot),\mathbf{b}(\cdot))$ of vector-valued functions which are defined on $[0,L]$ and which are assumed to be absolutely continuous. We set $\mathbf{a}:=-\mathbf{k}\times\mathbf{b}$ and by θ we denote the angle from $\mathbf{i}$ to $\mathbf{a}$ such that

$$\mathbf{a}=\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}, \quad \mathbf{b}=-\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j}.$$

Hence a configuration can be described alternatively by a pair $(\mathbf{r}(\cdot),\theta(\cdot))$. With the obvious choice of θ (modulo 2π) the absolute continuity of $\theta(\cdot)$ is implied by the absolute continuity of $\mathbf{b}(\cdot)$.

Let us decompose vector-valued functions with respect to the natural basis $\{\mathbf{a}, \mathbf{b}\}$. We set

$$\mathbf{r}' = \nu\mathbf{a} + \eta\mathbf{b}, \quad \mu := \theta'.$$

We call $\boldsymbol{\xi} := (\nu, \eta, \mu)$ the *strains* of a configuration. By the absolute continuity of $\mathbf{r}(\cdot)$, $\theta(\cdot)$ the strains must be integrable functions on $[0, L]$. The question of further regularity will be the subject of our investigation. For given strains $\nu(\cdot), \eta(\cdot), \mu(\cdot)$, $\mathbf{r}_0 \in \mathbb{R}^2$, and $\theta_0 \in \mathbb{R}$, we can represent a configuration by

$$\begin{aligned} \mathbf{r}(s) &= \mathbf{r}_0 + \int_0^s \Big[(\nu\cos\theta - \eta\sin\theta)\mathbf{i} + (\nu\sin\theta + \eta\cos\theta)\mathbf{j}\Big]\, d\tau, \\ \theta(s) &= \theta_0 + \int_0^s \mu\, d\tau. \end{aligned}$$

The natural undeformed state of the rod will be called *reference configuration* and all corresponding variables are identified by a superposed circle. We assume that

$$\mathring{\nu} = 1, \ \mathring{\eta} = 0, \quad \text{i.e.,} \quad \mathring{\mathbf{r}}' = \mathring{\mathbf{a}}\,.$$

This expresses that the cross-sections are orthogonal to the base curve and that s is the arc-length of the base curve in the reference configuration. An originally straight rod is obviously characterized by $\mathring{\mu} = 0$.

The requirement that deformations be locally *orientation-preserving* can be expressed by the condition that

$$\nu(s) > V(\mu(s), s) \ \text{ for } \ s \in [0, L], \quad \text{where} \quad V(\mu, s) := \begin{cases} h_2(s)\mu & \text{for } \mu \geq 0, \\ h_1(s)\mu & \text{for } \mu \leq 0. \end{cases} \tag{1.2}$$

Observe that $V(\cdot, s)$ is convex for all s. The curvature κ of a sufficiently smooth deformed base curve $\mathbf{r}(\cdot)$ is given by

$$\kappa = \frac{\Phi'}{\|\mathbf{r}'\|} = \frac{\mathbf{r}' \times \mathbf{r}''}{\|\mathbf{r}'\|^3} \cdot \mathbf{k}$$

where Φ denotes the angle from $\mathbf{i}$ to $\mathbf{r}'$ measured counter-clockwise.

Forces and equilibrium conditions. In consistency with the geometrical description of configurations let us identify subbodies $\breve{\mathcal{B}} \subset \mathcal{B}$ with the corresponding subset $\breve{\Omega} \subset \Omega$. In particular we define

$$\Omega_{\mathcal{I}} := \{(\tau, \zeta) \in \Omega : \ \tau \in \mathcal{I}\} \quad \text{for } \ \mathcal{I} \subset [0, L].$$

For a given configuration, the material of $\Omega_{[s,L]}$ exerts across section s a *resultant force* $\mathbf{n}(s)$ and a *resultant couple* $\mathbf{m}(s)$ on the material of $\Omega_{[0,s)}$. Naturally we have

$$\mathbf{n}(0) = \mathbf{0} \quad \text{and} \quad \mathbf{m}(0) = \mathbf{0}. \tag{1.3}$$

We suppose that all forces other than $\mathbf{n}$ acting at the body can be described by a finite vector-valued Borel measure

$$\breve{\Omega} \mapsto \mathfrak{f}(\breve{\Omega})$$

assigning the resultant force to subbodies $\breve{\Omega} \subset \Omega$. $\mathfrak{f}$ is called *external force*. It causes the *induced couple of* $\mathfrak{f}$ given by

$$\mathfrak{l}_{\mathfrak{f}}(\breve{\Omega}) \;:=\; \int_{\breve{\Omega}} \Big(\mathbf{p}(s,\zeta) - \mathbf{r}(s) \Big) \times d\mathfrak{f}(s,\zeta) \;=\; \int_{\breve{\Omega}} \zeta \mathbf{b}(s) \times d\mathfrak{f}(s,\zeta)$$

(recall (1.1)). Analogously we assume that all couples different from $\mathbf{m}$ and $\mathfrak{l}_{\mathfrak{f}}$ can be given by a finite vector-valued Borel measure

$$\breve{\Omega} \mapsto \mathfrak{l}(\breve{\Omega})$$

which we call *external couple*. For our planar theory we can suppose that all forces lie in the $\{\mathbf{i},\mathbf{j}\}$-plane and that all torques and couples are orthogonal to this plane. Thus we can introduce the so-called *stress resultants* $\boldsymbol{\Xi} := (N,H,M)$ by

$$\mathbf{n} \;=\; N\,\mathbf{a} + H\,\mathbf{b}\,, \quad \mathbf{m} \;=\; M\,\mathbf{k}\,.$$

A configuration of the rod is in equilibrium if the resultant force and the resultant torque about the origin vanish for each part of the rod. Using the distribution functions

$$\begin{aligned}
\mathbf{f}(s) \;&:=\; \int_{\Omega_{[s,L]}} d\mathfrak{f}(\tau,\zeta), \qquad \mathbf{l}(s) \;:=\; \int_{\Omega_{[s,L]}} d\mathfrak{l}(\tau,\zeta),\\
\mathbf{l}_f(s) \;&:=\; \int_{\Omega_{[s,L]}} d\mathfrak{l}_{\mathfrak{f}}(\tau,\zeta) \;=\; \int_{\Omega_{[s,L]}} \zeta \mathbf{b}(\tau) \times d\mathfrak{f}(\tau,\zeta)\,.
\end{aligned}$$

this is equivalent to the *equilibrium conditions* in integral form

$$\begin{aligned}
\mathbf{n}(s) \;-\; \mathbf{f}(s) \;&=\; \mathbf{0} \quad \text{for } s \in [0,L],\\
\mathbf{m}(s) \;-\; \int_s^L \mathbf{r}'(\tau) \times \mathbf{n}(\tau)\,d\tau \;-\; \mathbf{l}_f(s) \;-\; \mathbf{l}(s) \;&=\; \mathbf{0} \quad \text{for } s \in [0,L]\,.
\end{aligned}$$

Observe that by (1.3) the resultant external force and the resultant couple of all external actions must vanish for the whole body.

Constitutive relations. The material of the rod is taken to be *elastic*, i.e., there exist continuous *constitutive functions* $\hat{\boldsymbol{\Xi}} = (\hat{N},\hat{H},\hat{M})$ dependent on (ν,η,μ,s) such that the stress resultants are determined by the strains through

$$N = \hat{N}(\nu,\eta,\mu,s), \quad H = \hat{H}(\nu,\eta,\mu,s), \quad M = \hat{M}(\nu,\eta,\mu,s) \tag{1.4}$$

or, equivalently, in vector notation $\boldsymbol{\Xi} = \hat{\boldsymbol{\Xi}}(\boldsymbol{\xi},s)$. The domain of definition $\mathcal{D}(\hat{\boldsymbol{\Xi}})$ is obviously restricted by (1.2). As a consequence of the Strong Ellipticity Condition we can assume that $\boldsymbol{\xi} \to \hat{\boldsymbol{\Xi}}(\boldsymbol{\xi},s)$ is strictly monotone for each s, i.e.,

$$\Big(\hat{\boldsymbol{\Xi}}(\boldsymbol{\xi}_2,s) - \hat{\boldsymbol{\Xi}}(\boldsymbol{\xi}_1,s)\Big) \cdot (\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) \;>\; 0 \quad \text{for } \boldsymbol{\xi}_1 \neq \boldsymbol{\xi}_2$$

where $(\boldsymbol{\xi}_1,s),(\boldsymbol{\xi}_2,s) \in \mathcal{D}(\hat{\boldsymbol{\Xi}})$. Observe that this is equivalent with the positive-definiteness of the Jacobian $\hat{\boldsymbol{\Xi}}_{\boldsymbol{\xi}}$ in the case where $\hat{\boldsymbol{\Xi}}(\cdot,s)$ is smooth. The infinity conditions

$$\hat{N}(\nu,\eta,\mu,s) \to \begin{Bmatrix} +\infty \\ -\infty \end{Bmatrix} \quad \text{as} \quad \nu \to \begin{Bmatrix} +\infty \\ V(\mu,s) \end{Bmatrix},$$

$$\hat{H}(\nu,\eta,\mu,s) \to \pm\infty \quad \text{as} \quad \eta \to \pm\infty,$$
$$\hat{M}(\nu,\eta,\mu,s) \to \pm\infty \quad \text{as} \quad \mu \text{ approaches its positive and negative extremes of the region (1.2),}$$

ensure that extreme strains are accompanied by extreme reactions. Since we have chosen the natural undeformed state as reference configuration, we have that

$$\hat{\boldsymbol{\Xi}}(\overset{\circ}{\boldsymbol{\xi}},s) \;=\; \hat{\boldsymbol{\Xi}}(1,0,\overset{\circ}{\mu},s) \;=\; \mathbf{0} \quad \text{for all } s \in [0,L]\,.$$

The monotonicity and infinity conditions for $\hat{\boldsymbol{\Xi}}$ support a global implicit function theorem which ensures a unique solution of (1.4), i.e., we can find continuous functions $\hat{\boldsymbol{\xi}} = (\hat{\nu},\hat{\eta},\hat{\mu})$ such that

$$\nu = \hat{\nu}(N,H,M,s), \quad \eta = \hat{\eta}(N,H,M,s), \quad \mu = \hat{\mu}(N,H,M,s). \tag{1.5}$$

Furthermore $\boldsymbol{\Xi} \to \hat{\boldsymbol{\xi}}(\boldsymbol{\Xi},s)$ is also strictly monotone and inherits analog infinity conditions from $\hat{\boldsymbol{\Xi}}$ (cf. Antman [1]).

The material is said to be *hyperelastic* if there exists a *stored energy function* $(\nu,\eta,\mu,s) \to W(\nu,\eta,\mu,s)$ with the property that

$$\hat{N} \;=\; W_\nu, \quad \hat{H} \;=\; W_\eta, \quad \hat{M} \;=\; W_\mu.$$

We call a rod *unshearable* if the material is constrained in such a way that always $\eta = 0$. We get such a theory from the introduced Cosserat theory by simply defining $\hat{\eta} = 0$ in (1.5). In this case, however, H is a Lagrange multiplier corresponding to this material constraint and it is not determined constitutively (cf. Antman [1]). Moreover the mapping $\boldsymbol{\Xi} \to \hat{\boldsymbol{\xi}}(\boldsymbol{\Xi},s)$ is not strictly monotone anymore. In this sense we can consider unshearable rods as singular limit case of shearable materials. As we will see the loss of strict monotonicity causes interesting qualitative differences in regularity for shearable and unshearable materials. A first simple observation shows that

$$\frac{\mathbf{r}'}{\|\mathbf{r}'\|} \;=\; \mathbf{a} \quad \text{on } [0,L] \tag{1.6}$$

is always continuous, i.e., the direction of the tangent cannot jump for an unshearable rod.

For the rest of this paper we agree that we speak about the shearable case as long as nothing else is stated.

2 Obstacle problems

We define an *obstacle* $\mathcal{O}$ as a nonempty closed subset of $\mathbb{R}^2$. For our regularity investigations we consider the very rich class $\boldsymbol{\mathcal{O}}$ of obstacles $\mathcal{O} \neq \emptyset$ which are the closure of an open set and which have a Lipschitz boundary, i.e., $\partial\mathcal{O}$ is locally the graph of a Lipschitz function. As analytical tool we introduce the distance function

$$d(\mathbf{q}) \;:=\; \mathrm{dist}_{\mathcal{O}^c}\mathbf{q} \;-\; \mathrm{dist}_{\mathcal{O}}\mathbf{q} \quad \text{for } \mathbf{q} \in \mathbb{R}^2. \tag{2.1}$$

The obstacles $\mathcal{O} \in \boldsymbol{\mathcal{O}}$ have the properties that

$$\mathcal{O}^c \;=\; \operatorname{int}\left(\operatorname{cl}\mathcal{O}^c\right) \;\neq \emptyset\,,$$

and

$$d(\mathbf{q}) = 0 \quad \text{implies} \quad \mathbf{0} \notin \partial d(\mathbf{q}) \tag{2.2}$$

where $\partial d(\cdot)$ denotes Clarke's generalized gradient for locally Lipschitz continuous functionals (cf. the Appendix for a short introduction or Clarke [4]). This way we exclude certain singular cases which would bother our analysis by technicalities (for some discussion of the difficulties see Schuricht [13]).

We speak about an *obstacle problem* when the deformation of the rod is restricted by a given obstacle $\mathcal{O}$, i.e., when the admissible deformations are constrained by

$$\mathbf{p}(s,\zeta) \;\in\; \operatorname{cl}(\mathcal{O}^c) \quad \text{for all} \;\; (s,\zeta) \in \Omega$$

or, equivalently,

$$d(\mathbf{p}(s,\zeta)) \;\leq\; 0 \quad \text{on} \;\; \Omega\,.$$

The orientation-preserving condition (1.2) ensures that deformed configurations of the rod correspond to open mappings $(s,\zeta) \to \mathbf{p}(s,\zeta)$ on $\operatorname{int}\Omega$, i.e., images of open sets are open. This remains true if (1.2) only holds a.e. on $[0,L]$, as met for the solutions verified by general existence assertions (cf. Schuricht [13]). This implies the reasonable condition that the rod can touch an obstacle only with boundary points, i.e., points corresponding to $\partial\Omega$. We say that a configuration has *regular contact* if each cross-section has at most one contact point with the obstacle. We restrict our attention to such configurations in order to exclude the uninteresting cases where the behavior is governed by the constraints.

For the investigation of contact problems it is convenient to decompose the external force $\mathfrak{f}$ and the external couple $\mathfrak{l}$ such that

$$\mathfrak{f} \;=\; \mathfrak{f}_c + \mathfrak{f}_e, \quad \mathfrak{l}_f \;=\; \mathfrak{l}_{f_c} + \mathfrak{l}_{f_e}, \quad \mathfrak{l} \;=\; \mathfrak{l}_c + \mathfrak{l}_e,$$

where the subscript "c" refers to contact reactions and "e" to other external reactions. In analogy to the previous section we define the distribution functions $\mathbf{f}_c(\cdot)$, $\mathbf{f}_e(\cdot)$, $\mathbf{l}_{f_c}(\cdot)$, $\mathbf{l}_{f_e}(\cdot)$, $\mathbf{l}_c(\cdot)$, $\mathbf{l}_e(\cdot)$. Observe that all these functions belong to $\mathcal{BV}(0,L)$, because they are distributions of finite Borel measures (cf. Benedetto [3]). This also implies that for these functions the limits from the left and the right exist at every point. The equilibrium equations then readily imply the same property for the functions $\mathbf{m}$ and $\mathbf{n}$.

That we get a reasonable obstacle problem we have still to say something about the nature of contact reactions. Otherwise we could take any deformation of the rod and define the force and moment necessary to balance the system as contact reaction. However it is clear that an obstacle cannot balance all reactions. Thus we have to invoke some reasonable restrictions which we claim to do for frictionless contact.

Note. Let us shortly discuss the situation. There are essentially to ways to study obstacle problems.

In the hyperelastic case we can minimize the total energy under certain side conditions which is widely done in the literature. In the most cases simple models are used and the existence of generalized solutions, which satisfy some variational inequality, is verified. There are, however, no satisfactory results where the qualitative properties as, e.g., the direction of the contact force, are rigorously derived from such variational inequalities. On the one hand the models used are usually too primitive to describe the situation satisfactorily and, on the other hand, additional smoothness assumptions, which cannot be verified in general, have to be imposed to get some structure assertion about the contact reaction.

The second way to handle obstacle problems is to invoke some structural restriction for the contact reactions as hypotheses into the equilibrium conditions and then to solve the problem. Here the structural restrictions are usually motivated by experience as, e.g., that the tangential reactions vanish in frictionless contact. Such investigations are, however, mostly restricted to very special situations. Often only obstacles with smooth boundary can be handled or special techniques for corners are used. Sometimes additional assumptions as, e.g., the rough location of the contact region, are prescribed.

From the mathematical point of view both approaches have their justification. It is however unsatisfactorily that we usually cannot verify rigourously the equivalence between both. This is probably caused by the fact that the mostly used models and the analytical tools as, e.g., variational inequalities do not have enough structure. In Schuricht [13] a rigorous variational approach to general obstacle problems for Cosserat rods is presented where this gap can be closed for frictionless contact. The Euler-Lagrange equations of the variational problem, which are derived by tools of nonsmooth analysis, coincide with the equilibrium conditions and provide some natural structure of the contact reactions for the very large class of obstacles having Lipschitz boundary. We use these results as basis for our further regularity investigations.

Motivated by the results in Schuricht [13] for the hyperelastic case we formulate the

General contact condition. The *contact force* $\mathfrak{f}_c$ for frictionless contact has the form

$$\mathfrak{f}_c(\breve{\Omega}) = -\int_{\breve{\Omega}} \mathbf{d}^*(s,\zeta)\, d\rho(s,\zeta) \quad \text{for all Borel sets } \breve{\Omega} \subset \Omega \tag{2.3}$$

where ρ is a finite real non-negative Borel measure on Ω supported on the *contact set*

$$\Omega_c \;:=\; \{(s,\zeta) \in \Omega |\, d(\mathbf{p}(s,\zeta)) = 0\}$$

and

$$(s,\zeta) \to \mathbf{d}^*(s,\zeta) \in \partial d(\mathbf{p}(s,\zeta)) \qquad (\partial \text{ w.r.t. } d(\cdot))$$

is a ρ-integrable mapping on Ω (∂ denotes Clarke's generalized gradient, cf. Appendix). The corresponding *induced contact couple* $\mathfrak{l}_{f_c}$ is given by

$$\mathfrak{l}_{f_c}(\breve{\Omega}) \;=\; -\int_{\breve{\Omega}} \zeta \mathbf{b}(s) \times \mathbf{d}^*(s,\zeta)\, d\rho(s,\zeta) \quad \text{for Borel sets } \breve{\Omega} \subset \Omega\,.$$

The *pure contact couple* $\mathfrak{l}_c$ has to vanish, i.e.,

$$\mathfrak{l}_c(\breve{\Omega}) \;=\; \mathbf{0} \quad \text{for Borel sets } \breve{\Omega} \subset \Omega\,.$$

This general contact condition expresses that contact reactions only occur at points where the rod touches the obstacle, that in some sense the corresponding forces are directed normal to the boundary $\partial\mathcal{O}$ (cf. Appendix), and that no pure contact couple can occur. This is some generalization of the usual contact condition in problems with smooth boundary $\partial\mathcal{O}$ where the tangential components of contact reactions are assumed to vanish. In the following we claim to derive stronger properties for $\mathbf{d}^*$ and ρ in general and in special situations.

Since contact and other external reactions correspond to finite measures, the stress resultants must be bounded by the equilibrium conditions, i.e., for a solution of the obstacle problem there is a constant $c > 0$ such that

$$\|\boldsymbol{\Xi}(s)\| \;\leq\; c \quad \text{for} \;\; s \in [0, L].$$

The boundedness of the continuous function $\hat{\boldsymbol{\xi}}$ on compact sets implies that

$$\boldsymbol{\xi}(\cdot) \in \mathcal{L}^\infty(0, L)$$

(observe that the strains are determined only up to a set of measure zero). Thus $\mathbf{r}$, θ are Lipschitz continuous. For homogeneous materials we have that

$$\operatorname*{ess\,inf}_{s\in[0,L]} \Big(\nu(s) - V(\mu(s))\Big) \;>\; 0$$

(see also Schuricht [13]).

We now want to motivate a refined condition for the direction of $\mathbf{d}^*$. Let us consider the hyperelastic case and let $\mathbf{p}$ be an equilibrium state of the rod with regular contact which corresponds to a minimzer of the energy subjected to the obstacle $\mathcal{O}$. Assume that we have contact at the point $\mathbf{q}_0 := \mathbf{p}(s_0, \zeta_0) \in \partial\mathcal{O}$ for some $(s_0, \zeta_0) \in \partial\Omega$ with $s_0 \in]0, L[$. By

$$\mathcal{B}_{\mathbf{p}} \;:=\; \{\mathbf{q} \in \mathbb{R}^2 |\, \mathbf{q} = \mathbf{p}(s, \zeta), \;\; (s, \zeta) \in \Omega\}$$

we denote the closed set occupied by the deformed points according to the state $\mathbf{p}$. For some small closed ball B_{q_0} around $\mathbf{q}_0$ we define the new obstacle $\tilde{\mathcal{O}}$

$$\tilde{\mathcal{O}} \;:=\; \mathcal{O} \,\cup\, \Big(\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}}) \cap B_{q_0}\Big)\,.$$

We do not study the question whether $\tilde{\mathcal{O}} \in \boldsymbol{\mathcal{O}}$. But we readily see that the rod respects also the obstacle $\tilde{\mathcal{O}}$. Because $\mathbf{p}$ has regular contact with $\mathcal{O}$, it also has regular contact with $\tilde{\mathcal{O}}$ for sufficiently smallB_{q_0} (observe $s_0 \in]0, L[$). Obviously the same arguments are true for all obstacles $\breve{\mathcal{O}}$ with

$$\mathcal{O} \,\subset\, \breve{\mathcal{O}} \,\subset\, \tilde{\mathcal{O}}\,. \tag{2.4}$$

If we take the usual case where the external reactions $\mathfrak{f}_e$ and $\mathfrak{l}_e$ depend at most on the deformation $\mathbf{p}$ but not on the shape of the obstacle, then $\mathbf{p}$ corresponds also to a minimzer of the energy subjected to any of these $\breve{\mathcal{O}}$, since the set of admissible deformations becomes smaller by $\mathcal{O} \subset \breve{\mathcal{O}}$. Thus the above formulated general contact condition can be derived at least for all $\breve{\mathcal{O}} \subset \boldsymbol{\mathcal{O}}$ satisfying (2.4). In analogy to (2.1) we define

$$d_{\breve{\mathcal{O}}}(\mathbf{q}) \;:=\; \mathrm{dist}_{\breve{\mathcal{O}}^c}\mathbf{q} \;-\; \mathrm{dist}_{\breve{\mathcal{O}}}\mathbf{q} \quad \text{for} \;\; \mathbf{q} \in \mathbb{R}^2 \tag{2.5}$$

and we introduce the class of obstacles

$$\boldsymbol{\mathcal{O}}_{\mathcal{O},\mathbf{p}} \;:=\; \{\breve{\mathcal{O}} \in \boldsymbol{\mathcal{O}} |\, \mathcal{O} \subset \breve{\mathcal{O}} \subset \mathrm{cl}(\mathcal{B}^c_{\mathbf{p}})\}\,.$$

The generalized gradient clearly only depends on the local behavior of a function. Hence it is equivalent to evaluate $\partial d_{\breve{\mathcal{O}}}(\mathbf{p}(s_0,\zeta_0))$ either for all $\breve{\mathcal{O}} \in \boldsymbol{\mathcal{O}}$ satisfying (2.4) or for all $\breve{\mathcal{O}} \in \boldsymbol{\mathcal{O}}_{\mathcal{O},\mathbf{p}}$. Since $\mathbf{p}(s_0,\zeta_0)$ was an arbitrarily choosen contact point, we can formulate the

Refined contact condition. $\mathfrak{f}_c$ and $\mathfrak{l}_c$ satisfy the general contact condition given above and, in addition,

$$\mathbf{d}^*(s,\zeta) \;\in \bigcap_{\breve{\mathcal{O}} \in \boldsymbol{\mathcal{O}}_{\mathcal{O},\mathbf{p}}} \overline{\mathrm{cone}}\left(\partial d_{\breve{\mathcal{O}}}(\mathbf{p}(s,\zeta))\right) \quad \text{for } \;(s,\zeta) \in \mathrm{supp}\,\rho \subset \Omega_c \;\text{ with }\; s \in]0,L[\,, \tag{2.6}$$

where supp denotes the support of ρ.

Observe that we have to use the closed cone hull on the right hand side, because in the general contact condition a positive Lagrange multiplier is invoked into the measure ρ which could be different for different $\breve{\mathcal{O}}$. Roughly speaking, the obstacle $\breve{\mathcal{O}} \in \boldsymbol{\mathcal{O}}_{\mathcal{O},\mathbf{p}}$ where $\partial d_{\breve{\mathcal{O}}}(\mathbf{p}(s,\zeta))$ spans the smallest cone gives the most detailed information about the direction of the contact force. Of course such a refinement can also be derived for points $(s,\zeta) \in \Omega_c$ with $s = 0$ or $s = L$. The restriction to regular contact, however, causes some technicalities which we do not wont to carry out here.

This refined condition for the direction of the contact force expresses some combined normality both with respect to the obstacle and with respect to the deformed shape of the rod. Let us discuss three typical cases where we assume that $\mathbf{q}_0 = \mathbf{p}(s_0,\zeta_0)$ with $s_0 \in]0,L[$ is some a point of an equilibrium state $\mathbf{p}$.

Case 1. Let there exist two different half spaces $\mathcal{H}_1, \mathcal{H}_2 \subset \mathbb{R}^2$ and a small closed ball B_{q_0} around $\mathbf{q}_0$ such that

$$(\mathcal{O} \cap B_{q_0}) \;\subset\; (\mathcal{H}_i \cap B_{q_0}) \;\subset\; (\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}}) \cap B_{q_0}) \quad \text{for } \; i = 1,2\,.$$

Then, formally, $\mathbf{d}^*(s_0,\zeta_0) = \mathbf{0}$ by (2.6) and this gives a contradiction between the general contact condition and the regularity of $d(\cdot)$ according to (2.2). We however see that (2.3) and (2.6) have to hold only on supp ρ. Thus we conclude that $(s_0,\zeta_0) \notin \mathrm{supp}\,\rho$ in our case, i.e., there is no contact reaction at the contact point $\mathbf{q}_0 = \mathbf{p}(s_0,\zeta_0)$ (cf. Fig. 1).

In the next two cases we consider certain regular situations. A real locally Lipschitz continuous functional is called regular at some point in the sense of Clarke if all directional derivatives exist in the classical sense and equal Clarke's generelized directional derivatives at that point.

Case 2. Assume that $\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}}) \subset \boldsymbol{\mathcal{O}}$ and let there exist a half space $\mathcal{H} \subset \mathbb{R}^2$ with $\mathbf{q}_0 \in \partial\mathcal{H}$ and a small closed ball B_{q_0} around $\mathbf{q}_0$ such that

$$(\breve{\mathcal{O}} \cap B_{q_0}) \;\subset\; (\mathcal{H} \cap B_{q_0}) \quad \text{for all } \; \breve{\mathcal{O}} \in \boldsymbol{\mathcal{O}}_{\mathcal{O},\mathbf{p}}\,,$$

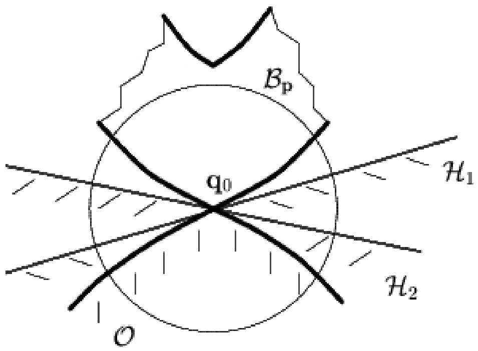

Fig. 1. There can be no contact reaction at the contact point.

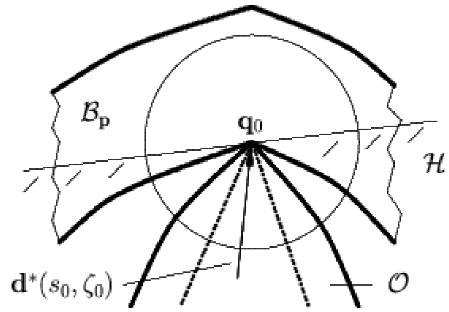

Fig. 2. The dashed cone, which is normal to the shape of the deformed rod, describes the possible directions of the contact force.

and assume that $-d_{\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}})}$ is regular at $\mathbf{q}_0$ in the sense of Clarke. Then (2.6) is equivalent with

$$\mathbf{d}^*(s_0, \zeta_0) \in \overline{\mathrm{cone}}\Big(\partial d_{\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}})}(\mathbf{q}_0)\Big)$$

(cf. Fig. 2).

Case 3. Assume that $\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}}) \subset \boldsymbol{\mathcal{O}}$ and let there exist a half space $\mathcal{H} \subset \mathbb{R}^2$ with $\mathbf{q}_0 \in \partial\mathcal{H}$ and a small closed ball B_{q_0} around $\mathbf{q}_0$ such that

$$(\breve{\mathcal{O}}^c \cap B_{q_0}) \supset (\mathcal{H} \cap B_{q_0}) \quad \text{for all } \breve{\mathcal{O}} \in \boldsymbol{\mathcal{O}}_{\mathcal{O},\mathbf{p}}\,,$$

and assume that $d_{\mathcal{O}}$ is regular at $\mathbf{q}_0$ in the sense of Clarke. Then (2.6) is equivalent to

$$\mathbf{d}^*(s_0, \zeta_0) \in \overline{\mathrm{cone}}\Big(\partial d_{\mathcal{O}}(\mathbf{q}_0)\Big)$$

(cf. Fig. 3). The proof of Cases 2+3 is given below.

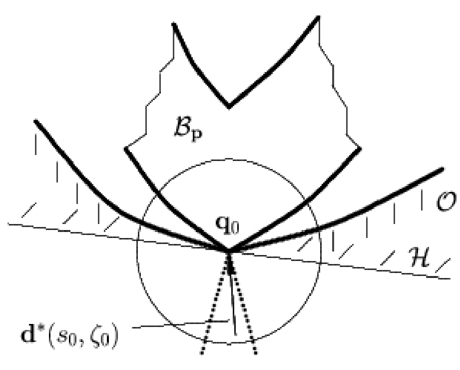

Fig. 3. The dashed cone, which is normal to the obstacle, describes the possible directions of the contact force.

The last two cases express the rough idea of the refined contact condition that the admissible obstacle which is locally the "closest to a half-space" gives the best condition for $\mathbf{d}^*(s_0, \zeta_0)$. But if, e.g., $-d_{\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}})}$ highly oscillates and is not regular at $\mathbf{q}_0$, then the assertion of Case 2 must not be true and the statement "closest to a half-space" cannot be taken in the sense of set inclusions. Let us still mention that convex Lipschitz functionals are always regular in the sense of Clarke. For nonsmooth concave functionals this is, however, wrong in general. This explains why it is reasonable to demand regularity of $-d_{\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}})}$ instead of $d_{\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}})}$ in Case 2.

While Case 2 and Case 3 look reasonable also from the mechanical point of view, Case 1 is a little surprising. Remember, however, that it is derived under the assumption about the equilibrium state to be a local minimizer of the energy, which expresses some kind of stability. A contact reaction at $\mathbf{q}_0$ in this case, however, should be highly unstable (cf. Fig. 1). Moreover observe that a "highly unstable" saddle point can disappear by making the admissible set smaller. Without carrying out this aspect in full detail we merely want to conclude from this argumentation the rough rule that the refined contact condition can be used in problems where we look for "reasonable stable" and "reasonable saddle" solutions. If this is too restrictive, then we should work with the general contact condition only.

Proof of Cases 2+3. In Case 2 we obviously have that $d_{\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}})}(\mathbf{q}) \geq d_{\breve{\mathcal{O}}}(\mathbf{q})$ on B_{q_0} for any $\breve{\mathcal{O}} \in \boldsymbol{\mathcal{O}}_{\mathcal{O},\mathbf{p}}$. Thus for any $\mathbf{w} \in \mathbb{R}^2$

$$
\begin{aligned}
\Big(-d_{\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}})}\Big)^0(\mathbf{q}_0;\mathbf{w}) &= \lim_{t\downarrow 0} \frac{-d_{\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}})}(\mathbf{q}_0+t\mathbf{w})}{t} \\
&\leq \liminf_{t\downarrow 0} \frac{-d_{\breve{\mathcal{O}}}(\mathbf{q}_0+t\mathbf{w})}{t} \;\leq\; \Big(-d_{\breve{\mathcal{O}}}\Big)^0(\mathbf{q}_0;\mathbf{w})\,.
\end{aligned}
$$

This readily implies that

$$
\partial\Big(-d_{\mathrm{cl}(\mathcal{B}^c_{\mathbf{p}})}\Big)(\mathbf{q}_0) \;\subset\; \partial\Big(-d_{\breve{\mathcal{O}}}\Big)(\mathbf{q}_0)\,.
$$

Since $\partial(-f)(u) = -\partial f(u)$ for any locally Lipschitz functional f, we obtain the assertion.

In Case 3 we use that $d_{\mathcal{O}}(\mathbf{q}) \leq d_{\breve{\mathcal{O}}}(\mathbf{q})$ on B_{q_0} for $\breve{\mathcal{O}} \in \boldsymbol{\mathcal{O}}_{\mathcal{O},\mathbf{p}}$ and we conclude analogously. $\square$

3 Concentrated contact reactions

In this section we study the consequences of concentrated contact reactions which, e.g., occur at corners of the obstacle. Furthermore we discuss a macroscopic and a microscopic view to the contact with sharp corners.

We assume that there is some isolated contact point $\mathbf{q}_0 = \mathbf{p}(s_0,\zeta_0) \in \partial\mathcal{O}$ with non-vanishing contact reaction and that this is the only contact point with the obstacle $\mathcal{O}$ for all cross-sections corresponding to a neighborhood of $s = s_0$. We call such a point $\mathbf{q}_0$ an *isolated active contact point*. Obviously $(s_0,\zeta_0) \in \partial\Omega$. That the behavior is not dominated by reactions different from the obstacle we impose the hypothesis that

(H1) $\mathbf{f}_e$, $\mathbf{l}_{f_e}$, and $\mathbf{l}_e$ are continuous at $s = s_0$.

An isolated non-vanishing contact reaction at exactly one point $\mathbf{p}(s_0,\zeta_0)$ means that the measure $\mathfrak{f}_c$ must have a concentration at (s_0,ζ_0) and the induced couple $\mathfrak{l}_{f_c}$ can have a corresponding concentration. By the equilibrium equations we have that

$$\tilde{\mathbf{n}}_0 \;:=\; \mathbf{n}(s_0+) - \mathbf{n}(s_0-) \;\neq\; \mathbf{0}, \quad \mathbf{m}(s_0+) - \mathbf{m}(s_0-) \;=\; \zeta_0 \mathbf{b}(s_0) \times \tilde{\mathbf{n}}_0,$$

where $\mathbf{n}(s_0\pm)$ and $\mathbf{m}(s_0\pm)$ denote the one-sided limits at s_0. We will see in the next section that this situation can only occur at contact points where either the boundary $\partial\mathcal{O}$ near $\mathbf{q}_0$ or the corresponding function h_1 or h_2 near s_0 is not smooth, i.e., roughly speaking, either $\partial\mathcal{O}$ has a corner at $\mathbf{q}_0$ or the boundary of the rod in the reference configuration, which is identified with $\partial\Omega$, has a corner at (s_0, ζ_0).

Proposition 3.1 *Let* (H1) *be satisfied and the equilibrium state* $\mathbf{p}$ *of the rod may have an isolated active contact point at* $\mathbf{q}_0 = \mathbf{p}(s_0, \zeta_0)$. *Then there exist* $\lambda > 0$ *and* $\mathbf{d}_0 \in \partial d(\mathbf{p}(s_0, \zeta_0))$ *(*$\not\ni \mathbf{0}$*) such that*

$$\mathbf{n}(s_0+) - \mathbf{n}(s_0-) \;=\; -\lambda \mathbf{d}_0\,, \quad \mathbf{m}(s_0+) - \mathbf{m}(s_0-) \;=\; -\lambda \zeta_0 \mathbf{b}(s_0) \times \mathbf{d}_0\,. \tag{3.2}$$

Moreover the strains $\boldsymbol{\xi}(s) = \hat{\boldsymbol{\xi}}(\boldsymbol{\Xi}(s), s)$ *have a finite jump at* $s = s_0$. *If, in particular,* (s_0, ζ_0) *lies on the base curve, then the tangent* $\mathbf{r}'(\cdot)$ *has a finite jump at* $s = s_0$.

Remark 3.3

(1) The above condition for $\mathbf{d}_0$ is based on the general contact condition and can be still strengthened by the refined contact condition taking into account the shape of the deformed rod. However the direction of the concentrated force will not be uniquely determined by the geometry of the obstacle and the deformed rod in general. This would be only the case, roughly speaking, if either $\partial\mathcal{O}$ or $\partial\mathcal{B}_{\mathbf{p}}$ is smooth near $\mathbf{q}_0$. But, by the regularity results below, this cannot be expected in general at an isolated active contact point. Furthermore observe that Case 1 of the previous section is impossible in that situation.

(2) The fact that the direction of the contact reaction is not uniquely determined by the geometry raises the expectation that solutions are not uniquely determined. Even more we have to expect a whole continuum of solutions for problems where contact with a sharp corner occurs. However we do not pause to study this effect in more detail, because it is behind the scope of this paper.

(3) The proof of the proposition essentially uses the strict monotonicity of the constitutive functions. Hence the assertion is not valid in the unshearable case. In Section 6 we will even show that the strains of an unshearable rod are continuous under a concentrated contact reaction, i.e., we have a qualitative difference to the shearable case.

(4) Sometimes the opinion arises that sharp corners do not occur in the real world, i.e., at least at a microscopic level we could sufficiently well approximate a sharp corner by a smooth boundary. Then, by the regularity results shown in the next section, such "nasty" things like concentrated forces where the direction is not even determined by the geometry cannot happen. However, the microscopically smoothened corner leads to a resultant contact force balanced by cross-sections in a very small neighborhood of s_0 only. But this looks like a concentrated force from

a macroscopic point of view and it in fact satisfies the same contact condition as the concentrated force corresponding to the sharp corner. The essential point now is that for the smoothened problems we must expect that the solutions and the corresponding resultant forces depend very sensitive on the special smooth approximation. On the other hand we do not know where the contact and the essential reactions at the smoothened corner really take place. Thus, at the end, a microscopic smoothening does not give more information than the macroscopic view at the sharp corner. Since we have sufficient nonsmooth tools today, it seems to be more efficient to attack such problems directly by nonsmooth arguments instead of a detour over a smooth approximation. Moreover it can be even useful and natural sometimes to approximate a "smooth corner" by a sharp one.

PROOF of Proposition 3.1. The existence and the structure of the jump of $\mathbf{n}$ and $\mathbf{m}$ at $s = s_0$ is a consequence of the equilibrium condition, the general contact condition, and (H1). The strict monotonicity of the constitutive function $\mathbf{\Xi} \to \hat{\boldsymbol{\xi}}(\Xi, s_0)$ and the continuity of $\hat{\mathbf{xi}}$ then imply a finite jump of $\boldsymbol{\xi}$ at $s = s_0$. If (s_0, ζ_0) lies on the base curve, then $\zeta_0 = 0$. Thus $M(\cdot)$ is continuous at s_0 by (3.2) and the strict monotonicity of $\mathbf{\Xi} \to \hat{\boldsymbol{\xi}}(\Xi, s_0)$ implies a jump of $\mathbf{r}'(\cdot)$ at s_0. $\square$

4 Contact with a $\mathcal{C}^1$ - obstacle

In the next two sections we claim to illuminate the regularity of contact reactions in the case of obstacles with smooth boundary.

Let us assume that

$$\mathbf{p}_0 := \mathbf{p}(s_0, \zeta_0) \in \mathcal{O} \qquad \text{for some } \; s_0 \in]0, L[, \;\; \zeta_0 \in [h_1(s_0), h_2(s_0)].$$

We call $\mathbf{p}_0$ a *contact point for the section* s_0. Clearly $\mathbf{p}_0 \in \partial\mathcal{O}$ and, without loss of generality, we have that $\zeta_0 = h_1(s_0)$. We suppose that the boundary $\partial\mathcal{O}$ *is of class* $\mathcal{C}^1$ *near* $\mathbf{p}_0$, i.e., $\partial\mathcal{O}$ locally coincides with a continuously differentiable curve

$$\sigma \to \mathbf{c}(\sigma), \quad \|\mathbf{c}'(\sigma)\| = 1 \quad \text{on }]\sigma_1, \sigma_2[, \quad \mathbf{c}(0) = \mathbf{p}_0$$

where $\sigma_1 < 0 < \sigma_2$.

To prevent that the behavior is dominated by singularities which are not caused by the obstacle we assume sufficient smoothness for all external reactions different from the obstacle and for the geometry of the rod. More precisely we invoke the hypotheses

(H1) $\mathbf{f}_e$, $\mathbf{l}_{f_e}$, and $\mathbf{l}_e$ are continuous at $s = s_0$.

(H2) h_1 is continuously differentiable near $s = s_0$.

Theorem 4.1 *Let* (H1), (H2) *be satisfied. Assume that* $\boldsymbol{\xi} = (\nu, \eta, \mu) \in (\mathcal{L}^1[0, L])^3$ *corresponds to an equilibrium configuration with regular contact where* $\mathbf{p}_0$ *is a contact point according to* $s_0 \in]0, L[$*. Let* $\partial\mathcal{O}$ *be of class* $\mathcal{C}^1$ *near* $\mathbf{p}_0$*. Then the contact force* $\mathfrak{f}_c$ *and the induced couple* $\mathfrak{l}_{f_c}$ *have no concentration at* $(s, \zeta) = (s_0, \zeta_0)$*, i.e., the distribution functions* $\mathbf{f}_c$ *and* $\mathbf{l}_{f_c}$ *are continuous at* $s = s_0$*. Moreover the force* $\mathbf{n}$ *and the couple* $\mathbf{m}$ *are continuous at* $s = s_0$*.*

Remark 4.2

(1) Theorem 4.1 is essentially based on the strict monotonicity of the constitutive functions and, therefore, it does not hold in the unshearable case. In Section 6 it will be shown that unshearable rods can have concentrated reactions in contact with arbitrarily smooth obstacles. This is again a qualitative difference to the shearable case. It is due, roughly speaking, to the stiffness of the unshearable material, while the richer structure of the shearable material allows the rod to snuggle up smoothly to the boundary $\partial\Omega$.

(2) If h_1' is not continuous, then in the simplest case the boundary of the undeformed rod has a corner. This obviously can cause an isolated active contact point even with a smooth obstacle. Arguing as in the previous section we see that this gives a concentrated reaction and therefore also a discontinuity in the strains.

(3) Let us mention that the proof of the Theorem only uses the general contact condition.

Corollary 4.3 *Under the assumptions of Theorem* 4.1 *we have that* ν, η, μ *are continuous at* $s = s_0$*, i.e.,* $\mathbf{r}'$ *and* θ' *are continuous at* $s = s_0$*.*

Proof of Theorem 4.1. Since $\mathbf{f}_c$ and $\mathbf{l}_{f_c}$ are distribution functions of finite Borel measures, we have that $\mathbf{f}_c, \mathbf{l}_{f_c} \in \mathcal{BV}[0,L]$. By (H1) and the forcebalance we get $\mathbf{n} \in \mathcal{BV}[0,L]$. Hence $\mathbf{n}$ is Lebesgue measurable and bounded (cf. Benedetto [3, Prop. 4.4]). Therefore $\mathbf{r}' \times \mathbf{n}$ is integrable and the momentbalance implies that $\mathbf{m} \in \mathcal{BV}[0,L]$. Thus the one-sided limits

$$\mathbf{n}_0^{\pm} := \mathbf{n}(s_0\pm), \quad \mathbf{m}_0^{\pm} := \mathbf{m}(s_0\pm), \quad \mathbf{f}_c(s_0\pm), \quad \mathbf{l}_{f_c}(s_0\pm)$$

exist (cf. Benedetto [3, Prop. 4.4]). We introduce the further notation

$$N_0^{\pm} := \mathbf{n}_0^{\pm}\cdot \mathbf{a}(s_0), \quad H_0^{\pm} := \mathbf{n}_0^{\pm}\cdot \mathbf{b}(s_0), \quad M_0^{\pm} := \mathbf{m}_0^{\pm}\cdot \mathbf{k}\,.$$

By the continuity of the constitutive function $\hat{\boldsymbol{\xi}}$ there exist also the one-sided limits of the strains $(\nu_0^{\pm}, \eta_0^{\pm}, \mu_0^{\pm}) := \lim_{s\to s_0\pm} \hat{\boldsymbol{\xi}}(\boldsymbol{\Xi}(s), s)$ where

$$\nu_0^{\pm} = \hat{\nu}(N_0^{\pm}, H_0^{\pm}, M_0^{\pm}, s_0), \quad \eta_0^{\pm} = \hat{\eta}(N_0^{\pm}, H_0^{\pm}, M_0^{\pm}, s_0), \quad \mu_0^{\pm} = \hat{\mu}(N_0^{\pm}, H_0^{\pm}, M_0^{\pm}, s_0)\,.$$

Using the forcebalance, basic properties of measures, (2.3), and the fact that the considered configuration has regular contact we obtain that

$$\begin{aligned}
\tilde{\mathbf{n}} &:= \mathbf{n}_0^+ - \mathbf{n}_0^- = \mathbf{f}_c(s_0+) - \mathbf{f}_c(s_0-) \\
&= \lim_{n\to\infty} -\mathfrak{f}_c(\Omega_{[s_0-\frac{1}{n}, s_0+\frac{1}{n}[}) = -\mathfrak{f}_c\Big(\bigcap_{n=1}^{\infty} \Omega_{[s_0-\frac{1}{n}, s_0+\frac{1}{n}[}\Big) \\
&= -\mathfrak{f}_c(\Omega_{[s_0,s_0]}) = \mathbf{d}^*(s_0, h_1(s_0))\,\rho(s_0, h_1(s_0))\,.
\end{aligned} \tag{4.4}$$

Analogously,

$$\begin{aligned}
\tilde{\mathbf{m}} &:= \mathbf{m}_0^+ - \mathbf{m}_0^- = -\mathfrak{l}_{f_c}(\Omega_{[s_0,s_0]}) \\
&= h_1(s_0)\mathbf{b}(s_0) \times \mathbf{d}^*(s_0, h_1(s_0))\,\rho(s_0, h_1(s_0)) = h_1(s_0)\mathbf{b}(s_0) \times \tilde{\mathbf{n}}\,.
\end{aligned}$$

Hence

$$\tilde{M} \;:=\; \tilde{\mathbf{m}}\cdot\mathbf{k} \;=\; h_1(s_0)\Big(\mathbf{k}\times\mathbf{b}(s_0)\Big)\cdot\tilde{\mathbf{n}} \;=\; -h_1(s_0)\mathbf{a}(s_0)\cdot\tilde{\mathbf{n}}\,.$$

Let us now assume that

$$\tilde{\mathbf{n}} \;\neq\; \mathbf{0}\,. \tag{4.5}$$

By the strict monotonicity of $\boldsymbol{\Xi}\to\hat{\boldsymbol{\xi}}(\boldsymbol{\Xi},s)$ we obtain with the notation $\boldsymbol{\Xi}^{\pm} := (N_0^{\pm},H_0^{\pm},M_0^{\pm})$ that

$$\begin{aligned} 0 \;&<\; \Big(\hat{\boldsymbol{\xi}}(\boldsymbol{\Xi}^{+},s_0)-\hat{\boldsymbol{\xi}}(\boldsymbol{\Xi}^{-},s_0)\Big)\cdot(\boldsymbol{\Xi}^{+}-\boldsymbol{\Xi}^{-}) \;=\; \begin{pmatrix}\nu_0^{+}-\nu_0^{-}\\ \eta_0^{+}-\eta_0^{-}\\ \mu_0^{+}-\mu_0^{-}\end{pmatrix}\cdot\begin{pmatrix}\tilde{\mathbf{n}}\cdot\mathbf{a}(s_0)\\ \tilde{\mathbf{n}}\cdot\mathbf{b}(s_0)\\ \tilde{\mathbf{m}}\cdot\mathbf{k}\end{pmatrix}\\ &=\; \Big(\mathbf{r}'(s_0+)-\mathbf{r}'(s_0-)\Big)\cdot\tilde{\mathbf{n}} \;-\; (\mu_0^{+}-\mu_0^{-})h_1(s_0)\mathbf{a}(s_0)\cdot\tilde{\mathbf{n}}\,. \end{aligned}$$

By (4.4) this is equivalent to

$$0 \;<\; \Big(\mathbf{r}'(s_0+)-\mathbf{r}'(s_0-)-(\mu_0^{+}-\mu_0^{-})h_1(s_0)\mathbf{a}(s_0)\Big)\cdot\mathbf{d}_0^{*} \tag{4.6}$$

where $\mathbf{d}_0^{*} := \mathbf{d}^{*}(s_0,h_1(s_0))$.

Using the fact that the rod respects the obstacle we show below by geometrical arguments that

$$\pm\Big(\mathbf{r}'(s_0\pm)+h_1'(s_0)\mathbf{b}(s_0)-\mu_0^{\pm}h_1(s_0)\mathbf{a}(s_0)\Big)\cdot\mathbf{d}_0^{*} \;\le\; 0. \tag{4.7}$$

But this contradicts (4.6). Thus (4.5) cannot hold and, hence, $\tilde{\mathbf{n}}=\mathbf{0}$ and $\tilde{\mathbf{m}}=\mathbf{0}$. This readily implies the assertion.

Let us still verify (4.7). The boundary $\partial\mathcal{O}$ coincides with the smooth curve $\mathbf{c}(\cdot)$ near the contact point $\mathbf{p}_0=\mathbf{c}(0)\in\partial\mathcal{O}$. Obviously $-\mathbf{d}_0^{*}$ is an outer normal of $\mathcal{O}$ at $\mathbf{p}_0$ and $\mathbf{t}_0 := \mathbf{c}'(0)$ is the normed tangent of $\mathbf{c}(\cdot)$ at the point $\mathbf{p}_0$. Clearly

$$\mathbf{d}_0^{*}\cdot\mathbf{t}_0 \;=\; 0\,.$$

Since the rod respects the obstacle, we can find some neighborhood $\mathcal{I}_0$ of s_0 and continuous functions $\check{\sigma}$, $\check{\gamma}$ on $\mathcal{I}_0$ such that

$$\mathbf{p}(s,h_1(s)) \;=\; \mathbf{c}(\check{\sigma}(s))-\check{\gamma}(s)\mathbf{d}_0^{*},\quad \check{\gamma}(s)\;\ge\;0\quad\text{for }\; s\in\mathcal{I}_0,\quad \check{\sigma}(s_0)=0,\quad \check{\gamma}(s_0)=0.$$

Recalling (1.1) we see that

$$\mathbf{p}(s,h_1(s))-\mathbf{p}(s_0,h_1(s_0)) \;=\; \int_{s_0}^{s}\Big(\mathbf{r}'(\tau)+h_1'(\tau)\mathbf{b}(\theta(\tau))-h_1(\tau)\mathbf{a}(\theta(\tau))\mu(\tau)\Big)\,d\tau\,. \tag{4.8}$$

On the other hand for $s\in\mathcal{I}_0$

$$\begin{aligned} \frac{\mathbf{p}(s,h_1(s))-\mathbf{p}(s_0,h_1(s_0))}{s-s_0}\cdot\mathbf{t}_0 \;&=\; \frac{\check{\sigma}(s)-\check{\sigma}(s_0)}{s-s_0}\;\frac{\mathbf{c}(\check{\sigma}(s))-\mathbf{c}(\check{\sigma}(s_0))}{\check{\sigma}(s)-\check{\sigma}(s_0)}\cdot\mathbf{t}_0\\ &=\; \frac{\check{\sigma}(s)-\check{\sigma}(s_0)}{s-s_0}\Big(1+o(1)\Big)\qquad(\text{as }\; s\to s_0). \end{aligned} \tag{4.9}$$

Because the integrand in (4.8) is essentially bounded, the left hand side of (4.9) must be bounded too. Thus

$$\frac{\check{\sigma}(s)-\check{\sigma}(s_0)}{s-s_0} \quad \text{is bounded for } s \text{ is a neighborhood of } s_0. \tag{4.10}$$

Suppose that there is some $\delta^+ \in \mathbb{R}$ such that

$$\Big(\mathbf{r}'(s_0+) + h_1'(s_0)\mathbf{b}(s_0) - \mu_0^+ h_1(s_0)\mathbf{a}(s_0)\Big) \cdot \mathbf{d}_0^* \;=\; \delta^+ \;>\; 0\,. \tag{4.11}$$

By considerations from the beginning of the proof we readily see that for the integrand in (4.8) the limit from the right at s_0 exists. Hence for $s > s_0$ close enough to s_0 we get that

$$\begin{aligned}
\frac{\delta^+}{2} \;&<\; \frac{\mathbf{p}(s,h_1(s)) - \mathbf{p}(s_0,h_1(s_0))}{s-s_0} \cdot \mathbf{d}_0^* \\
&=\; \frac{\check{\sigma}(s)-\check{\sigma}(s_0)}{s-s_0}\,\frac{\mathbf{c}(\check{\sigma}(s))-\mathbf{c}(\check{\sigma}(s_0))}{\check{\sigma}(s)-\check{\sigma}(s_0)} \cdot \mathbf{d}_0^* \;-\; \frac{\check{\gamma}(s)-\check{\gamma}(s_0)}{s-s_0}\,\mathbf{d}_0^* \cdot \mathbf{d}_0^* \\
&=\; \frac{\check{\sigma}(s)-\check{\sigma}(s_0)}{s-s_0}\,o(1) \;-\; \frac{\check{\gamma}(s)}{s-s_0}\|\mathbf{d}_0^*\|^2 \qquad (\text{as } s \to s_0+)\,.
\end{aligned} \tag{4.12}$$

Observing (4.10) and $\check{\gamma}(s) \geq 0$ the previous inequality yields a contradiction. Therefore (4.11) must be wrong and (4.7) is verified for "+". The other case proceeds analogously. Thus the proof is complete. □

Proof of Corollary 4.3. The assertion is a simple concequence of the continuity of the constitutive function $\hat{\boldsymbol{\xi}}$ and of $s \to \mathbf{a}(s), \mathbf{b}(s)$. □

5 Contact with a $\mathcal{C}^2$ - obstacle

Let us suppose that some connected boundary curve of the rod has contact with the obstacle $\mathcal{O}$. Since we consider only configurations with regular contact, we can restrict our attention, without loss of generality, to the case of contact with the bottom curve. More precisely, let there exist some interval $[s_l, s_r] \subset]0, L[$, $s_l < s_r$, such that

$$\mathbf{p}(s,h_1(s)) \in \partial\mathcal{O} \quad \text{for all } s \in [s_l, s_r]\,. \tag{5.1}$$

The boundary $\partial\mathcal{O}$ *be of class* $\mathcal{C}^2$, i.e., in a neighborhood of the contact area the boundary $\partial\mathcal{O}$ coincides with a twice continuously differentiable curve

$$\sigma \to \mathbf{c}(\sigma), \quad \|\mathbf{c}'(\sigma)\| = 1 \quad \text{on }]\sigma_1, \sigma_2[\,.$$

That the behavior is not dominated by singularities different from the contact reactions we again impose some hypotheses.

(H3) The constitutive function $\hat{\boldsymbol{\xi}}$ is continuously differentiable in (N, H, M, s).

(H4) $\mathbf{f}_e$, $\mathbf{l}_{f_e}$, and $\mathbf{l}_e$ are continuously differentiable in a neighborhood of $[s_l, s_r]$.

(H5) h_1 is twice continuously differentiable in a neighborhood of $[s_l, s_r]$.

Theorem 5.2 *Let* (H3) – (H5) *be satisfied. Assume that* $\boldsymbol{\xi} = (\nu, \eta, \mu) \in (\mathcal{L}^1[0, L])^3$ *corresponds to an equilibrium configuration with regular contact satisfying* (5.1)*. Let* $\partial\mathcal{O}$ *be of class* $\mathcal{C}^2$ *in a neighborhood of the contact area. Then the contact force* $\mathfrak{f}_c$ *and the induced couple* $\mathfrak{l}_{f_c}$ *have a continuous line density along the contact curve* $\{(s, h_1(s)) \in \Omega |\, s \in [s_l, s_r]\}$*, i.e., the distribution functions* $\mathbf{f}_c$ *and* $\mathbf{l}_{f_c}$ *are continuously differentiable on* $[s_l, s_r]$*. Moreover the force* $\mathbf{n}$ *and the couple* $\mathbf{m}$ *are continuously differentiable on* $[s_l, s_r]$*.*

Remark 5.3 (1) This theorem also essentially uses the strict monotonicity of the constitutive functions. Hence, it is not valid for unshearable materials where only continuity of the reactions in $]s_l, s_r[$ can be expected. Here we again see the qualitative difference in regularity between shearable and unshearable rods.

(2) In Section 6 we will see at a simple example that a concentrated contact reaction can occur at the ends of the rod for an arbitrarily smooth obstacle. Therefore it is natural that we have excluded the cases $s_l = 0$ or $s_r = L$ in the theorem.

Corollary 5.4 *Under the assumptions of Theorem* 5.2 *we have that* ν*,* η*,* μ *are continuously differentiable on* $[s_l, s_r]$*, i.e.,* $\mathbf{r}$ *and* θ *are twice continuously differentiable on* $[s_l, s_r]$*.*

Proof of Theorem 5.2. First we observe that at each contact point $\mathbf{p}(s, h_1(s))$ with $s \in [s_l, s_r]$ the assumptions of Theorem 4.1 are satisfied. Therefore $\mathbf{n}$, $\mathbf{m}$, $\mathbf{r}'$, θ', the strains (ν, η, μ), and the stress resultants (N, H, M) are continuous at all $s \in [s_l, s_r]$.

Since the measure $\mathfrak{f}_c$ is supported on the contact set, we merely have to verify the continuous differentiability of the distribution functions $\mathbf{f}_c$ and $\mathbf{l}_{f_c}$ on $[s_l, s_r]$. The properties of $\mathbf{n}$ and $\mathbf{m}$ are then simple consequences of the equilibrium conditions.

Obviously there exist some continuous function $\check{\sigma} : [s_l, s_r] \to \mathbb{R}$ such that

$$\mathbf{p}(s, h_1(s)) \;=\; \mathbf{c}(\check{\sigma}(s)) \quad \text{for } \; s \in [s_l, s_r]\,. \tag{5.5}$$

The tangent vector of the bottom curve of the rod is formally given by

$$\mathbf{t}(s) \;:=\; \frac{d}{ds}\, \mathbf{p}(s, h_1(s)) \;=\; \mathbf{r}' + h_1'\mathbf{b} - \mu h_1 \mathbf{a}\,. \tag{5.6}$$

Since the right hand side is continuous, we deduce the existence and continuity of $\mathbf{t}$ on $[s_l.s_r]$. Thus, by (5.5), $\check{\sigma}$ must be continuously differentiable and

$$\mathbf{t}(s) \;=\; \mathbf{c}'(\check{\sigma}(s))\, \check{\sigma}'(s) \quad \text{on } \; [s_l, s_r]\,.$$

The structure of $\mathfrak{f}_c$ is given in (2.3) and by the smoothness of the curve $\mathbf{c}$

$$\mathbf{d}(s) \;:=\; \mathbf{d}^*(s, h_1(s)) \;\in\; \partial d(\mathbf{p}(s, h_1(s)) \;=\; \{d'(s, h_1(s))\} \quad \text{for } \; s \in [s_l, s_r]\,. \tag{5.7}$$

Clearly $\mathbf{d}$ is a normal of the contact curve pointing into the obstacle, it is continuous on $[s_l, s_r]$, and $\mathbf{d}(s) \neq \mathbf{0}$.

We now fix any $s_0 \in [s_l, s_r]$ and agree that $s \to s_0$ is always to take for $s \in [s_l, s_r]$ only. Since the curve $\mathbf{c}$ is of class $\mathcal{C}^2$ and the unit tangent $\mathbf{c}'(\check\sigma(s))$ is orthogonal to $\mathbf{d}(s)$, we get

$$\begin{aligned}
\lim_{s\to s_0} \mathbf{d}(s_0)\cdot \frac{\mathbf{t}(s)-\mathbf{t}(s_0)}{s-s_0} & \\
&= \lim_{s\to s_0} \mathbf{d}(s_0)\cdot \frac{\mathbf{c}'(\check\sigma(s))\check\sigma'(s)-\mathbf{c}'(\check\sigma(s_0))\check\sigma'(s_0)}{s-s_0} \\
&= \lim_{s\to s_0} \mathbf{d}(s_0)\cdot \Big(\frac{\mathbf{c}'(\check\sigma(s))-\mathbf{c}'(\check\sigma(s_0))}{s-s_0}\,\check\sigma'(s) \;+\; \frac{\check\sigma'(s)-\check\sigma'(s_0)}{s-s_0}\,\mathbf{c}'(\check\sigma(s_0))\Big) \\
&= \mathbf{c}''(\check\sigma(s_0))\,\check\sigma'(s_0)^2\,.
\end{aligned} \tag{5.8}$$

On the other hand, using (5.6), we obtain

$$\begin{aligned}
\lim_{s\to s_0} \mathbf{d}(s_0)\cdot \frac{\mathbf{t}(s)-\mathbf{t}(s_0)}{s-s_0} & \\
= {}& \lim_{s\to s_0} \mathbf{d}(s_0)\cdot \Big(\frac{\mathbf{r}'(s)-\mathbf{r}'(s_0)}{s-s_0} \;-\; \frac{\mu(s)h_1(s)\mathbf{a}(s)-\mu(s_0)h_1(s_0)\mathbf{a}(s_0)}{s-s_0}\Big) \\
&+\mathbf{d}(s_0)\cdot\Big(h_1''(s_0)\mathbf{b}(s_0) \;-\; h_1'(s_0)\mu(s_0)\mathbf{a}(s_0)\Big) \\
= {}& \lim_{s\to s_0} \mathbf{d}(s_0)\cdot \Big(\frac{\mathbf{r}'(s)-\mathbf{r}'(s_0)}{s-s_0} \;-\; h_1(s_0)\mathbf{a}(s_0)\,\frac{\mu(s)-\mu(s_0)}{s-s_0}\Big) \\
&+\mathbf{d}(s_0)\cdot\Big(h_1''(s_0)\mathbf{b}(s_0) - h_1'(s_0)\mu(s_0)\mathbf{a}(s_0) - h_1'(s_0)\mu(s_0)\mathbf{a}(s_0) - h_1(s_0)\mu(s_0)^2\mathbf{b}(s_0)\Big).
\end{aligned}$$

Since the terms without limit on the right hand side are continuous functions of s_0 and since the right hand side in (5.8) is also continuous as function of s_0, the limit

$$\Delta_1(s_0) \;:=\; \lim_{s\to s_0} \mathbf{d}(s_0)\cdot \Big(\frac{\mathbf{r}'(s)-\mathbf{r}'(s_0)}{s-s_0} \;-\; h_1(s_0)\mathbf{a}(s_0)\,\frac{\mu(s)-\mu(s_0)}{s-s_0}\Big)$$

exists for each $s_0 \in [s_l, s_r]$ and $\Delta_1(\cdot)$ is continuous. Denoting differences like $\nu(s)-\nu(s_0)$ by $\triangle\,\nu$, $\triangle\,\eta$, etc., and $\triangle\, s := s - s_0$, we get

$$\begin{aligned}
\lim_{s\to s_0} \mathbf{d}(s_0)\cdot \Big(\frac{\mathbf{r}'(s)-\mathbf{r}'(s_0)}{\triangle\, s}\Big) \;&=\; \lim_{s\to s_0} \mathbf{d}(s_0)\cdot \Big(\frac{\nu(s)\,\triangle\,\mathbf{a}+\eta(s)\,\triangle\,\mathbf{b}+\mathbf{a}(s_0)\,\triangle\,\nu+\mathbf{b}(s_0)\,\triangle\,\eta}{\triangle\, s}\Big) \\
&=\; \mathbf{d}(s_0)\cdot\Big(\nu(s_0)\mu(s_0)\mathbf{b}(s_0) - \eta(s_0)\mu(s_0)\mathbf{a}(s_0)\Big) \\
&\quad\; + \lim_{s\to s_0} \mathbf{d}(s_0)\cdot \Big(\frac{\mathbf{a}(s_0)\,\triangle\,\nu+\mathbf{b}(s_0)\,\triangle\,\eta}{\triangle\, s}\Big)\,.
\end{aligned}$$

Therefore

$$\Delta_2(s_0) \;:=\; \lim_{s\to s_0} \mathbf{d}(s_0)\cdot \Big(\frac{\mathbf{a}(s_0)\,\triangle\,\nu+\mathbf{b}(s_0)\,\triangle\,\eta - h_1(s_0)\mathbf{a}(s_0)\,\triangle\,\mu}{\triangle\, s}\Big)$$

exists for each $s_0 \in [s_l, s_r]$ and $\Delta_2(\cdot)$ must be continuous. Using the notation

$$\mathbf{g}(s_0) \;:=\; \Big(\mathbf{d}(s_0)\cdot\mathbf{a}(s_0), \mathbf{d}(s_0)\cdot\mathbf{b}(s_0), -h_1(s_0)\mathbf{d}(s_0)\cdot\mathbf{a}(s_0)\Big)\,, \quad \triangle\,\boldsymbol{\xi} \;:=\; (\triangle\,\nu, \triangle\,\eta, \triangle\,\mu)\,,$$

and the continuous differentiability of $\hat{\boldsymbol{\xi}}$ we obtain that

$$\begin{aligned}
\Delta_2(s_0) \;&=\; \lim_{s\to s_0} \mathbf{g}(s_0)\cdot\frac{\triangle\,\boldsymbol{\xi}}{\triangle\, s} \;=\; \lim_{s\to s_0} \mathbf{g}(s_0)\cdot\frac{\hat{\boldsymbol{\xi}}(\boldsymbol{\Xi}(s),s)-\hat{\boldsymbol{\xi}}(\boldsymbol{\Xi}(s_0),s_0)}{\triangle\, s} \\
&=\; \lim_{s\to s_0}\Big(\mathbf{g}(s_0)\cdot\hat{\boldsymbol{\xi}}_{\boldsymbol{\Xi}}(\boldsymbol{\Xi}(s_0),s_0)\cdot\frac{\triangle\,\boldsymbol{\Xi}}{\triangle\, s}\Big) \;+\; \mathbf{g}(s_0)\cdot\hat{\boldsymbol{\xi}}_s(\boldsymbol{\Xi}(s_0),s_0)\,.
\end{aligned}$$

Hence

$$\Delta_3(s_0) \;:=\; \lim_{s\to s_0} \; \mathbf{g}(s_0)\cdot\mathbf{A}(s_0)\cdot\frac{\triangle\,\boldsymbol{\Xi}}{\triangle\, s}\,, \quad \text{where } \mathbf{A}(s_0) := \hat{\boldsymbol{\xi}}_{\boldsymbol{\Xi}}(\boldsymbol{\Xi}(s_0), s_0)\,,$$

must exist at all $s_0 \in [s_l, s_r]$ and $\Delta_3(\cdot)$ is continuous. Recalling the equilibrium conditions we see that

$$\begin{aligned}
\triangle\,\boldsymbol{\Xi} \;&=\; \begin{pmatrix} \mathbf{n}(s)\cdot\mathbf{a}(s) - \mathbf{n}(s_0)\cdot\mathbf{a}(s_0) \\ \mathbf{n}(s)\cdot\mathbf{b}(s) - \mathbf{n}(s_0)\cdot\mathbf{b}(s_0) \\ (\,\mathbf{m}(s) - \mathbf{m}(s_0)\,)\cdot\mathbf{k} \end{pmatrix} \\
&=\; \begin{pmatrix} \mathbf{a}(s_0)\cdot\triangle\,\mathbf{f}_c + \mathbf{a}(s_0)\cdot\triangle\,\mathbf{f}_e + \mathbf{n}(s)\cdot\triangle\,\mathbf{a} \\ \mathbf{b}(s_0)\cdot\triangle\,\mathbf{f}_c + \mathbf{b}(s_0)\cdot\triangle\,\mathbf{f}_e + \mathbf{n}(s)\cdot\triangle\,\mathbf{b} \\ -\int_{s_0}^{s}(\mathbf{r}'\times\mathbf{n})\cdot\mathbf{k}\,d\tau + \mathbf{k}\cdot\Big(\triangle\,\mathbf{l}_{f_c} + \triangle\,\mathbf{l}_{f_e} + \triangle\,\mathbf{l}_e\Big) \end{pmatrix}.
\end{aligned} \tag{5.9}$$

By the differentiability of $\mathbf{f}_e$, $\mathbf{l}_{f_e}$, $\mathbf{l}_e$, $\mathbf{a}$, and $\mathbf{b}$, we conclude that

$$\Delta_4(s_0) \;:=\; \lim_{s\to s_0} \mathbf{g}(s_0)\cdot\mathbf{A}(s_0)\cdot\frac{1}{\triangle\, s}\begin{pmatrix} \mathbf{a}(s_0)\cdot\triangle\,\mathbf{f}_c \\ \mathbf{b}(s_0)\cdot\triangle\,\mathbf{f}_c \\ \mathbf{k}\cdot\triangle\,\mathbf{l}_{f_c} \end{pmatrix}$$

exists and that $\Delta_4(\cdot)$ must be continuous. Because the rod has regular contact, the contact set Ω_c intersected with $\Omega_{[s_l,s_r]}$ coincides with the bottom curve of the rod in that interval. By (2.3) and (5.7) there exists a real non-negative Borel measure $\tilde\rho$ on $[s_l, s_r]$ such that for $s \in [s_l, s_r]$

$$\mathbf{f}_c(s) \;=\; -\int_s^L \mathbf{d}(\tau)\,d\tilde\rho(\tau) \;-\; \mathbf{f}_0\,, \quad \mathbf{l}_{f_c}(s) \;=\; -\int_s^L h_1(\tau)\mathbf{b}(\tau)\times\mathbf{d}(\tau)\,d\tilde\rho(\tau) \;-\; \mathbf{l}_0 \tag{5.10}$$

where $\mathbf{f}_0 := -\int_{\Omega_{]s_r,L]}} \mathbf{d}^*(\tau,\zeta)\,d\rho(\tau,\zeta)$ and $\mathbf{l}_0 := -\int_{\Omega_{]s_r,L]}} \zeta\mathbf{b}(\tau)\times\mathbf{d}^*(\tau,\zeta)\,d\rho(\tau,\zeta)$ are fixed vectors counting for contact reactions on $\Omega_{]s_r,L]}$. By the continuity of $\mathbf{d}$ we have that

$$\lim_{s\to s_0}\frac{\triangle\,\mathbf{f}_c}{\int_{s_0}^{s} d\tilde\rho(\tau)} \;=\; \mathbf{d}(s_0)\,, \quad \lim_{s\to s_0}\frac{\triangle\,\mathbf{l}_{f_c}}{\int_{s_0}^{s} d\tilde\rho(\tau)} \;=\; h_1(s_0)\mathbf{b}(s_0)\times\mathbf{d}(s_0)$$

Thus

$$\Delta_4(s_0) \;=\; \mathbf{g}(s_0)\cdot\mathbf{A}(s_0)\cdot\mathbf{g}(s_0)\;\lim_{s\to s_0}\frac{\int_{s_0}^{s} d\tilde\rho(\tau)}{s - s_0}.$$

Since Δ_4, $\mathbf{g}$, $\mathbf{A}$ are continuous, $\mathbf{g}\neq\mathbf{0}$ (because $\mathbf{d}\neq\mathbf{0}$), and $\mathbf{A}$ is positive definite,

$$\Delta(s_0) \;:=\; \lim_{s\to s_0}\frac{\int_{s_0}^{s} d\tilde\rho(\tau)}{s - s_0}$$

must exist as continuous function on $[s_l, s_r]$. Therefore (5.10) implies that

$$\mathbf{f}_c(s) \;=\; -\int_s^L \mathbf{d}(\tau)\Delta(\tau)\,d\tau \;-\; \mathbf{f}_0\,, \quad \mathbf{l}_{f_c}(s) \;=\; -\int_s^L h_1(\tau)\Delta(\tau)\mathbf{b}(\tau)\times\mathbf{d}(\tau)\,d\tau \;-\; \mathbf{l}_0\,.$$

But this readily gives the continuous differentiability of the the distribution functions $\mathbf{f}_c$ and $\mathbf{l}_{f_c}$ on $[s_l, s_r]$ and the proof is complete. $\square$

Proof of Corollary 5.4. Observing that $\mathbf{n}$, $\mathbf{m}$, $\mathbf{a}$, $\mathbf{b}$ are continuously differentible, we obtain the continuity of $\boldsymbol{\Xi}'(\cdot)$ on $[s_l, s_r]$. By (H3) we then get the continuous derivative

$$\boldsymbol{\xi}'(s) \;=\; \hat{\boldsymbol{\xi}}_{\boldsymbol{\Xi}}(\boldsymbol{\Xi}(s), s)\boldsymbol{\Xi}'(s) + \hat{\boldsymbol{\xi}}_s(\boldsymbol{\Xi}(s), s)$$

on $[s_l, s_r]$ and the assertion readily follows. $\square$

6 Comparison with the unshearable case

As we already mentioned there is a very interesting qualitative difference in regularity between shearable and unshearable materials. In some cases unshearable rods have more regularity and in other cases they have less regularity than shearable ones. This distinction is essentially based on the lack of strict monotone constitutive functions $\hat{\boldsymbol{\xi}}$ for unshearable materials. In this section we want to illuminate this aspect a little more detailed. That the behavior is not dominated by singularities different from the contact reactions we again assume that all other influences are smooth enough without specifying this precisely in each special case.

Concentrated contact reaction. Let us first consider the case where the rod has some isolated active contact point at cross-section $s = s_0$. As we have seen in Section 3 this causes a discontinuity in the strains in the shearable case. In particular the tangent of the reference curve passing through that contact point must have a discontinuity.

Let us now study this situation for an unshearable rod which we assume to have constant thickniss, i.e., h_1 and h_2 are constant. Clearly there must be a concentrated contact force $\tilde{\mathbf{n}}_0$, i.e., $\mathbf{n}(\cdot)$ has a jump of $\tilde{\mathbf{n}}_0$ at $s = s_0$. However, in contrast to the shearable case, the direction of $\mathbf{r}'$ cannot jump by (1.6). Therefore the boundary of $\mathcal{B}_{\mathbf{p}}$ has no corner at the contact point and the function $d_{cl\,(\mathcal{B}_{\mathbf{p}}^c)}$, defined according to (2.5), is continuously differentiable near that contact point. Consequently, by the refined contact condition the contact force $\tilde{\mathbf{n}}_0$ must be directed orthogonal to $\mathbf{r}'$ or, equivalently, $\mathbf{b}(s_0)$ and $\tilde{\mathbf{n}}_0$ are parallel. This readily implies that the induced couple of $\tilde{\mathbf{n}}_0$ vanishes. Thus N, M are continuous and H has a jump at s_0. In the hyperelastic unshearable case it is reasonable to assume that $\hat{\nu}$, $\hat{\mu}$ do not depend on H (cf. Antman [1, VIII.15.]). Then also ν, μ have to be continuous at s_0, i.e., $\mathbf{r}$ and θ are continuously differentiable at s_0. This readily shows that under an isolated concentrated force an unshearable rod has higher regularity than a shearable one which is caused, roughly speaking, by the rigidity of the unshearble material.

Contact with a smooth obstacle. Let us now study a special situation where the rod is partially in contact with a very smooth obstacle. We consider an originally straight rod with constant thickness $h > 0$. For technical convenience we choose $h_1(s) = 0$ on $[0, L]$, i.e., we use the bottom curve as base curve. The obstacle $\mathcal{O}$ be a half space and the point $\mathbf{r}(0)$ be confined to slide along a given line orthogonal to $\partial\mathcal{O}$. More precisely we assume that

$$\mathcal{O} \;=\; \{\mathbf{q} \in \mathbb{R}^2 |\, \mathbf{q} \cdot \mathbf{j} \le 0\}\,, \quad \mathbf{r}(0) \cdot \mathbf{i} \;=\; 0\,.$$

Now we apply a force $-\mathbf{n}_0 = -n_0\mathbf{j}$, $n_0 > 0$, and a couple $-M_0$, $M_0 > 0$, at the point $\mathbf{r}(0)$ such that for some $s_1 \in]0, L[$ the points $\mathbf{p}(s, 0)$ have contact with the obstacle for all $s \in [s_1, L]$ and all other points of the rod do not touch $\mathcal{O}$ (cf. Fig. 4).

Before we start more detailed investigations we provide some basic transformation properties concerning the base curve which are valid both for shearable and unshearable rods. Though it is convenient to study the problem with respect to the bottom curve, sometimes it is useful to argue with respect to the middle curve, because in that case the constitutive functions involve

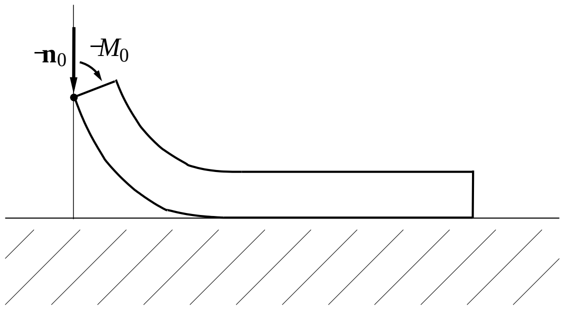

Fig. 4.

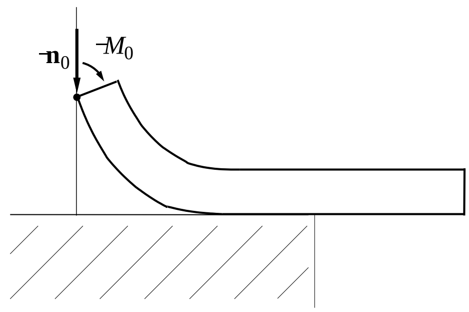

Fig. 5.

some additional symmetry. Let us mark values with respect to the middle curve by a subscript "m". We have $h_{1,m} = -\frac{h}{2}$ and $h_{2,m} = \frac{h}{2}$ and, obviously, $\mu_m = \mu$. By an exact derivation from the 3-dimensional theory it can be shown that $\hat{M}_m$ is odd in μ_m (cf. Antman & Marlow [2]). Moreover

$$M_m \;=\; M + \tfrac{h}{2} N\,. \tag{6.1}$$

Transformations according to a change of the base curve are investigated more detailed in a forthcoming paper.

First let us study the problem for the unshearable case. By (1.6) we must have that

$$\theta(s) = 0 \quad \text{and thus} \;\; \mu(s) = 0 \quad \text{on} \;\;]s_1, L]\,.$$

Since the obstacle can balance only forces normal to $\partial\mathcal{O}$, we get

$$\mathbf{n}\cdot\mathbf{i} \;=\; 0 \quad \text{on} \;\; [0,L], \quad N(s) \;=\; 0 \quad \text{on} \;\;]s_1, L]\,.$$

By (6.1) and the momentbalance we obtain

$$M(s) \;=\; 0\,, \quad M'(s) \;=\; 0 \quad \text{and, thus,} \quad \mathbf{n}(s) \;=\; \mathbf{0} \quad \text{on} \;\;]s_1, L]\,.$$

This means that we always have the trivial solution on $]s_1, L]$. On the other hand the forcebalance yields that

$$\mathbf{n}(s) \;=\; \mathbf{n}_0 \quad \text{on} \;\; [0, s_1[\,.$$

This finally tells us that $\mathbf{n}$ has a finite jump at $s = s_1$ which corresponds to a concentrated contact force exerted by the obstacle and balancing the prescribed force $-\mathbf{n}_0$ at the left end. Observe that there is no interaction between rod and obstacle on the interval $]s_1, L]$. This means that we could cut the obstacle outside a small neighborhood of the contact point $\mathbf{p}(s_1, 0)$ without changing the solution this way (cf. Fig. 5).

We now consider the same problem for a shearable rod. By Theorem 4.1 we do not have a concentrated reaction at the cross-section $s = s_0$. Theorem 5.2 tells us that the contact force $\mathbf{f}_c$ corresponds to a continuous function $\bar{f}_c(s)$ on $]0, L[$ with $f_c = 0$ on $[0, s_1[$ and $f_c \geq 0$ on $[s_1, L]$ and possibly a concentrated force $\mathbf{n}_L = n_L \mathbf{j}$, $n_L \geq 0$, acting at the right end such that

$$\mathbf{n}(s) \;=\; \mathbf{n}_L \;+\; \mathbf{j} \int_s^L \bar{f}_c(\tau)\, d\tau \quad \text{on} \;\; [0, L]\,. \tag{6.2}$$

Obviously $\mathbf{n}(s) = \mathbf{n}_0$ on $[0, s_1]$. In a forthcoming paper we study rod problems where the base curve is confined to remain straight. The solutions satisfy an ordinary differential equation in (θ, M) which easily implies that $\theta(L) = 0$ and $M(L) = 0$ would yield the trivial stress free solution for the straight part on $[s_0, L]$. This, however, would contradict the forcebalance for our contact problem and is, therefore, impossible. Hence $\theta(L) \neq 0$ and $\mathbf{n}_L \neq \mathbf{0}$. Clearly the moment $\mathbf{m}$ has also a continuous density, i.e., it is continuously differentiable. We must have that $\bar{f}_c(s) > 0$ on $]s_1, L[$ or at least on a subset with nonzero measure, i.e., there is a nonvanishing contact reaction along the contact area. This in particular implies that if we cut a part of the obstacle as in Fig. 5, then the solution changes as shown in Fig. 6. Here the free part at the right end corresponds to a trivial solution glued at the "last" cross-section having contact with the obstacle. This way

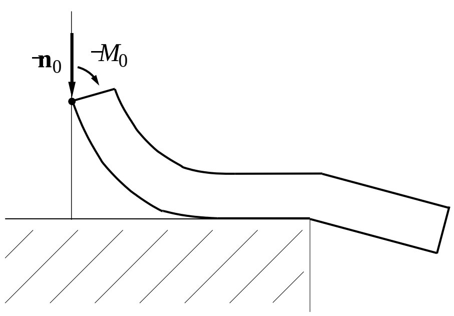

Fig. 6.

we obtain again an interesting qualitative difference in regularity between the shearable and the unshearable rod.

7 Appendix: Clarke's generalized gradients

A short introduction to Clarke's generalized gradients for locally Lipschitz continuous functionals, which is sufficient for our purposes, is given in this appendix. For a more comprehensive presentation the reader is refered to Clarke [4].

Let X be a Banach space and $f : X \mapsto \mathbb{R}$ a locally Lipschitz continuous functional. The *generalized directional derivative* $f^0(u;h)$ of f at u in the direction h is given by

$$f^0(u;h) := \limsup_{v \in X,\, v \to u,\, t \to +0} \frac{f(v+th) - f(v)}{t}\,.$$

We define the *generalized gradient* $\partial f(u)$ of f at u as the set

$$\partial f(u) := \{f^* \in X^* : \langle f^*, h\rangle \leq f^0(u;h) \text{ for all } h \in X\}.$$

$\partial f(u)$ is a nonempty, bounded, convex and weak*-compact subset of X^*. If f is continuously differentiable near u, then $\partial f(u)$ is the singleton $\{f'(u)\}$. For convex functionals, $f^0(u;h)$ is the usual one-sided directional derivative and $\partial f(u)$ is the subdifferential of convex analysis.

Let $\mathcal{A} \subset X$ be nonempty. The normal cone of $\mathcal{A}$ at u is given by

$$\mathcal{N}_{\mathcal{A}}(u) \ := \ \mathrm{cl}\left(\bigcup_{\lambda \geq 0} \lambda \partial \, \mathrm{dist}_{\mathcal{A}}(u)\right)$$

where cl denotes the weak*-closure. If $\mathcal{A}$ is convex, then $\mathcal{N}_{\mathcal{A}}(u)$ coincides with the cone of normals as defined in the convex analysis. For $u \in X$ with $0 \notin \partial f(u)$ let $\mathcal{A} = \{v \in X | \, f(v) \leq f(u)\}$. Then $\mathcal{N}_{\mathcal{A}}(u) \subset \left(\bigcup_{\lambda \geq 0} \lambda \partial f(u)\right)$. Equality holds under certain regularity.

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