INCOMPRESSIBILITY IN ROD AND SHELL THEORIES

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Résumé

On considère le problème de construire des théories exactes de poutres et coques pour les corps minces incompressibles. On utilise une méthode systématique qui consiste en l'imposition de contraintes pour reduire à un nombre fini donné les degrés de liberté de chaque section droite. On montre qu'il est très difficile de produire des théories qui conservent exactement l'incompressibilité; c'est impossible pour les théories naïves. Notamment, de nombreuses théories exactes ont des effets non locaux.

1. Introduction

Theories of rods and shells are the characterizations of the motion of thin solid bodies by finite numbers of equations in which there are respectively but one and two independent spatial variables. This paper gives a careful treatment of methods by which such geometrically exact nonlinear theories can account for the incompressibility of the three-dimensional bodies they model, exhibiting the subtle difficulties that arise (such as the appearance of nonlocal effects), the richness of the resulting theories, and their striking effects on the equations of motion.

We limit our attention to *induced* theories of rods and shells, which are constructed by a generalization of projection methods used in the numerical solution of partial differential equations that is similar to the theory of constraints in Lagrangian mechanics. We may regard the induced theories either as approximations of the three-dimensional theory or as constrained versions of it. The latter interpretation has the virtue that the governing equations are exact consequences of the three-dimensional theory obtained by the imposition of constitutive restrictions in the form of constraints and by the use of constitutive equations for appropriate stress resultants. (A discussion of convergence for hierarchies of induced theories is given by Antman (1997). Intrinsic theories of rods and shells consist in the direct postulation of the respective one- and two-dimensional equations. The interpretation of the variables in refined versions of intrinsic theories typically relies on an identification with the variables in induced theories. We do not treat theories constructed by asymptotic methods, for which Ciarlet (1997,1998), Trabucho and Viaño (1996), and the references cited therein should be consulted. In many cases, the variables that appear in such theories can be identified with those of the induced theories.)

After introducing background material from kinematics, we carefully study the planar deformation of rods. Then we describe the modifications necessary for the spatial deformation of rods (which is the most difficult of the theories we encounter here) and for shells. We conclude by exhibiting some of the governing partial differential equations for incompressible elastic rods and shells. All our techniques handle material constraints other than incompressibility.

2. Background and notation

We employ curvilinear coordinates and adhere to the convention that diagonal pairs of Latin indices are summed from 1 to 3 and diagonal pairs of Greek indices are summed from 1 to 2.

We study bodies $\mathcal{B}$ that are closures of domains. We identify material points of a body $\mathcal{B}$ by their positions $\boldsymbol{z}$ in its reference configuration. We suppose that there is a continuously differentiable invertible mapping $\mathcal{B} \ni \boldsymbol{z} \mapsto \tilde{\mathbf{x}}(\boldsymbol{z}) \in \mathbb{R}^3$, assigning a triple of curvilinear coordinates $\mathbf{x} \equiv (x^1, x^2, x^3)$ to each $\boldsymbol{z}$ in $\mathcal{B}$, such that the Jacobian

$$\det \frac{\partial \tilde{\mathbf{x}}}{\partial \boldsymbol{z}}(\boldsymbol{z}) > 0 \quad \forall\, \boldsymbol{z} \in \mathcal{B}. \tag{2.1}$$

We denote the inverse of $\tilde{\mathbf{x}}$ by $\tilde{\boldsymbol{z}}$. The Inverse Function Theorem implies that $\tilde{\boldsymbol{z}}$ is continuously differentiable and that

$$j(\mathbf{x}) \equiv \det \frac{\partial \tilde{\boldsymbol{z}}}{\partial \mathbf{x}}(\mathbf{x}) > 0 \quad \forall\, \mathbf{x} \in \tilde{\mathbf{x}}(\mathcal{B}). \tag{2.2}$$

The coordinates $\mathbf{x}$ range over

$$\mathcal{A} \equiv \tilde{\mathbf{x}}(\mathcal{B}) \tag{2.3}$$

as $\boldsymbol{z}$ ranges over $\mathcal{B}$.

We adopt the convention that $\frac{\partial f}{\partial x^k} \equiv f_{,k}$ for any function f and we adopt the standard abbreviations

$$\boldsymbol{g}_k(\mathbf{x}) \equiv \tilde{\boldsymbol{z}}_{,k}(\mathbf{x}), \quad \boldsymbol{g}^k(\mathbf{x}) \equiv \frac{\partial \tilde{x}^k}{\partial \boldsymbol{z}}(\tilde{\boldsymbol{z}}(\mathbf{x})), \quad j = (\boldsymbol{g}_1 \times \boldsymbol{g}_2) \cdot \boldsymbol{g}_3. \tag{2.4}$$

Thus $\boldsymbol{g}^k \cdot \boldsymbol{g}_l = \delta^k_l$, so that the bases $\{\boldsymbol{g}_k\}$ and $\{\boldsymbol{g}^k\}$ are dual to each other. (Indeed, $j\,\boldsymbol{g}^1 = \boldsymbol{g}_2 \times \boldsymbol{g}_3$, etc.)

Let $\boldsymbol{p}(\boldsymbol{z}, t)$ denote the position at time t of the material point $\boldsymbol{z}$. We define

$$\tilde{\boldsymbol{p}}(\mathbf{x}, t) := \boldsymbol{p}\big(\tilde{\boldsymbol{z}}(\mathbf{x}), t\big). \tag{2.5}$$

Then the (transposed) position gradient $\boldsymbol{F}$,the right Cauchy-Green deformation tensor $\boldsymbol{C}$, and the Jacobian $\det \boldsymbol{F}$ of the transformation $\boldsymbol{p}$ are given by

$$\boldsymbol{F} \equiv \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{z}} = \frac{\partial \tilde{\boldsymbol{p}}}{\partial \mathbf{x}} \cdot \frac{\partial \tilde{\mathbf{x}}}{\partial \boldsymbol{z}} = \tilde{\boldsymbol{p}}_{,k} \boldsymbol{g}^k, \quad \boldsymbol{C} \equiv \boldsymbol{F}^* \cdot \boldsymbol{F} = (\tilde{\boldsymbol{p}}_{,k} \cdot \tilde{\boldsymbol{p}}_{,l})\, \boldsymbol{g}^k \boldsymbol{g}^l, \tag{2.6}$$

$$\det \boldsymbol{F} = \frac{(\tilde{\boldsymbol{p}}_{,1} \times \tilde{\boldsymbol{p}}_{,2}) \cdot \tilde{\boldsymbol{p}}_{,3}}{j}. \tag{2.7}$$

We denote the values of kinematic variables in the reference configuration by superposed circles $^\circ$.

3. Kinematics for rods

We set $x^3 = s$ and take s to be the arc-length parameter of a distinguished material curve, called the *base curve*, with s ranging over the interval $[s_1, s_2]$ as $\mathbf{x}$ ranges over $\mathcal{A}$. Without loss of generality, we define the base curve by $x^1 = 0 = x^2$. The *material section* $\mathcal{B}(\sigma)$ of $\mathcal{B}$ is the set of all material points of $\mathcal{B}$ whose coordinates have the form (x^1, x^2, σ), i.e.,

$$\mathcal{B}(\sigma) \equiv \{\boldsymbol{z} \in \mathcal{B} \colon \tilde{\mathbf{x}}(\boldsymbol{z}) = (x^1, x^2, \sigma)\}. \tag{3.1}$$

We set

$$\mathcal{A}(\sigma) \equiv \tilde{\mathbf{x}}(\mathcal{B}(\sigma)) \equiv \{\mathbf{x} \in \mathcal{A} : \mathbf{x} = (x^1, x^2, \sigma)\}. \tag{3.2}$$

We endow $\mathcal{B}$ with the minimal rod-like character necessary for our development by requiring that (2.1) hold on $\mathcal{B}$ and that $\mathcal{B}(x^3)$ be bounded for each x^3. In order for the rod theories that we construct to be effective, $\mathcal{B}(s)$ should be small, so that the range of (x^1, x^2) is small relative to the range $[s_1, s_2]$ of s.

We generate rod theories by approximating the unknown $\tilde{\boldsymbol{p}}$ by an expression involving a finite number of unknown functions of s and t: We assume that there is a (thrice continuously differentiable) function $\mathbb{R}^N \times \mathcal{A} \times \mathbb{R} \ni (\mathsf{u}, \mathbf{y}, \tau) \mapsto \boldsymbol{\pi}(\mathsf{u}, \mathbf{y}, \tau)$ such that

$$\tilde{\boldsymbol{p}}(\mathbf{x}, t) = \boldsymbol{\pi}(\mathsf{u}(s, t), \mathbf{x}, t). \tag{3.3}$$

We shall give numerous concrete examples of (3.3). We regard (3.3) as imposing an infinite number of holonomic constraints on $\mathcal{A}(s) \ni (x^1, x^2) \mapsto \tilde{\boldsymbol{p}}(\mathbf{x}, t)$ that reduce the number of its degrees of freedom to N. In the terminology of rigid-body mechanics, u is the generalized position field for the constrained system. We denote partial derivatives with respect to s and t with subscripts.

When (3.3) holds, $\det \boldsymbol{F}$ reduces to $\delta(\mathsf{u}(s,t), \mathsf{u}_s(s,t), \mathbf{x}, t)$ where

$$\delta(\mathsf{u}, \mathsf{v}, \mathbf{y}, \tau) := \frac{[(\partial\boldsymbol{\pi}/\partial y^1) \times (\partial\boldsymbol{\pi}/\partial y^2)] \cdot [(\partial\boldsymbol{\pi}/\partial\mathsf{u}) \cdot \mathsf{v} + \partial\boldsymbol{\pi}/\partial y^3]}{j(\mathbf{y})} \tag{3.4}$$

where the arguments of the derivatives of $\boldsymbol{\pi}$ are $\mathsf{u}, \mathbf{y}, \tau$. The requirement that deformations of the form (3.3) locally preserve volume is that

$$\delta(\mathsf{u}, \mathsf{u}_s, \mathbf{x}, t) = 1 \quad \forall\, \mathbf{x} \in \mathcal{A}, \quad \forall\, t. \tag{3.5}$$

We now show how to incorporate the incompressibility constraint (3.5) into theories of rods based on (3.3). Because the theory has some surprising difficulties, we proceed from the particular to the general.

The foregoing exposition is a distillation of Sections XIV.1,2 of Antman (1995).

4. Planar deformations of incompressible rods

We analyze the mathematical structure of planar problems for rod theories generated by constraints of the form (3.3) and subject to the incompressibility condition (3.6). Let $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ be a fixed right-handed orthonormal basis. We define the deformed image $\boldsymbol{r}$ of the base curve by

$$\boldsymbol{r}(s,t) := \tilde{\boldsymbol{p}}(0,0,s,t). \tag{4.1}$$

In this section we limit our attention to deformations in which $\boldsymbol{r}$ is confined to the $\{\boldsymbol{i}, \boldsymbol{j}\}$-plane.

We first study theories in which the constraint (3.3) has the form

$$\tilde{\boldsymbol{p}}(x^1, x^2, s, t) = \boldsymbol{r}(s,t) + x^1 \gamma(s,t) \boldsymbol{b}(s,t) + x^2 \boldsymbol{k} \tag{4.2}$$

where $\boldsymbol{b}$ is a unit-vector valued function lying in the $\{\boldsymbol{i}, \boldsymbol{j}\}$-plane. We set

$$\begin{gathered} \boldsymbol{a} = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}, \quad \boldsymbol{b} = -\sin\theta\, \boldsymbol{i} + \cos\theta\, \boldsymbol{j}, \\ \boldsymbol{r}_s = \nu \boldsymbol{a} + \eta \boldsymbol{b}. \end{gathered} \tag{4.3}$$

We take $\nu = 1$, $\eta = 0$, $\theta = \theta^\circ$, $\gamma = 1$ in the reference configuration. We identify the $\mathbf{u}$ of (3.3) with γ, θ, and the components of $\boldsymbol{r}$.

Thus the incompressibility condition (3.5) reduces to

$$(\nu - x^1 \gamma \theta_s)\gamma = 1 - x^1 \theta_s^\circ \tag{4.4a}$$

or, equivalently,

$$\nu\gamma = 1, \quad \gamma^2 \theta_s = \theta_s^\circ. \tag{4.4b,c}$$

Condition (4.4b) is to be expected. But (4.4c) is disturbing: It says that the bending measured by θ_s is completely determined by ν, so that admissible strains are degenerate. In particular, if the rod is naturally straight, so that $\theta_s^\circ = 0$, then $\theta_s = 0$. This means that the permitted motions of a naturally straight rod are such that at any instant of time the shear is a constant function of s. If this shear is zero for all time, then the permitted motions consist merely in longitudinal stretching measured by ν and transverse stretching measured by $\gamma = 1/\nu$. (Such motions are not without intrinsic interest).

Since $|x^1|$ may be regarded as small, we could regard (4.4b) as a reasonable approximation to (4.4a), and ignore (4.4c). Before discussing such theories, let us first explain why the difficulty with (4.4c) is to be expected, and then how to overcome it.

Consider the two-dimensional plane-strain problem of bending an incompressible rectangular block into a sector of a circular tube, for which the deformation is given by

$$\boldsymbol{p}(x^1, x^2, s) = \sqrt{2x^1/\alpha + c}\,[\cos\alpha s \boldsymbol{i} + \sin\alpha s \boldsymbol{j}] + x^2 \boldsymbol{k} \tag{4.5}$$

where c is a constant. Let $h > 0$ and let x^1 range over $[-h, h]$. The deformation (4.5) does not take the line of centroids $x^1 = 0$ of the rectangular block into the circle of centroids of the deformed image of the block. Thus the transverse stretch in the two-dimensional flexure problem is never symmetric about the material line of centroids. On the other hand, (4.2) forces this stretch to be symmetric. This incompatibility is the source of the adverse consequences of (4.4c).

We might wish to compensate for the defects of (4.2) by replacing it with a finite Taylor expansion of higher order in x^1. The presence of $(x^1)^2$ would destroy the unwanted symmetry. But any given generalization of (4.2) can easily turn out to be ineffective. For example, let us take

$$\tilde{\boldsymbol{p}}(x^1, x^2, s, t) = \boldsymbol{r}(s,t) + [x^1\gamma(s,t) + (x^1)^2\zeta(s,t)]\boldsymbol{b}(s,t) + x^2\boldsymbol{k}. \tag{4.6}$$

Constructing the appropriate form of the incompressibility constraint (3.5) for (4.6), we find that $\zeta = 0$ wherever $\theta_s \neq 0$. Thus (4.6) reduces to (4.2). We shall explain the source of this difficulty below.

Let us now try to counteract the defects of (4.2) and (4.6) by replacing them with the most general expression that is quadratic in x^1:

$$\tilde{\boldsymbol{p}}(x^1, x^2, s, t) = \boldsymbol{r}(s,t) + x^1\boldsymbol{d}_{(1,0)}(s,t) + (x^1)^2\boldsymbol{d}_{(2,0)}(s,t) + x^2\boldsymbol{k}, \tag{4.7}$$

where we assume that $\boldsymbol{d}_{(1,0)}$ and $\boldsymbol{d}_{(2,0)}$ lie in the $\{\boldsymbol{i},\boldsymbol{j}\}$-plane. The substitution of (4.7) into (3.5) yields the system

$$\boldsymbol{k}\cdot[\boldsymbol{r}_s \times \boldsymbol{d}_{(1,0)}] = 1, \tag{4.8a}$$

$$\boldsymbol{k}\cdot[-\boldsymbol{d}_{(1,0)} \times \partial_s\boldsymbol{d}_{(1,0)} + 2\boldsymbol{r}_s \times \boldsymbol{d}_{(2,0)}] = \theta_s^\circ, \tag{4.8b}$$

$$\boldsymbol{k}\cdot[2\boldsymbol{d}_{(2,0)} \times \partial_s\boldsymbol{d}_{(1,0)} + \boldsymbol{d}_{(1,0)} \times \partial_s\boldsymbol{d}_{(2,0)}] = 0, \tag{4.8c}$$

$$\boldsymbol{k}\cdot[\boldsymbol{d}_{(2,0)} \times \partial_s\boldsymbol{d}_{(2,0)}] = 0, \tag{4.8d}$$

analogous to (4.4). Condition (4.8d) implies that $\partial_s\boldsymbol{d}_{(2,0)}$ is parallel to $\boldsymbol{d}_{(2,0)}$: There is a scalar-valued function ψ (which we take to be continuously differentiable whenever $\boldsymbol{d}_{(2,0)}$ is) such that $\partial_s\boldsymbol{d}_{(2,0)}(s,t) = \psi_s(s,t)\boldsymbol{d}_{(2,0)}(s,t)$. Thus

$$\boldsymbol{d}_{(2,0)}(s,t) = e^{\psi(s,t)-\psi(s_0,t)}\boldsymbol{d}_{(2,0)}(s_0,t) \tag{4.9e}$$

for some $s_0 \in [s_1, s_2]$, so that for each fixed t, the vector $\boldsymbol{d}_{(2,0)}(s,t)$ is parallel to a fixed vector. Thus, $\boldsymbol{d}_{(2,0)}$ either vanishes everywhere or vanishes nowhere. In the former case, (4.7) would reduce to (4.2), which we have shown to describe a degenerate class of motions. Otherwise, let us impose the very reasonable boundary condition that the section at s_0 be planar. Thus there is a constant unit vector $\boldsymbol{e} \in \operatorname{span}\{\boldsymbol{i},\boldsymbol{j}\}$ such that

$$[x^1\boldsymbol{d}_{(1,0)}(s_0,t) + (x^1)^2\boldsymbol{d}_{(2,0)}(s_0,t)] \times \boldsymbol{e} = 0, \tag{4.9f}$$

whence there are scalar-valued functions γ_1, γ_2 such that

$$\boldsymbol{d}_{(1,0)}(s_0,t) = \gamma_1(t)\boldsymbol{e}, \quad \boldsymbol{d}_{(2,0)}(s_0,t) = \gamma_2(t)\boldsymbol{e}. \tag{4.9g}$$

Then (4.8c) implies that there is a scalar-valued function ω such that

$$2\partial_s\boldsymbol{d}_{(1,0)}(s,t) - \psi_s(s,t)\boldsymbol{d}_{(1,0)}(s,t) = \omega(s,t)\boldsymbol{e}. \tag{4.9h}$$

The solution of this linear ordinary differential equation subject to initial condition (4.9g) says that there is a scalar-valued function ω_1 such that $\boldsymbol{d}_{(1,0)}(s,t) = \omega_1(s,t)\boldsymbol{e}$; we have already shown that there is a scalar-valued function ω_2 such that $\boldsymbol{d}_{(2,0)}(s,t) = \omega_2(s,t)\boldsymbol{e}$. Thus (4.7) reduces to an expression of the form (4.6) with $\boldsymbol{b}$ a constant. We accordingly conclude that (4.7), despite its generality, is, like (4.2) and (4.6), inadequate for incompressible materials.

To gain a better understanding of why the incompressible motions permitted by (4.2), (4.6), and (4.7) are degenerate, we replace them with a generalization that allows our two-dimensional version of (3.3) more freedom in responding to (3.5):

$$\tilde{\boldsymbol{p}}(x^1,x^2,s,t) = \boldsymbol{r}(s,t) + \alpha(x^1,s,t)\boldsymbol{a}(s,t) + \beta(x^1,s,t)\boldsymbol{b}(s,t) + x^2\boldsymbol{k}, \tag{4.10}$$

where $\alpha(0,s,t) = 0 = \beta(0,s,t)$ and where

$$\alpha = 0 \quad \text{and} \quad \beta(x^1,s,t) = x^1 \quad \text{in the reference configuration.} \tag{4.11}$$

Then the incompressibility condition (3.5) becomes

$$(\nu - \beta\theta_s + \alpha_s)\beta_{,1} - (\eta + \alpha\theta_s + \beta_s)\alpha_{,1} = 1 - x^1\theta_s^\circ. \tag{4.12}$$

This is a single partial differential equation for the two unknowns α and β. If one of these functions is given, then the other could be found at least locally by the method of characteristics or alternatively, under favorable regularity, by the use of a Taylor expansion.

Before discussing the full implications of (4.12), let us first impose the severe restriction that $\alpha = 0$. Then (4.12) reduces to an ordinary differential equation for $\beta(\cdot,s,t)$ whose solution subject to the initial condition $\beta(0,s,t) = 0$ is the continuous function given by $\beta(x^1,s,t) = \hat{\beta}(\nu(s,t),\theta_s(s,t),x^1,s)$ where

$$\hat{\beta}(\nu,\mu,y^1,y^3) \equiv \begin{cases} \frac{\nu - \sqrt{\nu^2 - (2y^1 - (y^1)^2\theta_s^\circ(y^3))\mu}}{\mu} & \text{for} \quad \mu \neq 0, \\ \frac{2y^1 - (y^1)^2\theta_s^\circ(y^3)}{2\nu} & \text{for} \quad \mu = 0. \end{cases} \tag{4.13}$$

Note that $\hat{\beta}(1,\theta_s^\circ(y^3),y^1,y^3) = y^1$ but that $\hat{\beta}$ in general is not odd in y^1. This β can be substituted into (4.10) to produce a version of (3.3). For $\mu \neq 0$, the Binomial Theorem implies that

$$\begin{aligned} \hat{\beta}(\nu,\mu,y^1,y^3) &= \frac{\nu}{\mu}\left[1 - \sqrt{1 - (2y^1 - (y^1)^2\theta_s^\circ(y^3))\frac{\mu}{\nu^2}}\right] \\ &= \frac{\nu}{\mu}\left[1 - 1 + y^1\frac{\mu}{\nu^2} + \cdots\right] = \frac{y^1}{\nu} + \cdots \end{aligned} \tag{4.14}$$

for sufficiently small $|y^1|$. The constraint induced by the choice $\hat{\beta}(\nu,\mu,y^1,y^3) = y^1/\nu$ is just the approximation (4.4b) to the actual constraint of incompressibility.

The failure of (4.6), when subject to (3.5), to allow for a rich collection of motions is not at all surprising because (4.6) has the form of (4.10) with $\alpha = 0$, and the only form of (4.10) with $\alpha = 0$ that satisfies (4.12) is given by (4.13), which is not quadratic in its third argument.

Now let us study (4.10) with the simplest nonzero α, namely, $\alpha(x^1, s, t) = (x^1)^2 u(s, t)$. (If α were to have a part that is linear in x^1, we could absorb it in the coefficient β of $\boldsymbol{b}$, i.e., $\boldsymbol{b}$, which derives all its kinematic significance from (4.10), could be defined as $\tilde{\boldsymbol{p}}_{,1}(0, x^2, s, t)$.) Then (4.12) becomes the following partial differential equation for β:

$$[\nu - \beta\theta_s + (x^1)^2 u_s]\beta_{,1} - 2x^1 u[\eta + (x^1)^2 u\theta_s + \beta_s] = 1 - x^1\theta_s^\circ. \tag{4.15}$$

Let us seek a solution β of (4.15) as a power series $\beta(x^1, s, t) = \sum_{k=1}^{\infty}(x^1)^k \beta_k(s, t)$ in x^1. Then the β_k are expressed in terms of the strains ν, η, θ_s, u by the recursions

$$\begin{aligned}
\nu\beta_1 &= 1,\\
2\nu\beta_2 &= \theta_s\beta_1^2 + 2u\eta - \theta_s^\circ,\\
3\nu\beta_3 &= \theta_s(3\beta_1\beta_2) - \beta_1 u_s + 2u\partial_s\beta_1,\\
4\nu\beta_4 &= \theta_s(4\beta_1\beta_3 + 2\beta_2^2) - 2\beta_2 u_s + 2u^2\theta_s + 2u\partial_s\beta_2,\\
(j+1)\nu\beta_{j+1} &= \theta_s\sum_{k=1}^{j}(1+j-k)\beta_k\beta_{1+j-k} - (j-1)u_s\beta_{j-1} + 2u\partial_s\beta_{j-1}\\
&\quad \text{for } j \geq 4.
\end{aligned} \tag{4.16}$$

Note that the β_k are determined uniquely by this recursion and that the series for β is typically infinite. Moreover, β_k depends on the derivatives of ν and u up to order $k-2$, and on derivatives of η and θ_s up to order $k-3$. When the series for β is infinite, β depends on an infinite number of derivatives of these strains.

The explanation for this unpleasant occurence is that (4.12) has a partial derivative of β in s, and therefore a solution would typically depend nonlocally on the argument s of ν, u, θ_s. Since (4.12) is a quasilinear partial differential equation for β, its solution would be given locally by the solution of its characteristic equations:

$$\begin{aligned}
\frac{dx^1}{d\tau} &= \nu(s) - \beta\theta_s(s) + \alpha_s(x^1, s),\\
\frac{ds}{d\tau} &= \alpha_{,1}(x^1, s),\\
\frac{d\beta}{d\tau} &= [\eta(s) + \alpha(x^1, s)\theta_s(s)]\alpha_{,1} + 1 - x^1\theta_s^\circ(s)
\end{aligned} \tag{4.17}$$

satisfying the initial conditions

$$x^1 = 0, \quad s = \sigma, \quad \beta = 0. \tag{4.18}$$

Here we have suppressed the argument t, which is merely a parameter in these equations. We know that given α, say in the form leading to (4.15), the problem (4.17), (4.18) has a unique solution when $|x^1|$ is sufficiently small, and this solution depends nonlocally on s. (In other words, (4.15) is a nonholonomic equation for β.) In this case, we handle (4.12) by replacing (4.10) with an expression of the form

$$\begin{aligned}
\tilde{\boldsymbol{p}}(x^1, x^2, s, t) = {}&\boldsymbol{r}(s, t) + (x^1)^2 u(s, t)\boldsymbol{a}(s, t)\\
&+ \hat{\beta}[\nu(\cdot, t), \eta(\cdot, t), \theta_s(\cdot, t), u(\cdot, t), x^1, s, t]\boldsymbol{b}(s, t) + x^2\boldsymbol{k},
\end{aligned} \tag{4.19}$$

where $\hat{\beta}$ is the solution of (4.12) that vanishes when $x^1 = 0$. In consequence, the resulting equations of motion would have this nonlocal character. Thus we have discovered a mechanism by which nonlocality in a spatial variable can enter continuum mechanics in a natural way. (Schwab and Wright (1995) encountered nonlocality in their numerical analysis of linearly elastic beams and plates.) We now see why (4.7) is ineffective: It has the form of (4.10) with an α depending in a special way on x^1, and the only form of (4.10) with such an α that satisfies (4.12) is given by (4.19).

It is clear that these same considerations apply to the case in which $\alpha(\cdot, s, t)$ is given as any finite Taylor expansion with $\alpha(0, s, t) = 0$, with the coefficients of its powers of x^1 together with $\boldsymbol{r}$ and θ regarded as constituting the N-tuple $\mathbf{u}$. Again, we find that β must typically be described by an infinite power series in x^1. It therefore follows that *in such cases no representation for $\boldsymbol{\pi}$ as a finite power series in x^1 can exactly describe incompressible motions.*

We could alternatively prescribe the form of β and solve (4.12) for α. In this case, however, there is no reasonable special case that leads to an ordinary differential equation for $\alpha(\cdot, s, t)$.

We, of course, can approximate (4.12) by taking a finite series approximation to the infinite series for β proposed to solve (4.15). Then our solution β depends on several derivatives of the strains ν, η, θ_s, u. This approach is a generalization of that which retains (4.4b) but ignores (4.4c) (an approach used in the literature).

Let us describe an alternative way of constructing a finite system of constraints approximating incompressibility, which offers its own difficulties. The requirement that the volume of the material between any pair of material sections $\mathcal{B}(s_1)$ and $\mathcal{B}(s_2)$ be constant when (3.3) holds is

$$\text{(4.20a)} \quad \int_{s_1}^{s_2} \int_{\mathcal{A}(s)} \delta(\mathbf{u}(s,t), \mathbf{u}_s(s,t), \mathbf{x}, t)\, j(\mathbf{x})\, dx^1\, dx^2\, ds = \int_{s_1}^{s_2} \int_{\mathcal{A}(s)} j(\mathbf{x})\, dx^1\, dx^2\, ds.$$

Since s_1 and s_2 are arbitrary, (4.20a) yields

$$\text{(4.20b)} \qquad \int_{\mathcal{A}(s)} \delta(\mathbf{u}(s,t), \mathbf{u}_s(s,t), \mathbf{x}, t)\, j(\mathbf{x})\, dx^1\, dx^2 = \int_{\mathcal{A}(s)} j(\mathbf{x})\, dx^1\, dx^2$$

for all s. To be specific, let us adopt (4.2), in which case $\delta = \gamma(\nu - x^1\gamma\theta_s)/(1 - x^1\theta_s^\circ)$, and (4.20b) reduces to the constraint

$$\text{(4.21)} \qquad \gamma(A(s)\nu - K(s)\gamma\theta_s) = A(s) - K(s)\theta_s^\circ(s)$$

where $A(s)$ is the area and $K(s)$ is the first moment of area of the section $\mathcal{B}(s)$:

$$\text{(4.22)} \qquad A(s) = \int_{\mathcal{A}(s)} dx^1\, dx^2 > 0, \quad K(s) = \int_{\mathcal{A}(s)} x^1\, dx^1\, dx^2.$$

The constraint (4.21) is an approximation of the incompressibility condition. If $K(s) = 0$, i.e., if $\boldsymbol{r}(s)$ locates the position of the centroid of the section $\mathcal{B}(s)$ in a deformed configuration, then (4.21) reduces to (4.4b). Otherwise, the situation is more complicated, as we now show. (These other cases are important because for certain contact problems, it is advantageous to choose $\boldsymbol{r}$ to give the position of a curve on the boundary.)

Since the right-hand sides of (4.20b) and (4.21) are positive, by (2.2), a γ satisfying (4.21) cannot vanish, and we can write (4.21) as

$$\nu = \nu^\sharp(\gamma, \theta_s, s) := k(s)\theta_s\gamma + \frac{1 - k(s)\theta_s^\circ(s)}{\gamma}, \qquad k := \frac{K}{A}. \tag{4.23a,b}$$

Thus for fixed s, (4.21) or (4.23a) describes surfaces in the three-dimensional space of (ν, γ, θ_s) that cannot intersect the plane $\gamma = 0$. Since $\gamma = 1$ in the reference configuration, since we want to limit our attention to the connected component of the constraint surface that contains the reference configuration, and since γ represents a stretch, we take $\gamma > 0$. (In this case, (4.23a) says that ν must be positive if $k(s)\theta_s \geq 0$.) Let us solve (4.23a) for γ:

$$\gamma^\pm(\nu, \theta_s, s) = \frac{\nu \pm \sqrt{\nu^2 - 4k\theta_s(1 - k\theta_s^\circ)}}{2k\theta_s} \tag{4.24$\pm$}$$

when $k\theta_s \neq 0$. We obtain $\gamma = [1 - k(s)\theta_s^\circ(s)]/\nu$ if $k\theta_s = 0$. That γ must be real means that the discriminant in (4.24) must be non-negative:

$$\nu^2 \geq 4k\theta_s(1 - k\theta_s^\circ). \tag{4.25}$$

If $k(s)\theta_s < 0$, the only positive solution of (4.23a) for γ (no matter what sign ν has) is γ^-. For $k(s)\theta_s > 0$, both $\gamma^\pm$ are positive. This means that for given ν and θ_s with $k(s)\theta_s > 0$, there are paradoxically two transverse stretches $\gamma^\pm$ with the area of a deformed cross section corresponding to γ^+ exceeding that corresponding to γ^-.

To ascertain the significance of these two deformations, we first observe that a deformation of the form (4.2) that locally preserves orientation satisfies $\delta > 0$ by definition, so that $\nu - x^1\gamma\theta_s > 0$ and $\gamma > 0$ (cf. (4.4a)) for each x^1 for which $(x^1, x^2) \in \mathcal{A}(s)$. In particular, if x^1 ranges over $[h^-(s), h^+(s)]$ when $(x^1, x^2) \in \mathcal{A}(s)$, then these inequalities imply that

$$\frac{\nu}{\gamma} > \begin{cases} h^+(s)\theta_s & \text{if} \quad \theta_s \geq 0, \\ h^-(s)\theta_s & \text{if} \quad \theta_s \leq 0. \end{cases} \tag{4.26}$$

To be specific, let us suppose that $k(s) > 0$ and let us fix $\theta_s > 0$. (The treatment for $k(s) < 0$ and $\theta_s < 0$ is analogous.) Then $\nu^\sharp(\cdot, \theta_s, s)$, given by (4.23a), has a unique minimum at $\gamma_{\mathrm{m}}(\theta_s, s) := \sqrt{\frac{1-k(s)\theta_s^\circ(s)}{k(s)\theta_s}}$. (Condition (4.25) mere states that $\nu^\sharp$ exceeds its minimum.) Let $\mathcal{G}^-(\theta_s, s)$ and $\mathcal{G}^+(\theta_s, s)$ be the graphs in the right half of the (γ, ν)-plane consisting of all points $(\gamma, \nu^\sharp(\gamma, \theta_s, s))$ with $\gamma \leq \gamma_{\mathrm{m}}(\theta_s, s)$ and $\gamma \geq \gamma_{\mathrm{m}}(\theta_s, s)$, respectively. Let $\mathcal{G}(\theta_s, s) := \mathcal{G}^-(\theta_s, s) \cup \mathcal{G}^+(\theta_s, s)$ denote the graph of $\nu^\sharp(\cdot, \theta_s, s)$. Since $k < h^+$ (under the tacit assumption that $\mathcal{A}$ has an interior point), the graph $\mathcal{G}(\theta_s, s)$ has a unique intersection at $\gamma_{\mathrm{i}}(\theta_s, s) := \sqrt{\frac{1-k(s)\theta_s^\circ(s)}{(h^+(s)-k(s))\theta_s}}$ with the line $\mathcal{L}(\theta_s, s)$ whose equation is $\nu = h^+(s)\theta_s\gamma$. The part of $\mathcal{G}$ lying above $\mathcal{L}$ corresponds to those orientation-preserving deformations satisfying the constraint (4.23a).

If $\gamma_{\mathrm{i}} < \gamma_{\mathrm{m}}$, i.e., if $2k < h^+$, then part of $\mathcal{G}^-$ and all of $\mathcal{G}^+$ lie below $\mathcal{L}$. If $\gamma_{\mathrm{m}} < \gamma_{\mathrm{i}}$, i.e., if $2k > h^+$, then all of $\mathcal{G}^-$ and part of $\mathcal{G}^+$ lie above $\mathcal{L}$. If $\mathcal{A}$ is a rectangle of the form $\{(x^1, x^2) : h^- \leq x^1 \leq h^+, |x^2| \leq b\}$ where b is a constant, then $2k = h^- + h^+$. Thus $\gamma_{\mathrm{i}} < \gamma_{\mathrm{m}}$ if and only if $h^- < 0$. If $\mathcal{A}$ is a triangle of the form $\{(x^1, x^2) : 0 = h^- \leq x^1 \leq h^+, |x^2| \leq c(h^+ - x^1)\}$ where c is a constant, then $2k = \frac{2}{3}h^+$, and $\gamma_{\mathrm{i}} < \gamma_{\mathrm{m}}$. If $\mathcal{A}$ is a triangle of the form $\{(x^1, x^2) : 0 =: h^- \leq x^1 \leq h^+, |x^2| \leq cx^1\}$ where c is a constant, then $2k = \frac{4}{3}h^+$, and $\gamma_{\mathrm{i}} > \gamma_{\mathrm{m}}$.

We can now explain the meaning of the two solutions $\gamma^\pm$ when $k\theta_s > 0$: We have approximated (4.4), which automatically ensures that the deformation (4.2) is orientation-preserving, with (4.21), which merely ensures that volumes between sections are conserved. We have thus admitted the unacceptable possibility that the conserved volumes can have negative contributions when γ is so large that the rays $x^1 \mapsto \boldsymbol{r}(s) + x^1\gamma(s)\boldsymbol{b}(s) + x^2\boldsymbol{k}$ intersect for nearby s's. (To prevent in well-set problems the occurrence of such orientation-reversing deformations, which correspond to those parts of $\mathcal{G}$ lying on and below $\mathcal{L}$, we must employ constitutive functions that blow up when $\delta \searrow 0$, as is done for compressible materials [1].) In particular, if $\gamma_{\mathrm{i}} \leq \gamma_{\mathrm{m}}$, then any deformation corresponding to γ^+ is unacceptable. (All the effects we discuss for deformations of the form (4.2) can be illustrated in the simple example of the bending and stretching a uniform straight rod or circular rod into a circular rod with perpendicular sections.)

Let us now examine the case that $\gamma_{\mathrm{m}} < \gamma_{\mathrm{i}}$. Here, for appropriate given values of ν, θ_s, the deformations corresponding to $\gamma^{\pm}$ each preserve orientation. Our third example in which $\mathcal{A}(s)$ is an isosceles triangle with vertex on $x^1 = h^- = 0$ and base on $x^1 = h^+$ shows why the deformation corresponding to γ^+ preserves both orientation and volume, even though its deformed cross sections are larger than those for γ^-: The deformation corresponding to γ^+ pushes the wide part of the triangle to a place where it suffers a more severe longitudinal compression due to the bending, which precisely compensates for the increased cross-sectional area.

We cannot declare one of the solutions $\gamma^{\pm}$ more natural than the others, because the graphs $\mathcal{G}^{\pm}(\theta_s, s)$ are merely part of the section $\mathcal{G}(\theta_s, s)$ of the surface defined by (4.23a) to which the strain (ν, θ_s, γ) are constrained. Any smooth motion of the rod, corresponding to a classical solution of the equations of motion, generates a curve on the surface defined by (4.23a). If we give (ν, θ_s) their reference values $(1, \theta_s^\circ)$ in (4.24), we find that γ^- equals its reference value 1 and $\gamma^+ = (1 - k\theta_s^\circ)/k\theta_s^\circ$ when $2k\theta_s^\circ < 1$, and that γ^+ equals its reference value 1 and $\gamma^- = (1 - k\theta_s^\circ)/k\theta_s^\circ$ when $2k\theta_s^\circ > 1$. Thus the branch $\mathcal{G}^-$ or $\mathcal{G}^+$ "containing" the reference state is determined by the parameter $k\theta_s^\circ$.

There are some pitfalls to be avoided in interpreting these results: Let us take $k(s) > 0$, $\theta_s > 0$, $h^+ > 0$, and $\gamma_{\mathrm{m}} < \gamma_{\mathrm{i}}$ so that certain deformations corresponding to γ^+ preserve orientation. If we substitute (4.24+) into (4.26), then we obtain from a careful computation that

$$(h^+ - k)\nu^2 < (1 - k\theta_s^\circ)(h^+)^2\theta_s. \tag{4.27}$$

Note that the coefficients on both sides of this inequality are positive. Now consider a one-parameter family of strains $[0,1] \ni \tau \mapsto (\tilde{\nu}(s,\tau), \tilde{\theta}_s(s,\tau), \gamma^+(\tilde{\nu}(s,\tau), \tilde{\theta}_s(s,\tau), s))$ in which $\tilde{\theta}_s(s,\tau) \searrow 0$ as $\tau \to 1$. It follows from (4.27) that $\tilde{\nu} \searrow 0$. This development falsely suggests that there is something inherently unnatural about γ^+. To see why this conclusion is false, consider an arbitrary one-parameter family of strains $[0,1] \ni \tau \mapsto (\nu(s,\tau), \theta_s(s,\tau), \gamma(s,\tau))$ also beginning at $(\tilde{\nu}(s,0), \tilde{\theta}_s(s,0), \gamma^+(\tilde{\nu}(s,0), \tilde{\theta}_s(s,0), s))$ for which $\theta_s(s,\tau) \searrow 0$ as $\tau \to 1$. The only constraint on this family is that it lies on the surface defined by (4.23a). There are many paths on this surface by which $\theta_s(s,\tau)$ can be decreased to 0. The solution of a well-set problem for the equations of motion would correspond to one such path. When we require that (4.27) hold, we are effectively limiting ourselves to paths of the special form $[0,1] \ni \tau \mapsto (\tilde{\nu}(s,\tau), \tilde{\theta}_s(s,\tau), \gamma^+(\tilde{\nu}(s,\tau), \tilde{\theta}_s(s,\tau), s))$. A study of the way $\mathcal{G}$ depends on θ_s, as given by (4.23a), shows that these special paths have the property that $\tilde{\nu}(s,\tau) \to 0$ and $\gamma^+(\tilde{\nu}(s,\tau), \tilde{\theta}_s(s,\tau), s)) \to \infty$ as $\tilde{\theta}_s(s,\tau) \to 0$. There is no reason to expect these special paths to correspond to the solution of any well-set problem with well-behaved data.

An analogous study of the case in which $k(s)\theta_s < 0$ shows that deformations corresponding to the part of the graph of $\nu^\sharp(\cdot, \theta_s, s)$ lying below the line given by (4.26) are unacceptable because they do not preserve orientation. Indeed, when $k = 0$, so that (4.21) and (4.23a) reduce to (4.4b), there can be orientation-reversing deformations consistent with this constraint. *This possibility (of importance in studies of existence and qualitative behavior) arises whenever the incompressibility constraint is only approximately satisfied.* In this case, appropriate constitutive assumptions are needed to prevent orientation-reversing deformations.

To refine (4.20b) or (4.21), we simply replace $dx^1\,dx^2$ in (4.20b) with $\omega(x^1)\,dx^1\,dx^2$ where ω is any function. In particular, if we take $\omega(x^1) = x^1$, then in place of (4.21) we get a similar equation with A, K replaced with K, J where $\det \begin{pmatrix} A & K \\ K & J \end{pmatrix} > 0$. Then this new equation and (4.21) yield (4.4b,c). These same ideas are applicable to (4.10).

5. Spatial deformations of incompressible rods

Since the theory for spatial problems is quite similar to that for planar problems, we just sketch the main ideas, pointing out the places where we encounter new sources of difficulty. The spatial analog of the degenerate planar representation

(4.2) is generated by the constraint

$$\tilde{\boldsymbol{p}}(x^1, x^2, s, t) = \boldsymbol{r}(s,t) + \varphi^1(\mathbf{x})\boldsymbol{d}_1(s,t) + \varphi^2(\mathbf{x})\boldsymbol{d}_2(s,t),$$
$$\boldsymbol{d}_1(s,t) = \gamma_1(s,t)\boldsymbol{a}_1(s,t), \quad \boldsymbol{d}_2(s,t) = \gamma_2(s,t)\boldsymbol{a}_2(s,t), \tag{5.1}$$

where φ^1 and φ^2 are given functions with $\{1, \varphi^1, \varphi^2\}$ independent (e.g., $\varphi^\mu(\mathbf{x}) = x^\mu$), and $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are unknown unit-vector-valued functions, not necessarily orthogonal. We identify u of (3.3) with components of $\boldsymbol{r}, \boldsymbol{d}_1, \boldsymbol{d}_2$. Let us set

$$\boldsymbol{a}^3 \equiv \boldsymbol{a}_3 := \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}. \tag{5.2}$$

We assume that

$$\boldsymbol{a}_3^\circ = \boldsymbol{r}_s^\circ, \quad \text{the } \boldsymbol{a}_k^\circ \text{ are orthonormal}, \quad \gamma_1^\circ = 1 = \gamma_2^\circ. \tag{5.3}$$

In this case, (3.5) reduces to

$$\gamma_1\gamma_2(\boldsymbol{a}_1 \times \boldsymbol{a}_2) \cdot (\boldsymbol{r}_s + \varphi^\mu \partial_s \boldsymbol{d}_\mu) = 1 + \varphi^\mu \boldsymbol{a}_3^\circ \cdot \partial_s \boldsymbol{a}_\mu^\circ. \tag{5.4}$$

Using the independence of the φ^α, we immediately obtain from (5.4) a set of three scalar equations, independent of x^1 and x^2, that restricts the derivatives of the nine independent components of u. But just as in (4.4), these equations unduly restrict the flexure of the rod. One could, however, adopt just one of these equations, namely

$$\gamma_1\gamma_2|\boldsymbol{a}_1 \times \boldsymbol{a}_2|\boldsymbol{a}^3 \cdot \boldsymbol{r}_s = 1, \tag{5.5}$$

as an approximation to the constraint of incompressibility and obtain a perfectly reasonable theory.

As we have observed in Section 4, a constraint for planar problems of the form

$$\tilde{\boldsymbol{p}}(x^1, x^2, s, t) = \sum_{m=0}^{M} (x^1)^m \boldsymbol{d}_{(m,0)}(s,t) + x^2\boldsymbol{k}, \tag{5.6}$$

where M is a positive integer and the $\boldsymbol{d}_{(m,0)}$ are taken to lie in the $\{\boldsymbol{i}, \boldsymbol{j}\}$-plane, cannot account exactly for incompressibility. (Nevertheless, (5.6) leads to useful approximations of incompressibility.) A fortiori, the generalization of (5.6) to spatial problems cannot account exactly for incompressibility. This generalization has the form

$$\tilde{\boldsymbol{p}}(x^1, x^2, s, t) = \sum_{(m,n)\in\mathcal{I}} (x^1)^m (x^2)^n \boldsymbol{d}_{(m,n)}(s,t) \tag{5.7}$$

where $\mathcal{I}$ is a collection of ordered pairs (m,n) of integers with $m \geq 0$, $n \geq 0$, $m + n \leq M$, with M a given positive integer.

Now(5.6) has $2(M+1)$ scalar unknowns. The constraint (3.5) corresponding to it is a polynomial equation in x^1 of degree $2M - 1$ and therefore provides $2M$ restrictions on the unknowns. On the other hand, (5.7) has $\frac{3}{2}(M+1)(M+2)$ scalar unknowns, and the corresponding (3.5) is a polynomial of degree $3M-2$ in the two variables x^1 and x^2 and therefore provides $\frac{1}{2}(3M-2)(3M-1)$ restrictions on the unknowns. For $M=4$, the number of restrictions for the spatial problem (at least formally) exceeds the number of unknowns. Consequently, there is a limit to the precision that can be obtained for spatial theories not having nonlocal terms.

Now let us examine what happens to (3.5) when we generalize the constraint (5.1) in analogy with (4.10):

$$\tilde{\boldsymbol{p}}(\mathbf{x},t)=\boldsymbol{r}(s,t)+\alpha(\mathbf{x},t)\boldsymbol{a}^3(s,t)+\beta^\gamma(\mathbf{x},t)\boldsymbol{a}_\gamma(s,t) \tag{5.8}$$

where

$$\alpha(0,0,s,t)=0=\beta^\mu(0,0,s,t) \tag{5.9}$$

and where

$$\alpha=0,\qquad \beta^\mu(\mathbf{x},t)=\varphi^\mu(\mathbf{x})\quad \text{in the reference configuration} \tag{5.10}$$

with (5.3) holding. (Without the further restrictions we shall impose on α and β^μ, (5.8) has virtually the generality of (3.3).)

We readily compute (3.5) for (5.8):

(5.11)
$$\begin{aligned}&(\alpha_{,1}\beta^1_{,2}-\alpha_{,2}\beta^1_{,1})(\boldsymbol{a}^3\times\boldsymbol{a}_1)\cdot(\boldsymbol{r}_s+\alpha\boldsymbol{a}^3_s+\beta^2_s\boldsymbol{a}_2+\beta^\gamma\partial_s\boldsymbol{a}_\gamma)\\&+(\alpha_{,1}\beta^2_{,2}-\alpha_{,2}\beta^2_{,1})(\boldsymbol{a}^3\times\boldsymbol{a}_2)\cdot(\boldsymbol{r}_s+\alpha\boldsymbol{a}^3_s+\beta^1_s\boldsymbol{a}_1+\beta^\gamma\partial_s\boldsymbol{a}_\gamma)\\&+(\beta^1_{,1}\beta^2_{,2}-\beta^2_{,1}\beta^1_{,2})|\boldsymbol{a}_1\times\boldsymbol{a}_2|(\boldsymbol{a}^3\cdot\boldsymbol{r}_s+\alpha_s+\beta^\gamma\boldsymbol{a}^3\cdot\partial_s\boldsymbol{a}_\gamma)=1+\varphi^\gamma\boldsymbol{a}^\circ_3\cdot\partial_s\boldsymbol{a}^\circ_\gamma.\end{aligned}$$

This generalization of (4.12) is a single first-order nonlinear partial differential equation for α,β^γ with independent variables $\mathbf{x}$ and parameter t. We could prescribe any two of these functions to depend on $\mathsf{u}(s,t),\mathbf{x},t$, where as usual u is a collection of generalized coordinate functions. The resulting equation could be solved at least locally subject to the initial condition (5.9) by the method of characteristics, but, in general, we could not expect the remaining functions to depend on the pointwise values of $\boldsymbol{r}(s,t),\boldsymbol{a}(s,t),\boldsymbol{b}(s,t),\mathsf{u}(s,t)$ and on the pointwise values of a finite number of their s-derivatives.

The development leading to (4.13) suggests that we might hope to find such a restricted dependence when $\alpha=0$. But in this case, (4.13) is still underdetermined, so we have to restrict β^1 and β^2 further. To avoid destroying the intrinsic symmetry between β^1 and β^2 by prescribing one of these we could restrict each of these functions to depend parametrically on a single function. Purely mathematical considerations might lead us to establish such a dependence by assuming that (β^1,β^2) and (φ^1,φ^2) are isotropic 2-vector functions of (x^1,x^2): There are functions $\sigma,s,\tau\mapsto\beta(\sigma,s,\tau),\varphi(\sigma,s)$ such that

$$\beta^\gamma(\mathbf{x},t)=\beta\big(\sqrt{(x^1)^2+(x^2)^2},s,t\big)x^\gamma,\quad \varphi^\gamma(\mathbf{x})=\varphi\big(\sqrt{(x^1)^2+(x^2)^2},s\big)x^\gamma. \tag{5.12}$$

This restriction on the representation of the position vector implies nothing about the isotropy of the body under study, but it does unduly restrict flexural motions just as (4.2) does, because its dependence on the x^α is too symmetric. This fact can be easily demonstrated directly.

The approximation described in the paragraphs containing (4.20)–(4.26) has obvious analogs for the spatial deformation of rods.

6. General considerations for incompressible rods

If the rod theory induced by (3.3) is to account for such three-dimensional material constraints as incompressibility, then (3.3) should either identically (or approximately) satisfy the three-dimensional constraint. In particular, if (3.3) is to describe an incompressible material, then u must satisfy (3.5). For each fixed $\mathbf{x}, t$, this is the equation for a $(2N-1)$-dimensional surface in the $2N$-dimensional space of $(\mathsf{u}, \mathsf{u}_s)$.

If δ depends on x^1, x^2 in a fairly simple way, e.g., so that (3.5) can be written as a polynomial equation in x^1, x^2, then we can immediately read off from (3.5) an equivalent finite number K of independent explicit restrictions relating $\mathsf{u}, \mathsf{u}_s, s, t$:

$$\kappa_1(\mathsf{u}, \mathsf{u}_s, s, t) = 0, \dots, \kappa_K(\mathsf{u}, \mathsf{u}_s, s, t) = 0. \tag{6.1}$$

In many cases we have found that (6.1) unduly restricts strains and thereby prevents motions that the rod theory should be capable of describing. A particularly unpleasant version of this difficulty occurs for certain theories governing the spatial deformation of rods, in which $K > N$. A simple remedy for these difficulties is to select from (6.1) a subset of conditions that do not lead to undue restrictions, at the cost of sacrificing the exact satisfaction of the constraint. When we use this approach, we presume that (6.1) has been appropriately reduced. We shall shortly describe methods for handling this reduced version of (6.1).

An alternative remedy for the inconsistencies inherent in (6.1) is to refrain from completely specifying the form of (3.3) a priori, but let it be adapted to handle (3.5). We followed this strategy in (4.10) and in (5.8). We then found that we could identically satisfy (3.5) in some cases by taking an appropriate form of (3.3) and in other cases by replacing (3.3) with a representation for $\boldsymbol{\pi}$ that depends on $\mathsf{u}(\cdot, t)$, rather than on $\mathsf{u}(s, t)$:

$$\tilde{\boldsymbol{p}}(\mathbf{x}, t) = \boldsymbol{\pi}^\star(\mathsf{u}(\cdot, t), \mathbf{x}, t) : \tag{6.2}$$

We also found that we could systematically approximate (6.2) with representations of the form

$$\tilde{\boldsymbol{p}}(\mathbf{x}, t) = \boldsymbol{\pi}^\star(\mathsf{u}(s, t), \partial_s \mathsf{u}(s, t), \dots, \partial_s^J \mathsf{u}(s, t), \mathbf{x}, t), \tag{6.3}$$

where J is a positive integer. (Of course, (6.3) generalizes (3.3).) In these cases, we do not have to concern ourselves with (3.5): We just substitute (3.3) or (6.2) or (6.3) into the Principle of Virtual Power, as in Section XIV.2 of Antman (1995), and produce in a consistent way the governing equations of motion. (See Section 9.)

We now show how to deal with (6.1). We could either retain some or all of (6.1) as side conditions for the equations of motion and then introduce corresponding Lagrange multipliers for these side conditions as described by Antman and Marlow (1991) and Antman (1995, Sec. XII.12). Suppose that we retain the last $K - M$ of the equations (6.1), $0 \le M \le K$. To treat the remaining first M equations of (6.1), we could, under favorable circumstances, solve these equations for M components of u as functions of the remaining components of u and of u_s, s, t. (If the circumstances were not favorable, we could relegate some of the offending equations from the first M equations of (6.1) to the remaining equations, and reduce M.) More generally, we could introduce generalized coordinates v so that these M equations would be equivalent to a system of the form

$$\mathsf{u} = \mathsf{u}^\star(\mathsf{v}, \mathsf{v}_s, s, t). \tag{6.4}$$

As the development leading to a truncated version of (4.16) indicates, we could obtain an alternative to (6.4) in which the dimension of v is reduced by having u depend on a different v together with several of its s-derivatives:

$$\mathsf{u} = \mathsf{u}^\star(\mathsf{v}, \partial_s \mathsf{v}, \dots, \partial_s^J \mathsf{v}, s, t) \tag{6.5}$$

where J is a positive integer. By substituting (6.5) into (3.3), we ensure that the first M equations of (6.1) are identically satisfied. If we now reinterpret u as v, then we recover (6.3).

7. Kinematics of shells

We give a brief account of the treatment of incompressibility for shells, emphasizing the features that differ from those for rods. We construct shell theories in analogy with our construction of rod theories by interchanging the roles of (x^1, x^2) with x^3. The curvilinear coordinates (x^1, x^2) are assumed to range over the closure $\mathcal{M}$ of a domain in $\mathbb{R}^2$ as $\mathbf{x}$ ranges over $\mathcal{A}$. We now assume that x^3 is bounded as $\mathbf{x}$ ranges over $\mathcal{A}$. For effective shell theories, x^3, which represents the thickness variable, should be small relative to the typical length scales of $\mathcal{B}$.

We generate shell theories by approximating the unknown $\tilde{\boldsymbol{p}}$ by an expression involving a finite number of unknown functions of x^1, x^2, t: We assume that there is a (thrice continuously differentiable) function $\mathbb{R}^N \times \mathcal{A} \times \mathbb{R} \ni (\mathsf{u}, \mathbf{y}, \tau) \mapsto \boldsymbol{\pi}(\mathsf{u}, \mathbf{y}, \tau)$ such that

$$\tilde{\boldsymbol{p}}(\mathbf{x}, t) = \boldsymbol{\pi}(\mathsf{u}(x^1, x^2, t), \mathbf{x}, t). \tag{7.1}$$

When (7.1) holds, $\det F(\mathbf{x}, t)$ reduces to $\delta(\mathsf{u}(x^1, x^2, t), \mathsf{u}_{,1}(x^1, x^2, t), \mathsf{u}_{,2}(x^1, x^2, t), \mathbf{x}, t)$ where

$$\delta(\mathsf{u}, \mathsf{v}_1, \mathsf{v}_2, \mathbf{y}, \tau) := \frac{\big\{[(\partial\boldsymbol{\pi}/\partial\mathsf{u}) \cdot \mathsf{v}_1 + \partial\boldsymbol{\pi}/\partial y^1] \times [(\partial\boldsymbol{\pi}/\partial\mathsf{u}) \cdot \mathsf{v}_2 + \partial\boldsymbol{\pi}/\partial y^2]\big\} \cdot (\partial\boldsymbol{\pi}/\partial y^3)}{j(\mathbf{y})} \tag{7.2}$$

where the arguments of the derivatives of $\boldsymbol{\pi}$ are $\mathsf{u}, \mathbf{y}, \tau$. The requirement that deformations of the form (7.1) locally preserve volume is that

$$\delta(\mathsf{u}, \mathsf{u}_{,1}, \mathsf{u}_{,2}, \mathbf{x}, t) = 1 \quad \forall\, \mathbf{x} \in \mathcal{A}, \quad \forall\, t. \tag{7.3}$$

8. Incompressible shells

Let us set $\tilde{\boldsymbol{p}}(x^1, x^2, 0, t) =: \boldsymbol{r}(x^1, x^2, t)$. For shell theories based upon the constraint

$$\tilde{\boldsymbol{p}}(\mathbf{x}, t) = \boldsymbol{r}(x^1, x^2, t) + x^3 \boldsymbol{d}, \quad \boldsymbol{d} = \gamma \boldsymbol{a}_3 \tag{8.1}$$

where $\boldsymbol{a}_3$ is a unit vector, (7.3) reduces to

$$(\boldsymbol{r}_{,1} \times \boldsymbol{r}_{,2}) \cdot \boldsymbol{d} \equiv (\boldsymbol{r}_{,1} \times \boldsymbol{r}_{,2}) \cdot \boldsymbol{a}_3 \gamma = 1, \tag{8.2a}$$

$$(\boldsymbol{r}_{,1} \times \boldsymbol{d}_{,2} - \boldsymbol{r}_{,2} \times \boldsymbol{d}_{,1}) \cdot \boldsymbol{d} = \boldsymbol{a}^\circ_{3,1} \cdot \boldsymbol{a}^\circ_1 - \boldsymbol{a}^\circ_{3,2} \cdot \boldsymbol{a}^\circ_2, \tag{8.2b}$$

$$(\boldsymbol{d}_{,1} \times \boldsymbol{d}_{,2}) \cdot \boldsymbol{d} = (\boldsymbol{a}^\circ_{3,1} \times \boldsymbol{a}^\circ_{3,2}) \cdot \boldsymbol{a}^\circ_3. \tag{8.2c}$$

Since the planar problems for rods discussed in Section 4 are but special cases of problems for shells, we find that (8.2) unduly restricts the deformations, just as (4.4) does. In many cases we can be content with imposing only (8.2a), and using it to represent γ in terms of the other strains.

The analog of (4.10) with $\alpha = 0$ is

$$\tilde{\boldsymbol{p}}(\mathbf{x}, t) = \boldsymbol{r} + \beta(\mathbf{x}, t)\boldsymbol{a}_3; \tag{8.3}$$

its substitution into (8.1) yields the analog of (4.12):

$$\begin{aligned}[]
&[(\boldsymbol{r}_{,1} \times \boldsymbol{r}_{,2}) + (\boldsymbol{r}_{,1} \times \boldsymbol{a}_{3,2} - \boldsymbol{r}_{,2} \times \boldsymbol{a}_{3,1})\beta + (\boldsymbol{a}_{3,1} \times \boldsymbol{a}_{3,2})\beta^2] \cdot \boldsymbol{a}_3 \beta_{,3} \\
&\qquad = 1 + x^3(\boldsymbol{a}^\circ_{3,1} \cdot \boldsymbol{a}^\circ_1 - \boldsymbol{a}^\circ_{3,2} \cdot \boldsymbol{a}^\circ_2) + (x^3)^2(\boldsymbol{a}^\circ_{3,1} \times \boldsymbol{a}^\circ_{3,2}) \cdot \boldsymbol{a}^\circ_3.
\end{aligned} \tag{8.4}$$

This ordinary differential equation for β as a function of x^3 subject to the initial condition that $\beta = 0$ when $x^3 = 0$ can immediately be integrated to define β implicitly as the solution of a cubic equation; the typical complexity of the solution restricts its utility. Even for axisymmetric deformations, (8.4) still yields a cubic equation for β.

9. Equations of motion

The equations of motion for incompressible rods and shells have a mathematical structure quite different from those based solely on constraints of the form (3.3) or (7.1) as can be expected from the discussion in Section 6. Here we illustrate this fact for the planar deformation of incompressible nonlinearly elastic rods governed by (4.10), (4.13) with $\alpha = 0$. For simplicity of exposition, let us assume that external forces are applied only at the ends of the rods. Then substituting (4.10), (4.13) into the Principle of Virtual Power, as explained by Antman (1995, Chap. XIV), we obtain the classical form of the equations of motion:

$$\begin{aligned}
-\partial_{ss}\boldsymbol{n}_{(2)} + \partial_s \boldsymbol{n}_{(1)} &= \frac{\partial}{\partial t}\int_{\mathcal{A}(s)} \tilde{\rho}[\boldsymbol{r}_t + (\hat{\beta}_\nu \nu_t + \hat{\beta}_\mu \theta_{st})\boldsymbol{b} - \hat{\beta}\theta_t \boldsymbol{a}]\, j\, dx^1\, dx^2 \\
&\quad \frac{\partial}{\partial s}\int_{\mathcal{A}(s)} \tilde{\rho}[\boldsymbol{r}_t + (\hat{\beta}_\nu \nu_t + \hat{\beta}_\mu \theta_{st})\boldsymbol{b} - \hat{\beta}\theta_t \boldsymbol{a}] \cdot (\hat{\beta}_\nu \boldsymbol{b}\boldsymbol{a})_t\, j\, dx^1\, dx^2, \\
&\quad - \frac{\partial^2}{\partial s \partial t}\int_{\mathcal{A}(s)} \boldsymbol{a}\tilde{\rho}[\boldsymbol{r}_t \cdot \boldsymbol{b} + \hat{\beta}_\nu \nu_t + \hat{\beta}_\mu \theta_{st}]\hat{\beta}_\nu\, j\, dx^1\, dx^2,
\end{aligned} \tag{9.1}$$

$$\begin{aligned}
&- \partial_{ss}M_{(2)} + \partial_s M_{(1)} - M_{(0)} \\
&\quad = - \int_{\mathcal{A}(s)} \tilde{\rho}[\boldsymbol{r}_t + (\hat{\beta}_\nu \nu_t + \hat{\beta}_\mu \theta_{st})\boldsymbol{b} - \hat{\beta}\theta_t \boldsymbol{a}] \cdot [\hat{\beta}_\nu \eta \boldsymbol{b} - \hat{\beta}\boldsymbol{a}]_t\, j\, dx^1\, dx^2 \\
&\qquad + \frac{\partial}{\partial t}\int_{\mathcal{A}(s)} \tilde{\rho}[\boldsymbol{r}_t + (\hat{\beta}_\nu \nu_t + \hat{\beta}_\mu \theta_{st})\boldsymbol{b} - \hat{\beta}\theta_t \boldsymbol{a}] \cdot [\hat{\beta}_\nu \eta \boldsymbol{b} - \hat{\beta}\boldsymbol{a}]\, j\, dx^1\, dx^2 \\
&\qquad + \frac{\partial}{\partial s}\int_{\mathcal{A}(s)} \tilde{\rho}[\boldsymbol{r}_t + (\hat{\beta}_\nu \nu_t + \hat{\beta}_\mu \theta_{st})\boldsymbol{b} - \hat{\beta}\theta_t \boldsymbol{a}] \cdot [\hat{\beta}_\mu \boldsymbol{b}]_t\, j\, dx^1\, dx^2 \\
&\qquad - \frac{\partial^2}{\partial s \partial t}\int_{\mathcal{A}(s)} \tilde{\rho}[\boldsymbol{r}_t \cdot \boldsymbol{b} + \hat{\beta}_\nu \nu_t + \hat{\beta}_\mu \theta_{st}]\, \hat{\beta}_\mu\, j\, dx^1\, dx^2,
\end{aligned} \tag{9.2}$$

where $\boldsymbol{n}_{(1)}, \boldsymbol{n}_{(2)} \in \operatorname{span}\{\boldsymbol{i}, \boldsymbol{j}\}$ and $M_{(0)}, M_{(1)}, M_{(2)}$ are weighted integrals of the Piola-Kirchhoff stress over a section and where $\tilde{\rho}(\mathbf{x})$ is the given mass density in the reference configuration. For elastic materials, $\boldsymbol{n}_{(1)} \cdot \boldsymbol{a}$, $\boldsymbol{n}_{(2)} \cdot \boldsymbol{a}$, $\boldsymbol{n}_{(1)} \cdot \boldsymbol{b}$, $\boldsymbol{n}_{(2)} \cdot \boldsymbol{b}$, $M_{(0)}, M_{(1)}, M_{(2)}$ are specified constitutive functions of $(\nu, \eta, \theta_s, \nu_s, \theta_{ss}, s)$. The integrands on the right-hand sides of (9.1) and (9.2) are simple explicit functions of x^1. For sections $\mathcal{A}(s)$ of simple form, the integrations can be carried out explicitly.

The essential features of (9.1) and (9.2) are that the highest-order partial derivatives, appearing on the left-hand sides, are fourth derivatives of $\boldsymbol{r}$ and θ and that the right-hand sides involve spatial derivatives of second time derivatives. These equations simplify considerably if (4.13) is replaced with its approximation $\hat{\beta}(\nu, \mu, x^1, s) = x^1/\nu$, which corresponds to (4.4b) and if $\int_{\mathcal{A}(s)} \tilde{\rho} x^1 \, dx^1 \, dx^2 = 0$ (so that $x^1 = 0$ corresponds to the mass center of each cross section). Nevertheless, the disposition of high derivatives persists.

These equations should be contrasted with those for a compressible nonlinearly elastic rod with the same kinematic structure. When $\int_{\mathcal{A}(s)} \tilde{x}^1 \rho \, dx^1 \, dx^2 = 0$,these equations have the form

$$\begin{aligned} \partial_s \boldsymbol{n} &= (\rho A)(s) \boldsymbol{r}_{tt}, \\ \partial_s M + (\boldsymbol{r}_s \times \boldsymbol{n}) \cdot \boldsymbol{k} &= (\rho J)(s) \theta_{tt} \end{aligned} \tag{9.3}$$

where $(\rho A)(s) := \int_{\mathcal{A}} \tilde{\rho} \, dx^1 \, dx^2$, $(\rho J)(s) := \int_{\mathcal{A}(s)} \tilde{\rho} (x^1)^2 \, j \, dx^1 \, dx^2$. Here $\boldsymbol{n} \cdot \boldsymbol{a}, \boldsymbol{n} \cdot \boldsymbol{b}, M$ are given constitutive functions of (ν, η, θ_s, s).

The complexity of the right-hand side of (9.1) can be seen in its specialization to purely longitudinal motions with concomitant thickness changes for a naturally straight, uniform incompressible elastic rod. In this case, (4.12) reduces to (4.4b), and $r := \boldsymbol{r} \cdot \boldsymbol{i}$ satisfies a scalar equation of the form

$$-\partial_{ss} N_{(2)}(r_s, r_{ss}) + \partial_s N_{(1)}(r_s, r_{ss}) = \rho A r_{tt} + \rho J \partial_s [2 r_s^{-5} {r_{st}}^2 - r_s^{-4} r_{stt}]. \tag{9.4}$$

(See Wright (1984). Saxton (1985) analyzed the version of (9.4) in which $N_{(2)} = 0$. His degenerate model, which implies that no energy is needed to change the thickness from section to section, is not a consequence of the use of the Principle of Virtual Power to derive the equations of motion. In many respects, (9.4) is easier to analyze than his model.)

10. Comments

In three-dimensional continuum mechanics, when the Piola-Kirchhoff stress at $\boldsymbol{z}, t$ depends only on the past history of $\boldsymbol{F}(\boldsymbol{z}, \cdot)$ and on $\boldsymbol{z}$, the material is called *simple*. Thus, in a simple material, this stress depends neither on higher spatial derivatives of $\boldsymbol{F}$ nor on $\boldsymbol{F}$ at other points. In this case, the equations of motion consist of a system of three scalar equations in which no derivative of position with respect to a spatial variable of order higher than 2 appears. If the Principle of Virtual Power is used to produce the equations of motion for rods (or shells) by invoking (3.3) (or (7.1)) when they are not subject to any material constraint such as incompressibility and if the three-dimensional constitutive equations are those for a simple material, then the resulting constitutive equations depend only on u and its first derivative(s) with respect to spatial variables, (i.e., they are constitutive equations for a simple material) and the equations of motion form a system (like (9.3)) in which no derivative of u with respect to a spatial variable of order higher than 2 appears. On the other hand, for incompressible materials, our work shows that the constitutive equations for rod and shell theories are not simple, even when those for the three-dimensional theory are. Moreover, we encounter rod and shell theories with nonlocal constitutive equations, which we can approximate by constitutive equations involving higher spatial derivatives.

The difficulties with incompressibility that we encountered in this paper are akin to those that would be encountered in the construction of finite-element methods for incompressible media described in a material formulation (cf. Le Tallec (1994)). The treatment of incompressibility by finite-element methods has been most extensively developed for the equations of fluid dynamics, in the standard spatial formulation of which incompressibility is characterized by the linear constraint that the divergence of the spatial velocity field vanish (cf. Temam (1977) and Girault & Raviart (1986), among others).

Presumably we could construct a rod theory for incompressible bodies by starting with the three-dimensional constitutive equations involving the Lagrange multiplier p. In this case, we would have to adopt a representation for it like (3.3) that is appropriately correlated with that for $\tilde{\boldsymbol{p}}$. We would still confront the issue of the consistency of (3.3) with (3.5).

Acknowledgment. P. G. Ciarlet proposed the subject of this paper to S. S. Antman many years ago. Antman's work on the research reported here was supported in part by NSF Grant DMS9623261 and by ARO-MURI97 Grant No. DAAG55-97-1-0114 to the Center for Dynamics and Control of Smart Structures.

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