# Variational approach to contact problems in nonlinear elasticity

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**Abstract**

We use variational methods to study problems in nonlinear 3-dimensional elasticity where the deformation of the elastic body is restricted by a rigid obstacle. For an assigned variational problem we first verify the existence of constrained minimizers whereby we extend previous results with different respects. Then we rigorously derive the Euler-Lagrange equation as necessary condition for minimizers, which was possible before merely with strong hypothetical smoothness assumptions for the solution. The Lagrange multiplier corresponding to the obstacle constraint provides structural information about the nature of frictionless contact and, in the case of contact with, e.g., a corner of the obstacle, we derive a qualitatively new contact condition taking into account the deformed shape of the elastic body. By our rigorous analysis it is shown the first time that energy minimizers really solve the mechanical contact problem.

## 1 Introduction

In nature we often meet the situation that the deformation of an elastic body is restricted by the presence of a rigid body. We call this an obstacle problem or, if we focus on the touching of the bodies, a contact problem. From the mathematical point of view, the fundamental difficulty inherent in this important class of problems is that they are not only nonlinear but even nonsmooth, which is due to the sudden change of mechanics at the borderline between free deformation and contact. However, for a long time powerful nonsmooth tools were not available. Thus contact problems were usually studied on the basis of essential simplifications in geometry and mechanics such that the admissible deformations form a convex set in a suitable function space. In that case a constrained minimzer of the energy satisfies a variational inequality and, for their treatment, a general theory was developed in the 1970s and 1980s. In fact still today it is the common opinion that the investigation of contact problems is related to variational inequalities. However, for general contact problems in nonlinear elasticity convexity is lost even in simple situations and, therefore, the classical approach is not applicable. Since the development of efficient methods for the treatment of nonconvex nonsmooth problems is just going on, there is not yet a general theory for nonlinear contact problems today. To be more precise, there are quite general results about the existence of solutions as constrained minimizers of the energy, but there is no rigorous derivation of the Euler-Lagrange equation, which corresponds to the mechanical equilibrium conditions for frictionless contact, and there are no further regularity results. This paper presents some contribution to close this gap.

Nonlinear contact problems in 3-dimensional elasticity are studied by Ciarlet & Nečas [7], [8] (cf. also Ciarlet [6]). In [7] the special situation that some part of the deformed boundary of the elastic body should not interpenetrate a rigid obstacle is considered. [8] treats the standard problem that the whole elastic body cannot interpenetrate some rigid obstacle and, in addition, self-penetration of the elastic body is excluded. The results in [7], [8] concerning the existence of a constrained minimizer in some Sobolev space $\mathcal{W}^{1,p}(\Omega)$ are quite general. But the necessary condition for these minimizers in form of a boundary value problem for a partial differential equation is based on strong hypothetical smoothness assumptions for the solution which cannot be verified in general. In the present paper we state existence results for contact problems which are based on further developments in nonlinear elasticity and which generalize the former results. Then, as main result, we derive the Euler-Lagrange equation for minimizers in waek form rigorously, i.e., for minimizers obtained from existence theory without further regularity assumptions. Here Clarke's calculus of generalized gradients is an essential tool. Finally we present qualitatively new contact conditions which are relevant for obstacles with corners and edges. Let us still mention that the used techniques were already successfully applied to obstacle problems for shearable nonlinearly elastic rods (cf. Schuricht [15], [16], and Schuricht & Degiovanni [12]).

After a brief introduction to nonlinear elasticity and the formulation of contact problems in Section 2, we show the existence of solutions of general obstacle problems as constrained energy minimizers in suitable Sobolev spaces $\mathcal{W}^{1,p}(\Omega)$ in Section 3. Theorem 3.3 is based on polyconvexity and generalizes previous results for contact problems with different respects as, e.g., less restrictive growth conditions, consideration of non-homogeneous materials, more general boundary conditions, and treatment of the case $p \leq 3$. Some of these extensions are just the application of new general results in nonlinear elasticity to contact problems, but also new aspects are contained. With Theorem 3.7 we provide a second existence result based on quasiconvexity. That we can derive the Euler-Lagrange equation in Section 4, we have to invoke a usual growth restriction from above to the elastic energy. This way we exclude the physically reasonable case that the energy blows up as $\det Du$ approaches zero. This case, however, cannot be treated rigorously even without unilateral constraints. But the generality of the arguments related to contact, which is the main aspect of this paper, is not restricted by such a growth assumption. Theorem 4.3 states the Euler-Lagrange equation in weak form for minimizers of general contact problems in $\mathcal{W}^{1,p}(\Omega)$. In contrast to a variational inequality we here even compute the Lagrange multiplier corresponding to the obstacle constraint which provides additional structural information about the nature of contact reactions. We in particular get by rigorous variational arguments that the contact forces for frictionless contact are directed normally to the boundary of the obstacle. This could be shown before merely by additional smooothnes assumptions for the solution or it was invoked into the theory just as hypotheses. The key idea of our proof is to formulate the obstacle constraint as inequality side condition with a locally Lipschitz continuous functional and then to apply a nonsmooth Lagrange multiplier rule from Clarke's calculus of generalized gradients. While most investigations of contact problems suppose a smooth boundary of the obstacle and of the reference configuration of the elastic body, our analysis even works for Lipschitz boundaries. This way we cover, e.g., general obstacles with corners and edges. In cases where the boundary of the obstacle is not smooth at some contact point (e.g., a corner) we derive a qualitatively new condition for the direction of the contact force which takes into account also the deformed shape of the elastic body. Finally the appendix gives a brief introduction to Clarke's calculus of generalized gradients and it provides the material we need for our analysis.

**Notation.**

For the set $A$ we denote by $A^c$, $\operatorname{cl} A$ or $\bar{A}$, $\partial A$, $\overline{\operatorname{co}}\, A$, and $|A|$ the complement, the closure, the boundary, the closed convex hull, and the Lebesgue measure. $\operatorname{dist}_A(\cdot)$ assigns the shortest distance to $A$ to each point. $\overline{\operatorname{cone}}\, A \equiv \operatorname{cl}\{tu|\; u \in A,\, t \geq\; 0\}$ is the closed cone hull of A.

For a matrix $F \in \mathbb{R}^{3\times 3}$ we express by $|F|$, $\det F$, and $\operatorname{adj} F$ any fixed norm, the determinant, and the adjugate (i.e., $F \operatorname{adj} F = \det F \operatorname{id}$).

If $X$ is a Banach space, then $X^*$ stands for its dual space, $\langle\cdot,\cdot\rangle$ for the duality form on $X^* \times X$, $u_n \to u$ for the strong convergence, and $u_n \rightharpoonup u$ for the weak convergence. $B_\varepsilon(x)$ is the open ball of radius $\varepsilon$ around $x$. For a locally Lipschitz continuous function $f : X \to \mathbb{R}$, Clarke's generalized gradient is denoted by $\partial f(u)$ and the generalized directional derivative by $f^0(u;v)$ (cf. appendix). For readers who are not familiar with the last notions it is sufficient for a rough understanding of the main ideas to consider $\partial f(u)$ as a set of "reasonable gradients" assigned to $f$ at $u$.

$\mathcal{L}^p$ stands for the Lebesgue space of $p$-integrable functions and $\mathcal{W}^{1,p}$ for the Sobolev space of all functions $u \in \mathcal{L}^p$ with generalized derivatives $Du \in \mathcal{L}^p$. $\mathcal{C}^k$ denotes the space of all $k$-times continuously differentiable functions and $\mathcal{C}^k_0$ the corresponding subspace of functions having compact support. $\mathcal{R}[M]$ is the set of regular finite Borel measures with support in the set $M$.

## 2 Formulation of contact problems

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary (cf. Evans & Gariepy [13, p. 127], Zeidler [17, p. 232]). We identify $\Omega$ with the reference configuration of an elastic body and its elastic deformation may be described by mappings $u \in \mathcal{W}^{1,p}(\Omega;\mathbb{R}^3)$ where $Du$ is also called deformation gradient of $u$. We are mainly interested in continuous deformations which is related to the case $p > 3$. The general nonlinear material response of an elastic material can be described by a stored energy density $(x, F) \to W(x, F)$ on $\mathbb{R}^{3\times3} \times \Omega$ such that the stored energy of a deformed configuration $u \in \mathcal{W}^{1,p}(\Omega)$ is given by

$$E_s(u) = \int_\Omega W(x, Du(x))\, dx\,.$$

Usually it is assumed that
(W1) $W(x,\cdot)$ is rank-1-convex and continuously differentiable on $\mathbb{R}^{3\times3}$ for all $x \in \Omega$, and
(W2) $W(\cdot, F)$ is measurable on $\Omega$ for all $F \in \mathbb{R}^{3\times3}$.

Prescribed external forces (as, e.g., gravity or boundary stresses) can be expressed in a very general way by means of a vector valued measure $f \in \mathcal{R}[\bar\Omega]$ such that the corresponding potential energy is given by

$$E_p(u) = \int_{\bar\Omega} u(x)\, df(x)\,.$$

Hence the total energy of a deformed body is

$$E(u) \equiv E_s(u) - E_p(u) = \int_\Omega W(x, Du(x))\, dx - \int_{\bar\Omega} u(x)\, df(x)\,.$$

We claim to study minimizers of that energy subject to additional side conditions modelling suitable restrictions of the deformation.

We prescribe Dirichlet boundary conditions on some part of the boundary $\Gamma_D \subset \partial\Omega$, i.e., there is some given $u_D \in \mathcal{W}^{1,p}(\Omega)$ such that

$$u(x) = u_D(x) \quad \text{on } \Gamma_D$$

for all admissible deformations $u$.

The deformation of the elastic body may be restricted by a rigid obstacle $\mathcal{O} \subset \mathbb{R}^3$. It is reasonable to assume that $\mathcal{O}$ is the closure of an open set and, for deformations not interpenetreting the rigid obstacle, we demand that

$$u(x) \in \overline{\mathbb{R}^3 \setminus \mathcal{O}} \text{ on } \bar\Omega\,. \tag{2.1}$$

Notice that "thin" obstacles as, e.g., points, curves, and surfaces, cannot be treated that way (cf. also Schuricht [15]). This quite abstract obstacle condition (2.1) is sufficient to get existence results. To derive the Euler-Lagrange equation as necessary condition for a constrained minimizer of the energy, we however will use an alternative formulation which is analytically better accessible. For that reason we impose the mild assumption that $\mathcal{O}$ has Lipschitz boundary. The signed distance function

$$d_{\mathcal{O}}(q) \equiv \mathrm{dist}_{\mathcal{O}^c} q - \mathrm{dist}_{\mathcal{O}} q \tag{2.2}$$

is obviously globally Lipschitz continuous on $\mathbb{R}^3$ with Lipschitz constant 1. The properties of $\mathcal{O}$ easily imply that

$$0 \notin \partial d_{\mathcal{O}}(q) \quad \text{for } \ q \in \partial\mathcal{O} \tag{2.3}$$

where $\partial d_{\mathcal{O}}(q)$ denotes Clarke's generalized gradient of $d_{\mathcal{O}}(\cdot)$ at $q$ (cf. appendix). (2.1) is then equivalent to

$$g(u) \equiv \max_{x\in\bar{\Omega}} d_{\mathcal{O}}(u(x)) \le 0\,. \tag{2.4}$$

Note that the functional $g$ is not smooth but merely locally Lipschitz continuous on $\mathcal{W}^{1,p}(\Omega)$ in general. By $\Omega_c(u) \subset \bar{\Omega}$ we denote the contact set of the deformed elastic body $u(\bar{\Omega})$, i.e., $x \in \Omega_c(u)$ if and only if $u(x) \in \mathcal{O}$. Obviously $\Omega_c(u) = \emptyset$ if no contact occurs.

From the requirement that deformations should be locally invertible and orientation preserving, the constraint

$$\det Du > 0 \ \text{ a.e. on } \ \Omega \tag{2.5}$$

enters the theory. It can be incorporated in a mechanically reasonable way by allowing $W$ to take the value $+\infty$ and by setting

$$W(x,F) = \infty \quad \text{if} \quad \det Du \le 0\,, \tag{2.6}$$

i.e., the elastic energy becomes infinite under total compression. Hence (2.5) is satisfied for deformations with finite stored energy $E_s(u) < \infty$. While we can invoke (2.6) in existence theory, it is still an open question in nonlinear elasticity to derive the Euler-Lagrange equation for an energy respecting (2.6) even in the case without obstacles.

We can now formulate a general class of obstacle problems as variational problem in the following way

$$\begin{aligned} E(u) &\;\to\; \text{Min!}\,, \quad u \in \mathcal{W}^{1,p}(\Omega;\mathbb{R}^3)\,, && (2.7)\\ u &\;=\; u_D \quad \text{on } \Gamma_D\,, && (2.8)\\ g(u) &\;\le\; 0\,. && (2.9) \end{aligned}$$

The existence of minimizers and the derivation of the Euler-Lagrange equation is the subject of this paper. We will in particular show that minimizers satisfy the mechanical equilibrium condition for frictionless contact without imposing hypothetical regularity assumptions. Though, at first glance, it seems that we merely invoke displacement boundary conditions to our variational problem, we in fact can treat mixed displacement traction boundary conditions by the general choice of the potential energy $E_p$.

While most investigations in elasticity are based on continuous deformations, it has been shown that also discontinuous states $u \in \mathcal{W}^{1,p}(\Omega)$, $1 < p < 3$, are possible as minimizer of some energy and cavitation can occur (cf. Ball [4]). Hence there is some considerable interest aslo in the case $p \leq 3$. For discontinuous deformations, however, we have to be more careful with the formulation of the side conditions in our variational problem. (2.8) is to take in the sense of trace. In (2.1) and (2.4) we have to choose the precise representative of $u$, i.e., (2.1) can merely be demanded for almost every $x \in \Omega$ and in (2.4) we have to take the essential supremum on $\Omega$. Though we do not intend to emphasize that case, we will cover it in our existence results for completeness. Let us finally mention that we tacitly choose the continuous representative in pointwise conditions for $u(x)$ in the case $p > 3$.

# 3 Existence of a minimizer

## 3.1 Formulation of the results

The existence of energy minimizing configurations of a nonlinearly elastic body restricted by a rigid obstacle was studied by Ciarlet and Nečas [7], [8]. In [7] the special case where some part of the boundary $\partial\Omega$ cannot occupy points inside the obstacle $\mathcal{O}$ is treated. [8] considers the usual unilateral problem as introduced in the previous section that the whole elastic body cannot interpenetrate some rigid obstacle and, in addition, self-penetration of the deformed body is prohibited by a suitable inequality side condition. Except for treating global injectivity of deformations the existence results of Ciarlet & Nečas are based on that of Ball [2]. Since by now less restrictive assumptions are standard in the case without contact (cf. Ball & Murat [5], Müller [14], Zhang [19]), let us formulate some general existence result for contact problems which generalizes the previous ones.

We invoke the following hypotheses:

(A0) $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary,

(A1) *polyconvexity*: there exists $g : \Omega \times \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that
$g(x, \cdot, \cdot, \cdot)$ is continuous and convex for all $x \in \Omega$,
$g(\cdot, F, \operatorname{adj} F, \det F)$ is measurable for all $F \in \mathbb{R}^{3\times3}$, and

$$W(x, F) = g(x, F, \operatorname{adj} F, \det F) \quad \text{for all } (x, F) \in \Omega \times \mathbb{R}^{3\times3},$$

(A2) *coercivity*: there are $\alpha > 0$, $p \geq 2$, $q \geq \frac{p}{p-1}$ such that

$$W(x, F) \geq \alpha\Big(|F|^p + |\operatorname{adj} F|^q\Big) \quad \text{for all } (x, F) \in \Omega \times \mathbb{R}^{3\times3}, \tag{3.1}$$

$$W(x, F) = +\infty \quad \text{if and only if} \quad \det F \leq 0, \tag{3.2}$$

(A3) If $p \leq 3$, then $E_p$ is continuous on $\mathcal{L}^{p^*}(\Omega)$ with $p^* \equiv \frac{3p}{3-p}$ for $p < 3$, $p^* \equiv +\infty$ for $p = 3$.

(A4) $\Gamma_D \subset \partial\Omega$, $\Gamma_D \neq \emptyset$, $|\Gamma_D| > 0$ if $p \leq 3$ ($|\Gamma_D|$ - two-dimensional Hausdorff measure), $u_D \in \mathcal{W}^{1,p}(\Omega)$ is given.

**Theorem 3.3** *Let* (A0) – (A4) *be satisfied and let* $E(\tilde{u}) < \infty$ *for some* $\tilde{u} \in \mathcal{W}^{1,p}$ *satisfying* (2.8), (2.9). *Then the variational problem* (2.7) – (2.9) *has a solution* $u \in \mathcal{W}^{1,p}(\Omega)$ *with* $\det Du > 0$ *a.e. on* $\Omega$.

The proof can be found in Section 3.2.

**Remark 3.4**

1) Notice that, in contrast to previous existence results, we impose weaker growth conditions, we consider non-homogeneous materials, we allow more general boundary conditions if $p > 3$, and we treat the case $p \leq 3$. Furthermore we do not demand that $u_D(\Gamma_D) \subset \partial\mathcal{O}$ (as in [8]) which is just the case excluded in Section 4.

2) In [8] it is shown that any $u \in \mathcal{W}^{1,p}(\Omega)$, $p > 3$, satisfying

$$\int_\Omega \det Du\, dx \leq |u(\Omega)| \tag{3.5}$$

is globally injective up to a set of measure zero, i.e.,

$$\operatorname{card} u^{-1}(z) = 1 \quad \text{for a.e. } z \in u(\bar{\Omega})$$

(card - number of elements of a set; $u^{-1}$ - inverse function of $u$). Since condition (3.5) defines a weakly closed set in $\mathcal{W}^{1,p}(\Omega)$ if $p > 3$ (cf. [8]), Theorem 3.3 remains true for $p > 3$ if we add (3.5) to our variational problem. This way we verify solutions which can have self-contact but which do not interpenetrate itself. For global injectivity of elastic deformations see also Ball [3].

3) The variational problem can also be studied within the space $\mathcal{W}^{1,1}(\Omega)$. Growth condition (3.1) however implies that merely deformations $u \in \mathcal{W}^{1,p}$ have finite energy $E(u)$ and, thus, Theorem 3.3 verifies a minimizer also within the larger space $\mathcal{W}^{1,1}(\Omega)$.

4) We already mentioned that the Euler-Lagrange equation cannot be derived for energies obeying (3.2). Thus a natural question is what happens if that condition fails. It can easily be seen from the proof that if we drop (3.2), then Theorem 3.3 remains true for $p \neq 3$ but without the assertion that $\det Du > 0$ a.e. on $\Omega$.

5) Using ideas developed in Ciarlet & Nečas [7] also the case $\Gamma_D = \emptyset$ can be treated by imposing additional restrictions to the obstacle $\mathcal{O}$ as, e.g., that $\mathbb{R}^3 \setminus \mathcal{O}$ is bounded.

Polyconvexity was introduced by Ball [2] to treat energies with the property (3.2) in existence theory. On the other hand, even in the case without obstacles we are able to derive the Euler-Lagrange equation merely for energies satisfying a growth restriction from above of the form (4.1) below which excludes (3.2). Thus we cannot expect to get the Euler-Lagrange equation in the case of contact with (3.2). But if we neglect (3.2), we can ask for existence results with a less restrictive convexity asssumption for $W$. For this reason, instead of (A1), (A2) we consider the following hypotheses:

(A1') *quasiconvexity:* let $W : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ be such that

$W(x,\cdot)$ is continuous for all $x$,

$W(\cdot,F)$ is measurable for all $F$, and

$$\int_{\tilde{\Omega}} W(\tilde{x},F)\,dx \leq \int_{\tilde{\Omega}} W(\tilde{x},F+D\varphi(x))\,dx$$

for all open subsets $\tilde{\Omega} \subset \Omega$, a.e. $\tilde{x} \in \Omega$, and all $F \in \mathbb{R}^{3\times 3}$, $\varphi \in \mathcal{C}_0^1(\Omega)$.

(A2') *coercivity:* there are $\alpha, \beta, \gamma > 0$, $1 < p < +\infty$ such that

$$\alpha|F|^p \leq W(x,F) \leq \beta + \gamma|F|^p \quad \text{for all } (x,F)\,. \tag{3.6}$$

**Theorem 3.7** *Let* (A0), (A1'), (A2'), (A3), (A4) *be satisfied and let* (2.8), (2.9) *be fulfilled by some* $\tilde{u} \in \mathcal{W}^{1,p}$. *Then the variational problem* (2.7) – (2.9) *has a solution in* $\mathcal{W}^{1,p}$.

Remarks 3.4.2), 3), and 5) also apply to Theorem 3.7. Furthermore, notice that polyconvexity and quasiconvexity imply rank-1-convexity and, hence, the previous existence results are consistent with condition (W1) in Section 2.

## 3.2 Proofs

Proof of Theorem 3.3. Since most arguments of the proof are known from the literature let us just sketch that parts.

Let $u_n$ denote a minimzing sequence of the variational problem. $\|Dv\|_{\mathcal{L}^p}$ is an equivalent norm on the space $\{v \in \mathcal{W}^{1,p}(\Omega) |\ v = u_D \text{ on } \Gamma_D\}$ by (A0), (A4) (cf. Zeidler [18, p.1032]). Hence, by (3.1), there is a subsequence (denoted the same way) such that

$$u_n \rightharpoonup u \text{ in } \mathcal{W}^{1,p}(\Omega)\,, \quad \operatorname{adj} Du_n \rightharpoonup A \text{ in } \mathcal{L}^q(\Omega).$$

We have $A = \operatorname{adj} Du$ and $\det Du_n \rightharpoonup \det Du$ in the sense of distributions (cf. Dacorogna [10, Ch. 4, Th. 2.6]). Furthermore $u_n \to u$ in $\mathcal{L}^p(\Omega)$ for $p \leq 3$ and in $\mathcal{C}(\bar{\Omega})$ for $p > 3$. Below we show that $u$ satisfies (2.8), (2.9), i.e., $u$ is admissible.

For $2 \leq p < 3$ the convergence $\det Du_n \rightharpoonup \det Du$ is actually in $\mathcal{L}^r$ with $r = \frac{2q}{3} > 1$ by Müller [14]. For $p > 3$ Hölder's inequality implies $\det Du_n$, $\det Du \in \mathcal{L}^{p/3}$ and hence $\det Du_n \rightharpoonup \det Du$ in $\mathcal{L}^{p/3}$. Then, by (A1), we can apply standard lower semicontinuity results for convex integrands (cf. [10, Ch. 3, Th. 3.4]) to obtain

$$E_s(u) \leq \liminf_{n\to\infty} E_s(u_n)\,. \tag{3.8}$$

Since $E_p$ is weakly continuous by (A3), $u$ minimizes $E$.

For $p=3$ we argue as in Müller [14]. By (A2), $\det Du_n > 0$ a.e. on $\Omega$ and, thus, it can be shown that $\det Du_n \rightharpoonup \det Du$ in $\mathcal{L}^1(K)$ for every compact $K \subset \Omega$. By the same lower semicontinuity result used above and by $W \geq 0$ we get

$$\int_K W(x, Du)\,dx \leq \liminf_{n\to\infty} \int_K W(x, Du_n)\,dx \leq \liminf_{n\to\infty} \int_\Omega W(x, Du_n)\,dx$$

for all compact $K \subset \Omega$. Choosing an increasing sequence of compact sets $K_m \subset \Omega$ with $|\Omega \setminus K_m| \to 0$ we obtain (3.8) by the monotone convergence theorem and, hence, $u$ minimizes $E$ also in that case.

It remains to show that $u$ respects (2.8), (2.9). For $p > 3$ that is trivial by $u_n \to u$ in $\mathcal{C}(\Omega)$. Let now $p \leq 3$ and, hence, $u_n \to u$ in $\mathcal{L}^p(\Omega)$. At least for a subsequence we have $u_n(x) \to u(x)$ a.e. on $\Omega$. Suppose that

$$d_{\mathcal{O}}(u(x)) > 0 \;\text{ on a set }\; \Omega_+ \subset \Omega \;\text{ with }\; |\Omega_+| > 0\,. \tag{3.9}$$

Hence not all of the pairwise disjoint sets $\Omega_n \equiv \{x \in \Omega | \frac{1}{n} \leq d_{\mathcal{O}}(u(x)) < \frac{1}{n-1}\}$, $n \in \mathbb{N}$ (identify $\frac{1}{0} = +\infty$), can have measure zero. Therefore, without loss of generality, we can even assume that

$$d_{\mathcal{O}}(u(x)) > \varepsilon \quad \text{on } \Omega_+ \text{ for some } \varepsilon > 0.$$

By Egoroff's theorem, $u_n \to u$ uniformly on a subset $\Omega_0 \subset \Omega$ with $|\Omega \setminus \Omega_0| < |\Omega_+|/2$. Obviously $|\Omega_{0+}| > 0$ for $\Omega_{0+} \equiv \Omega_0 \cap \Omega_+$ and

$$d_{\mathcal{O}}(u_n(x)) > \frac{\varepsilon}{2} \;\text{ on }\; \Omega_{0+}$$

for all $n$ sufficiently large. But this contradicts $g(u_n) \leq 0$ and (3.9) must be wrong. Consequently $u$ satifies (2.9). By the linearity of the trace operator $u$ also fulfils (2.8). $\diamondsuit$

Proof of Theorem 3.7. Let $u_n$ denote a minimizing sequence of the variational problem. By (A0), (A4) and the left inequality in (3.6) we can argue as in the previous proof to get a subsequence

$$u_n \rightharpoonup u \quad \text{in} \quad \mathcal{W}^{1,p}(\Omega)\,.$$

Also analogously as above we see that $u$ satisfies (2.8), (2.9). Using (A1') and the right inequality in (3.6) we obtain the lower semicontinuity of $E_s$ by a general result of Acerbi & Fusco [1], i.e.,

$$E_s(u) \leq \liminf_{n\to\infty} E_s(u_n)\,.$$

Since $E_p$ is weakly continuous on $\mathcal{W}^{1,p}(\Omega)$ by (A3), $u$ minimizes $E$. $\diamondsuit$

# 4 Euler-Lagrange equation

## 4.1 Formulation of the results

In this section we formulate the weak form of the Euler-Lagrange equation for minimizers of the variational problem (2.7) – (2.9). For this reason we impose the following hypotheses:

(B0) $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary,
$\mathcal{O} \subset \mathbb{R}^3$ is the closure of an open set and has Lipschitz boundary,

(B1) $W(x, \cdot)$ is continuously differentiable on $\mathbb{R}^{3\times 3}$ for all $x \in \Omega$,
$W(\cdot, F)$ is measurable on $\Omega$ for all $F \in \mathbb{R}^{3\times 3}$,

(B3) there are $c > 0$, $\gamma \in \mathcal{L}^1(\Omega)$ such that

$$|DW(x,F)| \le c|F|^p + \gamma(x) \text{ for all } F \in \mathbb{R}^{3\times 3},\ x \in \Omega \tag{4.1}$$

($DW$ - gradient of the function $W(x, \cdot)$),

(B4) we have

$$u_D(\Gamma_D) \cap \mathcal{O} = \emptyset\,, \tag{4.2}$$

i.e., Dirichlet boundary conditions are presribed at points not occupied by the obstacle.

While (B3) is a standard growth condition to ensure the differentiability of the stored energy function on a suitable Sobolev space, we use (B4) to get normality in the Lagrange multiplier rule in the proof of the following theorem.

**Theorem 4.3** *Let* (B0) – (B4) *be satisfied and let* $u \in \mathcal{W}^{1,p}(\Omega;\mathbb{R}^3)$, $p > 3$, *be a local minimizer of the variational problem* (2.7) – (2.9). *Then there exist a measure* $\mu_c \in \mathcal{R}[\Omega_c(u)]$ *and a* $\mu_c$*-integrable function* $d_c^* : \bar{\Omega} \to \mathbb{R}^3$ *such that*

$$d_c^*(x) \in \partial d_{\mathcal{O}}(u(x)) \quad \textit{for all } x \in \bar{\Omega} \tag{4.4}$$

*(*$\partial d_{\mathcal{O}}(\cdot)$ *- Clarke's generalized gradient, cf. appendix) and such that the following weak form of the Euler-Lagrange equation is satisfied:*

$$\int_\Omega DW(x, Du(x))D\varphi(x)\,dx - \int_{\bar{\Omega}} \varphi(x)\,df(x) + \int_{\bar{\Omega}} d_c^*(x)\varphi(x)\,d\mu_c(x) = 0 \tag{4.5}$$

*for all* $\varphi \in \mathcal{W}^{1,\infty}(\Omega;\mathbb{R}^3)$ *with* $\varphi = 0$ *on* $\Gamma_D$.

*In particular,* $\mu_c = 0$ *in the case where* $g(u) < 0$, *i.e., if* $\Omega_c(u) = \emptyset$.

The proof of the theorem is given in Section 4.2.

**Remark 4.6**

1) The usual method to derive a variational inequality as necessary condition for constrained minimizers in contact problems does not work for general contact problems in nonlinear elastostatics due to loss of convexity. Note that we have derived the Euler-Lagrange

equation (in weak form) rather than a variational inequality. This has the fundamental advantage that the Lagrange multiplier corresponding to the contact constraint occurs explicitly and we obtain more structural information about the nature of contact forces.

2) The vector valued measure $\tilde{\Omega} \to \int_{\tilde{\Omega}} d^* d\mu$ on $\bar{\Omega}$ with support on the contact set $\Omega_c(u)$ describes the contact force exerted by the rigid obstacle $\mathcal{O}$. Condition (4.4) says that it is directed normally to the boundary of the obstacle. In the case where the boundary $\partial\mathcal{O}$ is smooth, the set $\partial d_{\mathcal{O}}(u(x))$ is just the singelton containing the usual inner unit normal of $\mathcal{O}$ at $u(x)$ as long as $u(x) \in \partial\mathcal{O}$. If the boundary $\partial\mathcal{O}$ is not smooth (e.g., if it has a corner), then the right hand side in (4.4) denotes a closed convex set of generalized normals and relation (4.4) expresses normality for the contact force in a generalized sense (cf. (A.2) in the appendix). This way we are able to treat obstacles $\mathcal{O}$ with corners and edges in a general way. Notice that a normality condition like (4.4) is usually invoked as hypothesis for frictionless contact into the theory or it is derived under additional regularity assumptions for the solution $u$ which cannot be verified in general.

3) Let us emphasize that Theorem 4.3 can be applied directly to minimizers verified in Theorem 3.3 or Theorem 3.7 without further hypothetical smoothness assumptions for $u$. Hence, in contrast to previous results, we have shown rigorously that constrained minimizers of the energy really satisfy the mechanical equilibrium conditions for frictionless contact (see also the next remark).

4) Often problems in elasticity are formulated as partial differential equation which is considered as equilibrium condition. But, in our case, (4.5) implies only formally that

$$\operatorname{div} DW(x, Du) + f + f_c = 0 \quad \text{on } \Omega$$

where $f_c(\tilde{\Omega}) \equiv -\int_{\tilde{\Omega}} d_c^*(x)\, d\mu_c(x)$, $\tilde{\Omega} \subset \Omega$ measurable, is the measure determined by the last term in (4.5). Here $\tau(x) \equiv DW(x, Du(x))$ corresponds to the (first Piola-Kirchhoff) stress tensor. Now the question arises in which sense (4.5) is related to an equilibrium condition. From the mechanical point of view the resultant force exerted to subbodies of the elastic body has to vanish in equilibrium, i.e.,

$$\int_{\partial\tilde{\Omega}} \tau(x) \cdot n(x)\, da + \int_{\tilde{\Omega}} df + \int_{\tilde{\Omega}} df_c = 0 \tag{4.7}$$

for a sufficiently large class of subbodies $\tilde{\Omega} \subset \Omega$ ($n(x)$ - outer unit normal of $\tilde{\Omega}$ at $x$). In classical treatments, where $f$, $f_c$ are assumed to have integrable volume density, (4.7) is verified for all subbodies $\tilde{\Omega} \subset \Omega$ having piecewise smooth boundary. That is however too restrictive for our general contact problems where, e.g., concentrated forces can really occur in the case of contact at a "sharp" corner of $\mathcal{O}$. But, by a result of Degiovanni, Marzocchi & Musesti [11], (4.5) is equivalent to the validity of (4.7) for "almost all" subsets $\tilde{\Omega} \subset \Omega$ (for further details see [11]).

5) If $g(u) < 0$, then there is no contact and, of course, there is no contact reaction. Note that this naturally follows from $\mu_c = 0$ in that case.

The next corollary shows that the contact condition (4.4) can still be sharpend in the case where $\partial\mathcal{O}$ is not smooth, i.e., where the right hand side in (4.4) is not a singleton. As in the theorem above, let $u$ be a local minimizer of the variational problem (2.7) – (2.9). We define the set of obstacles $\boldsymbol{\mathcal{Q}}(u)$ which consists of all obstacles $\mathcal{Q}$ (i.e., each $\mathcal{Q}$ is the closure of an open set having Lipschitz boundary), with the additional property that

$$\mathcal{O} \subset \mathcal{Q} \subset \mathrm{cl}\, u(\bar{\Omega})^c .$$

If we now replace $\mathcal{O}$ with any $\mathcal{Q} \in \boldsymbol{\mathcal{Q}}(u)$ in problem (2.7) – (2.9), then the admissible set becomes smaller while $u$ is still admissible, i.e., $u$ remains a local minimizer of the problem. Thus we can apply Theorem 4.3 to each such modified problem and we always obtain a measure $\mu_{\mathcal{Q}}$ and a mapping $d^*_{\mathcal{Q}}$ such that $d^*_{\mathcal{Q}}(x) \in \partial d_{\mathcal{Q}}(u(x))$ on $\bar{\Omega}$ and such that the Euler-Lagrange equation is satisfied with $\mu_{\mathcal{Q}}, d^*_{\mathcal{Q}}$ instead of $\mu_c$, $d^*_c$. By (4.5) we therefore obtain that

$$\int_{\bar{\Omega}} d^*_{\mathcal{Q}} \varphi \, d\mu_{\mathcal{Q}} \;=\; \int_{\bar{\Omega}} d^*_c \varphi \, d\mu_c$$

for all test functions $\varphi \in \mathcal{W}^{1,\infty}(\Omega)$ with $\varphi = 0$ on $\Gamma_D$. Since $\Gamma_D$ does not belong to the support of $\mu_{\mathcal{Q}}$ and $\mu_c$ and since $\mathcal{W}^{1,\infty}(\Omega)$ is dense in the space of continuous functions, the Borel measures $\tilde{\Omega} \to \int_{\tilde{\Omega}} d^*_{\mathcal{Q}} \, d\mu_{\mathcal{Q}}$ and $\tilde{\Omega} \to \int_{\tilde{\Omega}} d^*_c \, d\mu_c$ are identical for all $\mathcal{Q} \in \boldsymbol{\mathcal{Q}}(u)$. This way we get the following sharper condition for $d^*$.

**Corollary 4.8** *Theorem* 4.3 *remains true if we replace* (4.4) *with the stronger condition*

$$d^*(x) \in \bigcap_{\mathcal{Q} \in \boldsymbol{\mathcal{Q}}(u)} \overline{\mathrm{cone}}\Big(\partial d_{\mathcal{O}}(u(x))\Big) . \tag{4.9}$$

Let us still explain why we have to use the intersection of the closed cone hulls instead of merely the intersection of the generelized gradients in the previous formula. Obviously, for two different obstacles $\mathcal{Q}_1, \mathcal{Q}_2 \in \boldsymbol{\mathcal{Q}}$ we can have gradients $d^*_{\mathcal{Q}_i}(x) \in \partial d_{\mathcal{Q}_i}(u(x))$, $i = 1, 2$, pointing into the same direction but having different length. Hence the intersection $\partial d_{\mathcal{Q}_1}(u(x)) \cap \partial d_{\mathcal{Q}_2}(u(x))$ may be empty and (4.9) would not make sense. On the other hand the variation of the measures $\mu_c$ and $\mu_{\mathcal{Q}}$ is not prescribed, and, thus the functions $d^*_c$ and $d^*_{\mathcal{Q}}$ may differ by a positive integrable factor in order to get the same vector-valued measures. This makes the use of closed cone hulls in (4.9) necessary.

**Remark 4.10**

1) Note that the refined contact condition (4.9) provides a closed convex cone for the direction of the contact force which can depend on the solution $u$. That is a new type of contact condition which was not considered in classical treatments yet. From the mechanical point of view such a condition seems to be reasonable, since it expresses somehow that we can exchange the role of the rigid obstacle and of the deformed body in order to study the contact force.

2) If the right hand side in (4.9) merely contains the origin, then the corresponding point $x$ cannot belong to the support of the measure $\mu_c$ describing the distribution of the contact force. This in particular means that we can have contact but no mechanical reaction at such a point (cf. also Case 3 below).

In order to illuminate the significance of condition (4.9) let us discuss some typical situations. We assume that $x_0 \in \Omega_c(u)$ and we set $q_0 \equiv u(x_0)$.

**Case 1.** Let $\mathrm{cl}\, u(\bar{\Omega})^c \in \boldsymbol{\mathcal{Q}}(u)$, let there exist a half space $\mathcal{H} \subset \mathbb{R}^3$ with $q_0 \in \partial\mathcal{H}$ and a closed ball $B_0$ around $q_0$ such that

$$\left(\mathrm{cl}\, u(\bar{\Omega})^c \cap B_0\right) \subset \left(\mathcal{H} \cap B_0\right),$$

and let $-d_{\mathrm{cl}\, u(\bar{\Omega})^c}$ be regular at $q_0$ in the sense of Clarke (cf. appendix). Then, at the point $x_0$, (4.9) is equivalent with

$$d^*(x_0) \in \overline{\mathrm{cone}}\left(\partial d_{\mathrm{cl}\, u(\bar{\Omega})^c}(q_0)\right),$$

i.e., the shape of the deformed elastic body gives the sharpest condition for the direction of the contact force (cf. Fig. 1).

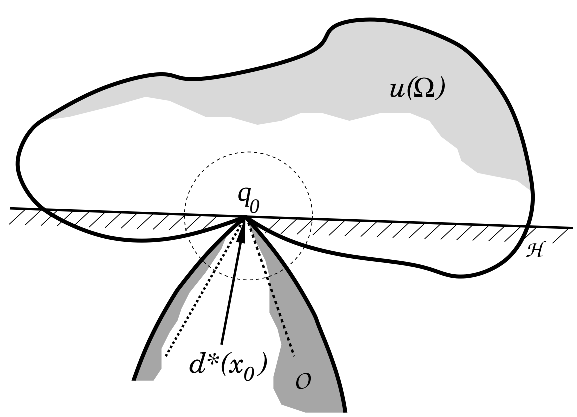


Figure 1. The dashed cone, which is normal to the deformed shape of the elastic body, provides the possible directions for the contact force $d^*(x_0)$.

**Case 2.** Let $\mathrm{cl}\, u(\bar{\Omega})^c \in \boldsymbol{\mathcal{Q}}(u)$, let there exist a half space $\mathcal{H} \subset \mathbb{R}^3$ with $q_0 \in \partial\mathcal{H}$ and a closed ball $B_0$ around $q_0$ such that

$$\left(\mathcal{O}^c \cap B_0\right) \subset \left(\mathcal{H} \cap B_0\right),$$

and let $d_{\mathcal{O}}$ be regular at $q_0$ in the sense of Clarke (cf. appendix). Then, at the point $x_0$, (4.9) is equivalent with

$$d^*(x_0) \in \overline{\mathrm{cone}}\left(\partial d_{\mathcal{O}}(q_0)\right),$$

i.e., (4.4) in Theorem 4.3, which is based on the shape of the obstacle $\mathcal{O}$, already gives the sharpest condition for $d^*(x_0)$ (cf. Fig. 2). The proof of Cases 1 and 2 is given in the next section.

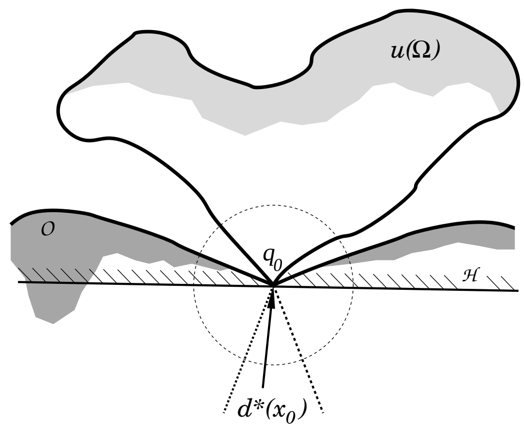


Figure 2. The dashed cone, which is normal to the shape of the obstacle, provides the possible directions for the contact force $d^*(x_0)$.

Cases 1 and 2 express the basic idea of the refined contact condition that, roughly speaking, the obstacle $\mathcal{Q} \in \boldsymbol{\mathcal{Q}}(u)$ which is locally the "closest to a half-space" gives the sharpest condition for the direction of the contact force $d^*(x_0)$. Observe, however, that if $-d_{\mathrm{cl}\, u(\bar{\Omega})^c}$ is not regular at $q_0$ (e.g., if $-d_{\mathrm{cl}\, u(\bar{\Omega})^c}$ highly oscillates), then the statement "closest to a half-space" cannot be taken in the sense of set inclusions. Let us still mention that convex Lipschitz continuous functions are always regular in the sense of Clarke. But for nonsmooth concave functions this is not true in general. This explains why it is reasonable to demand the regularity of $-d_{\mathrm{cl}\, u(\bar{\Omega})^c}$ instead of $d_{\mathrm{cl}\, u(\bar{\Omega})^c}$ in the first case.

**Case 3.** Let there exist two different half spaces $\mathcal{H}_1, \mathcal{H}_2 \subset \mathbb{R}^3$ and a closed ball $B_0$ around $q_0$ such that

$$\Big(\mathcal{O} \cap B_0\Big) \subset \Big(\mathcal{H}_i \cap B_0\Big) \subset \Big(\mathrm{cl}\, u(\bar{\Omega})^c \cap B_0\Big) \quad \text{for } \ i = 1, 2\,.$$

Then either $(\mathcal{H}_i \cap B_0) \cup \mathcal{O} \in \boldsymbol{\mathcal{Q}}(u)$ or we can find $\mathcal{Q}_i \in \boldsymbol{\mathcal{Q}}(u)$ which agree with $\mathcal{H}_i$ in a small neighborhood of $q_0$ (i=1,2). Hence $d^*(x_0) = 0$ by (4.9). This means that there is contact but no mechanical reaction at the point $u(x_0)$ (cf. Fig. 3).

While the first two cases seem to be reasonable also from the mechanical point of view, the last one is a little surprising. Of course the solution looks quite unstable, but it cannot be expected that the contact force has to vanish. Observe, however, that our arguments leading to (4.9) are based on the assumption that the equilibrium state is a local minimizer of the

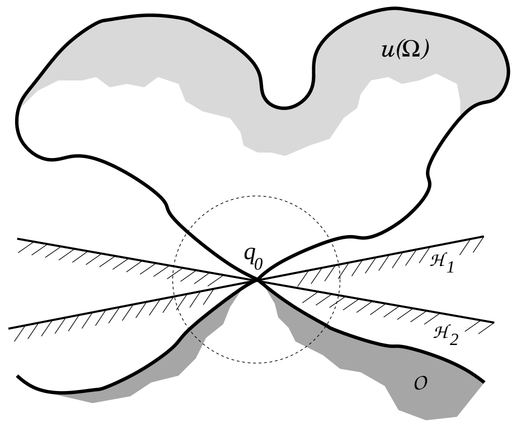


Figure 3. The elastic body touches the obstacle, but no contact force can be exerted.

energy, which expresses some kind of stability (recall the discussion in front of Corollary 4.8). Without carrying out this aspect in full detail we thus can interpret the situation of Case 3 in the following way: either the solution is stable and no contact force can occur or there is a nonvanishing contact reaction and the solution is highly unstable.

## 4.2 Proofs

Proof of Theorem 4.3. We present the proof in several steps.

a) *Modified problem.* Let us set

$$\hat{E}(v) \equiv E(u+v)\,, \;\; \hat{g}(v) \equiv g(u+v)\,, \;\; \hat{W}(x,F) \equiv W(x, Du(x)+F)\,.$$

Analogously we define $\hat{E}_p$ and $\hat{E}_s$. On the Banach space

$$X \equiv \{v \in \mathcal{W}^{1,\infty}(\Omega;\mathbb{R}^3) |\, v=0 \;\text{ on }\; \Gamma_D\}$$

we consider the modified variational problem:

$$\begin{aligned} \hat{E}(v) &\;\to\; \text{Min!}\,, \quad v \in X\,, && (4.11)\\ \hat{g}(v) &\;\leq\; 0\,. && (4.12)\end{aligned}$$

By the continuous embedding $\mathcal{W}^{1,\infty} \hookrightarrow \mathcal{W}^{1,p}$ we have that $v=0$ is a local minimizer of problem (4.11), (4.12).

b) *Differentiation of* $\hat{E}$ *in* $X$.

$\hat{E}_p$ is a linear continuous functional on the space of continuous functions and, thus, also on $\mathcal{W}^{1,\infty}$. Hence $\hat{E}_p$ is continuously differentiable on $X$ and we readily get that

$$\langle \hat{E}_p'(v), \varphi\rangle = \int_{\bar{\Omega}} \varphi(x)\, df(x) \quad \text{for all } \; \varphi \in X\,. \qquad (4.13)$$

Using the growth condition (4.1), we obtain by standard arguments that $\hat{E}_s$ is Gâteaux differentiable on $X$ with

$$\langle \hat{E}_s'(v), \varphi \rangle = \int_\Omega D\hat{W}(x, Dv(x)) D\varphi(x)\, dx \quad \text{for all } \varphi \in X\,. \tag{4.14}$$

Let $v_n \to v$ in $X$. Then there exist $\varphi_n \in X$, $\|\varphi_n\| \le 1$, such that

$$\begin{aligned} \|\hat{E}_s'(v) - \hat{E}_s'(v_n)\| &= \sup_{\varphi \in X,\, \|\varphi\| \le 1} \Big| \langle \hat{E}_s'(v) - \hat{E}_s'(v_n), \varphi \rangle \Big| \\ &\le \Big| \int_\Omega \Big( D\hat{W}(x, Dv(x)) - D\hat{W}(x, Dv_n(x)) \Big) D\varphi_n(x)\, dx \Big| + \frac{1}{n} \\ &\le \int_\Omega \Big| D\hat{W}(x, Dv(x)) - D\hat{W}(x, Dv_n(x)) \Big|\, dx + \frac{1}{n}\,. \end{aligned}$$

By the continuity of $DW(x,\cdot)$ and by (4.1), the dominated convergence theorem implies that the right hand side approaches zero as $n \to \infty$. Therefore $\hat{E}_s'(v_n) \to \hat{E}_s'(v)$ in $X^*$ which states the continuity of $\hat{E}_s'(\cdot)$ on $X$.

By the continuous differentiability of $\hat{E}_s$, $\hat{E}_p$, and thus also of $\hat{E}$, we obtain that

$$\{\hat{E}'(0)\} = \partial \hat{E}(0) = \partial \hat{E}_s(0) + \partial \hat{E}_p(0) = \{\hat{E}_s'(0)\} + \{\hat{E}_p'(0)\}$$

(cf. appendix).

c) *Generalized gradient $\partial \hat{g}(0)$.*

We claim to apply Proposition A.3 to the functional $\hat{g}$. For this reason we define $\beta : X \times \bar{\Omega} \to \mathbb{R}^3$ by

$$\beta(v,x) \equiv u(x) + v(x)\,.$$

Thus

$$\hat{g}(v) = \max_{x \in \bar{\Omega}} d_{\mathcal{O}}(u(x) + v(x)) = \max_{x \in \bar{\Omega}} d_{\mathcal{O}}(\beta(v,x))\,.$$

Let $(v_n, x_n) \to (v,x)$ in $X \times \bar{\Omega}$. Oviously

$$|\beta(v,x) - \beta(v_n,x_n)| \le |u(x) - u(x_n)| + |v(x) - v(x_n)| + |v(x_n) - v_n(x_n)|\,.$$

By the continuity of $u$, $v$ and by $v_n \to v$ in $\mathcal{W}^{1,\infty}(\Omega)$, the right hand side approaches zero as $n \to \infty$. Hence $\beta(\cdot,\cdot)$ is continuous on $X \times \bar{\Omega}$. Furthermore $\beta(\cdot, x)$ is linear and, thus, differentiable on $X$ for all $x \in \bar{\Omega}$ where

$$\beta_v(v,x)\varphi = \varphi(x) \quad \text{for all } \varphi \in X\,.$$

Again let $(v_n, x_n) \to (v,x)$ in $X \times \bar{\Omega}$. Then there exist $\varphi_n \in X$, $\|\varphi_n\| \le 1$, such that

$$\begin{aligned} \|\beta_v(v_n,x_n) - \beta_v(v,x)\| &= \sup_{\varphi \in X,\, \|\varphi\| \le 1} \Big| \Big( \beta_v(v_n,x_n) - \beta_v(v,x) \Big) \varphi \Big| \\ &\le |\varphi_n(x_n) - \varphi_n(x)| + \tfrac{1}{n} \;\le\; |x_n - x| + \tfrac{1}{n} \end{aligned} \tag{4.15}$$

(observe that all $\varphi_n$ are Lipschitz continuous with constant 1). The right-hand side tends to zero as $n \to \infty$ and, therefore, $\beta_v(\cdot,\cdot)$ is continuous. Since $d_{\mathcal{O}}$ is globally Lipschitz continuous with constant 1, the assumptions of Proposition A.3 are fulfilled and we can apply it to $\hat{g}$. Thus $\hat{g}$ is locally Lipschitz continuous on $X$ and for any $g^* \in \hat{g}(0)$ there exist a probability measure $\mu \in \mathcal{R}[\bar{\Omega}]$ and a $\mu$-integrable mapping

$$x \to d^*(x) \in \partial d_{\mathcal{O}}(u(x)) \tag{4.16}$$

such that

$$\langle g^*, \varphi \rangle = \int_{\bar{\Omega}} \langle d^*(x), \beta_v(0,x)\varphi \rangle \, d\mu(x) = \int_{\bar{\Omega}} d^*(x) \cdot \varphi(x) \, d\mu(x) \tag{4.17}$$

for all $\varphi \in X$. If $\hat{g}(0) = 0$, then $\mu \in \mathcal{R}[\Omega_c(u)]$.

Asssume now that $\hat{g}(0) = g(u) = 0$, i.e., $\Omega_c(u) \neq \emptyset$. We claim to show that

$$0 \notin \partial \hat{g}(0) \tag{4.18}$$

in this case. Let $g^* = g^*(d^*, \mu) \in \partial \hat{g}(0)$. Since $\mu$ is a probability measure with support in $\Omega_c(u)$, we can find some $\tilde{x} \in \Omega_c(u)$ such that

$$\mu(\, \bar{\Omega} \cap \overline{B_\varepsilon(\tilde{x})}\,) > 0 \quad \text{for all } \varepsilon > 0. \tag{4.19}$$

Obviously $\tilde{q} \equiv u(\tilde{x}) \in \partial\mathcal{O}$. By (2.3) and by the convexity and compactness of $\partial d_{\mathcal{O}}(\tilde{q}) \subset \mathbb{R}^3$, there is an open convex neighborhood of $\partial d_{\mathcal{O}}(\tilde{q})$ such that its closure does not contain the origin. Hence, by Proposition A.1.3, there is a neighborhood $U(\tilde{q})$ of $\tilde{q}$ such that

$$0 \notin \overline{\text{co}} \bigcup_{q \in U(\tilde{q})} \partial d_{\mathcal{O}}(q)\,.$$

Separation arguments for convex sets imply the existence of a vector $b \in \mathbb{R}^3$ such that

$$b \cdot b^* > 0 \quad \text{for all } b^* \in \partial d_{\mathcal{O}}(q), \;\; q \in U(\tilde{q})\,. \tag{4.20}$$

By the continuity of $u$ we can choose some small $\tilde{\varepsilon} > 0$ such that

$$u(\, \bar{\Omega} \cap \overline{B_{\tilde{\varepsilon}}(\tilde{x})}\,) \subset U(\tilde{q}) = U(u(\tilde{x}))\,. \tag{4.21}$$

With $\tilde{\Omega} \equiv \bar{\Omega} \setminus B_{\tilde{\varepsilon}}(\tilde{x})$ we clearly have that, for $x \in \bar{\Omega}$,

$$\tilde{\varphi}(x) \equiv b \, \text{dist}_{\tilde{\Omega}} x \in X = \mathcal{W}^{1,\infty}(\Omega; \mathbb{R}^3)$$

(note that $\mathcal{W}^{1,\infty}$ is just the space of Lipschitz continuous functions). Using (4.17), (4.19), (4.20), (4.21) and by $d^*(x) \in \partial d_{\mathcal{O}}(u(x))$, we readily obtain that

$$\begin{aligned} \langle g^*, \tilde{\varphi} \rangle &= \int_{\bar{\Omega}} d^*(x) \cdot \tilde{\varphi}(x) \, d\mu(x) \\ &= \int_{\bar{\Omega} \cap B_{\tilde{\varepsilon}}(\tilde{x})} d^*(x) \cdot b \, \text{dist}_{\tilde{\Omega}} x \; d\mu(x) \;\; > \;\; 0\,. \end{aligned} \tag{4.22}$$

Hence $g^* \neq 0$ and, since $g^* \in \hat{g}(0)$ was chosen arbitrarily, (4.18) is verified.

d) *Nonsmooth Lagrange multiplier rule.*

We now apply the Lagrange multiplier rule stated in Proposition A.1 to the modified variational problem (4.11), (4.12). Hence there exist $\lambda_0, \lambda_1 \geq 0$, not both zero, and a gradient $g^* \in \partial\hat{g}(0)$ such that

$$0 = \lambda_0 \hat{E}'(0) + \lambda_1 g^*$$

or, equivalently,

$$0 = \lambda_0 \langle \hat{E}'(0), \varphi \rangle + \lambda_1 \langle g^*, \varphi \rangle \quad \text{for all } \varphi \in X$$

and, in addition,

$$\lambda_1 \hat{g}(0) = \lambda_1 g(u) = 0\,.$$

Assume now that $\lambda_0 = 0$. Then $\lambda_1 > 0$, $g^* = 0$, and $g(u) = 0$ which contradicts (4.18). Thus, without lost of generality, we can choose $\lambda_0 = 1$ and we get $\lambda_1 = 0$ in the case where $g(0) < 0$. According to part c) of the proof, the gradient $g^*$ corresponds to a probability measure $\mu_1 \in \mathcal{R}[\bar{\Omega}]$ and a $\mu_1$-integrable mapping $d^*$ satisfying (4.16), while $\mu_1 \in \mathcal{R}[\Omega_c(u)]$ in the case of $g(u) = 0$. Using (4.13), (4.14), (4.17) and taking $\mu \equiv \lambda_1 \mu_1$ we finally obtain the assertion of Theorem 4.3. $\diamondsuit$

Proof of Cases 1 and 2. In Case 1 we obviously have that $d_{\mathrm{cl}\, u(\bar{\Omega})^c}(q) \geq d_{\mathcal{Q}}(q)$ on $B_0$ for any $\mathcal{Q} \in \boldsymbol{\mathcal{Q}}(u)$. Thus for any $w \in \mathbb{R}^3$ and any $\mathcal{Q} \in \boldsymbol{\mathcal{Q}}(u)$

$$\begin{aligned} \Big(-d_{\mathrm{cl}\, u(\bar{\Omega})^c}\Big)^0(q_0; w) &= \lim_{t \downarrow 0} \frac{-d_{\mathrm{cl}\, u(\bar{\Omega})^c}(q_0 + tw)}{t} \\ &\leq \liminf_{t \downarrow 0} \frac{-d_{\mathcal{Q}}(q_0 + tw)}{t} \;\leq\; \Big(-d_{\mathcal{Q}}\Big)^0(q_0; w)\,. \end{aligned}$$

This readily implies that

$$\partial\Big(-d_{\mathrm{cl}\, u(\bar{\Omega})^c}\Big)(q_0) \subset \partial\Big(-d_{\mathcal{Q}}\Big)(q_0) \;\text{ for all }\; \mathcal{Q} \in \boldsymbol{\mathcal{Q}}(u)\,.$$

By Proposition A.1 we can take out the minus and, since $\mathrm{cl}\, u(\bar{\Omega})^c \in \boldsymbol{\mathcal{Q}}(u)$, we obtain the assertion.

In Case 2 we use that $d_{\mathcal{O}}(q) \leq d_{\mathcal{Q}}(q)$ on $B_0$ for any $\mathcal{Q} \in \boldsymbol{\mathcal{Q}}(u)$ and we argue analogously as above to get

$$\partial d_{\mathcal{O}}(q_0) \subset \partial d_{\mathcal{Q}}(q_0) \;\text{ for all }\; \mathcal{Q} \in \boldsymbol{\mathcal{Q}}(u)\,,$$

which implies the assertion. $\diamondsuit$

## Appendix

A short introduction to Clarke's generalized gradients for locally Lipschitz continuous functionals, which is sufficient for our purposes, is given in this appendix. This calculus is a fundamental tool for handling nonsmooth problems. A comprehensive exposition can be found in Clarke [9].

Let $X$ be a Banach space and $f : X \mapsto \mathbb{R}$ a locally Lipschitz continuous functional. The *generalized directional derivative* $f^0(u;h)$ of $f$ at $u$ in the direction $h$ is given by

$$f^0(u;h) := \limsup_{v \in X,\, v \to u,\, t \to +0} \frac{f(v+th) - f(v)}{t}\,.$$

The mapping $h \to f^0(u;h)$ is positively homogeneous and subadditive. If $l_0$ is a local Lipschitz constant of $f$ near $u$, then $|f^0(u;h)| \leq l_0 \|h\|$ for all $h \in X$.

We define the *generalized gradient* $\partial f(u)$ of $f$ at $u$ as the set

$$\partial f(u) := \{f^* \in X^* |\, \langle f^*, h\rangle \leq f^0(u;h) \ \text{ for all } \ h \in X\}.$$

$\partial f(u)$ is a nonempty, bounded, convex and weak$^*$-compact subset of $X^*$. If $f$ is continuously differentiable, then $\partial f(u)$ is the singleton $\{f'(u)\}$. For convex functionals, $f^0(u;h)$ is the usual one-sided directional derivative and $\partial f(u)$ is the subdifferential of convex analysis.

Let us summerize some additional properties of the generalized gradient for our analysis (cf. Clarke [9]).

**Proposition A.1** *Let $f$ be Lipschitz continuous near $u \in X$ and let $l_0$ be its Lipschitz constant near $u$.*

1) $\|f^*\| \leq l_0$ *for all* $f^* \in \partial f(u)$.

2.1) $\partial(\alpha f)(u) = \alpha \partial f(u)$ *for all* $\alpha \in \mathbb{R}$.

2.2) $\partial \sum_{i=1}^n f_i(u) \subset \sum_{i=1}^n \partial f_i(u)$ *for locally Lipschitz continuous functionals* $f_i$.

3) *If* $\{u_i\} \subset X$ *and* $\{f_i^*\} \subset X^*$ *are sequences with* $f_i^* \in \partial f(u_i)$, $u_i \to u$ *and* $f_i^* \stackrel{*}{\rightharpoonup} f^*$ *for some* $f^* \in X^*$, *then* $f^* \in \partial f(u)$.

4.1) (*Minimum*). *If $f$ attains a local minimum* (*or maximum*) *at the point $u$, then* $0 \in \partial f(u)$.

4.2) (*Lagrange Multiplier Rule*). *Assume that* $g_0, g_1, \ldots, g_n : X \mapsto \mathbb{R}$ *are locally Lipschitz continuous. If $u$ is a local minimizer of $f$ subject to the restrictions* $g_0(v) \leq 0$ *and* $g_i(v) = 0$, $i = 1, \ldots, n$, *then there exist constants* $\lambda_f, \lambda_0 \geq 0$, *and* $\lambda_i \in \mathbb{R}$, *not all zero, such that*

$$0 \ \in \ \lambda_f \partial f(u) + \lambda_0 \partial g_0(u) + \sum_{i=1}^n \lambda_i \partial g_i(u)$$

*and* $\lambda_0 g_0(u) = 0$.

The function $f$ is called *regular* at $u \in X$ (in the sense of Clarke), if the usual one-sided directional derivative $f'(u;h)$ exists for each $h \in X$ and if it equals the generalized directional derivative, i.e., $f^0(u;h) = f'(u;h)$ for all $h \in X$. In particular, each Lipschitz continuous convex function $f$ is regular at any point.

For any nonempty subset $\mathcal{A} \subset X$ the normal cone of $\mathcal{A}$ at $u \in \mathcal{A}$ is given by

$$\mathcal{N}_{\mathcal{A}}(u) \;:=\; \mathrm{cl}^* \Big( \bigcup_{\lambda \geq 0} \lambda \partial \,\mathrm{dist}_{\mathcal{A}}(u) \Big)$$

where $\mathrm{cl}^*$ denotes the weak*-closure. If $\mathcal{A}$ is convex, then $\mathcal{N}_{\mathcal{A}}(u)$ coincides with the cone of normals as defined in the convex analysis. If $\mathcal{A} \equiv \{v \in X |\, f(v) \leq f(u)\}$ for some $u \in X$ with $0 \notin \partial f(u)$, then

$$\mathcal{N}_{\mathcal{A}}(u) \subset \Big( \bigcup_{\lambda \geq 0} \lambda \partial f(u) \Big) . \tag{A.2}$$

Equality holds under certain regularity. In the case where $f$ is the signed distance of $\mathcal{A}$ (cf. (2.2)) and $u \in \partial \mathcal{A}$ the right hand side of (A.2) can also be interpreted as some normal cone of $\mathcal{A}$ at $u$.

We now consider functionals of the following type

$$g(v) \equiv \max_{\xi \in \Omega} d(\beta(v, \xi)) .$$

Let us assume that

(i) $X, Y$ are Banach spaces where $Y$ is supposed to be reflexive and $\Omega$ is a metrizable sequentially compact topological space,

(ii) $\beta : X \times \Omega \to Y$ is continuous and $v \to \beta(v, \xi)$ is differentiable for all $\xi \in \Omega$ such that the derivative $\beta_v(\cdot, \cdot)$ is continuous on $X \times \Omega$,

(iii) $d : Y \to \mathbb{R}$ is Lipschitz continuous.

Since $\Omega$ is compact, $g$ is well defined and $\Omega(v) \equiv \{\xi \in \Omega |\, g(v) = d(\beta(v, \xi))\}$ is a nonempty closed subset of $\Omega$. Let us denote the set of all regular probability Borel measures on $\Omega$ supported on $\breve{\Omega} \subset \Omega$ by $\mathcal{R}[\breve{\Omega}]$. We now can describe the generalized gradient of $g$ as a composition of $\partial d(\cdot)$ and $\beta_v(\cdot, \cdot)$.

**Proposition A.3** *Suppose that* (i)–(iii) *hold. Then $g$ is locally Lipschitz continuous on $X$ and*

$$\partial g(v) \subset \Big\{ \int_{\Omega} \partial d(\beta(v, \xi)) \circ \beta_v(v, \xi)\, d\rho(\xi) \Big|\; \rho \in \mathcal{R}[\Omega(v)] \Big\} \quad \textit{for } v \in X,$$

*where the term on the right-hand side describes the subset of $X^*$ with the property that every element $g^*$ of this set corresponds to a mapping $d^* : \Omega \to Y^*$ with $d^*(\xi) \in \partial d(\beta(v, \xi))$ ($\partial$ with respect to $d(\cdot)$) and to a probability measure $\rho \in \mathcal{R}[\Omega(v)]$ such that*

$$\xi \;\to\; \langle d^*(\xi) \circ \beta_v(v, \xi),\, w \rangle_{X^* \times X} \;=\; \langle d^*(\xi), \beta_v(v, \xi)\, w \rangle_{Y^* \times Y}$$

*is $\rho$-integrable for all $w \in X$ and that*

$$\langle g^*, w\rangle \; = \; \int_\Omega \langle d^*(\xi), \beta_v(v,\xi)\, w\rangle \, d\rho(\xi) \quad \textit{for all } \; w \in X.$$

The proof of the proposition can be found in Schuricht [15, Prop. 6.10].

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