Global curvature for rectifiable loops

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Abstract

We study in detail the notion of *global curvature* defined on rectifiable closed curves, a concept which has been successfully applied in existence and regularity investigation regarding elastic self-contact problems in nonlinear elasticity. A bound on this purely geometric quantity serves as an *excluded volume constraint* to prevent selfintersections of slender elastic bodies modeled as elastic rods. Moreover, a finite global curvature characterizes simple closed curves, whose arc length parameterizations possess a Lipschitz continuous tangent field. The investigation of local and non-local properties of global curvature motivates, in particular, an extended definition of local curvature at any point of a rectifiable loop. Finally we show how a bound on global curvature can be used to define topological constraints such as a given knot type for closed loops or a prescribed linking number for closed framed curves, suitable to describe, e.g., supercoiling phenomena of biomolecules.

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1 Introduction

An elastic string, rope or wire being deformed in space cannot penetrate itself. It is surprisingly difficult to provide a mathematically precise and

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analytically tractable formulation for this physically evident phenomenon of self-avoidance, also called the *excluded volume constraint.* While most investigations in elasticity neglect this effect, it becomes crucial to exclude interpenetration in order to describe the physically relevant solutions, for instance, when dealing with closed knotted configurations where self-contact is to be expected.

In many situations the deformed state of a slender ring-shaped elastic body can be idealized as the tubular neighbourhood of a closed centrecurve. The excluded volume constraint is transferred to the centreline mathematically as a bound on the *global curvature.* This is a nonlocal quantity whose inverse, the *global radius of curvature,* was introduced by Gonzalez and Maddocks [11] in the context of ideal shapes of knots. Since this notion is not restricted to smooth curves (as is the case, e.g., for the classical normal injectivity radius), a bound on the global curvature is accessible by variational methods. In addition, the global curvature carries important information about the geometrical object. For example, we will prove here that a bounded global curvature *characterizes* the set of simple rectifiable loops, whose arc length parameterizations have a Lipschitz continuous tangent, see Theorem 2.1. This furnishes a remarkable equivalence between a purely geometric concept and analytic properties of closed space curves. Moreover, the relation between the pointwise global curvature function to the classical local curvature for smooth curves motivates our geometric definition of *generalized local curvature* at arbitrary points on curves that are merely rectifiable. To account for sophisticated topological constraints occurring in applications we consider *framed curves*, which are curves associated with an orthonormal frame at each point. A bound on the global curvature ensures that the linking number between the centrecurve and a curve generated by the frame is well-defined. The linking number allows us to identify the mechanically relevant solutions, which is important, for instance, to account for supercoiling phenomena of biomolecules modeled as elastic rods. The geometrically exact condition of bounded global curvature turns out to be suitable for the direct methods in the calculus of variations to prove existence results for highly nonlinear problems, see [12],[19]. Moreover, in contrast to alternative approaches, e.g., using repulsive potentials, where the necessary regularization can lead to serious mathematical and computational difficulties (see [9],[14],[18]), the geometrically exact condition of bounded global curvature allows us to derive the Euler-Lagrange equations including contact terms for minimizing elastic rods without shear and extension, see [16].

It should be emphasized that a rigorous derivation of variational equations in nonlinear elasticity taking into account self-contact has never been done before.

Crucial for all these applications is a thorough understanding of the purely geometric concept of global curvature. It is the objective of this paper to present a comprehensive investigation of this notion, both extending previous results in [12] and providing new and detailed analytical information about global curvature partly used in [16].

After defining the global curvature functions in Section 2 we present the central regularity result, Theorem 2.1. Part (ii) states that the tangent cone, as defined in geometric measure theory, reduces to a one-dimensional linear subspace at those points of a rectifiable loop, where the global curvature is finite. Part (iii) contains the remarkable fact that finite global curvature characterizes simple curves whose arc length parameterization possesses a Lipschitz continuous tangent field with Lipschitz constant equal to the global curvature. Geometrically, the global radius of curvature is the radius of the largest open ball that can be rotated tangentially about any point on the rectifiable loop without intersecting it, see part (iv) of Theorem 2.1.

Then the different cases of approaching the limit in the definition of the curvature functions are investigated, reflecting the local and nonlocal properties of global curvature. In particular, Theorem 2.3 (i) shows how the pointwise global curvature is related to the classical local curvature where the latter exists. This motivates the definition of the *generalized local curvature* at any point of a rectifiable loop, which turns out to be bounded from above by the approximate limes superior of $|\Gamma''|$ at that point (in the sense of geometric measure theory), and which equals $|\Gamma''|$ a.e., see Proposition 2.4. An alternative characterization of global curvature is given in Lemma 2.5. Specifically the geometric statement of Proposition 2.6 and the analytic information contained in Lemma 2.7 about the set of parameters, where global curvature is achieved, is essential for computing the structure of the contact term in the Euler-Lagrange equations in [16].

The excluded volume constraint in terms of global curvature is discussed in Section 3. Curves with finite global curvature $\mathcal{K}$ possess a tubular neighbourhood of uniform radius $\mathcal{K}^{-1}$, where the next-point projection onto the curve is single-valued and continuous, Proposition 3.1. Also the reverse statement is true, which constitutes a purely geometric justification for adopting the notion of global curvature as an excluded volume constraint. Furthermore, global injectivity for deformations of unshearable elastic rods is *equiva-*

lent to a finite bound on the global curvature of the centreline (Theorem 3.2). This means that a bound on the global curvature serves as an exact excluded volume constraint for unshearable rods also in the mechanical context.

Section 4 deals with functional properties of global curvature considered as a quantity defined on the space of rectifiable closed loops, and its role when dealing with topological constraints such as a given knot or link class. We prove in Lemma 4.1 that a bound on global curvature furnishes a closed condition with respect to uniform convergence, which is a sharpened version of [12, Lemma 4]. Proposition 4.2 states that isotopy classes, identifying the knot type for the centreline, are stable with respect to the C^0-topology on the set of loops with uniformly bounded global curvature. Thus the pull-tight phenomenon of small knots in the limit, not detected by the uniform topology alone, is excluded by an additional uniform upper bound on the global curvature. The closedness of isotopy classes of rectifiable loops with uniformly bounded global curvature follows easily, see Theorem 4.3. In contrast to, and supplementing the upper semicontinuity of the global radius of curvature proved in [4], we show that global curvature is continuous with respect to $C^{1,1}$-convergence of the arc length parameterizations (Theorem 4.4). This turns out to be quite useful when dealing with perturbations of a minimizer of a mechanical variational problem in order to derive the Euler-Lagrange equations in [16]. The *Gaussian linking number* is invoked via an analytic formula (avoiding topological degree theory) to define link classes for framed curves, which enables us to treat the case where the curve of the assigned orthonormal frames is not closed in $SO(3)$. With this notion we are able to distinguish the infinitely many components in the set of framed curves having the same boundary conditions, but where the number of rotations of the frame around the centreline differs. This distinction is necessary to select the correct solution in applications where supercoiling phenomena lead to a variety of configurations even for closed unknotted centrecurves. In Lemma 4.5 and Theorem 4.6 it is proved that a given link class constitutes a weakly closed condition and is stable with respect to small perturbations.

The proofs of all results are presented in Section 5.

Notation. We use $x \cdot y$ to denote the standard Euclidean inner product of x and y in $\mathbb{R}^3$, and $|\cdot|$ to denote the (intrinsic) distance between two points in $\mathbb{R}^3$ or in some parameter set $J \subset \mathbb{R}$ depending on the context. To denote the enclosed (smaller) angle between two non-zero vectors x and y in $\mathbb{R}^3$ we use $\sphericalangle(x, y) \in [0, \pi]$. The distance between a point $x \in \mathbb{R}^3$ and a subset $\Sigma \subset \mathbb{R}^3$ will be denoted by $\operatorname{dist}(x, \Sigma)$, the diameter of Σ will be denoted by $\operatorname{diam}(\Sigma)$. For any $\delta > 0$ we define open neighbourhoods of x and Σ by

$$B_\delta(x) = \{y \in \mathbb{R}^3 \mid |y - x| < \delta\} \quad \text{and} \quad B_\delta(\Sigma) = \{y \in \mathbb{R}^3 \mid \operatorname{dist}(y, \Sigma) < \delta\}.$$

The interior of a set Σ will be denoted by $\operatorname{int}\Sigma$ and its boundary by $\partial\Sigma$. The space of continuous functions on the closure of the interval $I = (a, b)$ will be denoted by $C^0(\bar{I})$, and $C^{k,1}(\bar{I})$, $k = 0, 1, 2, \ldots$, is the space of k-times continuously differentiable functions whose k-th derivative is Lipschitz continuous on $\bar{I}$. For Sobolev spaces of functions, whose weak derivatives up to order m are p-integrable, we use the standard notation $W^{m,p}(I)$. Notice that $C^{k,1}(\bar{I}) \cong W^{k+1,\infty}(I)$. The mean value integral of an integrable function f over a set E will be denoted by $\fint_E f := |E|^{-1} \int_E f$.

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2 Global curvature for rectifiable loops

We consider the set $\mathcal{L}$ of continuous and rectifiable closed curves $\gamma : \bar{I} \to \mathbb{R}^3$, with arc length parameterization $\Gamma_\gamma : S_L \to \mathbb{R}^3$. Here $I = (a, b)$ is an open interval, $L = L(\gamma) := \int_I |d\gamma| \geq 0$ denotes the length of γ, and S_L is the circle with perimeter L, which corresponds to the interval $[0, L]$ with identified endpoints, i.e., $S_L \cong \mathbb{R}/(L \cdot \mathbb{Z})$. The intrinsic distance on S_L and also the Euclidean distance in $\mathbb{R}^3$ will be denoted by $|\,.\,|$. In some instances we will need to use an ordering on the parameter set S_L to speak of inequalities between parameters $s_1, s_2 \in S_L$, and to consider one-sided limits. For this we tacitly assume that $s_1, s_2 \in (0, L)$, where, for instance, $s_1 < s_2$ is well-defined. To treat parameters near the endpoints $s = 0, s = L$ we simply shift the interval.

To simplify notation, we mostly omit the subscript γ and agree that Γ, Γ_k correspond to γ, γ_k and so on. According to [10, vol.II, p.255] this arc length parameterization Γ is Lipschitz continuous, i.e. of class $C^{0,1}([0,L],\mathbb{R}^3)$. Note that, by Rademacher's Theorem, Γ possesses a weak derivative Γ' a.e. on $[0,L]$ and $C^{0,1}([0,L],\mathbb{R}^3) \cong W^{1,\infty}([0,L],\mathbb{R}^3)$.

For a closed curve $\gamma \in \mathcal{L}$ the *global radius of curvature* $\rho_G[\gamma](s)$ *at the point* $s \in S_L$ is defined as

$$\rho_G[\gamma](s) := \begin{cases} \inf_{\substack{\sigma,\tau \in S_L \setminus \{s\} \\ \sigma \neq \tau}} R(\Gamma(s),\Gamma(\sigma),\Gamma(\tau)), & \text{if } L > 0, \\ 0, & \text{if } L = 0, \end{cases} \tag{1}$$

where $R(x,y,z) \geq 0$ is the radius of the smallest circle containing the points $x,\ y,\ z \in \mathbb{R}^3$. For collinear but distinct points x, y, z we set $R(x,y,z)$ to be infinite. When x, y and z are non-collinear (and thus distinct) there is a unique circle passing through them and

$$R(x,y,z) = \frac{|x-y|}{|2\sin[\sphericalangle(x-z,y-z)]|} = \frac{|x-y|}{2\left|\frac{x-z}{|x-z|} \wedge \frac{y-z}{|y-z|}\right|}. \tag{2}$$

If two points coincide, however, say $x = z$ or $y = z$, then there are many circles through the three points and we take $R(x,y,z)$ to be the smallest possible radius namely the distance $|x-y|/2$. We should point out that with this choice the function $R(x,y,z)$ fails to be continuous at points, where at least two of the arguments x, y, z, coincide. Notice nevertheless that, by definition, $R(x,y,z)$ is symmetric in its arguments. In Lemma 2.2 below we will see that $\rho_G[\gamma](s)$ is always finite for closed curves. In the case of smooth curves that have no or only transversal crossings, our definition of ρ_G agrees with that in [11], but it is different for curves with double covered regions. In particular, at parameters s, where Γ is not injective, our definition leads to $\rho_G[\gamma](s) = 0$, whereas this is not necessarily the case according to the definition in [11], see also the discussion in [12].

The *global radius of curvature of* γ is defined as

$$\mathcal{R}[\gamma] := \inf_{s \in S_L} \rho_G[\gamma](s). \tag{3}$$

The *global curvature of* γ *at* $s \in S_L$ is given by

$$\kappa_G[\gamma](s) := \sup_{\substack{\sigma,\tau \in S_L \setminus \{s\} \\ \sigma \neq \tau}} \frac{1}{R(\Gamma(s),\Gamma(\sigma),\Gamma(\tau))}, \tag{4}$$

which, of course, can take the value $+\infty$. In analogy to $\mathcal{R}[\gamma]$ we define the *global curvature of* γ by

$$\mathcal{K}[\gamma] := \sup_{s\in S_L} \kappa_G[\gamma](s). \tag{5}$$

Notice that for the global radius of curvature $\mathcal{R}[\gamma]$ and for the global curvature $\mathcal{K}[\gamma]$ we can also write

$$\mathcal{R}[\gamma] = \inf_{\substack{s,\sigma,\tau\in S_L\\ s\neq\sigma\neq\tau\neq s}} R(\Gamma(s),\Gamma(\sigma),\Gamma(\tau)), \tag{6}$$

$$\mathcal{K}[\gamma] = \sup_{\substack{s,\sigma,\tau\in S_L\\ s\neq\sigma\neq\tau\neq s}} \frac{1}{R(\Gamma(s),\Gamma(\sigma),\Gamma(\tau))}. \tag{7}$$

We will see in Lemma 2.2 that $\mathcal{R}[\gamma]\in[0,\infty)$ and hence (by the identity (9) below) $\mathcal{K}[\gamma]\in(0,\infty]$ for $\gamma\in\mathcal{L}$. If in addition, $\gamma\in\mathcal{L}$ has an injective and smooth[1] arc length parameterization, we infer from Theorem 2.1, part (iii), below that both $\mathcal{R}[\gamma]$ and $\mathcal{K}[\gamma]$ have values in $(0,\infty)$.

As an immediate consequence of the definitions of κ_G and ρ_G we observe for $\gamma\in\mathcal{L}$, that

$$\kappa_G[\gamma](s) = \frac{1}{\rho_G[\gamma](s)}, \tag{8}$$

$$\mathcal{K}[\gamma] = \frac{1}{\mathcal{R}[\gamma]}, \tag{9}$$

where we tacitly understand that the terms on the left-hand sides become infinite if the denominators on the right vanish.

The curve γ is said to be *simple* if its arc length parameterization $\Gamma: S_L\to\mathbb{R}^3$ is injective. Otherwise there exist $s,t\in S_L$ $(s\neq t)$ for which $\Gamma(s)=\Gamma(t)$. Any such pair will be called a *double point* of Γ.

In part (iii) of the following central regularity result Theorem 2.1 it is shown that the condition of finite global curvature $\mathcal{K}[\gamma]$ identifies curves with a $C^{1,1}$-arc length parameterization without double points. On the other hand, however, there are simple curves $\gamma\in\mathcal{L}$ with infinite global curvature $\mathcal{K}[\gamma]$, e.g., curves with corner points.

[1] It suffices that the corresponding arc length parameterization Γ is of class $C^{1,1}$.

Theorem 2.1. *Let $\gamma \in \mathcal{L}$ with arc length parameterization $\Gamma : S_L \to \mathbb{R}^3$.*

(i) *If Γ has a double point at the pair $s, t \in S_L$ ($s \neq t$), then $\rho_G[\gamma](s) = \rho_G[\gamma](t) = 0$.*

(ii) *At every point $s \in S_L$ where $\rho_G[\gamma](s) > 0$, Γ possesses a geometric unit tangent $T(s)$ satisfying*

$$T(s) := \lim_{\substack{\sigma \downarrow s \\ \sigma \neq s}} \frac{\Gamma(\sigma) - \Gamma(s)}{|\Gamma(\sigma) - \Gamma(s)|} = -\lim_{\substack{\tau \uparrow s \\ \tau \neq s}} \frac{\Gamma(\tau) - \Gamma(s)}{|\Gamma(\tau) - \Gamma(s)|}. \tag{10}$$

In addition, at points $s_0 \in S_L$, where Γ is differentiable, one has

$$\Gamma'(s_0) = T(s_0). \tag{11}$$

(iii) *$\mathcal{K}[\gamma] < \infty$ if and only if γ is simple and $\Gamma \in C^{1,1}([0, L], \mathbb{R}^3) \simeq W^{2,\infty}([0, L], \mathbb{R}^3)$.*

In particular, if $\mathcal{K}[\gamma]$ is finite, then

$$|\Gamma'(s_1) - \Gamma'(s_2)| \leq \mathcal{K}[\gamma]|s_1 - s_2| \quad \forall s_1, s_2 \in S_L, \tag{12}$$

i.e., Γ' has Lipschitz constant $\mathcal{K}[\gamma]$. Thus (11) is true for all $s_0 \in S_L$, if $\mathcal{K}[\gamma] < \infty$.

(iv) *For $\theta > 0$ let $D_\theta(z, z')$ denote the open planar disk in $\mathbb{R}^3$ of radius θ centred at $z \in \mathbb{R}^3$ perpendicular to $z' \in \mathbb{R}^3 \setminus \{0\}$. Let $s \in S_L$ be given with $\rho_G[\gamma](s) > 0$ and geometric tangent $T(s)$ as in (ii). Set*

$$C(s, \theta) = \partial D_\theta(\Gamma(s), T(s)) \quad \text{and} \quad M(s, \theta) = \bigcup_{z \in C(s,\theta)} B_\theta(z).$$

Then

(a) $\Gamma(S_L) \cap M(s, \rho_G[\gamma](s)) = \emptyset$.

(b) *Suppose $\mathcal{K}[\gamma] < \infty$, and let $\vartheta > 0$ be a given constant. Then the following holds:*

$$\Gamma(S_L) \cap M(s, \vartheta) = \emptyset \text{ for all } s \in S_L \iff \mathcal{K}[\gamma] \leq \vartheta^{-1}.$$

Remarks. 1. Let $\mathrm{Tan}\,(\gamma(\bar{I}),\Gamma(s))$ denote the *tangent cone* of $\gamma(\bar{I})$ at the point $\Gamma(s)$ in the sense of Federer [7, Sec. 3.1.21]. Then (ii) implies that

$$\mathrm{Tan}\,(\gamma(\bar{I}),\Gamma(s)) = \{\lambda T(s) : \lambda \in \mathbb{R}\}$$

for every $s \in S_L$ with $\rho_G[\gamma](s) > 0$. In other words, the tangent cone reduces to a 1-dimensional linear subspace at points with positive global radius of curvature.

2. The necessary condition for $\mathcal{K}[\gamma]$ to be finite in part (iii), which was shown in [12, Lemma 2], implies that the second derivative $\Gamma''(s)$ of Γ exists for a.e. $s \in S_L$, since $C^{1,1}([0,L],\mathbb{R}^3)$ is isomorphic to the Sobolev space $W^{2,\infty}([0,L],\mathbb{R}^3)$. Thus (12) actually implies

$$\|\Gamma''\|_{L^\infty} \le \mathcal{K}[\gamma] = \frac{1}{\mathcal{R}[\gamma]}\,. \tag{13}$$

3. Item (iv)(a) of the above result implies that an open ball of radius $\rho_G[\gamma](s)$ placed tangent at the point $\Gamma(s)$ may be rotated around the tangent vector $\Gamma'(s)$ without intersecting the curve. On the other hand, if $\mathcal{K}[\gamma] > \vartheta^{-1}$, then there is a point on the curve about which a similar rotation of a ball of radius ϑ could not be effected without intersecting the curve, according to part (iv)(b). Thus $\mathcal{R}[\gamma]$ is the radius of the largest open ball that can be rotated tangentially about every point of a curve γ without intersecting it.

The technical and lengthy proof of Theorem 2.1 as well as all the other proofs are deferred to Section 5. In the following lemma we provide bounds on $\mathcal{R}[\gamma]$ and $\mathcal{K}[\gamma]$ in terms of simple geometric quantities of the curve γ.

Lemma 2.2. *Let $\gamma \in \mathcal{L}$ with length $L(\gamma) = L$. Then*

$$\mathcal{R}[\gamma] \le \rho_G[\gamma](s) \le \frac{\mathrm{diam}\,(\gamma(\bar{I}))}{2} \le \frac{L}{4} \quad \textit{for all} \quad s \in S_L, \tag{14}$$

$$\mathcal{K}[\gamma] \ge \kappa_G[\gamma](s) \ge \frac{2}{\mathrm{diam}\,(\gamma(\bar{I}))} \ge \frac{4}{L} \quad \textit{for all} \quad s \in S_L. \tag{15}$$

The next theorem clarifies if and how the infima and suprema in (1),(3),(4) and (5) are realized on a closed curve. Later this will lead to the definition of generalized local curvature for rectifiable loops that are not necessarily differentiable, and also to an alternative characterization of global curvature in terms of the radius of a circle uniquely determined by two points and one tangent.

Theorem 2.3. (i) *For every* $\gamma \in \mathcal{L}$ *with arc length parameterization* $\Gamma \in C^{1,1}([0,L],\mathbb{R}^3)$, *there exists at least one parameter* $s \in S_L$, *such that either*

$$\mathcal{R}[\gamma] = \rho_G[\gamma](s) \text{ and } \mathcal{K}[\gamma] = \kappa_G[\gamma](s), \tag{16}$$

or there is a sequence $(s_j,\sigma_j,\tau_j) \to (s,s,s)$ *with* $s_j \neq \sigma_j \neq \tau_j \neq s_j$ *for all* $j \in \mathbb{N}$, *such that*

$$\mathcal{R}[\gamma] = \lim_{j\to\infty} R(\Gamma(s_j),\Gamma(\sigma_j),\Gamma(\tau_j)) \quad \text{and} \tag{17}$$

$$\mathcal{K}[\gamma] = \lim_{j\to\infty} \frac{1}{R(\Gamma(s_j),\Gamma(\sigma_j),\Gamma(\tau_j))}. \tag{18}$$

Moreover, if in the second alternative (17), (18), the parameter $s \in S_L$ *is a Lebesgue point of* Γ'', *and if there is a constant* $c > 0$, *such that*

$$\max\{|\tau_j - s|, |\sigma_j - s|, |s_j - s|\} \leq c \max\{|\tau_j - s_j|, |\tau_j - \sigma_j|, |s_j - \sigma_j|\} \tag{19}$$

for all $j \in \mathbb{N}$, *then*

$$\mathcal{R}[\gamma] = \rho_G[\gamma](s) = |\Gamma''(s)|^{-1} \text{ and} \tag{20}$$

$$\mathcal{K}[\gamma] = \kappa_G[\gamma](s) = |\Gamma''(s)|, \tag{21}$$

where Γ'' *is the precise representative in the sense of [6, p.46].*

(ii) *Let* $\gamma \in \mathcal{L}$ *with* $\mathcal{R}[\gamma] > 0$, *and let* $s \in S_L$. *Then there exists a sequence* $(\sigma_j,\tau_j) \to (\sigma,\tau)$ *in* $S_L \times S_L$ *with* $s \neq \sigma_j \neq \tau_j \neq s$ *for all* $j \in \mathbb{N}$ *satisfying*

$$\rho_G[\gamma](s) = \lim_{j\to\infty} R(\Gamma(s),\Gamma(\sigma_j),\Gamma(\tau_j)), \tag{22}$$

$$\kappa_G[\gamma](s) = \lim_{j\to\infty} \frac{1}{R(\Gamma(s),\Gamma(\sigma_j),\Gamma(\tau_j))}, \tag{23}$$

such that either

(a) $s = \sigma = \tau$, *or*

(b) $s \neq \sigma = \tau$.

Remarks. 1. The different options in Theorem 2.3 can occur simultaneously, as can, e.g., be seen for the planar circle. Here, at every point the local curvature and the global curvature coincide, i.e., the different limit cases in

(i) and (ii) hold simultaneously at every point due to the high degree of symmetry of the circle. Another interesting example is that of a *stadium curve*, which consists of two parallel straight line segments of equal length and distance d connected by two planar half circles of radius $d/2$. These curves have constant local curvature $\kappa = 0$ along the line segments and $\kappa = 2/d$ along the half circles, i.e., curvature jumps. The global curvature, however, is equal to $2/d$ and (16) holds everywhere along the curve. Stadium curves appear also in the examples of ideal $C^{1,1}$-links considered in [3] and [4].

2. If $s = s_j$ for infinitely many $j \in \mathbb{N}$ in (17),(18), then (16) holds true, too. Notice also that (19) holds (and therefore (20) and (21) as well) in the particular case, when $s = s_j$ for all $j \in \mathbb{N}$, where $s \in S_L$ is a Lebesgue point of Γ''.

3. Statement (ii) says, roughly speaking, that the infimum in the definition of $\rho_G[\gamma](s)$ as well as the supremum in $\kappa_G[\gamma](s)$ cannot exclusively be achieved by two distinct parameters $\sigma, \tau \in S_L$. Consequently, the infimum in (6) and the supremum in (7) cannot exclusively be achieved by three distinct parameters $s, \sigma, \tau \in S_L$. Moreover, due to the symmetry of $R(.,.,.)$ a third feasible option (c), namely $s = \sigma \neq \tau$, does not occur exclusively, that is, without option (b) at the same time. Observe, however, that R might not be continuous at the limit point (s, σ, τ).

4. Notice that the assumption $\mathcal{R}[\gamma] > 0$ is equivalent to demanding that γ possesses an injective arc length parameterization of class $C^{1,1}$ according to Theorem 2.1, (iii).

In case (ii)(a) of Theorem 2.3 the value $\rho_G[\gamma](s)$ expresses a local property of the curve at $s \in S_L$, which coincides with the classical local radius of curvature of $\gamma \in \mathcal{L}$ at s if γ is *smooth*.[2] For curves $\gamma \in \mathcal{L}$ which are only continuous and rectifiable in general, this observation motivates the following definition of the *generalized local radius of curvature* $\rho[\gamma](s)$ *of* $\gamma \in \mathcal{L}$ *at* $s \in S_L$ as

$$\rho[\gamma](s) := \liminf_{\substack{(\tau_j,\sigma_j)\to(s,s)\\ s\neq\tau_j\neq\sigma_j\neq s}} R(\Gamma(s), \Gamma(\tau_j), \Gamma(\sigma_j)). \tag{24}$$

Analogously, we define the *generalized local curvature of* $\gamma \in \mathcal{L}$ *at* $s \in S_L$ as

$$\kappa[\gamma](s) := \limsup_{\substack{(\tau_j,\sigma_j)\to(s,s)\\ s\neq\tau_j\neq\sigma_j\neq s}} \frac{1}{R(\Gamma(s), \Gamma(\tau_j), \Gamma(\sigma_j))}. \tag{25}$$

[2]This can easily be verified by expanding Γ about s when calculating the term $R(\Gamma(s), \Gamma(\sigma_j), \Gamma(\tau_j))$ in (22) in case (a) for j large, see also Lemma 5.2 in Section 5.

Note that for $\gamma \in \mathcal{L}$, both $\rho[\gamma]$ and $\kappa[\gamma]$ can take values in $[0, \infty]$, and that

$$\rho[\gamma](s) \geq \rho_G[\gamma](s) \text{ and } \kappa[\gamma](s) \leq \kappa_G[\gamma](s) \text{ for all } s \in S_L. \tag{26}$$

Proposition 2.4. *Let $\Gamma : S_L \to \mathbb{R}^3$ be the arc length parameterization of a simple curve $\gamma \in \mathcal{L}$ with $\Gamma \in C^{1,1}([0, L], \mathbb{R}^3)$. Then*

(i) $\kappa[\gamma](s) \leq \text{ap} \limsup_{\sigma \to s} |\Gamma''(\sigma)|$ *for all* $s \in S_L$.

(ii) $\kappa[\gamma](s) = |\Gamma''(s)| = \rho[\gamma](s)^{-1}$ *for a.e.* $s \in S_L$.

Here, ap lim sup denotes the approximate limes superior as defined, e.g. in [6, p. 47]. Suppose we have the local bound $|\Gamma''(\sigma)| \leq \kappa_0$ for a.e. σ in some open subinterval $J \subset S_L$, then, according to (i), $\kappa[\gamma](\sigma) \leq \kappa_0$ for all $\sigma \in J$. The essence of part (ii) is that for curves $\gamma \in \mathcal{L}$ with finite global curvature (hence γ simple and $\Gamma \in W^{2,\infty}$ by Theorem 2.1 (iii)), we can identify $\kappa[\gamma]$ with $|\Gamma''|$ a.e. on $[0, L]$, and from (13) we infer

$$\|\kappa[\gamma]\|_{L^\infty} = \|\Gamma''\|_{L^\infty} \leq \mathcal{K}[\gamma] \text{ for all } \gamma \in \mathcal{L} \text{ with } \mathcal{K}[\gamma] < \infty. \tag{27}$$

In the light of this inequality, we say for curves γ with $\mathcal{K}[\gamma] < \infty$, that *the global curvature $\mathcal{K}[\gamma]$ is locally not attained* if and only if

$$\|\Gamma''\|_{L^\infty} < \mathcal{K}[\gamma]. \tag{28}$$

Curves with this property are considered in Proposition 2.6 and Lemma 2.7 below and they play an essential role in [16], where the Euler-Lagrange equations for energy minimizing rods are derived.

For $\gamma \in \mathcal{L}$ with finite global curvature the alternative (b) in part (ii) of Theorem 2.3 expresses a nonlocal property of the curve. It motivates a different characterization of $\mathcal{K}[\gamma]$ which is analytically more tractable. Let $x, y, z \in \mathbb{R}^3$ be such that the vectors $x - y$ and z are linearly independent. By P we denote the plane spanned by $x - y$ and z. Then there is a unique circle contained in P through x and y, and tangent to z in the point y. We denote the radius of that circle by $r(x, y, z)$ and set $r(x, y, z) := \infty$, if $x - y$ and z are collinear. Using elementary geometric arguments r can be computed as

$$r(x, y, z) = \frac{|x - y|}{2\left|\frac{x-y}{|x-y|} \wedge \frac{z}{|z|}\right|}, \tag{29}$$

which shows that $r(x,y,z)$ is continuous on the set of triples (x,y,z) with the property, that $x-y$ and z are linearly independent. But it fails to be continuous at points, where, e.g., x and y coincide. Recall that for curves γ with $\mathcal{K}[\gamma]<\infty$, part (iii) of Theorem 2.1 says, that the corresponding arc length parameterization Γ possesses a Lipschitz continuous unit tangent field Γ' on $[0,L]$. Hence, for every pair $(s,\sigma)\in S_L\times S_L$, the radius $r(\Gamma(s),\Gamma(\sigma),\Gamma'(\sigma))$ is well defined, and we obtain the following identities for $\rho_G,\kappa_G,\mathcal{R}$ and $\mathcal{K}$:

Lemma 2.5. *Let $\gamma\in\mathcal{L}$ be such that $\mathcal{K}[\gamma]<\infty$, then at least one of the following statements (A),(B) is true:*

(A)

$$\begin{aligned}\rho_G[\gamma](s) &= \inf_{\substack{\sigma\in S_L\\ \sigma\neq s}} r(\Gamma(s),\Gamma(\sigma),\Gamma'(\sigma)) \quad \textit{and}\\ \kappa_G[\gamma](s) &= \sup_{\substack{\sigma\in S_L\\ \sigma\neq s}} \frac{1}{r(\Gamma(s),\Gamma(\sigma),\Gamma'(\sigma))},\end{aligned}$$

(B)

$$\begin{aligned}\rho_G[\gamma](s) &= \inf_{\substack{\sigma\in S_L\\ \sigma\neq s}} r(\Gamma(\sigma),\Gamma(s),\Gamma'(s)) = \rho[\gamma](s) \quad \textit{and}\\ \kappa_G[\gamma](s) &= \sup_{\substack{\sigma\in S_L\\ \sigma\neq s}} \frac{1}{r(\Gamma(\sigma),\Gamma(s),\Gamma'(s))} = \kappa[\gamma](s).\end{aligned}$$

If for $s\in S_L$ part (ii)(b) of Theorem 2.3 holds, then alternative (A) above is true.

In addition,

$$\mathcal{R}[\gamma] = \inf_{\substack{s,\sigma\in S_L\\ s\neq\sigma}} r(\Gamma(s),\Gamma(\sigma),\Gamma'(\sigma))\,, \tag{30}$$

$$\mathcal{K}[\gamma] = \sup_{\substack{s,\sigma\in S_L\\ s\neq\sigma}} \frac{1}{r(\Gamma(s),\Gamma(\sigma),\Gamma'(\sigma))}\,. \tag{31}$$

Because of the representation of $\mathcal{K}[\gamma]$ as a supremum in (31) the following set $A[\gamma]$, where the global curvature $\mathcal{K}[\gamma]$ is attained, is of particular interest.

$$A[\gamma] := \{(s,\sigma)\in[0,L]\times[0,L] : \mathcal{K}[\gamma] = \frac{1}{r(\Gamma(s),\Gamma(\sigma),\Gamma'(\sigma))}\}, \tag{32}$$

and one can show

Proposition 2.6. *Let $\gamma \in \mathcal{L}$ be such that $\mathcal{K}[\gamma] < \infty$. If $(s, \sigma) \in A[\gamma]$, then $s \neq \sigma$. If, in addition, $\mathcal{K}[\gamma]$ is locally not attained, then*

$$|\Gamma(s) - \Gamma(\sigma)| = 2\mathcal{R}[\gamma], \quad \textit{and} \tag{33}$$

$$\Gamma'(s) \cdot (\Gamma(s) - \Gamma(\sigma)) = \Gamma'(\sigma) \cdot (\Gamma(s) - \Gamma(\sigma)) = 0 \tag{34}$$

for all $(s, \sigma) \in A[\gamma]$.

Note that also $\Gamma(s) \neq \Gamma(\sigma)$ for $(s, \sigma) \in A[\gamma]$, if $\mathcal{K}[\gamma]$ is finite. The set $A[\gamma]$ can be empty, e.g., in the case when γ parameterizes a regular ellipse, where the local curvature $\kappa[\gamma]$ is maximal and equal to $\mathcal{K}[\gamma]$ at exactly the two vertices. In other words, for an ellipse, $\mathcal{K}[\gamma]$ is attained *exclusively* locally. On the other hand, if γ describes a circle, one has $A[\gamma] = [0, L] \times [0, L] \backslash$ diagonal.

For curves $\gamma \in \mathcal{L}$ such that $\mathcal{K}[\gamma]$ is locally not attained in the sense defined in (28), we have the following characterization of $\mathcal{K}[\gamma]$ as a maximum over pairs of parameters in a well defined compact subset of $[0, L] \times [0, L]$ away from the diagonal.

Lemma 2.7. *Let $\gamma \in \mathcal{L}$ with $\mathcal{K}[\gamma] < \infty$ such that $\mathcal{K}[\gamma]$ is locally not attained, and set*

$$\eta(\gamma) := \frac{1 - \mathcal{R}[\gamma] \cdot \|\Gamma''\|_{L^\infty}}{\|\Gamma''\|_{L^\infty}}, \tag{35}$$

$$\mathcal{Q} = \mathcal{Q}[\gamma] := \{(s, \sigma) \in [0, L] \times [0, L] : L - \eta(\gamma) \geq s - \sigma \geq \eta(\gamma)\}. \tag{36}$$

Then

(i) $0 < \eta(\gamma) < L/(2\pi)$,

(ii) $A[\gamma] \cap \mathcal{Q} \neq \emptyset$, *i.e.,*

$$\mathcal{K}[\gamma] = \max_{(s,\sigma) \in \mathcal{Q}} \frac{1}{r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma))},$$

(iii)

$$\mathcal{K}[\gamma] > \frac{1}{r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma))} \quad \textit{for all} \;\; (s, \sigma) \in [0, L]^2 \backslash \mathcal{Q}.$$

Note that the upper bound on $\eta(\gamma)$ in (i) can be improved to $L/(4\pi)$, if γ is non-trivially knotted by virtue of the Fáry-Milnor Theorem on the total curvature of knotted curves [13].

The significance of Lemma 2.7 is that $r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma))$ is continuously differentiable on a small neighbourhood of $\mathcal{Q}$, which is a crucial ingredient for the analysis in applications involving a bound on $\mathcal{K}$ as a constraint for variational problems, cf. [16].

3 Excluded volume constraint

Theorem 2.1, part (iv), establishes the fact that if a curve $\gamma \in \mathcal{L}$ has global curvature bounded by some constant θ^{-1}, then γ is restricted on how tightly it can bend locally, by the inequality $\kappa[\gamma](s) \leq \mathcal{K}[\gamma] \leq \theta^{-1}$ for a.e. $s \in I$, and on how close it can come to self-intersection globally. The interplay of local and global effects is strongly connected to the *excluded volume constraint*, which says that some tube surrounding the curve γ as a centreline, does not intersect itself.

To make this physically intuitive condition mathematically precise, we recall that for curves $\gamma \in \mathcal{L}$ with $\mathcal{K}[\gamma] < \infty$ we have a well-defined continuous tangent field Γ'. Thus we can speak of planar disk-shaped cross-sections of uniform size perpendicular to Γ' along the curve. The excluded volume constraint then consists of the condition that two different cross-sections do not intersect, which locally translates into a bound on the (local) curvature, and globally into a restriction on how close different curve points can approach each other in space.

One mathematical notion to describe such a constraint would be the *normal injectivity radius* from differential geometry, which, however, requires a certain amount of smoothness of the centreline γ, compare, e.g. [5]. In [11] it is shown that for smooth simple curves γ the normal injectivity radius coincides with $\mathcal{R}[\gamma]$. Proposition 3.1 below shows that the condition $\mathcal{K}[\gamma] \leq \theta^{-1}$ models in a geometrically exact way the excluded volume constraint for the tubular neighbourhood of $\gamma \in \mathcal{L}$ of uniform radius $\theta > 0$. An alternative characterization of the excluded volume constraint involving global injectivity of a deformation mapping will be discussed afterwards.

The set $B_\delta(\gamma(\bar{I})) = B_\delta(\Gamma(S_L))$ is said to be the *tubular neighbourhood of* γ of radius $\delta > 0$, where $\gamma \in \mathcal{L}$ has arc length parameterization $\Gamma : S_L \to \mathbb{R}^3$. We say that $B_\delta(\Gamma(S_L))$ is *non-self-intersecting* or *regular* if the closest-point projection map $\Pi_\Gamma : B_\delta(\Gamma(S_L)) \to \Gamma(S_L)$ is single-valued and continuous. That is to say, for any $x \in B_\delta(\Gamma(S_L))$ there is exactly one $s(x) \in S_L$ such that $\Pi_\Gamma(x) := \Gamma(s(x))$ satisfies

$$\mathrm{dist}(x, \Gamma(S_L)) = |\Gamma(s(x)) - x|,$$

and $\Pi_\Gamma(x)$ is a continuous function of $x \in B_\delta(\Gamma(S_L))$.

Proposition 3.1. *Consider $\gamma \in \mathcal{L}$ with $\mathcal{K}[\gamma] < \infty$ and let $\Gamma \in C^{1,1}(S_L, \mathbb{R}^3)$ denote its corresponding arc length parameterization. Let $\theta > 0$ be a given constant and $D_\theta(z, z')$ be the disk of radius θ centred at z, perpendicular to $z' \neq 0$. Then*

(i) *$B_\theta(\Gamma(S_L))$ is regular if and only if $\mathcal{K}[\gamma] \leq \theta^{-1}$,*

(ii) *Π_Γ has the property $\Pi_\Gamma^{-1}(\Gamma(s_0)) \cap B_\theta(\Gamma(S_L)) = D_\theta(\Gamma(s_0), \Gamma'(s_0))$ for every $s_0 \in S_L$, if $B_\theta(\Gamma(S_L))$ is regular.*

Items (i) and (ii) imply that the regularity of the tubular neighbourhood $B_\theta(\Gamma(S_L))$ is equivalent to the condition $\mathcal{R}[\gamma] \geq \theta$, and that $B_\theta(\Gamma(S_L))$ is the envelope of disjoint disks $D_\theta(\Gamma(s_0), \Gamma'(s_0))$. Since each point $x \in B_\theta(\Gamma(S_L))$ is in a unique disk $D_\theta(\Gamma(s_0), \Gamma'(s_0))$ normal to the curve, we deduce that $B_\theta(\Gamma(S_L))$ has the structure of a uniform tube of radius θ centred on γ. Moreover, according to item (i), any tubular neighbourhood of radius larger than $\mathcal{R}[\gamma]$ would fail to have this structure, which strongly justifies the formulation of the excluded volume constraint as a prescribed upper bound on the global curvature $\mathcal{K}[\gamma]$. In addition, continuity and compactness properties of $\mathcal{K}[.]$ on certain subsets of $\mathcal{L}$, which are treated in Section 4, turn out to be valuable in proving various existence results involving elastic rods and ideal knots, see [12].

The excluded volume constraint for $\gamma \in \mathcal{L}$ with arc length parameterization $\Gamma : S_L \to \mathbb{R}^3$, can be rephrased in terms of a deformation mapping $p : \Omega \to \mathbb{R}^3$ defined as

$$p(s, \xi_1, \xi_2) = \Gamma(s) + \xi_1 d_1(s) + \xi_2 d_2(s) \text{ for } (s, \xi_1, \xi_2) \in \Omega, \tag{37}$$

where Ω is the open[3] parameter set given by

$$\Omega := \{ (s, \xi_1, \xi_2) \in \mathbb{R}^3 \mid s \in S_L, \quad \xi_1^2 + \xi_2^2 < \theta^2 \}.$$

[3]Notice that in [16] we consider the closed parameter sets in accordance with the majority of works on elastic rods, see, e.g., [1].

Here, $\Gamma : S_L \to \mathbb{R}^3$ describes the closed centreline of the deformed ring-shaped body $p(\Omega)$, which might be considered as *elastic rod*, see [1]. $d_1(s), d_2(s)$ are orthogonal unit vectors describing the orientation of the deformed cross-section at $s \in S_L$. We interpret s as length parameter and ξ_1, ξ_2 as thickness parameters of the rod. With

$$d_3 := d_1 \wedge d_2$$

we get a right-handed orthonormal basis $\{d_1, d_2, d_3\}$ at each $s \in S_L$, whose vectors are called *directors*, and which can be identified with an orthogonal matrix $D = (d_1|d_2|d_3) \in SO(3)$ (the right-hand side denotes the matrix with columns d_1, d_2, d_3). We assume that $D : S_L \to SO(3)$ is continuous and that

$$\Gamma'(s) = d_3(s) \text{ for all } s \in S_L, \tag{38}$$

which models unshearability for the rod, i.e., cross sections remain orthogonal to the centreline. The pair (Γ, D), which we call *framed curve*, uniquely determines a deformed configuration of the rod.

A *closed framed curve* is given by the conditions

$$\Gamma(0) = \Gamma(L), \quad d_3(0) = d_3(L), \tag{39}$$

which means that the centreline Γ is a closed curve and that the cross-sections of the rod at $s = 0$ and $s = L$ coincide.

We have seen above that the constraint $\mathcal{K}[\gamma] \leq \theta^{-1}$ prevents the tube $B_\theta(\gamma(\bar{I}))$ from self-intersecting. However, as a model for a physical object, it is the points of $p(\Omega)$ that are naturally identified with material points, and the excluded volume constraint should guarantee the global injectivity of the mapping $p : \Omega \to \mathbb{R}^3$. The following theorem shows that the condition $\mathcal{K}[\gamma] \leq \theta^{-1}$ is in fact equivalent to global injectivity of p.

Theorem 3.2. *Consider a closed framed curve (Γ, D), where Γ is the arc length parameterization of $\gamma \in \mathcal{L}$ and where $D \in C^0(S_L, SO(3))$. Suppose that $\mathcal{K}[\gamma] < \infty$ and that (38) holds. Then $p : \Omega \to \mathbb{R}^3$ is globally injective if and only if $\mathcal{K}[\gamma] \leq \theta^{-1}$.*

Condition (38) allows us to identify the deformed rod $p(\Omega)$ with $B_\theta(\Gamma(S_L))$ and the result follows from the regularity of $B_\theta(\Gamma(S_L))$ as discussed above. When Γ' is not parallel to d_3, however, the condition $\mathcal{K}[\gamma] \leq \theta^{-1}$ does not imply global injectivity of p and vice versa. Notice that $p(\Omega)$ itself is not a uniform tube of radius θ if $\Gamma' \cdot d_1$ or $\Gamma' \cdot d_2$ is non-zero. An alternative approach to obtain global injectivity also for this general case is given in [15].

4 Continuity properties and topological constraints

Here we examine the properties of global curvature considered as a functional on the space $\mathcal{L}$ of continuous rectifiable closed curves as well as its impact on the topological constraint of a given isotopy class on a subset of curves in $\mathcal{L}$ and of a given link class on closed framed curves.

Since one is interested in compactness properties for sequences of loops in $\mathcal{L}$, compare [12], we are going to analyze the behaviour of the function $\mathcal{K}[.]$ on sequences in $\mathcal{L}$ that converge uniformly. Our first result in that direction states that the set of curves $\gamma \in \mathcal{L}$ with bounded length $L(\gamma) \leq L_0 < \infty$, satisfying $\mathcal{K}[\gamma] \leq \theta^{-1} < \infty$ is closed with respect to uniform convergence.

Lemma 4.1. *Let $L_0, \theta > 0$ be fixed. Then the set*

$$F_1 := \{\gamma \in \mathcal{L} : L(\gamma) \leq L_0 \ \text{ and } \ \mathcal{K}[\gamma] \leq \theta^{-1}\}$$

is closed in $C^0(\bar{I}, \mathbb{R}^3)$, i.e., with respect to uniform convergence.

In order to deal with minimization problems on isotopy (or knot) classes of closed curves, one has to investigate under which circumstances the isotopy relation is conserved. This is important for compactness arguments in the existence theory, see [12], but also for the variational aspects when deriving the Euler-Lagrange equations, see [16]. For that we recall the definition of isotopy:

Two continuous closed curves $K_1, K_2 \subset \mathbb{R}^3$ are *isotopic*, denoted as $K_1 \simeq K_2$, if there are open neighbourhoods N_1 of K_1, N_2 of K_2, and a continuous mapping $\Phi : N_1 \times [0, 1] \to \mathbb{R}^3$ such that $\Phi(N_1, \tau)$ is homeomorphic to N_1 for all $\tau \in [0, 1]$, $\Phi(x, 0) = x$ for all $x \in N_1$, $\Phi(N_1, 1) = N_2$, and $\Phi(K_1, 1) = K_2$.

Roughly speaking, two curves are in the same isotopy class if one can be continuously deformed onto the other.

Proposition 4.2. *Let $\gamma \in \mathcal{L}$ satisfy*

$$\mathcal{K}[\gamma] \leq \theta^{-1} \tag{40}$$

for some fixed constant $\theta > 0$. Then there exists $\epsilon = \epsilon(\gamma, \theta) > 0$, such that for all curves $\tilde{\gamma} \in \mathcal{L}$ with $\mathcal{K}[\tilde{\gamma}] \leq \theta^{-1}$ and

$$\|\gamma - \tilde{\gamma}\|_{C^0} \leq \epsilon, \tag{41}$$

one has $\gamma(\bar{I}) \simeq \tilde{\gamma}(\bar{I})$.

The statement of the lemma is no longer true if one removes the assumptions on the global curvature, small knotted regions might pull tight in the uniform topology. As an immediate consequence of Proposition 4.2 we get

Theorem 4.3. *Let $L_0, \theta > 0$ be fixed and $K \in \mathcal{L}$ be a given simple closed curve. Then the set*

$$F_2 := \{\gamma \in \mathcal{L} : L(\gamma) \le L_0, \gamma(\bar{I}) \simeq K, \mathcal{K}[\gamma] \le \theta^{-1}\}$$

is closed in $C^0(\bar{I}, \mathbb{R}^3)$, i.e., with respect to uniform convergence.

The next theorem provides an important continuity property of $\mathcal{K}[.]$ on some subset of $\mathcal{L}$ which is also an essential ingredient for our variational approach in [16].

Theorem 4.4. *Let $\mathcal{L}^* \subset \mathcal{L}$ be the set of curves $\gamma \in \mathcal{L}$ of fixed length $L(\gamma) = L > 0$, with the property that the corresponding arc length parameterization Γ is of class $C^{1,1}([0, L], \mathbb{R}^3)$. Then $\mathcal{K}[.]$ (and hence $\mathcal{R}[.]$) is continuous on $\mathcal{L}^*$ with respect to convergence of the corresponding arc length parameterizations in $C^{1,1}([0, L], \mathbb{R}^3)$.*

Cantarella et al. [4] have pointed out that $\mathcal{R}[.]$ is merely upper semicontinuous with respect to uniform convergence on the space of Lipschitz curves, and that $\mathcal{R}[.]$ can actually jump upwards in a limit even for C^1-convergence. For instance, consider a circle which is approximated in C^1 by a sequence of curves Γ_n such that $\|\Gamma_n''\|_{L^\infty} \to \infty$.

A closed framed curve satisfying (38) consists of a base curve Γ and an associated frame field D, both of which can be subjected to topological restrictions. We have discussed above how to prescribe knot classes for the base curve incorporating the concept of isotopy. Moreover, if we fix the rotation between the terminal frames $D(0)$ and $D(L)$ there are still infinitely many topologically distinct components in the space of closed framed curves within the same isotopy class for the base curve, which may be seen as follows.

If one prescribes the knot type for the centreline and if one glues together the terminal frames at $s = 0$ and $s = L$ such that $d_3(0) = d_3(L)$ and, for simplicity, $d_1(0) = d_1(L)$ hold, one has not entirely fixed the topological type of the framed curve. Indeed, every full rotation of the pair $d_1(L), d_2(L)$ within the cross section respects the boundary conditions, but changes the linking number between the centrecurve and the curve $\Gamma(.) + (\theta/2)d_1(.)$, which is a topological invariant. Since such a change of topological type is accompanied by an (often drastic) change of the equilibrium configuration for an elastic rod, we need to prescribe the linking number in order to identify particular solutions, see also the discussion in [2]. The approach in [12] using the concept of homotopies in $SO(3)$ distinguishes only two topologically different classes, since the fundamental group of $SO(3)$ is $\mathbb{Z}_2$.

One way to determine the link between two disjoint closed (but not necessarily simple) curves is to compute the *Gaussian linking number*, which is usually defined in terms of the topological degree, see, e.g., [17, p. 402]. For a pair of absolutely continuous disjoint curves, however, there is an analytically more convenient formula, which we adopt as definition for the linking number. For $\gamma_1, \gamma_2 \in \mathcal{L} \cap W^{1,1}(I, \mathbb{R}^3)$ with $\gamma_1(\bar{I}) \cap \gamma_2(\bar{I}) = \emptyset$, the *linking number* $l(\gamma_1, \gamma_2)$ is given by

$$l(\gamma_1, \gamma_2) := \frac{1}{4\pi} \int_I \int_I \frac{\gamma_1(s) - \gamma_2(t)}{|\gamma_1(s) - \gamma_2(t)|^3} \cdot [\gamma_1'(s) \wedge \gamma_2'(t)]\, ds\, dt. \tag{42}$$

One can show that $l(\gamma_1, \gamma_2)$ is integer-valued and stable with respect to smooth perturbations preserving the non-intersection property. Before we use the linking number to identify topological classes of closed framed curves we present some general properties.

Lemma 4.5. *(i) Let* $\gamma_{1,n}, \gamma_{2,n} \in \mathcal{L} \cap W^{1,q}(I, \mathbb{R}^3)$, $q > 1$, $n \in \mathbb{N}$, *with* $\gamma_{1,n} \rightharpoonup \gamma_1$, $\gamma_{2,n} \rightharpoonup \gamma_2$ *in* $W^{1,q}(I, \mathbb{R}^3)$, *and let*

$$\operatorname{dist}(\gamma_{1,n}(\bar{I}), \gamma_{2,n}(\bar{I})) \geq c \quad \text{for all} \quad n \in \mathbb{N}, \tag{43}$$

$$l(\gamma_{1,n}, \gamma_{2,n}) = l_0 \qquad \text{for all} \quad n \in \mathbb{N}, \tag{44}$$

where $c > 0$, $l_0 \in \mathbb{Z}$ *are given constants. Then* $l(\gamma_1, \gamma_2) = l_0$.

(ii) Let $\gamma_1, \gamma_2 \in \mathcal{L} \cap W^{1,q}(I, \mathbb{R}^3)$, $q > 1$, *such that*

$$\gamma_1(\bar{I}) \cap \gamma_2(\bar{I}) = \emptyset. \tag{45}$$

Then there is a constant $\epsilon = \epsilon(\gamma_1, \gamma_2) > 0$, *such that* $\beta_1(\bar{I}) \cap \beta_2(\bar{I}) = \emptyset$, *and* $l(\gamma_1, \gamma_2) = l(\beta_1, \beta_2)$ *for all* $\beta_1, \beta_2 \in \mathcal{L} \cap W^{1,q}(I, \mathbb{R}^3)$ *with* $\|\gamma_i - \beta_i\|_{W^{1,q}} \leq \epsilon$, $i = 1, 2$.

For a closed framed curve respecting (38) we want to consider the linking number of the curves $\Gamma(.)$ and $\Gamma(.) + (\theta/2)d_1(.)$. The problem here is that the second curve might not be closed and that the two curves might intersect each other. The first problem can be solved by closing the curve $\Gamma(.) + (\theta/2)d_1(.)$ up in a unique way, namely by

$$(46)\quad \beta^D(s) := \begin{cases} \Gamma(s) + \frac{\theta}{2}d_1(s) & \text{for } s \in [0, L], \\ \Gamma(L) + \frac{\theta}{2}[\cos(\phi_D(s-L))d_1(L) + \sin(\phi_D(s-L))d_2(L)] \\ \qquad\qquad\qquad\qquad \text{for } s \in [L, L+1], \end{cases}$$

where $\phi_D \in [0, 2\pi)$ is the angle between $d_1(0)$ and $d_1(L)$, such that $\phi_D - \pi$ has the same sign as $(d_1(0) \wedge d_1(L)) \cdot d_3(0)$. For technical reasons we identify Γ with its trivial extension onto $[0, L+1]$ according to

$$(47)\qquad \Gamma(s) := \Gamma(L) \;\text{ for }\; s \in [L, L+1].$$

Notice that $\Gamma, \beta^D \in W^{1,q}([0, L+1], \mathbb{R}^3)$, $1 \le q \le \infty$, if $\Gamma \in W^{1,q}([0, L], \mathbb{R}^3)$, and that Γ and β^D are closed. Demanding the global curvature bound $\mathcal{K}[\Gamma] \le \theta^{-1}$ we ensure that

$$(48)\qquad \Gamma([0, L+1]) \cap \beta^D([0, L+1]) = \emptyset$$

by Theorem 3.2. Thus the linking number of a closed framed curve (Γ, D) satisfying (38), $\mathcal{K}[\Gamma] \le \theta^{-1}$, $\Gamma \in W^{1,1}([0, L], \mathbb{R}^3)$ and $D \in W^{1,1}([0, L], \mathbb{R}^{3\times 3})$, is well-defined by

$$(49)\qquad l(\Gamma, D) := l(\Gamma, \beta^D).$$

Lemma 4.5 now readily implies the following properties of $l(\Gamma, D)$, which are needed to derive existence results along the lines of [12] and to verify the Euler-Lagrange equations as necessary minimality conditions as carried out in [16], see also [19].

Theorem 4.6. *(i) Let $\Gamma_n \rightharpoonup \Gamma$ in $W^{1,q}([0, L], \mathbb{R}^3)$, $q > 1$, $D_n \rightharpoonup D$ in $W^{1,p}([0, L], \mathbb{R}^{3\times 3})$, $p > 1$, for a sequence (Γ_n, D_n) of closed framed curves respecting (38), $\mathcal{K}[\Gamma_n] \le \theta^{-1}$, and*

$$(50)\qquad l(\Gamma_n, D_n) = l(\Gamma_1, D_1) \;\text{ for all }\; n \in \mathbb{N}.$$

Then $l(\Gamma, D)$ is well-defined and $l(\Gamma, D) = l(\Gamma_1, D_1)$.

(ii) Let $(\Gamma, D) \in W^{1,q}([0, L], \mathbb{R}^3) \times W^{1,p}([0, L], \mathbb{R}^{3\times 3})$, $p, q > 1$, be a closed framed curve satisfying (38). Then there is $\epsilon > 0$, such that

$$(51)\qquad l(\Gamma, D) = l(\tilde{\Gamma}, \tilde{D})$$

for all closed framed curves $(\tilde{\Gamma}, \tilde{D}) \in W^{1,q}([0,L],\mathbb{R}^3)\times W^{1,p}([0,L],\mathbb{R}^{3\times 3})$ *satisfying (38) and*

$$\|\Gamma - \tilde{\Gamma}\|_{W^{1,q}} \le \epsilon, \quad \|D - \tilde{D}\|_{W^{1,p}} \le \epsilon. \tag{52}$$

5 Proofs

We will first prove two technical lemmas, which will be quite useful in later proofs, too. The first one concerns the approximate limes superior and inferior of integrable bounded functions as defined, e.g., in [6, p.47].

Lemma 5.1. *Let* $f \in L^\infty(\Omega)$ *for some open set* $\Omega \subset \mathbb{R}^n$ *and let* $x \in \Omega$. *Suppose* $E_r \subset B_r(x)$ *such that there is a constant* c *with*

$$|B_r(x)| \le c|E_r| \text{ for all } r > 0. \tag{53}$$

Then

$$\begin{aligned} \operatorname{ap}\liminf_{z\to x} f(z) &\le \liminf_{r\to 0} \fint_{E_r} f(y)\,dy \\ &\le \limsup_{r\to 0} \fint_{E_r} f(y)\,dy \le \operatorname{ap}\limsup_{z\to x} f(z). \end{aligned} \tag{54}$$

If x *is a Lebesgue point of* f, *then (54) holds with equality everywhere, and the limes superior and limes inferior may be replaced by the limit.*

Proof. The approximate limes superior of f as $z \to x \in \Omega$ is defined as

$$a := \operatorname{ap}\limsup_{z\to x} f(z) := \inf\{s : \lim_{r\downarrow 0} \frac{|B_r(x)\cap\{f>s\}|}{|B_r(x)|} = 0\}.$$

We may assume that $0 \le a < \infty$ (otherwise consider $f + \|f\|_{L^\infty}$). Thus for any $\bar{a} > a$,

$$\begin{aligned} \fint_{E_r} f(y)dy &= \frac{1}{|E_r|}\left[\int_{E_r\cap\{f>\bar{a}\}} f(y)\,dy + \int_{E_r\cap\{f\le\bar{a}\}} f(y)\,dy\right] \\ &=: \mathrm{I} + \mathrm{II}. \end{aligned} \tag{55}$$

Then we estimate

$$
\begin{aligned}
\mathrm{I} &= \frac{1}{|E_r|}\int_{E_r\cap\{f>\bar a\}} f(y)\,dy \\
&\leq \|f\|_{L^\infty}\cdot\frac{|E_r\cap\{f>\bar a\}|}{|E_r|} \\
&\underset{(53)}{\leq} c\|f\|_{L^\infty}\cdot\frac{|B_r\cap\{f>\bar a\}|}{|B_r|} \\
&= o(1) \ \text{ as } \ r\to 0, \qquad (56)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\mathrm{II} &= \frac{1}{|E_r|}\int_{E_r\cap\{f\leq\bar a\}} f(y)\,dy \\
&\leq \bar a\cdot\frac{|E_r\cap\{f\leq\bar a\}|}{|E_r|}\leq \bar a. \qquad (57)
\end{aligned}
$$

Inserting (56) and (57) into (55) gives the right inequality in (54). One proceeds analogously for the left inequality. The last statement follows from the well-known fact that an integrable function is approximate continuous at all its Lebesgue points, in which case equality holds everywhere in (54), see [6, p.48]. □

The second lemma gives an explicit representation of the circumcircle radius R of three different points on a curve γ whose arc length parameterization Γ possesses weak second derivatives.

Lemma 5.2. *Let $\gamma\in\mathcal{L}$ be simple with arc length parameterization $\Gamma\in W^{2,1}([0,L],\mathbb{R}^3)$. For $s_1,s_2,s_3\in S_L$ with $s_1<s_2<s_3$, one has*

$$
R(\Gamma(s_1),\Gamma(s_2),\Gamma(s_3)) = \frac{\prod_{\substack{i,k=1\\ i<k}}^{3} C(s_i,s_k)}{2|A(s_1,s_2,s_3)+(s_2-s_1)B(s_1,s_2,s_3)|}, \qquad (58)
$$

where

$$C(s_i, s_k) := |\Gamma'(s_k) + \fint_{s_k}^{s_i} \int_{s_k}^{t} \Gamma''(\omega)\, d\omega\, dt|, \quad i \neq k, \tag{59}$$

$$A(s_1, s_2, s_3) := \Gamma'(s_2) \wedge \int_0^1 t \fint_{s_2 - t(s_2 - s_1)}^{s_2 + t(s_3 - s_2)} \Gamma''(\omega)\, d\omega\, dt, \tag{60}$$

$$\begin{aligned} B(s_1, s_2, s_3) := & \int_0^1 t \fint_{s_2}^{s_2 - t(s_2 - s_1)} \Gamma''(\omega)\, d\omega\, dt \\ & \wedge \int_0^1 t \fint_{s_2 - t(s_2 - s_1)}^{s_2 + t(s_3 - s_2)} \Gamma''(\omega)\, d\omega\, dt. \end{aligned} \tag{61}$$

If $s \in S_L$ is a Lebesgue point of Γ'', and if $s_j < \sigma_j < \tau_j$, with $(s_j, \sigma_j, \tau_j) \to (s, s, s)$ as $j \to \infty$, such that

$$r_j := \max\{|s - s_j|, |s - \tau_j|\} \leq c(\tau_j - s_j), \tag{62}$$

where $c > 0$ is a constant independent of j, then

$$\lim_{j \to \infty} R(\Gamma(s_j), \Gamma(\sigma_j), \Gamma(\tau_j)) = |\Gamma''(s)|^{-1}. \tag{63}$$

If $|\Gamma''(s)| = 0$, then (63) means that $R(\Gamma(s_j), \Gamma(\sigma_j), \Gamma(\tau_j)) \to \infty$ as $j \to \infty$.

Proof. Let us assume that $\Gamma(s_i), i = 1, 2, 3$, are not collinear. (They are distinct, since γ is simple and $s_i \neq s_k$ for $i \neq k$.) Without loss of generality we assume that $|s_1 - s_2| \leq |s_2 - s_3|$, (otherwise we choose a different substitution to derive (66) below). We consider the expansions

$$\begin{aligned} \Gamma(s_i) - \Gamma(s_k) &= \int_{s_k}^{s_i} (\Gamma'(\tau) - \Gamma'(s_k))\, d\tau + (s_i - s_k)\Gamma'(s_k) \\ &= (s_i - s_k) \left[\Gamma'(s_k) + \fint_{s_k}^{s_i} \int_{s_k}^{t} \Gamma''(\omega)\, d\omega\, dt \right], \end{aligned} \tag{64}$$

$$
\begin{aligned}
\Gamma(s_i)-\Gamma(s_k) &= \int_0^1 \Gamma'(s_k+t(s_i-s_k))\,dt(s_i-s_k)\\
&= (s_i-s_k)\int_0^1 \left[\Gamma'(s_k)+t(s_i-s_k)\int_0^1 \Gamma''(s_k+\sigma t(s_i-s_k))\,d\sigma\right]\,dt\\
&= \Gamma'(s_k)(s_i-s_k)+(s_i-s_k)^2\int_0^1 t\int_0^1 \Gamma''(s_k+\sigma t(s_i-s_k))\,d\sigma\,dt. \qquad (65)
\end{aligned}
$$

In particular, substituting $\rho(\sigma) := \sigma(s_1-s_2)/(s_3-s_2)$, we deduce

$$
\begin{aligned}
\Gamma(s_1)-\Gamma(s_2) &= \Gamma'(s_2)(s_1-s_2)\\
&\quad +(s_1-s_2)(s_3-s_2)\int_0^1 t\int_0^{\frac{s_1-s_2}{s_3-s_2}} \Gamma''(s_2+t\rho(s_3-s_2))\,d\rho\,dt. \qquad (66)
\end{aligned}
$$

Using (65) and (66) we compute

$$
\begin{aligned}
&|(\Gamma(s_1)-\Gamma(s_2))\wedge(\Gamma(s_3)-\Gamma(s_2))|\\
&= \Bigg|\Big[\Gamma'(s_2)(s_1-s_2)+(s_1-s_2)(s_3-s_2)\int_0^1 t\int_0^{\frac{s_1-s_2}{s_3-s_2}} \Gamma''(s_2+t\rho(s_3-s_2))\,d\rho\,dt\Big]\\
&\qquad \wedge\Big[\Gamma'(s_2)(s_3-s_2)+(s_3-s_2)^2\int_0^1 t\int_0^1 \Gamma''(s_2+t\rho(s_3-s_2))\,d\rho\,dt\Big]\Bigg|\\
&= \Bigg|\Gamma'(s_2)(s_3-s_2)^2(s_1-s_2)\wedge\int_0^1 t\int_{\frac{s_1-s_2}{s_3-s_2}}^1 \Gamma''(s_2+t\rho(s_3-s_2))\,d\rho\,dt\\
&\qquad +(s_1-s_2)(s_3-s_2)^3\int_0^1 t\int_0^{\frac{s_1-s_2}{s_3-s_2}} \Gamma''(s_2+t\rho(s_3-s_2))\,d\rho\,dt\\
&\qquad\quad \wedge\int_0^1 t\int_{\frac{s_1-s_2}{s_3-s_2}}^1 \Gamma''(s_2+t\sigma(s_3-s_2))\,d\sigma\,dt\Bigg|
\end{aligned}
$$

With a suitable substitution we get

$$
\begin{aligned}
&|(\Gamma(s_1)-\Gamma(s_2))\wedge(\Gamma(s_3)-\Gamma(s_2))| \\
&\quad = \Big|\Gamma'(s_2)(s_3-s_2)(s_1-s_2)\wedge\int_0^1\int_{s_2-t(s_2-s_1)}^{s_2+t(s_3-s_2)}\Gamma''(\omega)\,d\omega\,dt \\
&\qquad +(s_1-s_2)(s_3-s_2)\int_0^1\int_{s_2}^{s_2-t(s_2-s_1)}\Gamma''(\omega)\,d\omega\,dt \\
&\qquad\quad \wedge\int_0^1\int_{s_2-t(s_2-s_1)}^{s_2+t(s_3-s_2)}\Gamma''(\omega)\,d\omega\,dt\Big| \\
&\quad = |s_3-s_2||s_1-s_2||s_3-s_1|\Big|\Gamma'(s_2)\wedge\int_0^1 t \fint_{s_2-t(s_2-s_1)}^{s_2+t(s_3-s_2)}\Gamma''(\omega)\,d\omega\,dt \\
&\qquad +(s_2-s_1)\int_0^1 t\fint_{s_2}^{s_2-t(s_2-s_1)}\Gamma''(\omega)\,d\omega\,dt\wedge\int_0^1 t\fint_{s_2-t(s_2-s_1)}^{s_2+t(s_3-s_2)}\Gamma''(\omega)\,d\omega\,dt\Big|.
\end{aligned}
\tag{67}
$$

Inserting (67) and (64) in (2) one verifies (58).

If $\Gamma(s_i), i=1,2,3$, are collinear, then the right-hand side of (2) has vanishing denominator, which is then also true for the right-hand side in (58). That corresponds to an infinite value for $R(\Gamma(s_1),\Gamma(s_2),\Gamma(s_3))$ in its definition.

To prove (63) we apply (54) in Lemma 5.1 to the mean value expression in the term $A(s_j,\sigma_j,\tau_j)$ as defined in (60) of Lemma 5.2. We set $x=s$, $f:=(\Gamma^k)'', k=1,2,3$, replace E_r in Lemma 5.1 by the set

$$E_j^t := [\sigma_j - t(\sigma_j - s_j), \sigma_j + t(\tau_j-\sigma_j)]$$

and B_r by the $B_{r_j}(s)$. For given $\epsilon>0$ we choose $\delta := \epsilon\|\Gamma''\|_{L^\infty}^{-1}/2$. Notice that (62) implies $|B_{r_j}|\le c|E_j^t|/\delta$, i.e., condition (53) in Lemma 5.1 for each $t\ge\delta$. According to the last statement of Lemma 5.1 we get

$$\lim_{j\to\infty}\fint_{E_j^t}\Gamma''(\omega)\,d\omega = \Gamma''(s) \ \text{ for all } \ t\in[\delta,1].$$

Since $\Gamma'' \in L^\infty([0,L],\mathbb{R}^3)$ we may apply the Dominated Convergence Theorem to conclude

$$\lim_{j\to\infty}\int_\delta^1 t \fint_{E_j^t} \Gamma''(\omega)\,d\omega dt = \Gamma''(s)\left[\frac{1-\delta^2}{2}\right].$$

Hence there is j_0 such that for all $j \ge j_0$

$$\int_0^1 t \fint_{E_j^t} \Gamma''(\omega)\,d\omega dt = \Gamma''(s)\left[\frac{1-\delta^2}{2}\right] + \epsilon$$

by our choice of δ. Since $\delta \to 0$ as $\epsilon \to 0$ we have shown

$$\lim_{j\to\infty}\int_0^1 t \fint_{E_j^t} \Gamma''(\omega)\,d\omega dt = \Gamma''(s)/2,$$

hence

$$\lim_{j\to\infty} A(s_j,\sigma_j,\tau_j) = \Gamma'(s)\wedge\Gamma''(s)/2.$$

Since B is bounded by $\|\Gamma''\|^2_{L^\infty}/4$, and the numerator in (58) for $s_1 = s_j, s_2 = \sigma_j, s_3 = \tau_j$, converges to 1 as $j \to \infty$, the claim follows from the fact that $|\Gamma'(\sigma)| = 1$, i.e., $\Gamma'(\sigma) \perp \Gamma''(\sigma)$ for all $\sigma \in S_L$. $\square$

Proof of Theorem 2.1. Part (i) was proved in [12, Lemma 1], and can be seen as follows: If there are $s,t \in S_L, t \neq s$ with $\Gamma(s) = \Gamma(t)$, then, by definition of $\rho_G[\gamma]$ and $R(x,y,z)$, we have

$$\begin{aligned}\rho_G[\gamma](s) &= \inf\{\ R(\Gamma(s),\Gamma(\sigma),\Gamma(\tau)) \mid \sigma,\tau \in S_L\backslash\{s\},\ \sigma\neq\tau\ \}\\ &\le \inf\{\ R(\Gamma(s),\Gamma(t),\Gamma(\tau)) \mid \tau \in S_L\backslash\{s,t\}\ \}\\ &= \inf\{\ |\Gamma(s)-\Gamma(\tau)|/2 \mid \tau \in S_L\backslash\{s,t\}\ \}\\ &= 0,\end{aligned}$$

and similarly for $\rho_G[\gamma](t)$.

Part (ii) is shown in several steps. For simplicity we set $\rho := \rho[\gamma](s)$.

1. Consider a connected subarc $A_1 := \Gamma([s,\sigma_1])$ with fixed endpoints $P_0 := \Gamma(s)$ and $P_1 := \Gamma(\sigma_1)$, $s \neq \sigma_1$, and suppose that $\operatorname{diam} A_1 < 2\rho$ and $|P_1 - P_0| < \rho/2$, which is possible by choosing $|\sigma_1 - s|$ sufficiently small. Moreover, taking $|\sigma_1 - s|$ even smaller if necessary, we can assume that $\Gamma(\tau) \neq \Gamma(\sigma_1)$ for all $\tau \in (s,\sigma_1)$, otherwise we could find a sequence $\tau_i \to s$ with $\Gamma(\tau_i) = \Gamma(\sigma_1)$, which would imply that $\Gamma(s) = \Gamma(\sigma_1)$. But $\Gamma(s)$ is not a double point by virtue of part (i), because $\rho > 0$.

   Let l_1 be the lens-shaped intersection of all open balls of radius ρ containing P_0 and P_1 on their boundaries, i.e.,

   $$l_1 := \bigcap_{z\in C(P_0,P_1)} B_\rho(z), \tag{68}$$

   where $C(P_0,P_1) := \{z \in \mathbb{R}^3 \mid |z - P_0| = |z - P_1| = \rho\}$. We claim that

   $$A_1 \subset \overline{l_1}. \tag{69}$$

   To see this, suppose for contradiction that $A_1 \not\subset \overline{l_1}$ and consider the set

   $$\Xi := \bigcup_{z\in C(P_0,P_1)} B_\rho(z). \tag{70}$$

   Then, using the facts, that the open arc $\Gamma((s,\sigma_1))$ does not intersect $\Gamma(s)$ or $\Gamma(\sigma_1)$, that $\operatorname{diam} A_1 < 2\rho$ and that $|P_1 - P_0| < \rho/2$, we deduce that there must be a point $\bar{P} \in (A_1 \cap \Xi)\setminus\overline{l_1}$. (Notice that there is indeed such a point $\bar{P}$, otherwise we would have $\operatorname{diam} A_1 \geq 2\rho$, because any curve in $\mathbb{R}^3\setminus\Xi$ connecting P_0 and P_1 must have diameter at least as large as the great circle on $\partial B_\rho(z)$ connecting P_0 and P_1 outside of $\overline{l_1}$ for any of the balls $B_\rho(z)$ that generate Ξ. Moreover, since $|P_1 - P_0| < \rho/2$, the portion of such a great circle has diameter 2ρ.)

   Now we find that

   $$R(P_0,P_1,\bar{P}) = \frac{|P_1 - P_0|}{2\sin\bar{\alpha}} < \rho, \quad \text{where} \quad \bar{\alpha} := \sphericalangle(P_0 - \bar{P}, P_1 - \bar{P}). \tag{71}$$

   Since this contradicts the definition of $\rho_G[\gamma](s) = \rho$, we must have $A_1 \subset \overline{l_1}$ as claimed.

   The result in (71) may be seen by considering the intersection of Ξ with the plane containing the three non-collinear points P_0, P_1 and $\bar{P}$. This intersection may be described by two overlapping planar disks $D_\rho(z_1)$ and $D_\rho(z_2)$ of radius ρ, where $\partial D_\rho(z_1) \cap \partial D_\rho(z_2) = \{P_0, P_1\}$, and we may assume without loss of generality that $\bar{P} \in D_\rho(z_1) \backslash \overline{D_\rho(z_2)}$. ¿From elementary geometry we recall that, for any $\xi \in \partial D_\rho(z_1) \backslash \{P_0, P_1\}$, we have $\rho = |P_1 - P_0|/(2 \sin \beta)$ where $\beta := \sphericalangle(P_0 - \xi, P_1 - \xi)$. To establish (71), we first suppose that $\bar{\alpha} \in (0, \pi/2)$. In this case we may choose $\xi \in \partial D_\rho(z_1) \backslash \overline{D_\rho(z_2)}$ such that $\beta \in (0, \bar{\alpha})$, i.e., $\sin \beta < \sin \bar{\alpha}$, which implies (71). If we suppose that $\bar{\alpha} \in [\pi/2, \pi)$, then we may choose $\xi \in \partial D_\rho(z_1) \cap D_\rho(z_2)$ such that $\beta \in (\bar{\alpha}, \pi)$, i.e., $\sin \beta < \sin \bar{\alpha}$, which also implies (71).

2. Given $s, \sigma_1 \in S_L$ as above, we next consider a sequence $\sigma_n \downarrow s$ $(n \geq 1)$. We introduce $P_n := \Gamma(\sigma_n)$, $A_n := \Gamma([s, \sigma_n])$ and the lens-shaped region l_n defined by P_0, P_n and $\rho > 0$ in analogy to l_1 in step 1. Moreover, for each $n \geq 1$, we introduce the tangent cone T_n of l_n in P_0 as

$$T_n := \{\, x \in \mathbb{R}^3 \mid x = \lambda(q - P_0), \quad \lambda \geq 0,\ q \in \overline{l_n} \,\}.$$

Since $|P_n - P_0| < \rho/2$ and $\operatorname{diam} A_n < 2\rho$ we may use the same argument as in step 1 to conclude

$$A_n \subset \overline{l_n}, \quad \forall n \in \mathbb{N}. \tag{72}$$

Furthermore, by straightforward geometrical arguments we also find

$$l_{n+1} \subset l_n \quad \text{and} \quad T_{n+1} \subset T_n, \quad \forall n \in \mathbb{N}. \tag{73}$$

3. Let α_n be the opening angle of the cylindrical cone T_n. Since $0 < |P_n - P_0| < \rho/2$ and

$$\sin(\alpha_n/2) = \frac{|P_n - P_0|}{2\rho} \tag{74}$$

we deduce $\alpha_n \in (0, \pi/2)$. Moreover, since $|P_n - P_0| \to 0$ we deduce $\alpha_n \to 0$ as $n \to \infty$.

4. For each $n \geq 1$ we introduce a unit vector

$$t_n := (P_n - P_0)/|P_n - P_0| \in S^2,$$

which is well-defined since $|P_n - P_0| > 0$ for each $n \in \mathbb{N}$ by part (i). By definition of the cone T_n we have $t_n \in T_n$, and since $T_m \subset T_n$ $(m \geq n)$ and the opening angles satisfy $\alpha_n \to 0$, we deduce that $\{t_n\}_{n\in\mathbb{N}} \subset S^2$ is a Cauchy sequence. Therefore we find a vector

$$t_R(s) := \lim_{n\to\infty} \frac{\Gamma(\sigma_n) - \Gamma(s)}{|\Gamma(\sigma_n) - \Gamma(s)|} \in S^2.$$

Notice that $t_R(s)$ does not depend on the choice of sequence $\sigma_n \downarrow s$. In fact, assuming that a different sequence $\sigma_n' \downarrow s$ leads to a different unit vector $t_R'(s) \neq t_R(s)$, we arrive at a contradiction. In particular, the mixed sequence $\{\sigma_n''\} := \{\sigma_1, \sigma_1', \sigma_2, \sigma_2', \ldots\}$ would lead to a Cauchy sequence of unit vectors with no unique limit. Thus we must have $t_R'(s) = t_R(s)$. Analogously we can argue for $\tau_k \uparrow s$ to find a limit vector t_L with

$$t_L(s) := \lim_{k\to\infty} \frac{\Gamma(s) - \Gamma(\tau_k)}{|\Gamma(s) - \Gamma(\tau_k)|} \in S^2.$$

5. We claim that $t_R(s) = t_L(s)$. To see this, assume for contradiction that $t_R(s) \neq t_L(s)$. Consider the lens-shaped regions

$$l_n^R := \bigcap_{z\in C(P_0,\Gamma(\sigma_n))} B_\rho(z) \quad \text{and} \quad l_k^L := \bigcap_{z\in C(P_0,\Gamma(\tau_k))} B_\rho(z),$$

and the unit vectors

$$t_{L,k} := (\Gamma(\tau_k) - \Gamma(s))/|\Gamma(\tau_k) - \Gamma(s)|,$$
$$t_{R,n} := (\Gamma(\sigma_n) - \Gamma(s))/|\Gamma(\sigma_n) - \Gamma(s)|.$$

By similar arguments as in step 1 we deduce that

$$\Gamma([\tau_k, s]) \cap l_n^R = \emptyset, \tag{75}$$
$$\Gamma([s, \sigma_n]) \cap l_k^L = \emptyset \tag{76}$$

for all sufficiently large $n, k \in \mathbb{N}$. In fact, we may take τ_1 and σ_1 so close to s such that

$$\Gamma(t) \neq \Gamma(\sigma) \text{ for all } t \in [\tau_1, s) \text{ and } \text{ for all } \sigma \in (s, \sigma_1]. \tag{77}$$

Otherwise, by the definition of $R(.,.,.)$ we could find sequences $t_i \uparrow s$ and $\sigma_i \downarrow s$ with

$$R(\Gamma(s),\Gamma(t_i),\Gamma(\sigma_i)) = \frac{|\Gamma(s)-\Gamma(t_i)|}{2} \longrightarrow 0 \text{ as } i \to \infty,$$

contradicting the fact that $R(\Gamma(s),\Gamma(t_i),\Gamma(\sigma_i)) \geq \rho > 0$ for all $i \in \mathbb{N}$. Thus (77) is valid.

To prove (75),(76) we assume for contradiction that for all $k \in \mathbb{N}$ we find points $P_k \in \Gamma([\tau_k, s)) \cap l_n^R$, with $P_k \neq \Gamma(s)$. Note that $P_k \to \Gamma(s)$ as $k \to \infty$. Then for k sufficiently large one can find a sphere S with center on the straight line segment connecting $\Gamma(s)$ and $\Gamma(\sigma_n)$ with diameter $d(S)$ satisfying

$$d(S) \leq \frac{|\Gamma(s)-\Gamma(\sigma_n)|}{2} < \rho/4,$$

and such that S contains $\Gamma(s)$ and P_k. On the other hand, $\Gamma((s,\sigma_n)) \subset \overline{l_n^R}$, and $\Gamma(\sigma) \to \Gamma(s)$ as $\sigma \downarrow s$, hence by connectedness of the arc $\Gamma((s,\sigma_n))$ we can find $\bar{P} \in \Gamma((s,\sigma_n)) \cap S$ with $\bar{P} \neq \Gamma(s)$, and with $\bar{P} \neq P_k$ by (77). Intersecting the plane E spanned by $\Gamma(s), P_k$ and $\bar{P}$ with S produces a circle C with radius $R(C) \leq d(S)/2 \leq \rho/4$ containing the three curve points $\Gamma(s), \bar{P}, P_k$, which contradicts the definition of $\rho = \rho_G[\gamma](s)$. This proves (75), and (76) is shown in an analogous way.

Because of (75) and (76) the angle $\vartheta \in [0,\pi]$ between $t_R(s)$ and $-t_L(s)$ satisfies $0 < \vartheta < \pi$. Moreover, since

$$\lim_{k,n\to\infty} \sphericalangle(t_{L,k}, t_{R,n}) = \vartheta$$

and

$$\lim_{k\to\infty} \Gamma(\tau_k) = \lim_{n\to\infty} \Gamma(\sigma_n) = \Gamma(s)$$

we deduce that

$$\lim_{k,n\to\infty} R(\Gamma(\tau_k),\Gamma(\sigma_n),\Gamma(s)) = \lim_{k,n\to\infty} \frac{|\Gamma(\tau_k)-\Gamma(\sigma_n)|}{2\sin\sphericalangle(t_{L,k},t_{R,n})} = 0,$$

which contradicts the lower bound

$$R(\Gamma(s),\Gamma(\sigma_n),\Gamma(\tau_k)) = R(\Gamma(\tau_k),\Gamma(\sigma_n),\Gamma(s)) \geq \rho > 0.$$

Thus we must have $t_R(s) = t_L(s) = T(s)$ as claimed.

6. If s_0 is a parameter where Γ is differentiable, then $\Gamma'(s_0) = T(s_0)$. This follows from the fact that, if Γ is differentiable at s_0, then $|\Gamma'(s_0)| = 1$ and

$$\Gamma(\sigma_n) - \Gamma(s_0) = \Gamma'(s_0)(\sigma_n - s_0) + o(|\sigma_n - s_0|) \tag{78}$$

for any sequence $\sigma_n \downarrow s_0$. The result follows since

$$\frac{\Gamma(\sigma_n) - \Gamma(s_0)}{|\Gamma(\sigma_n) - \Gamma(s_0)|} = \frac{\Gamma'(s_0)(\sigma_n - s_0) + o(|\sigma_n - s_0|)}{|\sigma_n - s_0|} \cdot \Big[1 - \frac{o(|\sigma_n - s_0|)}{|\sigma_n - s_0|}\Big]$$

and $o(|\sigma_n - s_0|)/|\sigma_n - s_0| \to 0$ as $|\sigma_n - s_0| \to 0$. This concludes the proof of (11) and of part (ii).

Part (iii): Let $\mathcal{K}[\gamma] < \infty$, i.e., $\mathcal{R}[\gamma] > 0$, then by part (i) γ is simple. If Γ is differentiable at $\sigma_1, \sigma_2 \in S_L$, then

$$|\Gamma'(\sigma_1) - \Gamma'(\sigma_2)| \leq \mathcal{K}[\gamma]|\sigma_1 - \sigma_2|.$$

To establish this result, we consider first the case when $|\Gamma(\sigma_1) - \Gamma(\sigma_2)| < \mathcal{R}[\gamma]/2$. Let l_1 be the lens-shaped region as in (68) with ρ, P_0, P_1 replaced by $\mathcal{R}[\gamma], \Gamma(\sigma_1), \Gamma(\sigma_2)$, respectively. In this case we have $\Gamma'(\sigma_1) \in T_1$, and by symmetry $\Gamma'(\sigma_2) \in T_1$, where T_1 is the tangent cone of l_1 in $\Gamma(\sigma_1)$ with opening angle $\alpha_1 \in (0, \pi/2)$ as defined in the second step of the proof of part (ii). (Notice that for all $s \in S_L$ we have $\rho_G[\gamma](s) \geq \mathcal{R}[\gamma] > 0$ by definition of $\mathcal{R}[\gamma]$.) Using the fact that

$$\sin(\alpha_1/2) = |\Gamma(\sigma_1) - \Gamma(\sigma_2)|/2\mathcal{R}[\gamma]$$

together with the law of cosines we find

$$\begin{aligned} |\Gamma'(\sigma_1) - \Gamma'(\sigma_2)| &\leq \sqrt{2 - 2\cos\alpha_1} \\ &= |\Gamma(\sigma_1) - \Gamma(\sigma_2)|/\mathcal{R}[\gamma] \leq \mathcal{K}[\gamma]|\sigma_1 - \sigma_2|, \end{aligned} \tag{79}$$

as claimed. In the case when $|\Gamma(\sigma_1) - \Gamma(\sigma_2)| \geq \mathcal{R}[\gamma]/2$ the result is still true. In particular, the arc $[\sigma_1, \sigma_2] \subset S_L$ may be divided into subarcs $[\tau_i, \tau_{i+1}] \subset S_L$ ($i = 1, \dots m$) such that τ_i are points of differentiability (which is possible since Γ is Lipschitz continuous and hence differentiable almost everywhere), $\sigma_1 = \tau_1$, $\sigma_2 = \tau_{m+1}$ and $|\Gamma(\tau_i) - \Gamma(\tau_{i+1})| < \mathcal{R}[\gamma]/2$. Applying (79) to the subarcs $[\tau_i, \tau_{i+1}]$ and summing yields the required result.

We can now show that $\Gamma \in C^{1,1}(S_L, \mathbb{R}^3)$ and that Γ' has Lipschitz constant $\mathcal{K}[\gamma]$. To begin, we consider first the subset $\widetilde{S}_L$ of S_L where Γ is differentiable. Since $\widetilde{S}_L$ is dense in S_L and by (79) the map $\Gamma' : \widetilde{S}_L \to \mathbb{R}^3$ is uniformly continuous, we deduce that there is a unique uniformly continuous extension $V : S_L \to \mathbb{R}^3$. In particular, $V \in C^{0,1}(S_L, \mathbb{R}^3)$ with Lipschitz constant $\mathcal{K}[\gamma]$. To see that this implies $\Gamma \in C^{1,1}(S_L, \mathbb{R}^3)$, let $\sigma_0 \in S_L$ be given and note that since $\Gamma \in C^{0,1}(S_L, \mathbb{R}^3)$ is absolutely continuous we have

$$\Gamma(\sigma_n) - \Gamma(\sigma_0) = \int_{\sigma_0}^{\sigma_n} \Gamma'(\tau)\, d\tau = \int_{\sigma_0}^{\sigma_n} V(\tau)\, d\tau,$$

which implies

$$\frac{\Gamma(\sigma_n) - \Gamma(\sigma_0)}{\sigma_n - \sigma_0} = \frac{1}{\sigma_n - \sigma_0} \int_{\sigma_0}^{\sigma_n} V(\tau)\, d\tau$$

for any $\sigma_n \neq \sigma_0$. Since $V \in C^{0,1}(S_L, \mathbb{R}^3)$ the limit $\sigma_n \to \sigma_0$ is well-defined, i.e., $\Gamma'(\sigma_0)$ exists and

$$\Gamma'(\sigma_0) = V(\sigma_0), \quad \forall \sigma_0 \in S_L.$$

Thus $\Gamma' \in C^{0,1}(S_L, \mathbb{R}^3)$ with Lipschitz constant $\mathcal{K}[\gamma]$, which finishes the proof of necessity in part (iii).

Conversely, if γ is simple and $\Gamma \in C^{1,1}([0, L], \mathbb{R}^3)$, we take a minimal sequence of triples of distinct parameters $s_j, \sigma_j, \tau_j \in S_L$ for $\mathcal{R}[\gamma]$ using the characterization (6). Taking a subsequence we may assume that

$$(s_j, \sigma_j, \tau_j) \to (s, \sigma, \tau) \in S_L \times S_L \times S_L,$$

since S_L is compact, and we are going to look at the different possible limit cases, where we will repeatedly use (9).

Case I. If $s \neq \sigma \neq \tau \neq s$ then also $\Gamma(s) \neq \Gamma(\sigma) \neq \Gamma(\tau) \neq \Gamma(s)$, since Γ is simple by assumption. This means that according to the representation (2)

$$\begin{aligned} \mathcal{R}[\gamma] &= \lim_{j \to \infty} R(\Gamma(s_j), \Gamma(\sigma_j), \Gamma(\tau_j)) \\ &= \frac{|\Gamma(s) - \Gamma(\sigma)|}{\left| \frac{\Gamma(s) - \Gamma(\tau)}{|\Gamma(s) - \Gamma(\tau)|} \wedge \frac{\Gamma(\sigma) - \Gamma(\tau)}{|\Gamma(\sigma) - \Gamma(\tau)|} \right|}, \end{aligned} \tag{80}$$

unless $\Gamma(s_j), \Gamma(\sigma_j)$ and $\Gamma(\tau_j)$ are collinear for infinitely many $j \in \mathbb{N}$, in which case $\mathcal{R}[\gamma] = \infty$, i.e., $\mathcal{K}[\gamma] = 0$. (In fact, the latter cannot occur for closed curves $\gamma \in \mathcal{L}$ according to Lemma 2.2.) If $\Gamma(s_j), \Gamma(\sigma_j)$ and $\Gamma(\tau_j)$ are not collinear, the denominator on the right of (80) is positive and bounded from above by 2, whereas the numerator does not vanish, since γ is assumed to be simple, hence $\mathcal{R}[\gamma] > 0$, i.e., $\mathcal{K}[\gamma] < \infty$.

Case II. If two but not all of the three parameters s, σ, τ coincide, say $s = \sigma \neq \tau$, then we can use (2) again for $R(\Gamma(s_j), \Gamma(\sigma_j), \Gamma(\tau_j))$. Before going to the limit we select a subsequence such that the difference $s_j - \sigma_j$ has the same sign for all j. Then by the same argument as in step 6 of the proof for part (ii), in particular by (78), we obtain

$$\lim_{j\to\infty} \frac{\Gamma(s_j) - \Gamma(\sigma_j)}{|\Gamma(s_j) - \Gamma(\sigma_j)|} = \pm\Gamma'(s). \tag{81}$$

Consequently, using the symmetry of $R(.,.,.)$,

$$\begin{aligned}
\mathcal{R}[\gamma] &= \lim_{j\to\infty} R(\Gamma(s_j), \Gamma(\sigma_j), \Gamma(\tau_j)) \\
&= \lim_{j\to\infty} R(\Gamma(\tau_j), \Gamma(s_j), \Gamma(\sigma_j)) \\
&= \lim_{j\to\infty} \frac{|\Gamma(\tau_j) - \Gamma(s_j)|}{2\left|\frac{\Gamma(\tau_j)-\Gamma(\sigma_j)}{|\Gamma(\tau_j)-\Gamma(\sigma_j)|} \wedge \frac{\Gamma(s_j)-\Gamma(\sigma_j)}{|\Gamma(s_j)-\Gamma(\sigma_j)|}\right|} \\
&= \frac{|\Gamma(\tau) - \Gamma(s)|}{2\left|\frac{\Gamma(\tau)-\Gamma(s)}{|\Gamma(\tau)-\Gamma(s)|} \wedge \Gamma'(s)\right|}, \qquad (82)
\end{aligned}$$

where the denominator is bounded from above by 2, and the numerator is different from 0 as before, since γ is simple, hence $\mathcal{R}[\gamma] > 0$ and $\mathcal{K}[\gamma] < \infty$ in this case.

Case III. If $s = \sigma = \tau$, we apply Lemma 5.2. Notice that in our situation $\Gamma'' \in L^\infty([0, L], \mathbb{R}^3)$, and $\|\Gamma''\|_{L^\infty} > 0$, (otherwise $\Gamma' \equiv const.$ and γ could not be closed). So

$$|A(s_1, s_2, s_3)| \leq \|\Gamma''\|_{L^\infty}/2, \tag{83}$$

$$|B(s_1, s_2, s_3)| \leq \|\Gamma''\|^2_{L^\infty}/4, \tag{84}$$

$$|C(s_i, s_k)| \geq 1 - |s_i - s_k|, \quad \text{for} \quad i \neq k. \tag{85}$$

By symmetry of $R(.,.,.)$ we may assume that $s_j < \tau_j < \sigma_j$ for all $j \in \mathbb{N}$, and by (58),(83)–(85) we obtain

$$R(\Gamma(s_j), \Gamma(\sigma_j), \Gamma(\tau_j)) \geq \frac{1 + o(1)}{\|\Gamma''\|_{L^\infty} \cdot (1 + o(1))} \qquad \text{as} \quad j \to \infty. \tag{86}$$

This means that

$$\mathcal{R}[\gamma] = \lim_{j\to\infty} R(\Gamma(s_j), \Gamma(\sigma_j), \Gamma(\tau_j)) \geq \|\Gamma''\|^{-1}_{L^\infty} > 0, \tag{87}$$

i.e., $\mathcal{K}[\gamma] < \infty$ in this case, which concludes the proof of sufficiency in part (iii).

Part (iv) (b) was shown in [12, Lemma 3]. To prove (iv) (a), we argue as follows. Set $\rho := \rho_G[\gamma](s) > 0$ and let $s_n \to s$, $P_n := \Gamma(s_n)$, $P_0 := \Gamma(s)$ and

$$C_n := \{\, z\in\mathbb{R}^3 \mid |z - P_0| = |z - P_n| = \rho \,\}.$$

Notice that C_n is the circle of radius $\rho_n := \sqrt{\rho^2 - |P_n - P_0|^2/4}$ centred at $y_n := (P_n + P_0)/2$ and perpendicular to the unit vector $(P_n - P_0)/|P_n - P_0|$. We claim that

$$\mathrm{dist}_H(C_n, C(s,\rho)) \to 0 \quad \text{as} \quad n \to \infty, \tag{88}$$

where $C(s,\rho)$ is the circle defined in the statement of the theorem (for $\theta = \rho$), and $\mathrm{dist}_H(A,B)$ denotes the Hausdorff distance [7, p. 183] between two subsets A, B of $\mathbb{R}^3$. To establish this result, we note first that $\rho_n \to \rho$ and $y_n \to P_0$. Moreover, by part (ii), the only possible limits for $(P_n - P_0)/|P_n - P_0|$ as $n \to \infty$ are $\pm T(s)$. Thus C_n converges to a circle of radius ρ with centre P_0 in the plane perpendicular to $T(s)$. Since these properties completely characterize $C(s,\rho)$ our claim in (88) follows.

To establish (iv)(a), we consider the sets

$$\Xi_n := \bigcup_{z\in C_n} B_\rho(z)$$

as in the proof of part (ii). We assume for contradiction that there is a point $\bar{P} \in \Gamma(S_L) \cap M(s,\rho)$, which implies $\mathrm{dist}(\bar{P}, C(s,\rho)) < \rho$. For $n \in \mathbb{N}$ sufficiently large, we deduce from (88) that $\mathrm{dist}(\bar{P}, C_n) < \rho$, which implies $\bar{P} \in \Xi_n$, and moreover we have $|\bar{P} - P_0| > |P_n - P_0|$. These observations lead to the result $\bar{P} \in \Xi_n \backslash \overline{l_n}$, where

$$l_n := \bigcap_{z\in C_n} B_\rho(z).$$

By exactly the same arguments as in the proof of part (ii) we arrive at a statement of the form (71) with P_1 replaced by P_n. Since this contradicts the definition of $\rho = \rho_G[\gamma](s)$ claim (iv)(a) must be true. $\square$

Proof of Lemma 2.2. Only the second inequality in (14) requires a proof. (15) follows immediately from (8) and (9). If $\rho_G[\gamma](s) = 0$ the statement is trivial, so assume that $\rho := \rho_G[\gamma](s) > 0$, which in particular means that the curve has positive length. Notice that we did not assume that $\mathcal{R}[\gamma] > 0$, hence we may not have a tangent of Γ at s. However, the geometric tangent $T(s)$ exists according to part (ii) of Theorem 2.1, and we may apply part (iv) (a) of the same theorem to find that the curve Γ does not intersect $M(s,\rho)$. On the other hand, by (10), Γ intersects the plane E through $\Gamma(s)$ perpendicular to $T(s)$ transversally. Because $\Gamma(s)$ is not a double point ($\rho_G[\gamma](s) > 0$), Γ must intersect this plane in at least one different point $\Gamma(\sigma) \neq \Gamma(s)$, since γ is closed. Then $\Gamma(\sigma) \in E \backslash M(s,\rho)$, i.e., $|\Gamma(s) - \Gamma(\sigma)| \geq 2\rho$, which proves the lemma. $\square$

Proof of Theorem 2.3. It suffices to prove the first relation in (16) and the identities (17),(20) and (22), since the remaining identities follow from (8) and (9).

(i) If $\mathcal{R}[\gamma] = 0$, then we know by Theorem 2.1, part (iii), that γ is not simple, i.e., there exist $t \neq s$ such that $\Gamma(s) = \Gamma(t)$. But then, by Theorem 2.1, part (i), $\rho_G[\gamma](s) = \rho_G[\gamma](t) = 0$, thus (i) is trivially true in this case.

If $\mathcal{R}[\gamma] > 0$, we argue as follows. Taking a minimizing sequence of triples of distinct parameters $s_j, \sigma_j, \tau_j \in S_L$ with $R(\Gamma(s_j), \Gamma(\sigma_j), \Gamma(\tau_j)) \to \mathcal{R}[\gamma]$, which is possible according to (6), we may assume that $(s_j, \sigma_j, \tau_j) \to (s, \sigma, \tau)$ as $j \to \infty$ by compactness. Note that $\mathcal{R}[\gamma]$ is bounded by Lemma 2.2 and that γ is simple, by Theorem 2.1 (iii), hence we may assume that $\Gamma(s_j), \Gamma(\sigma_j)$ and $\Gamma(\tau_j)$ are distinct and non-collinear for all $j \in \mathbb{N}$.

As in the proof of part (iii) of Theorem 2.1 we distinguish three different cases.

Case I. If $s \neq \sigma \neq \tau$ then we can use (2) to obtain

$$\begin{aligned} \mathcal{R}[\gamma] &= \lim_{j\to\infty} R(\Gamma(s_j), \Gamma(\sigma_j), \Gamma(\tau_j)) \\ &= R(\Gamma(s), \Gamma(\sigma), \Gamma(\tau)) \\ &\geq \rho_G[\gamma](s) \geq \mathcal{R}[\gamma], \end{aligned} \tag{89}$$

hence equality holds everywhere, which proves (16) in this case. Observe that also $\Gamma(s), \Gamma(\sigma), \Gamma(\tau)$ are distinct and non-collinear here.

Case II. If two but not all of the three parameters coincide, say $s = \sigma \neq \tau$ we can repeat the calculation that led to (82) in the proof of Theorem 2.1, part (iii) to obtain

$$
\begin{aligned}
\mathcal{R}[\gamma] &= \lim_{j\to\infty} R(\Gamma(s_j), \Gamma(\sigma_j), \Gamma(\tau_j)) \\
&= \frac{|\Gamma(\tau) - \Gamma(s)|}{2\left|\frac{\Gamma(\tau)-\Gamma(s)}{|\Gamma(\tau)-\Gamma(s)|} \wedge \Gamma'(s)\right|}. \qquad (90)
\end{aligned}
$$

Notice that Γ is differentiable everywhere, since $\mathcal{K}[\gamma] < \infty$ now. According to Theorem 2.1 we may insert in (90)

$$\Gamma'(s) = T(s) = \lim_{\substack{t_k \downarrow s \\ t_k \neq s}} \frac{\Gamma(t_k) - \Gamma(s)}{|\Gamma(t_k) - \Gamma(s)|}$$

to obtain by the definition of $\rho_G[\gamma](\tau)$

$$
\begin{aligned}
\mathcal{R}[\gamma] &= \lim_{\substack{t_k \downarrow s \\ t_k \neq s}} \frac{|\Gamma(\tau) - \Gamma(s)|}{2\left|\frac{\Gamma(\tau)-\Gamma(s)}{|\Gamma(\tau)-\Gamma(s)|} \wedge \frac{\Gamma(t_k)-\Gamma(s)}{|\Gamma(t_k)-\Gamma(s)|}\right|} \\
&= \lim_{\substack{t_k \downarrow s \\ t_k \neq s}} R(\Gamma(\tau), \Gamma(s), \Gamma(t_k)) \\
&\geq \rho_G[\gamma](\tau) \geq \mathcal{R}[\gamma],
\end{aligned}
$$

which implies equality, i.e.,(16).

Case III. If $s = \sigma = \tau$ we have (17), which finishes the proof of the first statement in (i).

In order to apply (63) of Lemma 5.2 in the case when $s \in S_L$ is a Lebesgue point of Γ'' we may assume a fixed ordering, say $s_j < \sigma_j < \tau_j$, by taking subsequences, because we know from (17) that the limit as $j \to \infty$ exists. Condition (19) then implies (62), hence we infer from (63) and (17) that

$$\mathcal{R}[\gamma] = |\Gamma''(s)|^{-1}. \qquad (91)$$

As pointed out in the first remark following Theorem 2.3, any sequence of the form $(s, \sigma_j, \tau_j) \to (s, s, s)$ respects (19) as well. Applying (91) to a minimizing sequence (s, σ_j, τ_j) with

$$\lim_{j\to\infty} R(\Gamma(s), \Gamma(\sigma_j), \Gamma(\tau_j)) = \rho_G[\gamma](s)$$

leads to (20).

(ii) Fix $s \in S_L$ and take a minimal sequence of pairs (u_k, t_k) with $s \neq u_k \neq t_k \neq s$ realizing the infimum in the definition of $\rho := \rho_G[\gamma](s)$, i.e., with

$$\lim_{k\to\infty} R(\Gamma(s), \Gamma(u_k), \Gamma(t_k)) = \rho_G[\gamma](s). \tag{92}$$

As before by taking a subsequence we can assume that $(u_k, t_k) \to (\sigma, \tau) \in S_L \times S_L$. We are going to distinguish between three different cases.

Case I. $s = \sigma = \tau$, this is case (a) of the statement of the theorem and we are done.

Case II. $s \neq \sigma \neq \tau$. In this case we will show, that γ intersects or touches non-transversally the unique sphere of radius ρ, which contains the circumcircle of $\Gamma(s), \Gamma(\sigma)$ and $\Gamma(\tau)$ as a great circle. (Note that $\Gamma(s), \Gamma(\sigma)$ and $\Gamma(\tau)$ are distinct and non-collinear, since γ is simple ($\mathcal{R}[\gamma] > 0$) and because $\mathcal{R}[\gamma]$ is bounded by Lemma 2.2.) In other words, if $y \in \mathbb{R}^3$ is the centre of that sphere, then we claim that

$$[(\Gamma(s) - y) \cdot \Gamma'(s)][(\Gamma(\sigma) - y) \cdot \Gamma'(\sigma)][(\Gamma(\tau) - y) \cdot \Gamma'(\tau)] = 0. \tag{93}$$

In fact, if (93) were false, we could find a slightly (but strictly) smaller sphere within the previous one and with center $y_1 \in [y, \Gamma(s)]$, which touches the original sphere in $\Gamma(s)$ and still contains two other distinct points $\Gamma(\sigma_1)$, $\Gamma(\tau_1)$ close to $\Gamma(\sigma)$ and $\Gamma(\tau)$, respectively. Intersecting this smaller sphere with the plane through $\Gamma(s), \Gamma(\sigma_1)$ and $\Gamma(\tau_1)$ gives a circle with radius strictly smaller than ρ contradicting the definition of $\rho = \rho_G[\gamma](s)$, which proves our claim. (Notice that also $\Gamma(s), \Gamma(\sigma_1)$ and $\Gamma(\tau_1)$ are non-collinear, hence span a plane for $\Gamma(\sigma_1)$ and $\Gamma(\tau_1)$ sufficiently close to $\Gamma(\sigma)$ and $\Gamma(\tau)$, respectively.)

Now by (93) we know that γ is tangential to the original sphere centred at $y \in \mathbb{R}^3$ with radius ρ in at least one of the points $\Gamma(s), \Gamma(\tau), \Gamma(\sigma)$, say in $\Gamma(\tau)$. Intersecting this sphere with the plane P spanned by the (non-collinear) vectors $\Gamma(s) - \Gamma(\tau)$ and $\Gamma'(\tau)$ we obtain a circle C with radius $r(C)$, which satisfies

$$\rho \geq r(C) = \frac{|\Gamma(s) - \Gamma(\tau)|}{2\left|\frac{\Gamma(s)-\Gamma(\tau)}{|\Gamma(s)-\Gamma(\tau)|} \wedge \Gamma'(\tau)\right|} = \lim_{j\to\infty} R(\Gamma(s), \Gamma(\tau), \Gamma(\sigma_j)) \geq \rho, \tag{94}$$

for a sequence $\sigma_j \to \tau$ as $j \to \infty$. This follows from (29) and the calculation that led to (82) in the proof of Theorem 2.1. Hence we have equality everywhere in (94), which proves that in this situation we have (22) (b) with the sequence $(\sigma_j, \tau_j) \to (\tau, \tau)$ as $j \to \infty$, where $\tau_j := \tau$ for all $j \in \mathbb{N}$.

Case III. If two but not all of the three parameters s, σ, τ coincide we have two subcases.

IIIa If $\tau \neq s = \sigma$ (the case $\sigma \neq \tau = s$ can be treated analogously) one can go to the limit in (92) as in the calculation that led to (82) to get

$$\rho_G[\gamma](s) = \lim_{k\to\infty} R(\Gamma(s), \Gamma(u_k), \Gamma(t_k)) = \frac{|\Gamma(\tau) - \Gamma(s)|}{2\left|\frac{\Gamma(\tau)-\Gamma(s)}{|\Gamma(\tau)-\Gamma(s)|} \wedge \Gamma'(s)\right|}.$$

By (29), the expression on the right equals the radius of the unique circle C through $\Gamma(s)$ and $\Gamma(\tau)$ tangent to Γ in $\Gamma(s)$. If $\partial B_\rho(y)$ is the unique sphere containing C as a great circle then $B_\rho(y) \subset M(s, \rho)$ as defined in Theorem 2.1, part (iv), which implies that Γ must also be tangential to $\partial B_\rho(y)$ in the point $\Gamma(\tau)$, i.e., $\Gamma'(\tau) \cdot (\Gamma(\tau) - y) = 0$. Now we look at the circle C^* obtained by intersecting $\partial B_\rho(y)$ with the plane spanned by the vectors $\Gamma(s) - \Gamma(\tau)$ and $\Gamma'(\tau)$. Then we get for the radius $r(C^*)$ of C^*

$$\begin{aligned} \rho \;&\geq\; r(C^*) = \lim_{l\to\infty} R(\Gamma(s), \Gamma(\tau), \Gamma(v_l)) \\ &\geq\; \rho_G[\gamma](s) = \rho, \end{aligned} \tag{95}$$

for a sequence $v_l \to \tau$ as $l \to \infty$. Hence we have equality everywhere, which implies that we have (22) with (b) with the sequence (v_l, τ) converging to (τ, τ) as $l \to \infty$.

IIIb If $s \neq \tau = \sigma$ we are immediately in case (b) of the statement in part (ii) of the theorem, and we are done, which finishes the proof. □

Proof of Proposition 2.4. (i) We take a sequence $(\tau_j, \sigma_j) \to (s, s)$, $s \neq \tau_j \neq \sigma_j \neq s$, realizing the limes superior in the definition of $\kappa[\gamma](s)$, see (25). If $s < \tau_j < \sigma_j$ for infinitely many $j \in \mathbb{N}$ we apply (54) of Lemma 5.1 to the mean value expression in the term $A(s, \tau_j, \sigma_j)$ as defined in (60) of Lemma 5.2. For that set $x = s$, replace E_r in Lemma 5.1 by the set

$$E_j^t := [\tau_j - t(\tau_j - s), \tau_j + t(\sigma_j - \tau_j)],$$

and B_r by $B_j := B_{\sigma_j - s}(s)$ and set $f := |\Gamma''|$. Then for given $\epsilon > 0$ we choose

$$\delta := \epsilon \|\Gamma\|_{L^\infty}^{-1}/2. \tag{96}$$

Notice that according to Lemma 5.1 we can find j_0 sufficiently large, such that for all $j \geq j_0$ and $t \in [\delta, 1]$

$$
\tag{97} \fint_{\tau_j - t(\tau_j - s)}^{\tau_j + t(\sigma_j - \tau_j)} |\Gamma''(\omega)|\, d\omega \leq \text{ap} \limsup_{z \to s} |\Gamma''(z)| + \epsilon/2,
$$

Combining (96) and (97) we obtain

$$
\left| \int_0^1 t \fint_{E_j^t} \Gamma''(\omega)\, d\omega dt \right| \leq \text{ap} \limsup_{z \to s} |\Gamma''(z)| + \epsilon
$$

for all $j \geq j_0$. Consequently,

$$
\begin{aligned}
\kappa[\gamma](s) &= \lim_{j \to \infty} \frac{1}{R(\Gamma(s), \Gamma(\tau_j), \Gamma(\sigma_j))} \\
&\underset{(58)}{=} \lim_{j \to \infty} \frac{2|A(s, \tau_j, \sigma_j) + (\tau_j - s)B(s, \tau_j, \sigma_j)|}{C(s, \tau_j)C(s, \sigma_j)C(\tau_j, \sigma_j)} \\
&\leq \limsup_{j \to \infty}\ 2 \int_0^1 \fint_{E_t^j} |\Gamma''(\omega)|\, d\omega\, dt \\
&\leq \text{ap} \limsup_{z \to s} |\Gamma''(z)|.
\end{aligned}
$$

Here we have used the fact that $|B(s, \tau_j, \sigma_j)| \leq \|\Gamma''\|_{L^\infty}^2/4$ and that $C(s, \tau_j)$, $C(s, \sigma_j)$ and $C(\tau_j, \sigma_j)$ converge to 1 as $j \to \infty$.

If $\tau_j < s < \sigma_j$ for infinitely many $j \in \mathbb{N}$, use $A(\tau_j, s, \sigma_j)$ and $B(\tau_j, s, \sigma_j)$ instead; all the other possible orderings of an infinite number of the parameters s, σ_j, τ_j can be treated similarly.

(ii) One simply needs to recall that a.e. point $s \in S_L$ is a Lebesgue point of Γ'', since $\Gamma \in C^{1,1} \simeq W^{2,\infty}$. This means that we can use (63) in Lemma 5.2 for $s_j = s$ for all $j \in \mathbb{N}$, to replace the limes superior in the definition of $\kappa[\gamma](s)$, (25), by the limit as $j \to \infty$ to obtain (ii). Notice that for the sequence $(\tau_j, \sigma_j) \to (s, s)$ with $s \neq \tau_j \neq \sigma_j \neq s$ realizing the limes superior we may assume a fixed ordering ($\tau_j < \sigma_j < s$, or $\sigma_j < s < \tau_j$, etc. for all $j \in \mathbb{N}$) by taking suitable subsequences. Thus Lemma 5.2 is applicable, because (62) is automatically satisfied if $s = s_j$ for all $j \in \mathbb{N}$, as pointed out in the first remark following Theorem 2.3. □

Proof of Lemma 2.5. Notice first that it suffices to prove the statements for ρ_G and $\mathcal{R}$, the relations for κ_G, and $\mathcal{K}$ follow from (8) and (9).

It is easy to see that the right-hand side of (A) (and (B)) is bounded from above by $\operatorname{diam}(\gamma(\bar{I}))/2$, since we may consider the continuously differentiable function, (recall that the assumption $\mathcal{K}[\gamma] < \infty$ leads to a Lipschitz continuous tangent field Γ' by Theorem 2.1),

$$f(\sigma) := |\Gamma(s) - \Gamma(\sigma)|^2,$$

bounded by $(\operatorname{diam}(\gamma(\bar{I})))^2$. Here $s \in S_L$ is fixed. Notice that f attains its maximum on S_L by compactness. Let $\tau \in S_L$ be such that $f(\tau)$ is maximal. Then $\Gamma(S_L)$ is tangent to the sphere of radius $\delta := |\Gamma(s) - \Gamma(\tau)|/2$ centred at

$$m := \frac{\Gamma(s) + \Gamma(\tau)}{2}$$

in the point $\Gamma(\tau)$, since $f'(\tau) = 0$. Any great circle through $\Gamma(s)$ and $\Gamma(\tau)$ and tangent to $\Gamma'(\tau)$ has radius $\delta \leq \operatorname{diam}(\gamma(\bar{I}))/2$, hence also the infimum of such radii must have that upper bound.

Taking a minimal sequence $\{\sigma_j\} \subset S_L$ for the right-hand side of (A), i.e., with $r(\Gamma(s), \Gamma(\sigma_j), \Gamma'(\sigma_j))$ tending to the infimum in (A), we find for a given $\epsilon > 0$ an index j_0 such that for all $j \geq j_0$

$$r(\Gamma(s), \Gamma(\sigma_j), \Gamma'(\sigma_j)) \leq \inf_{\substack{\sigma \in S_L \\ \sigma \neq s}} r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma)) + \epsilon/2. \tag{98}$$

Since the right-hand side of (A) is finite, the same is true for the left-hand side of (98) for all $j \geq j_0$, which means that the vectors $\Gamma(s) - \Gamma(\sigma_j)$ and $\Gamma'(\sigma_j)$ are non-collinear for $j \geq j_0$. Hence we can apply (29). On the other hand, approximating the left-hand side of (98) for $j = j_0$ as we did to derive (82), we find for the above ϵ some $t \in S_L$ close to σ_{j_0} such that

$$R(\Gamma(s), \Gamma(\sigma_{j_0}), \Gamma(t)) \leq \inf_{\substack{\sigma \in S_L \\ \sigma \neq s}} r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma)) + \epsilon,$$

which immediately gives

$$\rho_G[\gamma](s) \leq \inf_{\substack{\sigma \in S_L \\ \sigma \neq s}} r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma)). \tag{99}$$

Analogously one obtains

$$\rho_G[\gamma](s) \leq \inf_{\substack{\sigma \in S_L \\ \sigma \neq s}} r(\Gamma(\sigma), \Gamma(s), \Gamma'(s)). \tag{100}$$

For the reverse inequalities we first note that according to (ii) of Theorem 2.3 we can represent $\rho_G[\gamma](s)$ as limit as in (22) with two possible cases. In case (ii)(b) of that theorem we are done, since then by the same calculation that led to (82),

$$\begin{aligned} \rho_G[\gamma](s) &= r(\Gamma(s), \Gamma(\tau), \Gamma'(\tau)) \\ &\geq \inf_{\substack{\sigma \in S_L \\ \sigma \neq s}} r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma)). \end{aligned} \tag{101}$$

In case (ii)(a) there is a sequence $(\sigma_j, \tau_j) \to (s, s)$, such that

$$\rho_G[\gamma](s) = \lim_{j\to\infty} R(\Gamma(s), \Gamma(\sigma_j), \Gamma(\tau_j)) \geq \rho[\gamma](s) \underset{(26)}{\geq} \rho_G[\gamma](s), \tag{102}$$

by definition of $\rho[\gamma]$, see (24), hence equality holds.

Remaining in case (ii)(a), we renumber the minimal sequence $(\sigma_j, \tau_j) \to (s, s)$ appropriately, so that we can assume that for each $j \in \mathbb{N}$

$$\theta_j := R(\Gamma(s), \Gamma(\sigma_j), \Gamma(\tau_j)) \leq \rho_G[\gamma](s) + \frac{1}{j}\,. \tag{103}$$

Then for fixed $j \in \mathbb{N}$,

$$\theta_j^* := \inf_{\substack{t \in S_L \\ t \neq s, \sigma_j}} R(\Gamma(s), \Gamma(\sigma_j), \Gamma(t)) \leq \theta_j, \tag{104}$$

we can choose a minimal sequence t_k converging w.l.o.g. to some τ_j^*, such that

$$\theta_j^* = \lim_{k\to\infty} R(\Gamma(s), \Gamma(\sigma_j), \Gamma(t_k)), \tag{105}$$

where $s \neq \sigma_j \neq t_k \neq s$ for all $k \in \mathbb{N}$. There are three cases in this situation:

Case I. If $\tau_j^* = \sigma_j$ ($\neq s$), then with the same calculation that led to (82) before, we obtain

$$\theta_j^* = r(\Gamma(s), \Gamma(\sigma_j), \Gamma'(\sigma_j)) \geq \inf_{\substack{\sigma \in S_L \\ \sigma \neq s}} r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma)), \tag{106}$$

and by (103)–(106) we get as $j \to \infty$

$$\rho_G[\gamma](s) \geq \inf_{\substack{\sigma \in S_L \\ \sigma \neq s}} r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma)). \tag{107}$$

Case II. If $\tau_j^* = s$ ($\neq \sigma_j$), then we obtain in the same way as in Case I

$$\rho_G[\gamma](s) \geq \inf_{\substack{\sigma \in S_L \\ \sigma \neq s}} r(\Gamma(\sigma), \Gamma(s), \Gamma'(s)). \tag{108}$$

Case III. If $\tau_j^* \neq s \neq \sigma_j \neq \tau_j^*$, then we can go to the limit $k \to \infty$ in formula (2) for $R(.,.,.)$ to obtain $\theta_j^* = R(\Gamma(s), \Gamma(\sigma_j), \Gamma(\tau_j^*))$. Let $\partial B_{\theta_j^*}(y_j)$ be the unique sphere of radius θ_j^* containing the unique circumcircle of $\Gamma(s), \Gamma(\sigma_j), \Gamma(\tau_j^*)$ as a great circle. Then we claim that

$$\Gamma'(\tau_j^*) \perp \Gamma(\tau_j^*) - y_j. \tag{109}$$

This is in fact true, since otherwise we could find $\Gamma(\tilde{\tau})$ with $\tilde{\tau}$ close to τ_j^*, such that

$$R(\Gamma(s), \Gamma(\sigma_j), \Gamma(\tilde{\tau})) < \theta_j^*, \tag{110}$$

which would contradict the definition of θ_j^* in (105) as infimum over such radii. (110) holds for the following reason: Consider the triangle $\triangle := \triangle(\Gamma(s), \Gamma(\sigma_j), \Gamma(\tau_j^*))$. Recall that we are in case (ii) (a) of Theorem 2.3 now, which means that by taking j sufficiently large from the beginning on we may assume that $|\Gamma(s) - \Gamma(\sigma_j)| < \mathcal{R}[\gamma]/2$, i.e., that the angle α of $\triangle$ in $\Gamma(\tau_j^*)$ is either in $(0, \pi/2)$ or in $(\pi/2, \pi)$. In either case, considering the perturbed triangle $\tilde{\triangle} := \triangle(\Gamma(s), \Gamma(\sigma_j), \Gamma(\tilde{\tau}))$ for $\Gamma(\tilde{\tau}) \in B_{\theta_j^*}(y_j)$ with $\tilde{\tau}$ close to τ_j^*, leads to (110). In fact, intersecting the plane spanned by $\Gamma(s), \Gamma(\sigma_j), \Gamma(\tilde{\tau})$ with the $\partial B_{\theta_j^*}(y_j)$ gives a circle ∂D_ϑ with radius $\vartheta \leq \theta_j^*$, and with $\Gamma(\tilde{\tau})$ contained in the open disk D_ϑ. (Notice that, since θ_j^* is finite, the points $\Gamma(s), \Gamma(\sigma_j), \Gamma(\tau_j^*)$ are non-collinear, which remains true if we replace $\Gamma(\tau_j^*)$ by $\Gamma(\tilde{\tau})$.) Also the angle $\tilde{\alpha}$ of $\tilde{\triangle}$ in $\Gamma(\tilde{\tau})$ is in the same interval as α (either $(0, \pi/2)$ or $(\pi/2, \pi)$), for $|\tilde{\tau} - \tau_j^*|$ sufficiently small. But then

$$R(\Gamma(s), \Gamma(\sigma_j), \Gamma(\tilde{\tau})) < R(\Gamma(s), \Gamma(\sigma_j), \xi) = \vartheta \leq \theta_j^*$$

for any comparison point $\xi \in \partial D_\vartheta$, as one easily checks with (2). Thus we have shown (110) and hence also (109).

Now we intersect the plane P_j spanned by the two vectors $\Gamma(s) - \Gamma(\tau_j^*)$ and $\Gamma'(\tau_j^*)$ with $\partial B_{\theta_j^*}(y_j)$ to obtain a circle C_j^* through $\Gamma(s)$ and $\Gamma(\tau_j^*)$ and tangent to $\Gamma'(\tau_j^*)$ with radius $r(C_j^*) \le \theta_j^*$. On the other hand, we may write

$$\theta_j^* \ge r(C_j^*) = r(\Gamma(s), \Gamma(\tau_j^*), \Gamma'(\tau_j^*)) \ge \inf_{\substack{\sigma\in S_L \\ \sigma\neq s}} r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma)),$$

which together with (103)–(105) gives the desired inequality

$$\rho_G[\gamma](s) \ge \inf_{\substack{\sigma\in S_L \\ \sigma\neq s}} r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma)). \tag{111}$$

Summarizing (99), (100), (101), (107), (108), (111) leads to the first equality in (A) and (B), respectively. The second equality in (B) was shown in (102). Notice that $\rho_G[\gamma](s) = \rho[\gamma](s)$ may or may not hold in alternative (A). Taking the infimum over $s \in S_L$ on both sides of the equations (A) and (B) gives (30). □

Proof of Proposition 2.6. By Lemma 2.2 we have $\mathcal{K}[\gamma] > 0$. Hence $r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma))$ is finite for all $(s, \sigma) \in A[\gamma]$, therefore $s \neq \sigma$ by definition of $r(.,.,.)$.

Let $(s, \sigma) \in A[\gamma]$. If (33) is true, then $\Gamma(s), \Gamma(\sigma)$ lie diametrically on the unique circle C through these points, which is tangential to $\Gamma'(\sigma)$ in $\Gamma(\sigma)$, with radius

$$r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma)) = \mathcal{K}[\gamma]^{-1} = \mathcal{R}[\gamma].$$

Theorem 2.1, part (iv) (b) gives

$$M(\sigma, \mathcal{R}[\gamma]) \cap \Gamma(S_L) = \emptyset,$$

which means that in the points $\Gamma(s)$ and $\Gamma(\sigma)$, the curve Γ is tangential to the unique sphere of radius $\mathcal{R}[\gamma]$ that contains C as a great circle. Now (34) follows.

It remains to show (33). We start with a technical lemma.

Lemma 5.3. *Let $\|\Gamma''\|_{L^\infty} \le l$, and assume that $\Gamma(s), \Gamma(t) \in C_R$ where $s \neq t$, and where C_R is a circle of radius $R < l^{-1}$ tangential to $\Gamma'(t)$. Then*

$$|s - t| > \frac{1 - Rl}{l}. \tag{112}$$

Proof. Without loss of generality we may assume $s > t$, and choose the coordinate system such that $\Gamma(t) = 0$, $\Gamma'(t) = (1,0,0)$, and that the center of C_R is the point $(0,0,R)$. Then we have the expansions

$$\begin{aligned}
\Gamma^3(s) &= \int_t^s \Gamma^{3\prime}(\tau)\,d\tau = \int_t^s (\Gamma^{3\prime}(\tau) - \Gamma^{3\prime}(t))\,d\tau,\\
\Gamma^1(s) &= \Gamma^{1\prime}(t)(s-t) + \int_t^s (\Gamma^{1\prime}(\tau) - \Gamma^{1\prime}(t))\,d\tau,\\
\int_t^s (\Gamma^{j\prime}(\tau) - \Gamma^{j\prime}(t))\,d\tau &= \int_t^s \int_t^\tau \Gamma^{j\prime\prime}(\sigma)\,d\sigma d\tau,
\end{aligned}$$

which immediately gives the estimates

$$|\Gamma^3(s)| \leq l|s-t|^2/2, \tag{113}$$

$$|\Gamma^1(s)| \geq |s-t| - l|s-t|^2/2, \tag{114}$$

If the right-hand side of (114) is non-positive then

$$|s-t| \geq \frac{2}{l} > \frac{1-Rl}{l},$$

and we are done. If on the other hand, the right-hand side of (114) is positive, we estimate for $\Gamma(s) \in C_R$

$$\begin{aligned}
R^2 &\geq (\Gamma^1(s))^2 + (\Gamma^3(s) - R)^2\\
&\geq [1 - Rl - l|s-t|]|s-t|^2 + R^2 + l^2|s-t|^4/4\\
&> R^2,
\end{aligned}$$

unless the term in brackets is negative, which means

$$|s-t| > \frac{1-Rl}{l}\,.$$

□

To prove (33) we assume the contrary, i.e., that for $(s,t) \in A[\gamma]$, $P := \Gamma(s), Q := \Gamma(t)$, $\theta := \mathcal{R}[\gamma]$,

$$2\rho := |P - Q| < 2\theta. \tag{115}$$

Then

$$m := \frac{P+Q}{2} \in B_\theta(\Gamma(S_L)),$$

where $B_\theta(\Gamma(S_L))$ is regular according to Proposition 3.1, which is proved independently. Hence we can look at the continuous projection Π_Γ and conclude that $\Pi_\Gamma(m) \neq P, Q$, since $|P-m| = |Q-m| = \rho$, $P \neq Q$, since $s \neq t$, but the projection $\Pi_\Gamma(m)$ is a unique point on Γ. Hence there is a unique $S \neq P, Q$, $S \in \Gamma(S_L) \cap B_\rho(m)$ with $\Pi_\Gamma(m) = S$. Let $\partial B_\theta(y)$ be the sphere of radius θ containing the unique circle through $\Gamma(s), \Gamma(t)$, tangent to $\Gamma'(t)$ in $\Gamma(t)$, as a great circle. Then by Theorem 2.1, part (iv) (b), we know that $S \notin B_\theta(y) \subset M(t, \theta)$.

Case I. Let $S \notin \overline{B_\theta(y)}$. The angle of the triangle $\triangle(P, Q, S)$ in S lies in $(\pi/2, \pi)$, since $S \in B_\rho(m)$. Consequently,

$$\begin{aligned} R(Q, P, S) &= \frac{|Q-P|}{2|\sin \sphericalangle(Q-S, P-S)|} \\ &< \frac{|Q-P|}{2|\sin \sphericalangle(Q-S^*, P-S^*)|} = R(P, Q, S^*) \leq \theta, \end{aligned} \tag{116}$$

for any $S^* \in \partial B_\theta(y) \cap E^*$, $S^* \neq P, Q$, where E^* is the plane spanned by P, Q, S. (116) contradicts the definition of $\theta = \mathcal{R}[\gamma]$.

Case II. If $S \in \partial B_\theta(y)$, then we claim that we find $\tilde{S} \in \Gamma(S_L)$ arbitrarily close to S with $\tilde{S} \notin \overline{B_\theta(y)}$ and apply the argument as in Case I to $\tilde{S}$ instead of S to arrive at a contradiction. To show that $\tilde{S}$ exists, we assume not, i.e, we assume that for $S = \Gamma(\sigma)$ we have $\Gamma(B_\delta(\sigma)) \subset \partial B_\theta(y)$, for some small $\delta > 0$. For $(\tau_j, s_j) \to (\sigma, \sigma)$ one gets $R(\Gamma(\sigma), \Gamma(\tau_j), \Gamma(s_j)) \leq \theta$, for j sufficiently large. Hence $\kappa[\gamma](\sigma) \geq \theta^{-1}$ by (25). This is also true for any $\bar{\sigma} \in B_\delta(\sigma)$, which means that $\|\Gamma''\|_{L^\infty} \geq \theta^{-1} = \mathcal{K}[\gamma]$, contradicting the assumption that $\mathcal{K}[\gamma]$ is locally not attained. □

Proof of Lemma 2.7. (i). By the classical theorem of Fenchel [8] we have

$$\int_0^L |\Gamma''(t)|\, dt \geq 2\pi,$$

which readily implies

$$\|\Gamma''\|_{L^\infty} \geq \frac{2\pi}{L},$$

hence

$$\eta(\gamma) \le \frac{1 - \mathcal{R}[\gamma] \cdot (2\pi/L)}{(2\pi/L)} < \frac{L}{2\pi}.$$

On the other hand, $\eta(\gamma) > 0$, since $\mathcal{K}[\gamma]$ is locally not attained, see (28).

(ii)&(iii). We take a maximizing sequence $\{s_j\}_{j\in\mathbb{N}} \subset S_L$ for $\mathcal{K}[\gamma]$ in the definition (5). In other words, for every $\epsilon > 0$, we find j_0 such that for all $j \ge j_0$,

$$\mathcal{K}[\gamma] \le \kappa_G[\gamma](s_j) + \epsilon. \tag{117}$$

Take $0 < \epsilon < \mathcal{K}[\gamma] - \|\Gamma''\|_{L^\infty}$, which is possible by (28). According to Theorem 2.3 (ii) we have two possible cases (a) and (b) for $\kappa_G[\gamma](s_j)$ for fixed $j \ge j_0$. In case (a) there is a sequence $(\sigma_k, \tau_k) \to (s_j, s_j)$ with

$$\kappa_G[\gamma](s_j) = \lim_{k\to\infty} \frac{1}{R(\Gamma(s_j), \Gamma(\sigma_k), \Gamma(\tau_k))}$$

by (23). We use Proposition 2.4 to conclude that

$$\kappa_G[\gamma](s_j) \le \limsup_{\substack{(t_l,u_l)\to(s_j,s_j)\\ s_j\ne t_l\ne u_l\ne s_j}} \frac{1}{R(\Gamma(s_j), \Gamma(t_l), \Gamma(u_l))} = \kappa[\gamma](s_j) \le \|\Gamma''\|_{L^\infty}.$$

Hence by choice of ϵ we get

$$\mathcal{K}[\gamma] \le \|\Gamma''\|_{L^\infty} + \epsilon < \mathcal{K}[\gamma],$$

which is absurd, hence Case (a) cannot occur.

If on the other hand, Case (b) holds, then we know by the calculation that led to (82) that

$$\kappa_G[\gamma](s_j) = \frac{1}{r(\Gamma(s_j), \Gamma(\tau_j), \Gamma'(\tau_j))} =: r_j^{-1}$$

for some $\tau_j \ne s_j$. Now apply Lemma 5.3 for $l = \|\Gamma''\|_{L^\infty}$, $R = r_j$, which is applicable since by choice of ϵ,

$$r_j^{-1} \ge \mathcal{K}[\gamma] - \epsilon > \|\Gamma''\|_{L^\infty} = l.$$

We get

$$|s_j - \tau_j| \ge \eta(\gamma) \ \text{ for all } \ j \ge j_0.$$

Then we can go to the limit for a subsequence $(s_j, \tau_j) \to (s, \tau)$ in (117) to get

$$\mathcal{K}[\gamma] = \frac{1}{r(\Gamma(s), \Gamma(\tau), \Gamma'(\tau))}, \tag{118}$$

where $|s - \tau| \geq \eta(\gamma)$.

This in fact furnishes the proof of (iii), since assuming (118) we can apply Lemma 5.3 directly to get $|s - \tau| \geq \eta(\gamma)$. Recall that for $s, \tau \in S_L$, the term $|s - \tau|$ denotes the intrinsic distance on S_L, whence the other statement, $|s - \tau| \leq L - \eta(\gamma)$ for $s, \tau \in [0, L]$ is automatically proved as well.

(ii) It remains to show that we can restrict our attention to the set $\mathcal{Q}$ as stated in the Lemma. In fact, if

$$\mathcal{R}[\gamma] = r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma)), \tag{119}$$

for some $(s, \sigma) \notin \mathcal{Q}$ (but with $|s - \sigma| \geq \eta(\gamma)$ as we have just shown), then we will show that the reversed pair $(\sigma, s) \in \mathcal{Q}$ also satisfies (119), which shows that statement (ii) is true. Indeed, we know by Theorem 2.1, part (iv) (b), that for $\theta := \mathcal{R}[\gamma]$

$$M(\sigma, \theta) \cap \Gamma(S_L) = \emptyset. \tag{120}$$

If $\partial B_\theta(y)$ is the sphere that contains the circle through $\Gamma(s), \Gamma(\sigma)$ tangent to $\Gamma'(\sigma)$ as a great circle, then (120) implies, that $(\Gamma(s) - y) \cdot \Gamma'(s) = 0$. Thus we can also look at the circle through $\Gamma(s)$, $\Gamma(\sigma)$ tangent to $\Gamma'(s)$ with radius $\rho \leq \theta$. But (30) actually implies $\rho = \theta$, hence

$$\mathcal{R}[\gamma] = r(\Gamma(\sigma), \Gamma(s), \Gamma'(s)). \tag{121}$$

□

Proof of Proposition 3.1. For completeness of presentation we recall the proof that was already carried out in [12, Lemma 3].

(i) The first claim is that if $\mathcal{R}[\gamma] \geq \theta > 0$, then the tubular neighbourhood $B_\theta(\Gamma(S_L))$ is regular as defined above. To show that the closest-point projection map Π_Γ is well-defined for $x \in B_\theta(\Gamma(S_L))$, we note that if $\text{dist}(x, \Gamma(S_L)) = 0$, then $x = \Pi_\Gamma(x)$ is well-defined since γ is simple by Theorem 2.1. If $0 < \text{dist}(x, \Gamma(S_L)) < \theta$, then there is at least one point $s \in S_L$ such that $|x - \Gamma(s)| = \mathrm{dist}(x, \Gamma(S_L))$ since $\Gamma(S_L)$ is a compact set. For any such s the differentiable function $f(t) := |x - \Gamma(t)|^2$ has the property $f(t) \geq f(s) := \delta^2$ for all $t \in S_L$ where $\delta < \theta$. Thus $0 = f'(s) = 2(x - \Gamma(s)) \cdot \Gamma'(s)$. If there were another point $\sigma \in S_L$ with $f(\sigma) = f(s)$ $(s \neq \sigma)$ then

$$\Gamma(\sigma) \in \partial B_\delta(x) \backslash \{\Gamma(s)\} \subset B_\theta(y) \subset M(s, \theta)$$

where $y := \Gamma(s) + \theta(x - \Gamma(s))/|x - \Gamma(s)|$, which contradicts item (iv)(b) of Theorem 2.1. Hence $\Pi_\Gamma : B_\theta(\Gamma(S_L)) \to \Gamma(S_L)$ given by $\Pi_\Gamma(x) := \Gamma(s(x))$ for $x \in B_\theta(\Gamma(S_L))$ is well-defined. Assuming for contradiction that Π_Γ is not continuous, we could find a sequence $x_n \to x \in B_\theta(\Gamma(S_L))$ and a constant $c > 0$ with $|\Pi_\Gamma(x_n) - \Pi_\Gamma(x)| \geq c$. Since $\Gamma(S_L)$ is compact, we may assume that $\Pi_\Gamma(x_n) \to p \in \Gamma(S_L)$ with $|p - \Pi_\Gamma(x)| \geq c$. Using the continuity of the distance function $\mathrm{dist}(\cdot, \Gamma(S_L))$ and the uniqueness of $s(x)$ we obtain

$$\begin{aligned} \mathrm{dist}(x, \Gamma(S_L)) = |x - \Pi_\Gamma(x)| < |x - p| &= \lim_{n\to\infty} |x_n - \Pi_\Gamma(x_n)| \\ &= \lim_{n\to\infty} \mathrm{dist}(x_n, \Gamma(S_L)) = \mathrm{dist}(x, \Gamma(S_L)), \end{aligned}$$

which is a contradiction. Thus Π_Γ is also continuous and the regularity of $B_\theta(\Gamma(S_L))$ is established.

The second claim is that if $B_\theta(\Gamma(S_L))$ is regular, then $\mathcal{K}[\gamma] \leq \theta^{-1}$. To establish this claim, we assume $B_\theta(\Gamma(S_L))$ is regular which, by definition, implies that γ is simple. We assume for contradiction that $\mathcal{K}[\gamma] > \theta^{-1}$, i.e., $\mathcal{R}[\gamma] < \theta$, which implies that there is a point $s_0 \in S_L$ such that $\rho_G[\gamma](s_0) < \theta$. Then, by the Definition of ρ_G in (1), there exist distinct points $s_1, s_2 \in S_L$ different from s_0 such that $0 < \rho_G[\gamma](s_0) \leq \delta < \theta$ where $\delta = R(\Gamma(s_0), \Gamma(s_1), \Gamma(s_2))$. Moreover, since γ is simple, the points $\Gamma(s_0)$, $\Gamma(s_1)$ and $\Gamma(s_2)$ are distinct. These points define a unique circle C of radius δ, and we denote the centre of C by p. Without loss of generality we assume $0 = s_0 < s_1 < s_2 < L$ and we consider the disjoint, open subarcs of S_L defined by $E_0 = (s_0, s_1)$, $E_1 = (s_1, s_2)$ and $E_2 = (s_2, s_0)$.

Since $|p - \Gamma(s_i)| = \delta$ $(i = 0, 1, 2)$ we have $\mathrm{dist}(p, \Gamma(S_L)) \leq \delta < \theta$ which implies $p \in B_\theta(\Gamma(S_L))$. Moreover, we must have the strict inequality $\mathrm{dist}(p, \Gamma(S_L)) < \delta$ since by hypothesis there is a unique $s(p) \in S_L$ such that $\mathrm{dist}(p, \Gamma(S_L)) = |p - \Gamma(s(p))|$. Thus $s(p) \neq s_i$ $(i = 0, 1, 2)$ and we may assume $s(p) \in E_0$.

We next consider the subarc $E_* = E_1 \cup \{s_2\} \cup E_2$ so that $S_L = E_0 \cup E_* \cup \{s_0, s_1\}$, and we consider the line segment between p and $\Gamma(s_2)$, i.e.,

$$x(\alpha) = (1-\alpha)p + \alpha\Gamma(s_2), \quad \alpha \in [0,1].$$

This segment has the properties that $x(0) = p$, $x(1) = \Gamma(s_2)$,

$$|x(\alpha) - \Gamma(s_2)| < |x(\alpha) - \Gamma(s_i)|, \quad 0 < \alpha \leq 1 \quad (i = 0,1),$$

and $x(\alpha) \in B_\theta(\Gamma(S_L))$ for $0 \leq \alpha \leq 1$. To obtain the required contradiction, notice that

$$\begin{aligned} \operatorname{dist}(x(\alpha), \Gamma(S_L)) &\leq |x(\alpha) - \Gamma(s_2)| \\ &< |x(\alpha) - \Gamma(s_i)|, \quad 0 < \alpha \leq 1 \quad (i = 0,1), \end{aligned}$$

which implies $\Pi_\Gamma(x(\alpha)) \neq \Gamma(s_i)$ for $0 < \alpha \leq 1$ $(i = 0,1)$. However, $\Pi_\Gamma(x(0)) = \Gamma(s(p)) \in \Gamma(E_0)$ and $\Pi_\Gamma(x(1)) = \Gamma(s_2) \in \Gamma(E_*)$. Thus the image of the line segment $x(\alpha)$ under the map Π_Γ is disconnected. Since this contradicts the hypothesis that $B_\theta(\Gamma(S_L))$ is regular we must have $\mathcal{K}[\gamma] \leq \theta^{-1}$ as claimed.

(ii) We assume that $B_\theta(\Gamma(S_L))$ is regular. Then for each $x \in B_\theta(\Gamma(S_L))$ there is a unique $s = s(x) \in S_L$ such that $|x - \Gamma(s)| < \theta$ and $(x - \Gamma(s)) \cdot \Gamma'(s) = 0$. Notice that for each point x in a given normal disk $D_\theta(s_0) := D_\theta(\Gamma(s_0), \Gamma'(s_0))$ the point s_0 has these properties, which implies $s(x) = s_0$ for all $x \in D_\theta(s_0)$. Thus $\Pi_\Gamma(D_\theta(s_0)) = \Gamma(s_0)$. Assuming for contradiction that there is a point $y \in B_\theta(\Gamma(S_L)) \backslash D_\theta(s_0)$ such that $\Pi_\Gamma(y) = \Gamma(s_0)$, we must have $(y - \Gamma(s_0)) \cdot \Gamma'(s_0) = 0$, which implies $y \in \overline{D_\mu(s_0)} \backslash D_\theta(s_0)$ for some $\mu \geq \theta$. However, for such a point we would have $\operatorname{dist}(y, \Gamma(S_L)) \geq \theta$, which is a contradiction. The claim follows. □

Proof of Theorem 3.2. Theorem 3.2 was proven in [12, Lemma 7], for the convenience of the reader we include the proof here as well.

Notice that, for each fixed $s \in S_L$, the map $p(t, \cdot, \cdot)$ is injective and that the image of $p(t, \cdot, \cdot)$ is the open disk $D_\theta(\Gamma(s(t)), \Gamma'(s(t)))$ as considered in Proposition 3.1.

Our first claim is that if $\mathcal{K}[\gamma] \leq \theta^{-1}$, then $p : \Omega \to \mathbb{R}^3$ is globally injective. To see this, assume for contradiction that p does not have this property. Then there exists $s_1, s_2 \in S_L$, $s_1 \neq s_2$, such that $D_\theta(\Gamma(s_1), \Gamma'(s_1)) \cap D_\theta(\Gamma(s_2), \Gamma'(s_2)) \neq \emptyset$. We denote by x any point in this intersection. Since

$\mathcal{K}[\gamma] \leq \theta^{-1}$ we may apply Proposition 3.1 (i) to conclude that the projection $\Pi_\Gamma : B_\theta(\Gamma(S_L)) \to \Gamma(S_L)$ is single-valued, and apply Proposition 3.1 (ii) to conclude that $\Pi_\Gamma(x) = \Gamma(s_1)$ and $\Pi_\Gamma(x) = \Gamma(s_2)$, which is a contradiction. Thus $p : \Omega \to \mathbb{R}^3$ must be globally injective.

Our second claim is that if $p : \Omega \to \mathbb{R}^3$ is globally injective, then $\mathcal{K}[\gamma] \leq \theta^{-1}$. To see this, assume for contradiction that $\theta^{-1} < \mathcal{K}[\gamma] < \infty$, or equivalently, $0 < \mathcal{R}[\gamma] < \theta$ and consider any η such that $\mathcal{R}[\gamma] < \eta < \theta$. Then by Theorem 2.1 (iv)(b) there is a parameter $s_* \in S_L$ such that $\Gamma(S_L) \cap M(s_*, \eta) \neq \emptyset$. This implies there is a point $z_* \in C(s_*, \eta) = \partial D_\eta(\Gamma(s_*), \Gamma'(s_*))$ such that $\operatorname{dist}(z_*, \Gamma(S_L)) < \eta$. By compactness, there is a point $\Gamma(\bar{s})$ such that $\operatorname{dist}(z_*, \Gamma(S_L)) = |z_* - \Gamma(\bar{s})|$, and $\bar{s} \neq s_*$ since $|z_* - \Gamma(\bar{s})| < \eta$. Moreover, $(z_* - \Gamma(\bar{s})) \cdot \Gamma'(\bar{s}) = 0$. Since $\eta < \theta$ we have $z_* \in D_\theta(\Gamma(\bar{s}), \Gamma'(\bar{s}))$ and also $z_* \in D_\theta(\Gamma(s_*), \Gamma'(s_*))$, which contradicts the global injectivity of $p : \Omega \to \mathbb{R}^3$. Thus $\mathcal{K}[\gamma] \leq \theta^{-1}$ as claimed. □

Proof of Lemma 4.1. Let $\{\gamma_i\} \subset F_1$ be a sequence with

$$\|\gamma_i - \gamma\|_{C^0} \to 0 \text{ as } i \to \infty. \tag{122}$$

Since $L(\gamma_i) \leq L_0$ for all $i \in \mathbb{N}$, we know that $\gamma \in BV(I, \mathbb{R}^3) \cap \mathcal{L}$ with length $L(\gamma) \leq L_0$. The remaining part of the proof is identical to the proof (in terms of $\mathcal{R}[.]$ rather than $\mathcal{K}[.]$) of [12, Lemma 4]. One simply needs to check that the uniform convergence (122) suffices to carry out the arguments in [12, pp. 29,30]. □

Proof of Proposition 4.2. Assuming the contrary we can find a sequence $\gamma_j \to \gamma$ in $C^0(\bar{I}, \mathbb{R}^3)$, with $\mathcal{K}[\gamma_j] \leq K_0$ for all $j \in \mathbb{N}$, i.e., by (9), $\mathcal{R}[\gamma_j] \geq K_0^{-1}$ for all $j \in \mathbb{N}$, such that $\gamma \not\simeq \gamma_j$ for all $j \in \mathbb{N}$. This contradicts the proof of [12, Lemma 5], where under the above conditions an isotopy mapping is constructed explicitly to get $\gamma_j \simeq \gamma$ for sufficiently large $j \in \mathbb{N}$. □

Proof of Theorem 4.3. By virtue of Lemma 4.1 it suffices to prove that for $\{\gamma_i\}_{i\in\mathbb{N}} \subset F_2$ with

$$\|\gamma_i - \gamma\|_{C^0} \to 0 \text{ as } i \to \infty, \tag{123}$$

one has $\gamma(\bar{I}) \simeq k_0$. This was shown in [12, Lemma 5] but follows here as a simple corollary of the stronger result Proposition 4.2. In fact, for i sufficiently large we have condition (41), because of (123), and Lemma 4.1 implies $\mathcal{K}[\gamma] \le K_0$, i.e., assumption (40) in particular. $\square$

Proof of Theorem 4.4. If $\mathcal{R}[\gamma] = 0$, then by Theorem 2.1, part (iii), γ must have a double point, i.e., $\Gamma(s) = \Gamma(t)$ for some $s \neq t$ in S_L. Then any sequence $\{\gamma_i\} \subset \mathcal{L}$ of fixed length $L(\gamma_i) = L$ with $\|\Gamma_i - \Gamma\|_{C^0}$ satisfies $\mathcal{R}[\gamma_i] \to 0$ as $i \to \infty$. This is true, because assuming the contrary would lead to an immediate contradiction to Lemma 4.1. Now assume $\mathcal{R}[\gamma] > 0$ for some $\gamma \in \mathcal{L}$ with length $L(\gamma) = L$, hence γ is simple in particular. We take a sequence of curves γ_l with arc length parameterizations Γ_l converging to Γ in $C^{1,1}([0, L], \mathbb{R}^3)$, where Γ is the arc length parameterization of γ. We proceed in several steps.

1. We claim that there is $l_0, \rho > 0$, such that

$$|\Gamma_l(s) - \Gamma_l(t)| \ge \frac{|s-t|}{2} \quad \text{for all } \; l \ge l_0, |s-t| < \rho. \tag{124}$$

In fact, expanding Γ_l about $t \in S_L$, we obtain

$$\begin{aligned} |\Gamma_l(s) - \Gamma_l(t)| &\ge |s-t| - \left| \int_t^s (\Gamma_l'(\sigma) - \Gamma_l'(t))\, d\sigma \right| \\ &\ge |s-t| - \left| \int_t^s (\Gamma'(\sigma) - \Gamma'(t))\, d\sigma \right| \\ &\qquad - 2\|\Gamma_l' - \Gamma'\|_{C^0} |s-t|. \end{aligned}$$

Choosing first l_0 so large that $\|\Gamma_l - \Gamma\|_{C^1} < 1/8$ for all $l \ge l_0$, and then $\rho > 0$ (depending on the Lipschitz constant $\mathcal{K}[\gamma]$ of Γ') so small that

$$|\Gamma'(t_1) - \Gamma'(t_2)| \le 1/4 \; \text{ for all } \; |t_1 - t_2| < \rho,$$

we arrive at (124). For later purposes we take ρ so small that

$$\rho \le \frac{1}{4\|\Gamma''\|_{L^\infty}}\,. \tag{125}$$

2. We claim that there are constants $d > 0, l_1 \ge l_0$ such that for all $l \ge l_1$ and all $s, \sigma \in S_L$ with $|s - \sigma| \ge \rho$ one has

$$|\Gamma_l(s) - \Gamma_l(\sigma)| \ge d. \tag{126}$$

It suffices to prove the claim for the fixed simple limit curve γ, (with some constant $d = d_\gamma$), since then the claim follows from the simple estimate

$$|\Gamma_l(s) - \Gamma_l(\sigma)| \geq d_\gamma - 2\|\Gamma_l - \Gamma\|_{C^0} \geq d_\gamma/2$$

for all $|s - \sigma| \geq \rho$ and for all $l \geq l_1$, with $l_1 \geq l_0$ sufficiently large. Assuming that (126) is not true for Γ, one finds by compactness $(s_k, \sigma_k) \to (s, \sigma)$ with $|s - \sigma| \geq \rho$, as $k \to \infty$, with

$$|\Gamma(s_k) - \Gamma(\sigma_k)| \leq \frac{1}{k}\,.$$

Hence by continuity of Γ we infer $\Gamma(s) = \Gamma(\sigma)$, i.e., $\mathcal{R}[\gamma] = 0$, which is excluded here.

3. Combining (124) and (126) we get for l sufficiently large, that γ_l is simple, which by Theorem 2.1, part (iii) implies that $\mathcal{R}[\gamma_l] > 0$ for sufficiently large $l \in \mathbb{N}$.

4. The assumption that

$$\lim_{l\to\infty} \mathcal{K}[\gamma_l] < \mathcal{K}[\gamma], \tag{127}$$

will lead to a contradiction. Indeed, then by (27) we have

$$\|\Gamma_l''\|_{L^\infty} \leq \mathcal{K}[\gamma] - \epsilon$$

for some fixed $\epsilon > 0$ and l sufficiently large. But due to the convergence $\|\Gamma - \Gamma_l\|_{C^{1,1}} \to 0$ as $l \to \infty$, we then get

$$\|\Gamma''\|_{L^\infty} \leq \mathcal{K}[\gamma] - \epsilon.$$

This means that $\mathcal{K}[\gamma]$ is locally not attained, see (28), which implies by Lemma 2.7, part (ii), that

$$\mathcal{K}[\gamma] = \frac{1}{r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma))} \tag{128}$$

for some $s, \sigma \in S_L$ with $|s - \sigma| \geq \eta(\gamma)$, where

$$\eta(\gamma) = \frac{1 - \mathcal{R}[\gamma] \cdot \|\Gamma''\|_{L^\infty}}{\|\Gamma''\|_{L^\infty}} = \frac{1 - \mathcal{R}[\gamma] \cdot (\mathcal{K}[\gamma] - \epsilon)}{\mathcal{K}[\gamma] - \epsilon} \geq \epsilon \mathcal{R}[\gamma]^2 > 0.$$

But the right-hand side in (128) may be approximated as

$$\mathcal{K}[\gamma_l] \underset{(31)}{\geq} \frac{1}{r(\Gamma_l(s),\Gamma_l(\sigma),\Gamma_l'(\sigma))} \longrightarrow \frac{1}{r(\Gamma(s),\Gamma(\sigma),\Gamma'(\sigma))} = \mathcal{K}[\gamma]$$

contradicting (127).

5. Assuming on the other hand

$$\lim_{l\to\infty} \mathcal{K}[\gamma_l] > \mathcal{K}[\gamma], \tag{129}$$

we find $\epsilon > 0$ and $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$ (by (27) applied to Γ),

$$\mathcal{K}[\gamma_l] \geq \mathcal{K}[\gamma] + \epsilon \geq \|\Gamma''\|_{L^\infty} + \epsilon. \tag{130}$$

This together with $\|\Gamma_l - \Gamma\|_{C^{1,1}} \to 0$ as $l \to \infty$ gives us

$$\mathcal{K}[\gamma_l] \geq \|\Gamma_l''\|_{L^\infty} + \epsilon/2 \tag{131}$$

for all l sufficiently large. Hence for all l sufficiently large we know that $\mathcal{K}[\gamma_l]$ is locally not attained. Moreover $\mathcal{R}[\gamma_l]$ is positive, hence $\mathcal{K}[\gamma_l]$ finite, and by Lemma 2.7 we conclude that for each such l there exist $s_l, \sigma_l \in S_L$ with $|s_l - \sigma_l| \geq \eta(\gamma_l) > 0$, such that

$$\mathcal{K}[\gamma_l] = \frac{1}{r(\Gamma_l(s_l),\Gamma_l(\sigma_l),\Gamma_l'(\sigma_l))}. \tag{132}$$

We claim that there is a uniform constant $c > 0$, such that for all l sufficiently large

$$\eta(\gamma_l) \geq c. \tag{133}$$

Indeed, assuming $\eta(\gamma_l) \to 0$ as $l \to \infty$, we get

$$\mathcal{R}[\gamma_l]\|\Gamma_l''\|_{L^\infty} \to 1, \tag{134}$$

since $\|\Gamma_l''\|_{L^\infty} \to \|\Gamma''\|_{L^\infty} \in (0,\infty)$. (134) together with (9) implies $\mathcal{K}[\gamma_l] \to \|\Gamma''\|_{L^\infty}$ contradicting (130), which proves the claim.

By compactness we may assume that $s_l \to s$ and $\sigma_l \to \sigma$ in S_L for $l \to \infty$, with $|s - \sigma| \geq c > 0$, by (133), which means that we can go to the limit in (132) to obtain

$$\lim_{l\to\infty} \mathcal{K}[\gamma_l] = \lim_{l\to\infty} \frac{1}{r(\Gamma_l(s_l),\Gamma_l(\sigma_l),\Gamma_l'(\sigma_l))} = \frac{1}{r(\Gamma(s),\Gamma(\sigma),\Gamma'(\sigma))} \leq \mathcal{K}[\gamma],$$

where we used (31) applied to γ to obtain the last inequality. But this contradicts (129), hence if $\lim_{l\to\infty}\mathcal{K}[\gamma_l]$ exists it must be equal to $\mathcal{K}[\gamma]$.

6. We actually proved with the previous argument that the sequence $\{\mathcal{K}[\gamma_l]\}_{l\in\mathbb{N}}\subset\mathbb{R}$ must be bounded, since otherwise we deduce a contradiction exactly as we have just done. Hence there exists a convergent subsequence $l_k, k\in\mathbb{N}$, with

$$\lim_{k\to\infty}\mathcal{K}[\gamma_{l_k}]=\mathcal{K}[\gamma].$$

By the subsequence principle, the whole sequence must actually converge to $\mathcal{K}[\gamma]$, since we have proved that if a limit exists it must equal $\mathcal{K}[\gamma]$. This proves (sequential) continuity. □

Proof of Lemma 4.5. (i) The weak convergence implies

$$\gamma_{1,n}\to\gamma_1 \text{ and } \gamma_{2,n}\to\gamma_2 \text{ in } C^0(\bar{I},\mathbb{R}^3) \tag{135}$$

as $n\to\infty$. Hence we can assume that $\gamma_1,\gamma_2\in\mathcal{L}\cap W^{1,q}(I,\mathbb{R}^3)$ and by (43)

$$\operatorname{dist}(\gamma_1(\bar{I}),\gamma_2(\bar{I}))\geq c. \tag{136}$$

Thus $l(\gamma_1,\gamma_2)$ is well-defined, and we can estimate

$$\begin{aligned}|l(\gamma_{1,n},\gamma_{2,n}-l(\gamma_1,\gamma_2)|\leq&\\ &\left|\int_I\int_I\left(\frac{\gamma_{1,n}(s)-\gamma_{2,n}(t)}{|\gamma_{1,n}(s)-\gamma_{2,n}(t)|^3}-\frac{\gamma_1(s)-\gamma_2(t)}{|\gamma_1(s)-\gamma_2(t)|^3}\right)\cdot[\gamma_{1,n}'(s)\wedge\gamma_{2,n}'(t)]\,ds\,dt\right|\\ &+\left|\int_I\int_I\frac{\gamma_1(s)-\gamma_2(t)}{|\gamma_1(s)-\gamma_2(t)|^3}\cdot[(\gamma_{1,n}'(s)-\gamma_1'(s))\wedge\gamma_{2,n}'(t)]\,ds\,dt\right|\\ &+\left|\int_I\int_I\frac{\gamma_1(s)-\gamma_2(t)}{|\gamma_1(s)-\gamma_2(t)|^3}\cdot[\gamma_1'(s)\wedge(\gamma_{2,n}'(t)-\gamma_2(t)]\,ds\,dt\right| \qquad (137)\\ &=:\mathrm{I}_n+\mathrm{II}_n+\mathrm{III}_n.\end{aligned}$$

We estimate the three terms on the right separately.

$$\begin{aligned}\mathrm{I}_n&=\left|\int_I\gamma_{2,n}'(t)\cdot\left[\int_I\left(\frac{\gamma_{1,n}(s)-\gamma_{2,n}(t)}{|\gamma_{1,n}(s)-\gamma_{2,n}(t)|^3}-\frac{\gamma_1(s)-\gamma_2(t)}{|\gamma_1(s)-\gamma_2(t)|^3}\right)\wedge\gamma_{1,n}'(s)\,ds\right]dt\right|\\ &\leq\sup_{s,t\in I}\left|\frac{\gamma_{1,n}(s)-\gamma_{2,n}(t)}{|\gamma_{1,n}(s)-\gamma_{2,n}(t)|^3}-\frac{\gamma_1(s)-\gamma_2(t)}{|\gamma_1(s)-\gamma_2(t)|^3}\right|\|\gamma_{1,n}'\|_{L^1}\|\gamma_{2,n}'\|_{L^1}\to 0, \qquad (138)\end{aligned}$$

as $n \to \infty$, by (135),(136) and the fact that because of the weak convergence the $W^{1,q}$-norms of $\gamma_{1,n}$ and $\gamma_{2,n}$ are uniformly bounded.

$$\begin{aligned} \mathrm{II}_n &= \left| \int_I \gamma_{2,n}'(t) \cdot \left[\int_I \frac{\gamma_1(s) - \gamma_2(t)}{|\gamma_1(s) - \gamma_2(t)|^3} \wedge (\gamma_{1,n}'(s) - \gamma_1'(s))\, ds \right] dt \right| \\ &=: \left| \int_I \gamma_{2,n}'(t) \cdot \xi_n(t)\, dt \right|, \end{aligned} \tag{139}$$

where we have denoted the inner integral by $\xi_n(t)$. By the weak convergence $\gamma_{1,n}' \rightharpoonup \gamma_1'$ in $L^q(I, \mathbb{R}^3)$, one finds

$$|\xi_n(t)| \to 0 \ \text{ as } \ n \to \infty \ \text{ for all } \ t \in I,$$

hence also

$$|\xi_n(t)|^{q^*} \to 0 \ \text{ as } \ n \to \infty \ \text{ for all } \ t \in I, \tag{140}$$

where $q^{-1} + q^{*-1} = 1$. Furthermore, using (136),

$$|\xi_n(t)|^{q^*} \leq \frac{|b-a|}{C^{2q^*}} (\|\gamma_{1,n}'\|_{L^q}^{q^*} + \|\gamma_1'\|_{L^q}^{q^*}) \leq \tilde{C} < \infty; \tag{141}$$

here $\tilde{C}$ does not depend on n. Thus by Lebesgue's Dominated Convergence Theorem we conclude that $\xi_n \to 0$ in L^{q^*}. This together with the weak convergence $\gamma_{2,n} \rightharpoonup \gamma_2$ in $W^{1,q}(I, \mathbb{R}^3)$ leads to $\mathrm{II}_n \to 0$ as $n \to \infty$.

The third term III_n has the same structure as II_n.

(ii) The embedding $W^{1,q} \hookrightarrow C^{0,1-(1/q)}$ implies that for $\|\gamma_i - \beta_i\|_{W^{1,q}}$ sufficiently small one has by (45)

$$\beta_1(\bar{I}) \cap \beta_2(\bar{I}) = \emptyset. \tag{142}$$

Hence the linking number $l(\beta_1, \beta_2)$ is well-defined. Going back to (138) and (139) with $\gamma_{i,n}$ replaced by β_i, $i = 1, 2$, one observes that the terms $\mathrm{I}_n, \mathrm{II}_n$ (and analogously III_n) can be made arbitrarily small by choosing ϵ sufficiently small. Notice that by (45) and (142) we can estimate the denominators on the right-hand side in (138) and (139) from below by some constant and that the norms $\|\beta_i'\|_{L^1}$ are bounded by, say $2\|\gamma_i'\|_{L^1}$ for $i = 1, 2$, for ϵ sufficiently small. □

Proof of Theorem 4.6. To apply Lemma 4.5 for $\gamma_{1,n} := \Gamma_n$, $\gamma_{2,n} := \Gamma_n + (\theta/2)d_{1,n}$, $\gamma_1 := \Gamma$ and $\gamma_2 := \Gamma + (\theta/2)d_1$, we merely need to verify that hypothesis (43) holds. In fact, part (iv)(b) of Theorem 2.1 implies that

$$\text{dist}\,(\Gamma_n([0,L]), (\Gamma_n + (\theta/2)d_{1,n})([0,L])) \geq \theta/2 \text{ for all } n \in \mathbb{N},$$

since $\mathcal{K}[\Gamma_n] \leq \theta^{-1}$ and because (38) was assumed to hold for all $n \in \mathbb{N}$. Thus the first part of Lemma 4.5 implies (i), and part (ii) follows immediately from the second part of that lemma. □

References

[1] S.S. ANTMAN, *Nonlinear Problems of Elasticity,* Springer, New York (1995).

[2] J.C. ALEXANDER, S.S. ANTMAN, The ambiguous twist of Love, *Quarterly Appl. Math.* **40**, (1982) 83–92.

[3] J. CANTARELLA, R.B. KUSNER, J.M. SULLIVAN, Crossing Numbers of Tight Knots, *GANG preprint* (Oct 1999).

[4] J. CANTARELLA, R.B. KUSNER, J.M. SULLIVAN, On the minimum ropelength of knots and links, *eprint arXiv:math.* GT/0103224 (2001).

[5] M. DO CARMO, *Riemannian Geometry,* Birkhäuser, Boston, Basel, Berlin (1992).

[6] L.C. EVANS, R.F. GARIEPY, *Measure Theory and Fine Properties of Functions,* Studies in Advanced Mathematics, CRC Press Boca Raton, New York, London, Tokyo (1992).

[7] H. FEDERER, *Geometric Measure Theory*, Grundlehren der math. Wissenschaften **153**, Springer, Berlin, Heidelberg, New York (1969).

[8] W. FENCHEL, Über die Krümmung und Windung geschlossener Raumkurven, *Math. Ann.* **101**, (1929) 238–252.

[9] M.H. FREEDMAN, ZHENG-XU HE, ZHENGHAN WANG, On the Möbius Energy of Knots and Unknots, *Annals of Math.* **139** no. 1, (1994) 1–50.

[10] M. GIAQUINTA, S. HILDEBRANDT, *Calculus of Variations*, Vols I, II, Grundlehren der math. Wissenschaften **310, 311**, Springer, Berlin, Heidelberg, New York (1996).

[11] O. GONZALEZ, J.H. MADDOCKS, Global Curvature, Thickness, and the Ideal Shape of Knots, *The Proceedings of the National Academy of Sciences, USA* **96** no. 9, (1999) 4769–4773.

[12] O. GONZALEZ, J.H. MADDOCKS, F. SCHURICHT, H. VON DER MOSEL, Global curvature and self-contact of nonlinearly elastic curves and rods, *Calc. Var.* online publ. DOI 10.1007/s005260100089, (2001).

[13] J. MILNOR, On the total curvature of knots, *Ann. of Math.* (2) **52**, (1950) 248–257.

[14] J. O'HARA, Energy of Knots, in A. Stasiak, V. Katritch, L.H. Kauffman (eds.), *Ser. on Knots and Everything* **19**, World Scientific, Singapore (1988) 288–314.

[15] F. SCHURICHT, Global injectivity and topological constraints for spatial nonlinearly elastic rods, preprint **1**, MPI f. Math. Leipzig (2001).

[16] F. SCHURICHT, H. VON DER MOSEL, Euler Lagrange equations for nonlinearly elastic rods with self-contact, manuscript.

[17] M. SPIVAK, *Differential Geometry* vol. I, Lecture Notes, Brandeis Univ. (1970).

[18] H. VON DER MOSEL, Minimizing the elastic energy of knots, *Asymptotic Anal.* **18**, (1998) 49–65.

[19] H. VON DER MOSEL, *Existence and Regularity for Nonlinear Elastic Self-Contact Problems,* Habilitation thesis, Universität Bonn (2001).