Ordinary differential equations with measurable right-hand side and parameters in metric spaces

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Abstract

We consider initial value problems for ordinary differential equations with a measurable right-hand side containing parameters in general metric spaces. We prove existence and uniqueness of solutions and the continuous dependence of the solution on the initial data and the parameters under weak assumptions on the right-hand side. In the case where the parameters belong to a normed linear space we show that the solution depends differentiably on the parameters and initial values. These generalizations of classical results in the theory of ordinary differential equations have applications in optimal control theory and other variational problems with nonholonomic constraints.

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Introduction

In nature, engineering and economics many processes are governed by systems of ordinary differential equations of the form

$$\dot{x} = f(t, x(t), \lambda) \text{ on } I \subset I\!R, \quad x(\tau) = \xi\,, \tag{1}$$

where the right-hand side f depends on certain parameters λ. In addition to the fundamental questions concerning existence and uniqueness of solutions of the initial

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value problem (1), the continuous dependence of the solution $x(t;\tau,\xi,\lambda)$ on the initial data (τ,ξ) and the parameters λ is a central issue in the applications. Due to inaccuracies of practically all measurements in the Sciences, the investigation of how a solution of a differential system depends on the data and parameters is of great importance.

Usually one is interested in the solution of the initial value problem (1) in its own, however, an ordinary differential equation such as (1) may also occur as part of a more comprehensive problem, e.g., as a nonholonomic constraint for a variational problem, which is a typical situation in optimal control theory. It may also happen that derivatives of x with respect to initial values and parameters enter explicitely the Euler–Lagrange equations of a variational problem as will be described at the end of this introduction. Hence for a variety of applications it turns out to be necessary to study the differentiability properties of the solution $x(t;\tau,\xi,\lambda)$ with respect to initial values and parameters.

The dependence of solutions of ordinary differential systems (1) on parameters is extensively treated in the literature in the case where f is a smooth function in all of its variables, and where the parameter λ belongs to some subset Λ of the Euclidean space $\mathbb{R}^n$, see e.g. the monograph by Walter, [6]. In many applications, however, the function $t \mapsto f(t,x,\lambda)$ is merely measurable and the parameter space may be some infinite dimensional function space. For this more general case only results concerning *continuous* dependence on parameters can be found in the literature, whereas the *differentiable* dependence on parameters is investigated only for parameter sets $\Lambda \subset \mathbb{R}^n$ and smooth right-hand sides, (cf. the appendix in the treatise of Hestenes [3]). With the present paper we want to close this gap.

In Section 1 we treat the continuous dependence of solutions $x(t;\tau,\xi,\lambda)$ on the initial data and parameters (τ,ξ,λ). Section 2 is concerned with the differentiable dependence on initial values and parameters (ξ,λ). The main results are stated as Theorem 1.1 and Theorem 2.1 in Sections 1.1 and 2.1, respectively. The proofs can be found in Sections 1.2 and 2.2. In both cases we proceed basically as in the smooth case, i.e. we replace the ordinary differential system (1) with an equivalent integral equation and look for a solution in a suitable Banach space by means of Banach's fixed point theorem. However, the proofs are much more technical than in the smooth case because of our weak assumptions on f.

Lemma 1.2 in Section 1.2 presents a result which can basically be found in Hestenes [3], where it is, however, not completely correctly stated. Hestenes considers compact subsets $S \subset \mathbb{R} \times \mathbb{R}^n$ with elements (t,x), where each slice of S with $t = const.$ is convex. One argument in the proof is based on the erroneous conclusion that for each sufficiently small $\delta > 0$ also the δ-neighbourhood $B_\delta(S)$ of S has only convex slices $t = const.$ Our counterexample in the appendix demonstrates that this is not true in general.

Let us finally mention that we were led to questions concerning the parameter dependence of solutions of ordinary differential equations in the course of investigating mechanical problems. In particular, we are interested in the mechanical behaviour of DNA-molecules which can be modeled as elastic rods. Using a general rod theory we are able to describe self-contact phenomena of such long slender molecules by a rigorous variational approach. Here, ordinary differential equations of the form (1) occur naturally as side conditions in the variational problem, and the differentiable dependence of the solution on parameters is needed to derive the Euler–Lagrange equations, [2],[5]. The problem of self-contact in elasticity could not be treated rigorously before.

Notation: We use the standard notation $B_R(x)$ for open balls with radius $R > 0$ and center x, where x is a point in a metric space X. If for an open, bounded set Ω contained in $\Omega_0 \subset X$ also its closure $\overline{\Omega}$ is contained in Ω_0, then we simply write $\Omega \subset\subset \Omega_0$. The space of continuous functions defined on X with values in a metric space Y will be denoted by $C^0(X, Y)$. In addition, if X and Y are normed linear spaces, we define $C^k(X, Y)$ to be the space of functions from X to Y whose k-th derivative exists and is continuous on X. $L^1(I, Y)$ is the space of integrable Y-valued functions on some interval $I \subset \mathbb{R}$ with respect to the one-dimensional Lebesgue-measure, and we write $L^1(I, \mathbb{R}) = L^1(I)$.

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1 Local existence, uniqueness and continuous dependence of the solution

1.1 Formulation of the result

Consider the initial value problem

$$\dot{x}(t) = f(t, x(t), \lambda), \quad x(\tau) = \xi \in \mathbb{R}^n, \tag{2}$$

where $\lambda \in \Lambda$ is a parameter in a metric space Λ. Here, the function f maps $\Omega_0 \times \Lambda$ into $\mathbb{R}^n$, where Ω_0 is a region in $\mathbb{R} \times \mathbb{R}^n$.

Our hypotheses on the right-hand side of (2) are: For any $\omega = (\tau, \xi) \in \Omega_0$ there is $\delta = \delta(\omega) > 0$, and there are measurable functions M_ω, K_ω, such that the cylinder $C_\delta(\omega) := B_\delta(\tau) \times B_\delta(\xi)$ is contained in Ω_0, and $M_\omega, K_\omega \in L^1(B_\delta(\tau))$ with the following properties:

(f1) For every $x \in B_\delta(\xi) \subset I\!R^n$ and $\lambda \in \Lambda$ the function $f(.,x,\lambda)$ is measurable on $B_\delta(\tau) \subset I\!R$ with

$$|f(t,x,\lambda)| \le M_\omega(t) \quad \text{for a.e. } t \in B_\delta(\tau). \tag{3}$$

Note that this implies that $f(.,x,\lambda) \in L^1(B_\delta(\tau), I\!R^n)$ for all $x \in B_\delta(\xi), \lambda \in \Lambda$.

(f2) For every $x, y \in B_\delta(\xi) \subset I\!R^n$ and for each $\lambda \in \Lambda$ one has

$$|f(t,x,\lambda) - f(t,y,\lambda)| \le K_\omega(t)|x-y| \quad \text{for a.e. } t \in B_\delta(\tau). \tag{4}$$

(f3) For each $x \in B_\delta(\xi), \lambda_0 \in \Lambda$

$$\lim_{\lambda \to \lambda_0} \int_{B_\delta(\tau)} |f(s,x,\lambda) - f(s,x,\lambda_0)|\, ds = 0. \tag{5}$$

Using standard arguments one can easily check that the conditions (f2) and (f3) follow from the following stronger condition, where we assume that $\Lambda \subset V$ is an open subset of some normed linear space V, compare with slightly stronger condition (f5) in Section 2.1:

(f4) $f(t,.,\lambda)$ is differentiable on $B_\delta(\xi)$ for a.e. $t \in B_\delta(\tau)$ and for each $\lambda \in \Lambda$. Moreover $f(t,x,.)$ is Fréchet–differentiable on Λ for a.e. $t \in B_\delta(\tau)$ and for each $x \in B_\delta(\xi)$. There is a function $P_\omega \in L^1(B_\delta(\tau))$, such that

$$|f_x(t,x,\lambda)| + \|f_\lambda(t,x,\lambda)\|_{L(V,I\!R^n)} \le P_\omega(t) \tag{6}$$

for all $\lambda \in \Lambda$, $x \in B_\delta(\xi)$, and for a.e. $t \in B_\delta(\tau)$.

We consider open bounded subsets $\Omega \subset\subset \Omega_0$ with the property that there is some $\delta_\Omega > 0$, such that

$$\Omega \text{ and } B_\delta(\Omega) \quad \text{are } x\text{-convex for all } 0 < \delta < \delta_\Omega. \tag{7}$$

A set $S \subset I\!R \times I\!R^n$ is *convex in x* or *x-convex,* if for all pairs $(t,x),(t,y) \in S$ the straight line segments $[(t,x),(t,y)]$ lie in S, compare Hestenes [3, p.377]. Moreover we choose Ω for simplicity in such a way, that

$$J := \{t \in I\!R : (t,x) \in \Omega \text{ for some } x \in I\!R^n\} \quad \text{is an open interval.} \tag{8}$$

Remark. In the appendix we present an example of a bounded open set Ω, such that Ω and even $\overline{\Omega}$ are x-convex, but such that there is no δ_Ω, such that (7) holds. Hence the reasoning in Hestenes [3, p.377] under the weaker assumption that only

Ω is x-convex, contains a gap, see our proof of Lemma 1.2. Note that if the set Ω is convex, then $B_\delta(\Omega)$ is convex (hence also x-convex) for all $\delta > 0$.

Now we are in the position to formulate a local existence and uniqueness theorem for an initial value problem of the form (2) including the continuous dependence on parameters $\lambda \in \Lambda$ and initial data $(\tau, \xi) \in \Omega$.

Theorem 1.1. *Let f satisfy (f1)-(f3) (or (f1) and (f4)), and let $\Omega \subset\subset \Omega_0$ be an open, bounded set, such that (7) and (8) hold. Then there exists a constant $\rho > 0$, such that for each $w = (\tau, \xi, \lambda) \in \Omega \times \Lambda$ there is a unique solution $x = x(t; \tau, \xi, \lambda)$ on $B_\rho(\tau)$ of the initial value problem*

$$\dot{x}(t) = f(t, x(t), \lambda), \quad x(\tau) = \xi. \tag{9}$$

Moreover, x depends continuously on the initial data and the parameter, i.e., for any fixed $\tau^* \in J$, one has

$$x \in C^0(B_\rho(\tau^*) \times \Omega^* \times \Lambda, \mathbb{R}^n)$$

where $\Omega^* := \{(\tau, \xi) \in \Omega : \tau \in B_\rho(\tau^*)\}$.

Note that for each $(\tau, \xi) \in \Omega_0$ one can choose an open, bounded *convex* set $\Omega \subset\subset \Omega_0$ containing (τ, ξ). Hence also (7) holds for all $\delta > 0$, in particular for all $\delta > 0$, such that $B_\delta(\Omega) \subset \Omega_0$. Thus, Theorem 1.1 gives a local statement for each $(\tau, \xi) \in \Omega_0$. Note also that the radius ρ in Theorem 1.1 does not depend on (τ, ξ, λ), but merely on Ω and f.

1.2 Proofs

First we will see that, since $\Omega \subset\subset \Omega_0$ satisfies (7) and is bounded, we can find integrable functions M, K, and a constant $\delta > 0$ all independent of the point $\omega \in \Omega$, such that (f1)–(f3) hold uniformly with M, K in place of M_ω, K_ω. This result can be found in Hestenes [3, p.377], but for completeness we will give a proof here. As pointed out above we actually need the assumption (7) on Ω, which is stronger than just x-convexity of Ω or $\overline{\Omega}$, see our example in the appendix.

Lemma 1.2. *In addition to the assumptions (f1)–(f3), let $\Omega \subset\subset \Omega_0$ be an open, bounded set, such that (7) and (8) hold. Then there is a number $\delta > 0$, such that $B_\delta(\Omega) \subset \Omega_0$, and there are functions $M, K \in L^1(\mathbb{R})$, such that*

$$|f(t, x, \lambda)| \le M(t) \quad \textit{for a.e. } t \in B_\delta(J), \tag{10}$$

$$|f(t, x, \lambda) - f(t, y, \lambda)| \le K(t)|x - y| \quad \textit{for a.e. } t \in B_\delta(J), \tag{11}$$

for each $\lambda \in \Lambda$ and all $x, y \in I\!R^n$ with $(t,x),(t,y) \in B_\delta(\Omega)$. Furthermore, for $(\tau, x) \in \overline{\Omega}$ one has

$$\lim_{\lambda \to \lambda_0} \int_{B_\delta(\tau)} |f(s,y,\lambda) - f(s,y,\lambda_0)| \, ds = 0 \tag{12}$$

for any $y \in B_\delta(x), \lambda_0 \in \Lambda$.

Note that $\{t \in I\!R : (t,x) \in B_\delta(\Omega) \text{ for some } x \in I\!R^n\} = B_\delta(J)$.

Proof. For given $\omega = (\tau, \xi) \in \overline{\Omega}$ consider the concentric cylinders $C_{\delta(\omega)}(\omega)$, $C_{\delta(\omega)/2}(\omega)$, $C_{\delta(\omega)/4}(\omega) \subset \Omega_0$, such that (f1)-(f3) hold on $C_{\delta(\omega)}(\omega)$. Then

$$\overline{\Omega} \subset \bigcup_{\omega \in \overline{\Omega}} B_{\delta(\omega)/4}(\omega), \tag{13}$$

and according to the theorem of Heine-Borel there is a finite number of points $\omega_i = (\tau_i, \xi_i) \in \overline{\Omega}$, $i = 1, \ldots, N$, such that for $\delta_i := \delta(\omega_i)$

$$\overline{\Omega} \subset \bigcup_{i=1}^{N} B_{\delta_i/4}(\omega_i). \tag{14}$$

Take $0 < 4\delta < \min\{\delta_1, \ldots, \delta_N, 4\delta_\Omega\}$, where δ_Ω is the constant in condition (7). Then

$$B_\delta(\Omega) \subset \bigcup_{i=1}^{N} B_{\delta_i/2}(\omega_i), \tag{15}$$

since for $\omega \in B_\delta(\Omega)$ there is $\omega^* \in \Omega$, such that $|\omega - \omega^*| < \delta$. Then by (14) we find $i^* \in \{1, \ldots, N\}$, such that $|\omega^* - \omega_{i^*}| < \delta_{i^*}/4$, which implies

$$|\omega - \omega_{i^*}| \le |\omega - \omega^*| + |\omega^* - \omega_{i^*}| < \delta + \delta_{i^*}/4 \le \delta_{i^*}/2.$$

Let $M_i := M_{\omega_i}, K_i := K_{\omega_i} \in L^1(B_{\delta_i}(\tau_i))$ for $i \in \{1, \ldots, N\}$ be the functions of our hypotheses. Extending these finitely many functions to all of $I\!R$ by 0 and denoting these extensions by $\tilde{M}_i, \tilde{K}_i, i = 1, \ldots, N$, we can define

$$M := \sum_{i=1}^{N} \tilde{M}_i, \quad K := \sum_{i=1}^{N} \tilde{K}_i \quad \text{on } I\!R. \tag{16}$$

Since $0 \le \tilde{M}_i(t) \le M(t)$, we immediately verify (10). Indeed, for $(t,x) \in B_\delta(\Omega)$ we find by (15) some $i \in \{1, \ldots, N\}$ with $(t,x) \in B_{\delta_i/2}(\omega_i) \subset B_{\delta_i}(\omega_i)$, hence by assumption (f1) $|f(t,x,\lambda)| \le M_i(t)$, unless t belongs to the null-set excluded in (f1). In that case we ignore that point. To be more precise, let $J_0 := \bigcup_{i=1}^N \{t \in B_\delta(\tau_i) :$ (3) or (4) does not hold $\}$, then we obtain (10) for all $t \in B_\delta(J) - J_0$. Note that J_0 has measure zero as the finite union of null-sets.

To show (11) take $(t,x),(t,y) \in B_\delta(\Omega)$ with $t \in B_\delta(J) - J_0$. If $|x-y| < \delta$, we are done, since then by (15) we find $i \in \{1,\dots,N\}$, such that $(t,x) \in B_{\delta_i/2}(\omega_i)$, in particular $x \in B_{\delta_i/2}(\xi_i) \subset I\!R^n$. But then $y \in B_{\delta_i}(\xi_i)$, since $\delta \le \delta_i/4$, and (f2) on $B_{\delta_i}(\xi_i)$ implies the desired estimate (11) for K as defined in (16). If $|x-y| \ge \delta$, however, take $k \in I\!N$, such that $|x-y| < k\delta$. Setting $y_j := y + (j/k)(x-y)$ for $j = 0,\dots,k$, we obtain

$$y_0 = y, \quad y_k = x, \quad |y_j - y_{j-1}| = |x-y|/k < \delta. \tag{17}$$

The set $B_\delta(\Omega)$ is x-convex, since $\delta < \delta_\Omega$, hence all the points $(t,y_j), j = 0,\dots,k$, are contained in $B_\delta(\Omega)$. For each $j \in \{1,\dots,k\}$ one can find $i_j \in \{1,\dots,N\}$ by (15), such that $(t,y_j) \in B_{\delta_{i_j}/2}(\omega_{i_j})$, hence by (17) $(t,y_{j-1}) \in B_{3\delta_{i_j}/4}(\omega_{i_j})$, since $\delta < \delta_{i_j}/4$.

Now (f2) implies for all $\lambda \in \Lambda$

$$\begin{aligned} |f(t,x,\lambda) - f(t,y,\lambda)| &\le \sum_{j=1}^k |f(t,y_j,\lambda) - f(t,y_{j-1},\lambda)| \\ &\le \sum_{j=1}^k K_{i_j}(t)|y_j - y_{j-1}| \\ &\le K(t)\sum_{j=1}^k |y_j - y_{j-1}| \\ &\underset{(17)}{=} K(t)\sum_{j=1}^k |x-y|/k = K(t)|x-y|, \end{aligned}$$

which proves (11).

Now, (14) implies that for $(\tau,x) \in \Omega$ we can find $i \in \{1,\dots,N\}$ such that $(\tau,x) \in B_{\delta_i/4}(\omega_i)$, where $\omega_i = (\tau_i,\xi_i) \in \overline{\Omega}$. Since $\delta < \delta_i/4$ one obtains $B_\delta((\tau,x)) \subset B_{\delta_i/2}(\omega_i)$, i.e., $B_\delta(x) \subset B_{\delta_i/2}(\xi_i)$ and $B_\delta(\tau) \subset B_{\delta_i/2}(\tau_i)$. Thus we can apply (5) to get

$$\begin{aligned} &\lim_{\lambda\to\lambda_0} \int_{B_\delta(\tau)} |f(s,y,\lambda) - f(s,y,\lambda_0)|\,ds \\ &\qquad\le \lim_{\lambda\to\lambda_0} \int_{B_{\delta_i/2}(\tau_i)} |f(s,y,\lambda) - f(s,y,\lambda_0)|\,ds = 0 \end{aligned}$$

for all $y \in B_\delta(x) \subset B_{\delta_i/2}(\xi_i)$. □

For a subinterval $J^* \subset B_\delta(J)$ let $D^0(J^*, I\!R^n)$ be the space of piecewise continuous functions x on J^* with

$$\text{graph}\, x := \{(t, x(t)) : t \in J^*\} \subset B_\delta(\Omega) \tag{18}$$

equipped with the norm given by $\sup_{J^*} |x(.)|$. Here $\delta > 0$ is the constant appearing in Lemma 1.2. We introduce the integral expression

$$E(x, y, \lambda, \mu) := \int_{J^*} |f(s, x(s), \lambda) - f(s, y(s), \mu)|\, ds \tag{19}$$

for $x, y \in D^0(J^*, I\!R^n), \lambda, \mu \in \Lambda$. By (11), for all $x, y \in D^0(J^*, I\!R^n), \lambda \in \Lambda$,

$$0 \leq E(x, y, \lambda, \lambda) \leq \sup_{J^*} |x(.) - y(.)| \int_{J^*} K(s)\, ds. \tag{20}$$

Lemma 1.3. *(i) Let $x \in C^0(J^*, I\!R^n) \cap D^0(J^*, I\!R^n)$, then $f(., x(.), \lambda) \in L^1(J^*, I\!R^n)$ for each $\lambda \in \Lambda$. Moreover, if $\lambda_0 \in \Lambda$, then*

$$\lim_{\lambda \to \lambda_0} \int_{J^*} |f(s, x(s), \lambda) - f(s, x(s), \lambda_0)|\, ds = 0. \tag{21}$$

(ii) For given $x \in D^0(J^*, I\!R^n), \lambda_0 \in \Lambda, \epsilon > 0$, there exists $\sigma = \sigma(x, \lambda_0, \epsilon) > 0$, such that

$$B_\sigma(x) \subset D^0(J^*, I\!R^n), \quad B_\sigma(\lambda_0) \subset \Lambda, \tag{22}$$

and such that for all $y, z \in B_\sigma(x), \lambda \in B_\sigma(\lambda_0)$ one has the inequality

$$E(y, z, \lambda, \mu) < \epsilon. \tag{23}$$

Proof. Approximate x with a sequence of step functions $\{x_k\} \subset D^0(J^*, I\!R^n)$ in the L^∞-sense. Apply (11) now for $\lambda \in \Lambda$ for each $k \in I\!N$ to get

$$|f(t, x_k(t), \lambda) - f(t, x(t), \lambda)| \leq K(t)|x_k(t) - x(t)| \text{ for a.e. } t \in J^*. \tag{24}$$

Since for a.e. $t \in J^*$ we have $K(t) < \infty$, we get the pointwise limit

$$\lim_{k \nearrow \infty} f(t, x_k(t), \lambda) = f(t, x(t), \lambda) \text{ for a.e. } t \in J^*.$$

Since x_k is a step function, we have that $f(., x_k(.), \lambda)$ is measurable for every $k \in I\!N$. Indeed, the image $x_k(J^*)$ consists of finitely many points in $I\!R^n$. Look at the set $X_B := \{x \in I\!R^n : f(t, x, \lambda) \in B\}$ for some Borel set $B \in \mathcal{B}(I\!R^n)$, where $\mathcal{B}(I\!R^n)$

denotes the Borel algebra of $I\!R^n$. We find that $X_B \cap x_k(J^*)$ is finite, hence measurable. Since x_k is measurable, we know that

$$\{t \in J^* : x_k(t) \in X_B\} = \{t \in J^* : x_k(t) \in X_B \cap x_k(J^*)\} \in \mathcal{B}(I\!R)$$

Hence also the set $\{t \in J^* : f(t, x_k(t), \lambda) \in B\}$ is in the Borel algebra $\mathcal{B}(I\!R)$, which is what we needed to prove that $f(., x_k(.), \lambda)$ is measurable.

As a pointwise limit of measurable functions we obtain $f(., x(.), \lambda)$ as a measurable function for each $\lambda \in \Lambda$. Since by (10) $f(., x(.), \lambda)$ is bounded by $M \in L^1(I\!R)$, we get that $f(., x(.), \lambda) \in L^1(J^*, I\!R^n)$.

Statement (21) follows from (23) for $y = z = x$, since $\epsilon > 0$ can be chosen arbitrarily small.

To prove (23), let $\epsilon > 0$ be given. Select a step function $x^* \in D^0(J^*, I\!R^n)$ such that $\sup_{J^*} |x(.) - x^*(.)| < \epsilon/(8\gamma)$ for some positive constant $\gamma > \int_{J^*} K(s)\, ds$. Then apply (20) to get

$$E(x, x^*, \mu, \mu) < \epsilon/8 \text{ for all } \mu \in \Lambda. \tag{25}$$

Since x^* is a step function with $\operatorname{graph} x^* \subset B_\delta(\Omega)$, we may apply (12) to find $\sigma = \sigma(x, \lambda_0, \epsilon) > 0$, such that

$$E(x^*, x^*, \lambda, \lambda_0) < \epsilon/4 \text{ for all } \lambda \in B_\sigma(\lambda_0) \subset \Lambda \tag{26}$$

and also $B_\sigma(x) \subset D_0(J^*, I\!R^n)$, $B_\sigma(\lambda_0) \subset \Lambda$, i.e., such that (22) holds.

Now we use

$$\begin{aligned} E(x, x, \lambda, \lambda_0) &\leq E(x, x^*, \lambda, \lambda) + E(x^*, x^*, \lambda, \lambda_0) + E(x^*, x, \lambda_0, \lambda_0) \\ &\underset{(25),(26)}{<} \epsilon/8 + \epsilon/4 + \epsilon/8 = \epsilon/2 \qquad \text{for all } \lambda \in B_\sigma(\lambda_0). \end{aligned} \tag{27}$$

Diminish σ if necessary to have also $\sigma < \epsilon/(4\gamma)$. For $y, z \in B_\sigma(x), \lambda \in B_\sigma(\lambda_0)$ we conclude with (20),(22) and (27)

$$\begin{aligned} E(y, z, \lambda, \lambda_0) &\leq E(y, x, \lambda, \lambda) + E(x, x, \lambda, \lambda_0) + E(x, z, \lambda_0, \lambda_0) \\ &\leq \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon. \end{aligned}$$

□

For $J^* \subset B_\delta(J)$ let $\Omega^* := \{(\tau, \xi) \in \Omega : \tau \in J^*\}$, where Ω is as before. We define the function space

$$C^0_{J^*} := \{x \in C^0(J^* \times \Omega^* \times \Lambda, I\!R^n) : \|x\|_\infty < \infty\}, \tag{28}$$

where

$$\|x\|_\infty := \sup_{(t,w)\in J^*\times\Omega^*\times\Lambda} |x(t,w)| \ \text{ for } \ x\in C^0(J^*\times\Omega^*\times\Lambda, I\!R^n). \tag{29}$$

It is well known that $(C^0_{J^*}, \|.\|_\infty)$ is in fact a Banach space, see e.g. Dunford-Schwartz [1, Lemma I.4.18, Corollary I.7.7].

From now on we focus on functions $x\in C^0_{J^*}$, such that

$$\operatorname{graph} x(.,w) = \{(t,x(t,w)) : t\in J^*\} \subset B_\delta(\Omega) \ \text{ for all } \ w\in\Omega^*\times\Lambda, \tag{30}$$

and we define the operator A on such functions x as

$$Ax(t,w) := \xi + \int_\tau^t f(s,x(s,w),\lambda)\,ds \ \text{ for } \ (t,w)=(t,\tau,\xi,\lambda)\in J^*\times\Omega^*\times\Lambda. \tag{31}$$

Lemma 1.4. *(i) $Ax\in C^0_{J^*}$ for all $x\in C^0_{J^*}$ satisfying (30).*

(ii) If $|J^*|$ is so small that

$$\int_{J^*} K(s)\,ds \le 1/2, \tag{32}$$

then

$$\|Ax_1 - Ax_2\|_\infty \le \frac{1}{2}\|x_1-x_2\|_\infty \tag{33}$$

for all $x_1,x_2\in C^0_{J^*}$ satisfying (30).

Proof. Let $x\in C^0_{J^*}$, such that (30) holds. For $(t,w)=(t,\tau,\xi,\lambda)$ and $(t',w')=(t',\tau',\xi',\lambda')$ in $J^*\times\Omega^*\times\Lambda$, we estimate

$$\begin{aligned}
&|Ax(t,w)-Ax(t',w')| \\
&\qquad\le\ |\xi-\xi'| + \Big|\int_\tau^t f(s,x(s,w),\lambda)\,ds - \int_{\tau'}^{t'} f(s,x(s,w'),\lambda')\,ds\Big| \\
&\qquad\le\ |\xi-\xi'| + \Big|\int_\tau^t f(s,x(s,w),\lambda)\,ds - f(s,x(s,w'),\lambda')\,ds\Big| \\
&\qquad\qquad + \Big|\int_\tau^{\tau'} f(s,x(s,w'),\lambda')\,ds\Big| + \Big|\int_{t'}^t f(s,x(s,w'),\lambda')\,ds\Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \quad |\xi - \xi'| + \int_{J^*} |f(s, x(s,w), \lambda) - f(s, x(s,w), \lambda')| \, ds \\
&\quad + \int_{J^*} |f(s, x(s,w), \lambda') - f(s, x(s,w'), \lambda')| \, ds \\
&\quad + \Big| \int_{\tau}^{\tau'} f(s, x(s,w'), \lambda') \, ds \Big| + \Big| \int_{t'}^{t} f(s, x(s,w'), \lambda') \, ds \Big| \\
&\leq \quad |\xi - \xi'| + E(x(.,w), x(.,w), \lambda, \lambda') + E(x(.,w), x(.,w'), \lambda', \lambda') \qquad (34) \\
&\quad + \Big| \int_{\tau}^{\tau'} M(s) \, ds \Big| + \Big| \int_{t'}^{t} M(s) \, ds \Big|.
\end{aligned}$$

For any given $\epsilon > 0$ we find some $\eta = \eta(\epsilon, t, w) > 0$, such that by (23) the right-hand side of (34) is less than 5ϵ for $|\xi - \xi'| < \eta$, $\lambda' \in B_\eta(\lambda) \subset \Lambda$, $\tau' \in B_\eta(\tau) \cap J^*$ and $t' \in B_\eta(t) \cap J^*$. Note that we also use the continuity of x to get $x(.,w')$ sufficiently close to $x(.,w)$ in $D^0(J^*, \mathbb{R}^n)$. This concludes the proof of (i).

To prove (33) choose $x_1,\ x_2 \in C^0_{J^*}$ satisfying (30). In particular, $x_1(.,w)$, $x_2(.,w) \in D^0(J^*, \mathbb{R}^n)$. Now we conclude

$$\begin{aligned}
|Ax_1(t,w) - Ax_2(t,w)| &\leq E(x_1(.,w), x_2(.,w), \lambda, \lambda) \\
&\leq \sup_{s \in J^*} |x_1(.,w) - x_2(.,w)| \int_{J^*} K(s) \, ds \quad \text{by (20)} \\
&\leq \frac{1}{2} \|x_1 - x_2\|_\infty \quad \text{by (32).}
\end{aligned}$$

Taking the supremum over all $(t,w) \in J^* \times \Omega^* \times \Lambda$ on the left gives the desired estimate (33). □

Proof of Theorem 1.1 . Let $\delta > 0, M, K \in L^1(\mathbb{R})$ as in Lemma 1.2. Fix any $\tau^* \in J$. We choose $\rho \in (0, \delta)$ so small that

$$\int_{B_\rho(\tau^*)} K(s) \, ds < \frac{1}{2}, \tag{35}$$

$$\int_{B_\rho(\tau^*)} M(s) \, ds < \frac{\delta}{2}. \tag{36}$$

Set

$$J^* := B_\rho(\tau^*) \tag{37}$$

$$\Omega^* := \{(\tau, \xi) \in \Omega : \tau \in J^*\} \tag{38}$$

$$C^* := \{y \in C^0_{J^*} : \operatorname{graph} y(.,w) \subset \overline{B_{\delta/2}(\Omega)} \ \text{ for all } \ w \in \Omega^* \times \Lambda\}. \tag{39}$$

Obviously, C^* is closed in $C^0_{J^*}$.

By (10),(11), (35) and (36) we obtain for all $y, z \in C^*$ and all $(t, w) = (t, \tau, \xi, \lambda) \in J^* \times \Omega^* \times \Lambda$

$$\tag{40} \Big| \int_\tau^t f(s, y(s, w), \lambda)\, ds - \int_\tau^t f(s, z(s, w), \lambda)\, ds \Big| < \frac{\|y - z\|_\infty}{2},$$

$$\tag{41} \Big| \int_\tau^t f(s, y(s, w), \lambda)\, ds \Big| < \frac{\delta}{2}.$$

We now fix any $x_1 \in C^*$ and define a sequence $\{x_k\} \subset C^* \subset C^0_{J^*}$ by

$$\tag{42} x_1(t, w) := x_1,$$

$$\tag{43} x_k(t, w) := A x_{k-1}(t, w) \text{ for all } (t, w) \in J^* \times \Omega^* \times \Lambda.$$

Clearly $x_1 \in C^*$. If $x_{k-1} \in C^*$ for $k \geq 2$, then $x_k \in C^0_{J^*}$ by Lemma 1.4 (i). Furthermore, by (31) and (41) one has

$$|x_k(t, w) - \xi| = \Big| \int_\tau^t f(s, x_{k-1}(s, w), \lambda)\, ds \Big| < \frac{\delta}{2}$$

for all $(t, w) = (t, \tau, \xi, \lambda) \in J^* \times \Omega^* \times \Lambda$. Since $(\tau, \xi) \in \Omega^*$, we get $x_k \in C^* \subset C^0_{J^*}$ for all $k \in I\!N$. By (35) and because C^* is a closed subset of $C^0_{J^*}$, the contraction property (33) together with the Banach Fixed Point Theorem leads to a fixed point $x \in C^0_{J^*}$ of the equation $x = Ax$, i.e.,

$$\tag{44} x(t, w) = x(t; \tau, \xi, \lambda) = \xi + \int_\tau^t f(s, x(s, w), \lambda)\, ds$$

for all $(t, w) \in J^* \times \Omega^* \times \Lambda$.

Obviously $t \mapsto x(t; \tau, \xi, \lambda)$ solves (9) on $B_\rho(\tau^*)$ for arbitrary $(\tau, \xi, \lambda) \in \Omega^* \times \Lambda$. In particular, $t \mapsto x(t; \tau^*, \xi, \lambda)$ provides a solution of (9) on $B_\rho(\tau^*)$ for any $(\tau^*, \xi, \lambda) \in \Omega \times \Lambda$. On the other hand, $(t, \tau, \xi, \lambda) \mapsto x(t; \tau, \xi, \lambda)$ is continuous on $J^* \times \Omega^* \times \Lambda$, where $J^* = B_\rho(\tau^*)$, i.e.,

$$x \in C^0(B_\rho(\tau^*) \times \Omega^* \times \Lambda, I\!R^n).$$

It remains to show uniqueness. Let $t \mapsto y(t)$ also be a solution of (9) for given $w^* = (\tau^*, \xi, \lambda) \in \Omega^* \times \Lambda$ in a small open neighbourhood $\tilde{J} \subset J^*$ of τ^*. By continuity one finds that graph $y \subset B_{\delta/2}(\Omega)$. for sufficiently small $\tilde{J}$, because $(\tau^*, \xi) \in \Omega^*$. Since y satisfies the integral equation as (44) for $\tau = \tau^*$ on $\tilde{J}$, we get by (11) and (35)

$$\begin{aligned} |x(t; \tau^*, \xi, \lambda) - y(t)| &\leq \int_{\tau^*}^t |f(s, x(s, w^*), \lambda) - f(s, y(s), \lambda)|\, ds \\ &\leq \frac{1}{2} \sup_{s \in \tilde{J}} |x(s, w^*) - y(s)| \text{ for all } t \in \tilde{J}. \end{aligned}$$

Taking the supremum also on the left-hand side leads to

$$x(t, w^*) = y(t) \quad \text{for all} \quad t \in \tilde{J},$$

and we get uniqueness on $\tilde{J}$. But this is in fact sufficient for uniqueness on all of $B_\rho(\tau^*)$, since we have constructed a solution for each initial value (τ, ξ), which is locally unique. □

2 Differentiability properties of the solution

2.1 Formulation of the result

We again consider the system of differential equations

$$(1) \qquad \dot{x}(t) = f(t, x(t), \lambda), \quad x(\tau) = \xi \in I\!R^n,$$

but now we are interested in the *differentiable* dependence of the solution on $\xi \in I\!R^n$ and $\lambda \in \Lambda$. While in Section 1 Λ was a metric space, we assume here that Λ is a open subset of a normed linear space V. In addition, we extend our set of assumptions (f1)–(f3) on the right-hand side f of (2): For any $\omega = (\tau, \xi) \in \Omega_0$ there is a constant $\delta = \delta(\omega) > 0$ such that $C_\delta(\omega) := B_\delta(\tau) \times B_\delta(\xi) \subset \Omega_0$, and such that the following properties hold:

(f5) $f(t, ., .)$ is continuously Fréchet–differentiable on $B_\delta(\xi) \times \Lambda$ for a.e. $t \in B_\delta(\tau)$. There is a function $P_\omega \in L^1(B_\delta(\tau))$, such that

$$(45) \qquad |f_x(t, x, \lambda)| + \|f_\lambda(t, x, \lambda)\|_{L(V, \boldsymbol{R}^n)} \le P_\omega(t)$$

for all $\lambda \in \Lambda$, $x \in B_\delta(\xi)$, and for a.e. $t \in B_\delta(\tau)$.

(f6) There is a function $Q_\omega : B_\delta(\tau) \times I\!R \to I\!R$, such that

- (Q1) $Q_\omega(., s) \in L^1(B_\delta(\tau))$ for all $s \in I\!R$,
- (Q2) $Q_\omega(t, .)$ is monotone for a.e. $t \in B_\delta(\tau)$,
- (Q3) $Q_\omega(t, s) \to 0$ for $s \to 0$ for a.e. $t \in B_\delta(\tau)$,

and such that

$$(46) \qquad |f_x(t, x, \lambda) - f_x(t, y, \lambda)| + \|f_\lambda(t, x, \lambda) - f_\lambda(t, y, \lambda)\|_{L(V, \boldsymbol{R}^n)} \le Q_\omega(t, |x - y|)$$

for all $\lambda \in \Lambda$, $x, y \in B_\delta(\xi)$, and for a.e. $t \in B_\delta(\tau)$. Here $\|.\|_{L(V, \boldsymbol{R}^n)}$ denotes the usual operator norm on the space of linear mappings from V to $I\!R^n$.

As in Section 1.1 one can easily check that (f6) follows from the following stronger condition

(f6*) $f(t,.,.)$ is twice differentiable on $B_\delta(\xi) \times \Lambda$ for a.e. $t \in B_\delta(\tau)$. There is a function $R_\omega \in L^1(B_\delta(\tau))$, such that

$$\|D^2 f(t,x,\lambda)\| \leq R_\omega(t) \tag{47}$$

for all $\lambda \in \Lambda$, $x \in B_\delta(\xi)$, and for a.e. $t \in B_\delta(\tau)$. Here $D^2 f$ denotes the operator consisting of all second (Fréchet–) derivatives of f.

In the next theorem we formulate the differentiability properties of the solution $x = x(t;\tau,\xi,\lambda)$ of (2) with respect to the initial value $\xi \in \mathbb{R}^n$ and the parameter $\lambda \in \Lambda$ for any fixed $\tau \in J$, where, as in Section 1,

$$J := \{t \in \mathbb{R} : (t,x) \in \Omega \text{ for some } x \in \mathbb{R}^n\}$$

is an open interval by choice of an open, bounded set $\Omega \subset\subset \Omega_0$ with the properties (7) and (8). We use the notation

$$\Xi(\tau) := \{\xi \in \mathbb{R}^n : (\tau,\xi) \in \Omega\}.$$

Theorem 2.1. *Let $\Omega \subset\subset \Omega_0$ be open and bounded, such that (7) and (8) hold, and let f satisfy (f1),(f5),(f6). Then there exists a constant $\rho > 0$, such that for each $\tau \in J$, $\xi \in \Xi(\tau)$ and $\lambda \in \Lambda$ there is a unique solution $x = x(t;\xi,\lambda)$ on $B_\rho(\tau)$ of the initial value problem*

$$\dot{x}(t) = f(t,x(t),\lambda), \quad x(\tau) = \xi. \tag{9}$$

Moreover, x is continuously differentiable with respect to the initial value and the parameter, more precisely

$$x \in C^1(B_\rho(\tau) \times \Xi(\tau) \times \Lambda, \mathbb{R}^n).$$

For notational convenience we write $w = (\xi,\lambda) \in \Xi(\tau) \times \Lambda$. The Fréchet–derivative of x with respect to w will be denoted by $D_w x$.

Corollary 2.2. *Under the assumptions of Theorem 2.1 one has*

$$D_w x(t,w)v = \zeta + \int_\tau^t [f_x(s,x(s,w),\lambda) D_w x(s,w)v + f_\lambda(s,x(s,w),\lambda)\mu]\, ds \tag{48}$$

for all $v = (\zeta,\mu) \in \mathbb{R}^n \times V$. Moreover, $\dot{x}$ is Fréchet–differentiable with respect to $w \in \Xi \times \Lambda$ and $D_w x$ is differentiable with respect to t a.e. on $B_\rho(\tau)$, and one obtains

$$\frac{d}{dt} D_w x(t,w)v = D_w\left(\frac{d}{dt} x(t,w)\right)v \tag{49}$$

for all $v \in \mathbb{R}^n \times V$, $w \in \Xi \times \Lambda$ and for a.e. $t \in B_\rho(\tau)$.

Remarks. 1. The careful reader will ask why the theorem does not treat the dependence of x on the initial time τ. The classical case where f is continuously differentiable in all variables tells us that the partial derivative $x_\tau(t;\tau,\xi,\lambda)$ is a sum containing the term $f(\tau,\xi,\lambda)$, see e.g., Walter [6, p.154]. The function f, however, is not continuous but merely measurable in τ in our situation. For simplicity we have omitted the formulation of our differentiability results in that generality, instead we refer the interested reader to Hestenes [3, pp. 390–397] for results in that direction.

2. If the set of parameters $\Lambda \subset V$ is compact (or locally compact), then one can omit (f6) in Theorem 2.1, since a continuous function is uniformly continuous on a compact set.

2.2 Proofs

As in Section 1.2 we can prove a global version of (45) for open, bounded subsets $\Omega \subset\subset \Omega_0$ satisfying (8). Since we will not need a global version of (46), condition (7) is not necessary for proving the following Lemma.

Lemma 2.3. *In addition to (f5),(f6) assume that $\Omega \subset\subset \Omega_0$ is an open and bounded set, such that (8) holds. Then there is a number $\delta_0 > 0$ with $B_{\delta_0}(\Omega) \subset \Omega_0$, such that $f(t,.,.)$ is continuously Fréchet–differentiable on $\{x \in \mathbb{R}^n : (t,x) \in B_{\delta_0}(\Omega)\} \times \Lambda$ for a.e. $t \in B_{\delta_0}(J)$. Moreover, there is a function $P \in L^1(\mathbb{R})$, such that*

$$|f_x(t,x,\lambda)| + \|f_\lambda(t,x,\lambda)\|_{L(V,\mathbb{R}^n)} \le P(t) \tag{50}$$

for all $\lambda \in \Lambda$, for a.e. $t \in \mathbb{R}$, such that $(t,x) \in B_{\delta_0}(\Omega)$.

Remark. Note that (50) together with the continuity of f_x, f_λ imply that $f_x(.,x(.),\lambda) \in L^1(J^*,\mathbb{R}^n)$ and $f_\lambda(.,x(.),\lambda) \in L^1(J^*, L(V,\mathbb{R}^n))$ for any subinterval $J^* \subset J$, $x \in D^0(J^*,\mathbb{R}^n)$ and any $\lambda \in \Lambda$, where $D^0(J^*,\mathbb{R}^n)$ is the space of piecewise continuous functions x on J^* with

$$\operatorname{graph} x := \{(t,x(t)) : t \in J^*\} \subset B_{\delta_0}(\Omega). \tag{51}$$

In fact, one may argue exactly as in the proof of Lemma 1.3 (i), using (50) and the continuity of f_x, f_λ instead of (10) and (11).

Proof of Lemma 2.3 . We proceed analogously to the proof of Lemma 1.2 using a finite covering as in (14),(15). For given $\omega = (\tau,\xi) \in \overline{\Omega}$ consider the concentric cylinders $C_{\delta(\omega)}(\omega)$, $C_{\delta(\omega)/2}(\omega)$, $C_{\delta(\omega)/4}(\omega) \subset \Omega_0$, such that (f5),(f6) hold on $C_{\delta(\omega)}(\omega)$. Then

$$\overline{\Omega} \subset \bigcup_{\omega\in\overline{\Omega}} B_{\delta(\omega)/4}(\omega), \tag{52}$$

and according to the theorem of Heine-Borel there is a finite number of points $\omega_i = (\tau_i, \xi_i) \in \overline{\Omega}$, $i = 1, \dots, \tilde{N}$, such that for $\delta_i := \delta(\omega_i)$

$$\overline{\Omega} \subset \bigcup_{i=1}^{\tilde{N}} B_{\delta_i/4}(\omega_i). \tag{53}$$

Then one obtains

$$B_{\delta_0}(\Omega) \subset \bigcup_{i=1}^{\tilde{N}} B_{\delta_i/2}(\omega_i) \tag{54}$$

for $\delta_0 := \min\{\delta_1, \dots, \delta_{\tilde{N}}\}/4$. In order to show (50) take the function $P \in L^1(I\!R)$ defined as

$$P(t) := \sum_{i=1}^{\tilde{N}} \tilde{P}_i(t),$$

where $\tilde{P}_i := P_{\omega_i}$ on $B_{\delta(\omega_i)}(\tau_i)$ and $\tilde{P}_i := 0$ on $I\!R - B_{\delta(\omega_i)}(\tau_i)$. The continuity of $f_x(t, ., .)$ and $f_\lambda(t, ., .)$ being a pointwise property follows from (46). $\square$

As in Section 1.2 we consider a subinterval $J^* \subset B_{\delta^*}(J)$ for $\delta^* := \min\{\delta, \delta_0\}$, where δ and δ_0 are the positive constants in Lemma 1.2 and Lemma 2.3, respectively. We fix a parameter $\tau^* \in J^*$ and consider the slice $\Xi^* := \Xi(\tau*)$. Let us define the function spaces

$$C^0_{J^*} := \{x \in C^0(J^* \times \Xi^* \times \Lambda, I\!R^n) : \|x\|_\infty < \infty\}, \tag{55}$$

$$C^0_{J^*,L} := \{X \in C^0(J^* \times \Xi^* \times \Lambda, L(I\!R^n \times V, I\!R^n)) : \|X\|_\infty < \infty\}, \tag{56}$$

$$C^1_{J^*} := \{x \in C^0_{J^*} : D_w x \in C^0_{J^*,L}\}, \tag{57}$$

where we used the norms

$$\|x\|_\infty := \sup_{(t,w)\in J^*\times\Xi^*\times\Lambda} |x(t,w)| \quad \text{for} \quad x \in C^0(J^* \times \Xi^* \times \Lambda, I\!R^n), \tag{58}$$

$$\|X\|_\infty := \sup_{(t,w)\in J^*\times\Xi^*\times\Lambda} \|X(t,w)\|_{L(\boldsymbol{R}^n\times V,\boldsymbol{R}^n)} \tag{59}$$

$$\text{for} \quad X \in C^0(J^* \times \Xi^* \times \Lambda, L(I\!R^n \times V, I\!R^n)).$$

Recall that for $w = (\xi, \lambda) \in \Xi^* \times \Lambda$

$$\|X(t,w)\|_{L(\boldsymbol{R}^n\times V,\boldsymbol{R}^n)} := \sup_{\|v\|_{R^n\times V}\le 1} |X(t,w)v|$$

for $v = (\zeta, \mu) \in I\!R^n \times V$ with

$$\|v\|_{\boldsymbol{R}^n \times V} := |\zeta| + \|\mu\|_V .$$

We introduce the norm

$$\|x\|_{1,\infty} := \|x\|_\infty + \|D_w x\|_\infty \text{ on } C^1_{J^*}. \tag{60}$$

Lemma 2.4. *The spaces* $(C^0_{J^*}, \|.\|_\infty)$, $(C^0_{J^*,L}, \|.\|_\infty)$ *and* $(C^1_{J^*}, \|.\|_{1,\infty})$ *are Banach spaces.*

Proof. For $(C^0_{J^*}, \|.\|_\infty)$ and $(C^0_{J^*,L}, \|.\|_\infty)$ we refer to [1] as before. For $C^1_{J^*}$ take a Cauchy sequence $\{x_n\} \subset C^1_{J^*}$, which means that $x_n \to x$ in $C^0(J^* \times \Xi^* \times \Lambda, I\!R^n)$ and $D_w x_n \to X$ in $C^0(J^* \times \Xi^* \times \Lambda, L(I\!R^n \times V, I\!R^n))$. We have to show that the Fréchet–derivative $D_w x$ exists and satisfies

$$D_w x = X. \tag{61}$$

In order to do that, observe that for $v \in I\!R^n \times V$ with $\|v\|_{\boldsymbol{R}^n \times V}$ sufficiently small,

$$\begin{aligned}
x_n(t, w+v) - x_n(t,w) &= \int_0^1 \frac{d}{ds} x_n(t, w+sv)\, ds \\
&= \int_0^1 D_w x_n(t, w+sv)\, ds\, v \\
&= \int_0^1 [D_w x_n(t, w+sv) - X(t, w+sv)]\, ds\, v \\
&\quad + \int_0^1 X(t, w+sv)\, ds\, v \\
&= \int_0^1 [D_w x_n(t, w+sv) - X(t, w+sv)]\, ds\, v \\
&\quad + \int_0^1 [X(t, w+sv) - X(t,w)]\, ds\, v \\
&\quad + \int_0^1 X(t,w)\, ds\, v.
\end{aligned}$$

Now we estimate

$$\begin{aligned}
|x_n(t, w+v) - x_n(t,w) - \int_0^1 [X(t, w+sv) - X(t,w)]\, ds\, v - \int_0^1 X(t,w)\, ds\, v| \\
\le \|D_w x_n - X\|_\infty \|v\|_{\boldsymbol{R}^n \times V} \text{ for all } n \in I\!N.
\end{aligned}$$

Going to the limit $n \to \infty$ on both sides of this inequality we obtain

$$x(t, w + v) - x(t, w) = X(t, w)v + o(\|v\|_{\boldsymbol{R}^n \times V}) \text{ for } \|v\|_{\boldsymbol{R}^n \times V} \to 0,$$

where we used the uniform convergence $x_n \to x$ in $C^0(J^* \times \Xi^* \times \Lambda, I\!R^n)$, $D_w x_n \to X$ in $C^0(J^* \times \Xi^* \times \Lambda, L(I\!R^n \times V, I\!R^n))$ and the continuity of X. This implies (61). $\square$

We consider $x \in C^1_{J^*}$, such that

$$\operatorname{graph} x(., w) = \{(t, x(t, w)) : t \in J^*\} \subset B_{\delta^*}(\Omega) \tag{62}$$

for all $w = (\xi, \lambda) \in \Xi^* \times \Lambda$. Similarly as in (31) we define the operator A on functions $x \in C^1_{J^*}$ which satisfy (62).

$$Ax(t, w) := \xi + \int_{\tau^*}^{t} f(s, x(s, w), \lambda)\, ds \tag{63}$$

for all $(t, w) = (t, (\xi, \lambda)) \in J^* \times \Xi^* \times \Lambda$. Note that, in contrast to (31), τ^* is fixed here.

Lemma 2.5. *For all $x \in C^1_{J^*}$ with (62) we have that $Ax \in C^1_{J^*}$, and*

$$\begin{aligned} D_w Ax(t, w)v = \zeta &+ \int_{\tau^*}^{t} f_x(s, x(s, w), \lambda) D_w x(s, w)v\, ds \\ &+ \int_{\tau^*}^{t} f_\lambda(s, x(s, w), \lambda)\mu\, ds \end{aligned} \tag{64}$$

for all $(t, w) \in J^* \times \Xi^* \times \Lambda$ and $v = (\zeta, \mu) \in I\!R^n \times V$.

Proof. One can show that $Ax \in C^0_{J^*}$ for $x \in C^1_{J^*} \subset C^0_{J^*}$ satisfying (62) by a reasoning analogous to the proof of Lemma 1.4 (i), by simply fixing $\tau := \tau^*$. Let us now show that $Ax \in C^1_{J^*}$ and that (64) holds. For $w = (\xi, \lambda) \in \Xi^* \times \Lambda$ and $v = (\zeta, \mu) \in I\!R^n \times V$ we consider the function

$$\alpha(\sigma) := Ax(t, w + \sigma v) = \xi + \sigma\zeta + \int_{\tau^*}^{t} f(s, x(s, w + \sigma v), \lambda + \sigma\mu)\, ds$$

for all $|\sigma|$ so small, that $w + \sigma v \in \Xi^* \times \Lambda$. The assumption (f5) on f implies that the integrand on the right-hand side is differentiable with respect to σ for a.e. $s \in J^*$, and we can estimate

$$\begin{aligned} &\Big|\frac{d}{d\sigma} f(s, x(s, w + \sigma v), \lambda + \sigma\mu)\Big| \\ &= \big|f_x(s, x(s, w + \sigma v), \lambda + \sigma\mu) D_w x(s, w + \sigma v)v + f_\lambda(s, x(s, w + \sigma v), \lambda + \sigma\mu)\mu\big| \\ &\le P(s)(\|D_w x\|_\infty \|v\|_{\boldsymbol{R}^n \times V} + \|\mu\|_V) \quad \text{for a.e. } s \in J^*, \quad \text{by (50).} \end{aligned}$$

Hence we may apply Zeidler [8, p.1018] to obtain

$$\begin{aligned} \alpha'(0) &= \zeta + \int_{\tau^*}^{t} \big[f_x(s,x(s,w),\lambda)D_w x(s,w)v + f_\lambda(s,x(s,w),\lambda)\mu\big]\,ds \\ &=: D_w Ax(t,w)v. \end{aligned} \tag{65}$$

Analogously to the previous estimate we get

$$\begin{aligned} |D_w Ax(t,w)v| &\le |\zeta| + \int_{\tau^*}^{t} P(s)(\|D_w x\|_\infty \|v\|_{\boldsymbol{R}^n\times V} + \|\mu\|_V)\,ds \\ &\le \|v\|_{\boldsymbol{R}^n\times V}\big[1 + (\|D_w x\|_\infty + 1)\int_{J^*} P(s)\,ds\big], \end{aligned}$$

which implies that $D_w Ax(t,w) \in L(I\!R^n \times V, I\!R^n)$, i.e., that $D_w Ax(t,w)$ is the Gâteaux–derivative of $Ax(t,.)$. Below we show the continuity of $D_w Ax(.,.)$ on $J^* \times \Xi^* \times \Lambda$. Hence $D_w Ax(t,w)$ is the Fréchet–derivative of $Ax(t,.)$ (cf. Zeidler [7, p.137]), and $D_w Ax \in C^0_{J^*,L}$. Since $Ax \in C^0_{J^*}$ we conclude that $Ax \in C^1_{J^*}$.

Let us finally show that $D_w Ax(.,.)$ is continuous on $J^* \times \Xi^* \times \Lambda$. Indeed, taking sequences $(t_n, w_n) \to (t,w) \in J^* \times \Xi^* \times \Lambda$ one estimates for $v = (\zeta,\mu) \in I\!R^n \times V$:

$$\begin{aligned} &|D_w Ax(t_n,w_n)v - D_w Ax(t,w)v| \\ &\quad\le\; |D_w Ax(t_n,w_n)v - D_w Ax(t_n,w)v| + |D_w Ax(t_n,w)v - D_w Ax(t,w)v| \\ &\quad\le\; \Big|\int_{\tau^*}^{t_n} |f_x(s,x(s,w_n),\lambda_n)D_w x(s,w_n)v - f_x(s,x(s,w),\lambda)D_w x(s,w)v|\,ds\Big| \\ &\qquad + \Big|\int_{\tau^*}^{t_n} |f_\lambda(s,x(s,w_n),\lambda_n)\mu - f_\lambda(s,x(s,w),\lambda)\mu|\,ds\Big| \\ &\qquad + \Big|\int_{t}^{t_n} |f_x(s,x(s,w),\lambda)D_w x(s,w)v + f_\lambda(s,x(s,w),\lambda)\mu|\,ds\Big| \\ &\quad\le\; \int_{J^*} |f_x(s,x(s,w_n),\lambda_n) - f_x(s,x(s,w),\lambda)|\|D_w x(s,w_n)\|_{L(\boldsymbol{R}^n\times V,\boldsymbol{R}^n)}\|v\|_{\boldsymbol{R}^n\times V}\,ds \\ &\qquad + \int_{J^*} |f_x(s,x(s,w),\lambda)|\|D_w x(s,w_n) - D_w x(s,w)\|_{L(\boldsymbol{R}^n\times V,\boldsymbol{R}^n)}\|v\|_{\boldsymbol{R}^n\times V}\,ds \\ &\qquad + \int_{J^*} |f_\lambda(s,x(s,w_n),\lambda_n) - f_\lambda(s,x(s,w),\lambda)|\|\mu\|_V\,ds \\ &\qquad + \Big|\int_{t}^{t_n} \big[|f_x(s,x(s,w),\lambda)|\|D_w x(s,w)\|_{L(\boldsymbol{R}^n\times V,\boldsymbol{R}^n)}\|v\|_{\boldsymbol{R}^n\times V} \\ &\qquad\qquad + |f_\lambda(s,x(s,w),\lambda)|\|\mu\|_V\big]\,ds\Big| \end{aligned}$$

$$
\begin{aligned}
&\leq \quad \|D_w x\|_\infty \|v\|_{\boldsymbol{R}^n \times V} \int_{J^*} |f_x(s, x(s, w_n), \lambda_n) - f_x(s, x(s, w), \lambda)|\, ds \\
&\quad +\|v\|_{\boldsymbol{R}^n \times V} \sup_{J^*} \|D_w x(., w_n) - D_w x(., w)\|_{L(\boldsymbol{R}^n \times V, \boldsymbol{R}^n)} \int_{J^*} P(s)\, ds \\
&\quad +\|\mu\|_V \int_{J^*} |f_\lambda(s, x(s, w_n), \lambda_n) - f_\lambda(s, x(s, w), \lambda)|\, ds \\
&\quad +(\|D_w x\|_\infty \|v\|_{\boldsymbol{R}^n \times V} + \|\mu\|_V) \cdot \Big| \int_t^{t_n} P(s)\, ds \Big| .
\end{aligned} \tag{66}
$$

Note that by (50) the integrands on the right-hand side are all dominated by $2P \in L^1(J^*)$, and since $f_x(s, ., .)$ and $f_\lambda(s, ., .)$ are both continuous on $\{x \in I\!\!R^n : (s, x) \in B_{\delta_0}(\Omega)\} \times \Lambda$, we obtain pointwise for a.e. $s \in J^*$:

$$
\begin{aligned}
\lim_{n\to\infty} |f_x(s, x(s, w_n), \lambda_n) - f_x(s, x(s, w), \lambda)| = 0, \\
\lim_{n\to\infty} |f_\lambda(s, x(s, w_n), \lambda_n) - f_\lambda(s, x(s, w), \lambda)| = 0,
\end{aligned}
$$

where we have also used that $x(., w_n) \to x(., w)$ as $n \to \infty$. After taking the supremum in (66) over all $v \in I\!\!R^n \times V$ with $\|v\|_{\boldsymbol{R}^n \times V} \leq 1$ we can apply Lebegue's Dominated Convergence Theorem to get

$$
\begin{aligned}
&\|D_w Ax(t_n, w_n) - D_w Ax(t, w)\|_{L(\boldsymbol{R}^n \times V, \boldsymbol{R}^n)} \\
&\qquad \leq \quad \|D_w x\|_\infty \int_{J^*} |f_x(s, x(s, w_n), \lambda_n) - f_x(s, x(s, w), \lambda)|\, ds \\
&\qquad\quad + \sup_{J^*} \|D_w x(., w_n) - D_w x(., w)\|_{L(\boldsymbol{R}^n \times V, \boldsymbol{R}^n)} \int_{J^*} P(s)\, ds \\
&\qquad\quad + \int_{J^*} |f_\lambda(s, x(s, w_n), \lambda_n) - f_\lambda(s, x(s, w), \lambda)|\, ds \\
&\qquad\quad + (\|D_w x\|_\infty + 1) \cdot \Big| \int_t^{t_n} P(s)\, ds \Big| \longrightarrow 0 \quad \text{as } \ n \to \infty,
\end{aligned} \tag{67}
$$

by the integrability of P, and since for $x \in C^1_{J^*}$ one has

$$
\sup_{J^*} \|D_w x(., w_n) - D_w x(., w)\|_{L(\boldsymbol{R}^n \times V, \boldsymbol{R}^n)} \to 0 \ \text{ as } \ n \to \infty.
$$

□

We define an operator B on $C^0_{J^*} \times C^0_{J^*,L}$, which assigns to each $(x, X) \in C^0_{J^*} \times C^0_{J^*,L}$ a mapping

$$
B(x, X) : J^* \times \Xi^* \times \Lambda \longrightarrow L(I\!\!R^n \times V, I\!\!R^n)
$$

given by

$$B(x,X)(t,\xi,\lambda)v := \zeta + \int_{\tau^*}^t f_x(s,x(s,\xi,\lambda),\lambda)X(s,\xi,\lambda)v\,ds$$
$$+\int_{\tau^*}^t f_\lambda(s,x(s,\xi,\lambda),\lambda)\mu\,ds \tag{68}$$

for $v=(\zeta,\mu)\in I\!R^n\times V$. Note that in fact $B(x,X)(t,\xi,\lambda)\in L(I\!R^n\times V, I\!R^n)$ for all $(t,\xi,\lambda)\in J^*\times\Xi^*\times\Lambda$.

Lemma 2.6. *(i) $B(x,X)\in C^0_{J^*,L}$ for all $x\in C^0_{J^*}$ satisfying (62) and all $X\in C^0_{J^*,L}$, i.e., the mapping $(t,\xi,\lambda)\mapsto B(x,X)(t,\xi,\lambda)$ is continuous on $J^*\times\Xi^*\times\Lambda$ for each $(x,X)\in C^0_{J^*}\times C^0_{J^*,L}$, where x satisfies (62).*

(ii) If, in addition, $|J^*|$ is so small that

$$\int_{J^*} P(s)\,ds < \frac{1}{2}, \tag{69}$$

then

$$\|B(x,X_1)-B(x,X_2)\|_\infty \le \frac{1}{2}\|X_1-X_2\|_\infty \tag{70}$$

for each $x\in C^0_{J^*}$ with (62), and all $X_1,X_2\in C^0_{J^*,L}$. Here, $\|.\|_\infty$ is the sup-norm as defined in (59).

(iii) Let $\{x_k\}\subset C^0_{J^*}$ be a sequence satisfying (62), $x_k\to x\in C^0_{J^*}$, such that x satisfies (62) and $X\in C^0_{J^*,L}$. Then $\|B(x_k,X)-B(x,X)\|_\infty\to 0$ as $k\to\infty$.

Remark. By Lemma 2.5 we readily see that for all $x\in C^1_{J^*}$ satisfying (62) the identity

$$D_wAx(t,w)v = B(x,D_wx)(t,w)v$$

holds for all $(t,w)\in J^*\times\Xi^*\times\Lambda$, $v\in I\!R^n\times V$, or, equivalently,

$$D_wAx = B(x,D_wx). \tag{71}$$

Proof of Lemma 2.6 . We already realized that $B(x,X)(t,\xi,\lambda)\in L(I\!R^n\times V, I\!R^n)$ for all $(t,\xi,\lambda)\in J^*\times\Xi^*\times\Lambda$, $(x,X)\in C^0_{J^*}\times C^0_{J^*,L}$. The continuity of the mapping $(t,\xi,\lambda)\mapsto B(x,X)(t,\xi,\lambda)$ on $J^*\times\Xi^*\times\Lambda$ can be shown as that of $(t,w)\mapsto D_wAx(t,w)$ in the previous proof as long as $(x,X)\in C^0_{J^*}\times C^0_{J^*,L}$ with x satisfying (62).

For the contraction property (70) we estimate with (50)

$$
\begin{aligned}
&|B(x,X_1)(t,w)v - B(x,X_2)(t,w)v| \\
&\qquad \leq \Big| \int_{\tau^*}^{t} |f_x(s,x(s,w),\lambda)(X_1(s,w)-X_2(s,w))v|\, ds \Big| \\
&\qquad \leq \|X_1 - X_2\|_\infty \|v\|_{\mathbb{R}^n\times V} \int_{J^*} P(s)\, ds \\
&\qquad \leq \frac{1}{2}\|X_1 - X_2\|_\infty \|v\|_{\mathbb{R}^n\times V} \qquad \text{by (69)},
\end{aligned}
$$

for all $(t,w) := (t,\xi,\lambda) \in J^*\times\Xi^*\times\Lambda$ and $v \in \mathbb{R}^n\times V$. Since $B(x,X_j)(t,w) \in L(\mathbb{R}^n\times V, \mathbb{R}^n)$, $j = 1,2$, we can take the supremum over $v \in \mathbb{R}^n \times V$ with $\|v\|_{\mathbb{R}^n\times V} \leq 1$, and get

$$
\|B(x,X_1)(t,w) - B(x,X_2)(t,w)\|_{L(\mathbb{R}^n\times V,\mathbb{R}^n)} \leq \frac{1}{2}\|X_1 - X_2\|_\infty.
$$

Note that $B(x,X_j) \in C^0_{J^*,L}$ for $j = 1,2$, hence we can take the supremum over $J^*\times\Xi^*\times\Lambda$ on the left-hand side to conclude that (70) holds.

In order to show (iii) we estimate similarly as before

$$
\begin{aligned}
&|B(x_k,X)(t,w)v - B(x,X)(t,w)v| \\
&\qquad \leq \Big| \int_{\tau^*}^{t} |(f_x(s,x_k(s,w),\lambda) - f_x(s,x(s,w),\lambda))X(s,w)v|\, ds \Big| \\
&\qquad\quad + \Big| \int_{\tau^*}^{t} |(f_\lambda(s,x_k(s,w),\lambda) - f_\lambda(s,x(s,w),\lambda))\mu|\, ds \Big| \\
&\qquad \leq \|X\|_\infty \|v\|_{\mathbb{R}^n\times V} \int_{J^*} |f_x(s,x_k(s,w),\lambda) - f_x(s,x(s,w),\lambda)|\, ds \\
&\qquad\quad + \|\mu\|_V \int_{J^*} |f_\lambda(s,x_k(s,w),\lambda) - f_\lambda(s,x(s,w),\lambda)|\, ds.
\end{aligned}
$$

First we take the supremum over all $v \in \mathbb{R}^n \times V$ with $\|v\|_{\mathbb{R}^n\times V}$ to obtain

$$
\begin{aligned}
&\|B(x_k,X)(t,w) - B(x,X)(t,w)\|_{L(\mathbb{R}^n\times V,\mathbb{R}^n)} \\
&\qquad \leq \|X\|_\infty \int_{J^*} |f_x(s,x_k(s,w),\lambda) - f_x(s,x(s,w),\lambda)|\, ds \\
&\qquad\quad + \int_{J^*} |f_\lambda(s,x_k(s,w),\lambda) - f_\lambda(s,x(s,w),\lambda)|\, ds. \qquad (72)
\end{aligned}
$$

Now we use the finite covering (54) to define the sets

$$
J_l := \{s \in J^* : (s,x(s,w)) \in B_{\delta_l}(\omega_l)\} \ \text{ for } \ l = 1,\ldots,\tilde{N}.
$$

Note that since x satisfies (62) we have

$$J^* \subset \bigcup_{l=1}^{\tilde{N}} J_l.$$

On each J_l for $l = 1, \ldots, \tilde{N}$, we can apply (46) on the integrands on the right-hand side of (72) to obtain

$$\begin{aligned}
&\|B(x_k, X)(t,w) - B(x,X)(t,w)\|_{L(\boldsymbol{R}^n \times V, \boldsymbol{R}^n)} \\
&\quad \leq \|X\|_\infty \sum_{l=1}^{\tilde{N}} \int_{J_l} |f_x(s, x_k(s,w), \lambda) - f_x(s, x(s,w), \lambda)|\, ds \\
&\quad\quad + \sum_{l=1}^{\tilde{N}} \int_{J_l} |f_\lambda(s, x_k(s,w), \lambda) - f_\lambda(s, x(s,w), \lambda)|\, ds \\
&\quad \leq (\|X\|_\infty + 1) \sum_{l=1}^{\tilde{N}} \int_{J_l} Q_{\omega_l}(s, r_k)\, ds, \qquad (73)
\end{aligned}$$

where $r_k := \|x_k - x\|_\infty$. Note that we used the monotonicity property (Q2) for each $Q_{\omega_i}, i = 1, \ldots, \tilde{N}$. Finally, taking the supremum over $J^* \times \Xi^* \times \Lambda$ on the left-hand side we arrive at

$$\|B(x_k, X) - B(x,X)\|_\infty \longrightarrow 0 \;\text{ as }\; k \to \infty,$$

where we have applied Lebegue's Dominated Convergence Theorem using $Q_{\omega_l}(., 2\|x\|_\infty)$, as integrable majorizing functions for the integrands $Q_{\omega_l}(s, r_k), l = 1, \ldots, \tilde{N}$ on the right-hand side of (73) for sufficiently large $k \in I\!N$. □

Proof of Theorem 2.1 . Let $\delta^* = \min\{\delta, \delta_0\}$, where $\delta_0 > 0, P \in L^1(I\!R)$ as in Lemma 2.3 and $\delta > 0$ and the functions $M, K \in L^1(I\!R)$ as in Lemma 1.2. Fix any $\tau^* := \tau \in J$ and choose $\rho \in (0, \delta^*)$, such that

$$\int_{B_\rho(\tau^*)} K(s)\, ds < \frac{1}{2}, \qquad (74)$$

$$\int_{B_\rho(\tau^*)} M(s)\, ds < \frac{\delta}{2}, \qquad (75)$$

$$\int_{B_\rho(\tau^*)} P(s)\, ds < \frac{1}{2}. \qquad (76)$$

Set

$$J^* := B_\rho(\tau^*), \tag{77}$$

$$\Xi^* := \{\xi \in I\!R^n : (\tau^*, \xi) \in \Omega\} = \Xi(\tau^*). \tag{78}$$

Define $C^0_{J^*}, C^0_{J^*,L}$ and $C^1_{J^*}$ according to (55),(56) and (57), and set

$$C^{1*} := \{y \in C^1_{J^*} : \operatorname{graph} y \subset \overline{B_{\delta/2}(\Omega)}\}, \tag{79}$$

(compare also with (37),(38) and (39).) Fix any $x_1 \in C^{1*}$, set $x_{k+1} := Ax_k$, which implies that $x_k \in C^1_{J^*}$ for all $k \in I\!N$ by Lemma 2.5. Moreover, by the same reasoning as in the proof of Theorem 1.1 one shows that $x_k \in C^{1*}$ for all $k \in I\!N$. By the contraction property (33), which therefore holds on the sequence $\{x_k\}$ by the definition of C^{1*}, and the Banach Fixed Point Theorem we obtain

$$x_k \to x \in C^0_{J^*} \text{ with } Ax = x.$$

For this fixed x the operator $B(x, .)$ on $C^0_{J^*,L}$ is a contraction mapping according to (70) of Lemma 2.6, hence has also a fixed point $X \in C^0_{J^*,L}$, i.e., $B(x, X) = X$. Now set $X_1 := D_w x_1$, and consider the sequence $X_{k+1} := B(x_k, X_k)$ with the x_k from above. Observe that $X_k = D_w x_k$ for all $k \in I\!N$ by (71). We claim that

$$X_k \to X \quad \text{in}\, C^0(J^* \times \Xi^* \times \Lambda, L(I\!R^n \times V, I\!R^n)). \tag{80}$$

In fact, we can use (70) and Lemma 2.6 (iii) to get

$$\begin{aligned}
\|X_{k+1} - X\|_\infty &= \|B(x_k, X_k) - B(x, X)\|_\infty \\
&\le \|B(x_k, X_k) - B(x_k, X)\|_\infty + \|B(x_k, X) - B(x, X)\|_\infty \\
&\le \frac{1}{2}\|X_k - X\|_\infty + \beta_k \text{ with } \beta_k \to 0 \text{ for } k \to \infty.
\end{aligned} \tag{81}$$

For a given $\epsilon > 0$ there is a $k_0 \in I\!N$, such that $\beta_k \le \epsilon/2$ for all $k \ge k_0$. Thus from (81) we infer for $k > k_0 + 1$

$$\begin{aligned}
\|X_{k+1} - X\|_\infty &\le \frac{1}{2}(\|X_k - X\|_\infty + \epsilon) \\
&\le \frac{1}{2}\big(\frac{1}{2}(\|X_{k-1} - X\|_\infty + \epsilon) + \epsilon\big) \\
&\le \frac{1}{2^{k-k_0+1}}\|X_{k_0} - X\|_\infty + \epsilon \sum_{i=1}^{k-k_0+1} \frac{1}{2^i} \\
&\le \frac{1}{2^{k-k_0+1}}\|X_{k_0} - X\|_\infty + \epsilon.
\end{aligned}$$

Consequently, $\|X_{k+1} - X\|_\infty \leq 2\epsilon$ for k sufficiently large, since the first term on the right-hand side can be made arbitrarily small choosing k large enough. This finishes the proof of (80).

Finally we observe that $D_w x = X$, since $C^1_{J^*}$ is a Banach space according to Lemma 2.4. This implies that x is in fact in $C^1_{J^*}$. Theorem 1.1 shows that x is the unique solution of (9) on $B_\rho(\tau^*)$, which finishes the proof of Theorem 2.1. □

Proof of Corollary 2.2 . (68),(71) and the fact that $Ax = x$ from the previous proof imply

$$\begin{aligned} D_w x(t,w)v &= D_w Ax(t,w)v = B(x, D_w x)(t,w)v \\ &= \zeta + \int_\tau^t [f_x(s, x(s,w), \lambda) D_w x(s,w)v + f_\lambda(s, x(s,w), \lambda)\mu]\, ds \end{aligned}$$

for $v = (\zeta, \mu) \in I\!R^n \times V$, which proves (i). Hence, we can differentiate $D_w x$ with respect to t (a.e. on $B_\rho(\tau)$) and obtain

$$\frac{d}{dt} D_w x(t,w)v = f_x(t, x(t,w), \lambda) D_w x(t,w)v + f_\lambda(t, x(t,w), \lambda)\mu \tag{82}$$

for a.e. $t \in B_\rho(\tau)$ and for $v = (\zeta, \mu) \in I\!R^n \times V$. On the other hand, looking at the differential system (9) for $x = x(t,w)$, we see that under the conditions (f1)-(f6) on f together with Lemma 2.3, that $\dot{x}$ is Fréchet–differentiable with respect to $w \in \Xi \times \Lambda$, since the right-hand side of (9) is. Computing this Fréchet derivative using the chain rule one obtains exactly the expression in (82), hence

$$\frac{d}{dt} D_w x(t,w)v = D_w(\frac{d}{dt} x(t,w))v,$$

i.e., (49) as claimed. □

Appendix: A counterexample

We are going to construct an open, bounded x-convex set $\Omega \subset I\!R^3$, such that also its closure $\overline{\Omega}$ is x-convex, with the property that for every $\delta > 0$ sufficiently small the corresponding δ-neighbourhood $B_\delta(\Omega)$ fails to be x-convex. This set Ω clearly violates condition (7). In the following we write $x = (\xi, \eta) \in I\!R^2$.

Consider the curve $\gamma : (0,1) \to I\!R^3$ defined as

$$\gamma(t) := \begin{pmatrix} t \\ \xi(t) \\ \eta(t) \end{pmatrix} := \begin{pmatrix} t \\ t\cos\phi(t) \\ t\sin\phi(t) \end{pmatrix}$$

where $\phi(t) := -2\pi \log_{3/2} t$ for $t \in (0,1)$. Then we have

$$\eta(t) = 0 \text{ and } \xi(t) = t \text{ for } t = 1, \frac{2}{3}, \frac{4}{9}, \dots, \frac{2^k}{3^k}, \dots .$$

For the planar disk $D_\rho := \{(0,\xi,\eta) \in I\!R^3 : \xi^2 + \eta^2 < \rho\}$ and for $\epsilon > 0$ set

$$\Omega(\epsilon) := \{(t,\xi,\eta) \in I\!R^3 : t \in (0,1), (t,\xi,\eta) \in \gamma(t) + D_{t\epsilon}\}.$$

Obviously, $\Omega(\epsilon)$ and $\overline{\Omega(\epsilon)}$ are bounded and (ξ,η)-convex. For $0 < \epsilon < 1$ the set $\Omega(\epsilon)$ is a spiral, which does not intersect the t-axis, and the closure $\overline{\Omega(\epsilon)}$ intersects the t-axis only in the origin. It will turn out that for $\epsilon > 0$ sufficiently small $\Omega := \Omega(\epsilon)$ is the desired set.

Given any $0 < \delta < \sqrt{2}/3$, set $t_0 := \sqrt{2}\delta$, and $t_1 := 3t_0/2$, hence $[t_0, t_1] \subset (0,1)$. We claim that the circle

$$C_{\tilde t} := \{(\tilde t, \xi, \eta) : \xi^2 + \eta^2 = \tilde t^{\,2}\}, \quad \tilde t := \frac{t_0 + t_1}{2},$$

is contained in the δ-neighbourhood of $\Gamma := \gamma((0,1))$, i.e.,

$$C_{\tilde t} \subset B_\delta(\Gamma) \subset B_\delta(\Omega(\epsilon)) \text{ for all } \epsilon > 0. \tag{83}$$

Indeed, if one parametrizes $C_{\tilde t}$ by the mapping $c : [t_0, t_1] \to I\!R^3$ defined by

$$\sigma \mapsto c(\sigma) := \tilde t(1, \cos\phi(\sigma), \sin\phi(\sigma)),$$

one easily checks that

$$\operatorname{dist}(\Gamma, c(\sigma)) \le |\gamma(\sigma) - c(\sigma)| \le \delta/2 \text{ for all } \sigma \in [t_0, t_1].$$

Now observe that Γ lies on the boundary of the cone K, whose axis is equal to the t-axis and which is centered at the origin with opening angle $\pi/4$, since $|\gamma(t) - (t,0,0)| = t$ for all $t \in (0,1)$. Consequently, we have that

$$\Omega(\epsilon) \subset \Omega_K(\epsilon), \quad \text{where } \Omega_K(\epsilon) := \bigcup_{(t,\xi,\eta)\in\partial K} (t,\xi,\eta) + D_{t\epsilon}. \tag{84}$$

By elementary geometric arguments one can easily deduce (see Figure 1) that

$$\operatorname{dist}((t,0,0), \Omega_K(\epsilon)) = t\sin(\arctan(1-\epsilon)).$$

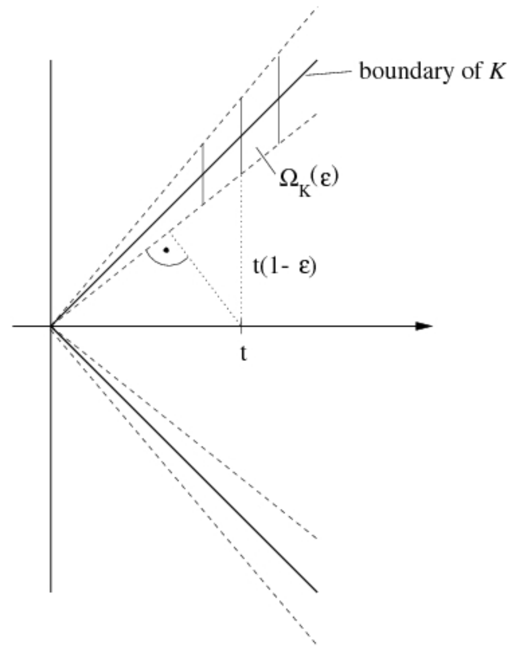

Fig. 1.

Since $\sin(\arctan(1-\epsilon)) \rightarrow \frac{1}{\sqrt{2}}$ as $\epsilon \rightarrow 0$, there is some $\epsilon_0 > 0$ such that

$$\sin(\arctan(1-\epsilon_0)) > \frac{4}{5\sqrt{2}}.$$

Thus for $t = \tilde{t} = 5t_0/4 = 5\sqrt{2}\delta/4$

$$\operatorname{dist}\left((\tilde{t},0,0),\Omega_K(\epsilon_0)\right) = \frac{5}{4}\delta\sqrt{2}\sin(\arctan(1-\epsilon_0)) > \delta.$$

That implies that the point $(\tilde{t},0,0)$, which is the center of the circle $C_{\tilde{t}}$, is not contained in $B_\delta(\Omega_K(\epsilon_0))$. By (84) we get $(\tilde{t},0,0) \notin B_\delta(\Omega(\epsilon_0))$. This together with (83) shows that $B_\delta(\Omega(\epsilon_0)$ is not (ξ,η)-convex for all $\delta \in (0,\sqrt{2}/3)$.

References

[1] N. DUNFORD, J.T. SCHWARTZ, *Linear Operators I,* Interscience, New York, 1958.

[2] O. GONZALEZ, J.H. MADDOCKS, F. SCHURICHT, H. VON DER MOSEL, Global Curvature and Self-Contact of Nonlinearly Elastic Rods, Manuscript.

[3] M.R. HESTENES, *Calculus of Variations and Optimal Control Theory,* J. Wiley & Sons, New York, 1966.

[4] E. KAMKE, *Das Lebesgue–Stieltjes Integral,* Teubner, Leipzig, 1960.

[5] F. SCHURICHT, H. VON DER MOSEL, Euler–Lagrange Equations for Nonlinearly Elastic Rods with Self-Contact, in preparation.

[6] W. WALTER, *Ordinary differential equations,* (Graduate texts in math. 182, Readings in mathematics), Springer, New York Berlin Heidelberg, 1998.

[7] E. ZEIDLER, *Nonlinear functional analysis and its applications,* vol.I: *Fixed–Point Theorems,* Springer, New York Berlin Heidelberg, 1985.

[8] E. ZEIDLER, *Nonlinear functional analysis and its applications,* vol.IIB: *Nonlinear Monotone Operators,* Springer, New York Berlin Heidelberg, 1990.