Characterization of ideal knots

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Abstract

We present a characterization of ideal knots, i.e., of closed knotted curves of prescribed thickness with minimal length, where we use the notion of global curvature for the definition of thickness. We show with variational methods that for an ideal knot γ, the normal vector $\gamma''(s)$ at a curve point $\gamma(s)$ is given by the integral over all contact chords $\gamma(\tau)-\gamma(s)$ against a Radon measure μ_s, where $|\gamma(\tau)-\gamma(s)|/2$ realizes the given thickness. As geometric consequences we obtain in particular, that points without contact lie on straight segments of γ, and for points $\gamma(s)$ with exactly one contact point $\gamma(\tau)$ we have that $\gamma''(s)$ points exactly into the direction of the contact chord $\gamma(\tau)-\gamma(s)$. Moreover, isolated contact points lie on straight segments of γ, and curved segments of γ consist of contact points only, all realizing the prescribed thickness with constant (maximal) global curvature. The set of contact parameters is closed due to the continuity of the global curvature function on this set. These results confirm various conjectures found in the literature.

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1 Introduction

Consider a closed knotted curve $\gamma : [0,1] \to \mathbb{R}^3$ such that the tubular neighborhood with given radius $\theta > 0$ does not intersect itself. Then we shrink the length $L(\gamma)$ of the curve until we arrive at some minimal length L_0 where the tube surrounding the shortened curve $\gamma_0([0,1])$ forms a completely tight knot which cannot be shortened anymore. The middle curve $\gamma_0([0,1])$ of such a tight tube is an *ideal knot* in the sense of [2], [9] and [11].

For a long time an accessible formulation of the problem was not avaiable. The main difficulty was to find a suitable analytical condition preventing self-intersection of the tubular neighborhood of the curve. In [10] several notions of thickness based e.g. on the critical self-distance introduced in [11], or on Gromov's concept of distortion were investigated and related to each other. Today, however, it seems to be more convenient to use a lower bound on the *global radius of curvature* to prescribe the thickness for a curve, a notion which was introduced in [7] and further analyzed in [8] and [13], see also [3].

Numerical simulations based on various notions of thickness indicated some interesting features of possible ideal configurations, see, e.g., the contributions in [15]. However, there was no satisfactory interpretation of these numerical results, since

(a) the existence of ideal knots was not verified yet, and

(b) there was no equation (or any other analytical or geometric characterization) of ideal knots, which could be checked to verify that the numerically computed configurations are indeed close to being ideal.

Meanwhile the existence of ideal knots for each knot type was shown independently by [3] and [8]. Furthermore it turned out that the minimizing curves belong to $C^{1,1}$. The present paper provides some necessary optimality conditions as a characterization of ideal knots and confirms a number of conjectures found in the literature (cf. [10],[3],[16]). In [7] such conditions are studied under the hypothetical smoothness assumption that ideal knots were C^2-curves, which we believe cannot be expected in general, since there are examples of ideal links, exhibited in [3], which are only $C^{1,1}$, and there seems to be strong numerical evidence that one can also find ideal knots which are not C^2-smooth, see [16]. Here we show that, for a.e. $s \in [0,1]$, the curvature vector $\gamma''(s)$ of the ideal knot γ belongs to the convex cone spanned by the chords realizing the supremum in the definition of the *global curvature function* at s, the inverse of the global radius of curvature function at s. In fact, we prove an integral representation formula for $\gamma''(s)$ in terms of the contact chords, see Theorem 3.1 in Section 3. This result implies several geometric consequences for the shape of an ideal knot. Parts of the ideal configuration where the tubular neighborhood does not exhibit self-contact have to be straight (which may be considered as special case of the cone condition for γ'' with a trivial cone). Consequently, there has to be self-contact on curved parts of the tubular neighborhood which implies a constant global curvature function there. These results are derived under the assumption that the local curvature of the ideal knot is strictly less than its *global curvature*, and it seems that this condition is not violated for a large class of ideal knots. Let us finally mention that our methods apply also to *ideal links* (i.e., several closed knotted curves that are linked with each other) and provide analogous results in this more general situation.

The proof essentially adapts the variational methods developed in [14], where the Euler-Lagrange equations are derived for elastic rods with self-contact. There, a similar variational problem involving an upper bound on global curvature as a nonsmooth side condition was treated by a Lagrange multiplier rule of Clarke's nonsmooth calculus of generalized gradients (cf. [4]). Furthermore the concept of framed curves, i.e., curves associated with an orthonormal frame at each point, played an important role. The key idea to transfer these methods to the present setting of ideal knots is, that $C^{1,1}$-curves can be represented as framed curves. This way the freedom of different parameterizations of a curve is removed in a natural way, thus avoiding unpleasant expressions for derivatives and artificial singularities. Speaking in mechanical terms we identify $C^{1,1}$-curves with a tubular elastic string which resists extension but without bending and torsional stiffness. From that point of view the above mentioned optimality conditions express that the resultant self-contact force at a cross section is directed as the curvature vector of the middle curve, and that bending can only occur at points of self-contact.

Section 2 introduces the notion of global curvature and summarizes a number of properties needed for our analysis. In particular a continuity result for "pointwise" global curvature is shown which ensures that the length parameters of the ideal shape corrresponding to "contact" form a closed set. The precise definition of ideal knots is given in Section 3, which also contains the main results about their characterization. In Section 4 we discuss the representation of $C^{1,1}$-curves as framed curves, which is an essential ingredient in the subsequent proof. Section 5 provides the proof of the main results, which is carried out in several steps.

Notation. The strictly postive real numbers are denoted by $\mathbb{R}_+$. We use $x \cdot y$ to denote the standard Euclidean inner product of x and y in $\mathbb{R}^3$, and $|\cdot|$ to denote the (intrinsic) distance between two points in $\mathbb{R}^3$ or in some parameter set $J \subset \mathbb{R}$ depending on the context. To denote the enclosed (smaller) angle between two non-zero vectors x and y in $\mathbb{R}^3$ we use $\sphericalangle(x, y) \in [0, \pi]$. The space of continuous functions on the closure of the interval $I = (0, 1)$ will be denoted by $C^0(\bar{I})$, and $C^{k,1}(\bar{I})$, $k = 0, 1, 2, \ldots$, is the space of k-times continuously differentiable functions whose k-th derivative is Lipschitz continuous on $\bar{I}$. $L^p(I)$, $1 \leq p \leq \infty$, stands for the Lebesgue space of p-integrable functions. For Sobolev spaces of functions, whose weak derivatives up to order m are p-integrable, we use the standard notation $W^{m,p}(I)$. Notice that $C^{k,1}(\bar{I}) \cong W^{k+1,\infty}(I)$. For general Banach spaces X with dual space X^* we denote the duality pairing on $X^* \times X$ by $\langle ., . \rangle_{X^* \times X}$.

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2 Global curvature

We consider the set $\mathcal{L}$ of continuous and rectifiable closed curves $\gamma : \bar{I} \to \mathbb{R}^3$, with arc length parameterization $\Gamma_\gamma : S_L \to \mathbb{R}^3$. Here $I = (0, 1)$ is the open unit interval, $L = L(\gamma) := \int_I |d\gamma| \geq 0$ denotes the length of γ, and S_L is the circle with perimeter L, which corresponds to the interval $[0, L]$ with identified endpoints, i.e., $S_L \cong \mathbb{R}/(L \cdot \mathbb{Z})$. The intrinsic distance on S_L and also the Euclidean distance in $\mathbb{R}^3$ will be denoted by $|\,.\,|$. To simplify notation, we mostly omit the subscript γ and agree that Γ, Γ_k correspond to γ, γ_k and so on. It is well-known that this arc length parameterization Γ is Lipschitz continuous, i.e. of class $C^{0,1}([0, L], \mathbb{R}^3)$. Note that, by Rademacher's Theorem, Γ possesses a derivative Γ' a.e. on $[0, L]$ and $C^{0,1}([0, L], \mathbb{R}^3) \cong W^{1,\infty}([0, L], \mathbb{R}^3)$.

For a closed curve $\gamma \in \mathcal{L}$ of positive length L the *global curvature* $\kappa_G[\gamma](s)$ *at the point* $s \in S_L$ is defined as

$$\kappa_G[\gamma](s) := \sup_{\substack{\sigma,\tau \in S_L \setminus \{s\} \\ \sigma \neq \tau}} \frac{1}{R(\Gamma(s), \Gamma(\sigma), \Gamma(\tau))}, \tag{1}$$

where $R(x, y, z) \geq 0$ is the radius of the smallest circle containing the points x, y, $z \in \mathbb{R}^3$. For collinear but pairwise distinct points x, y, z we set $R(x, y, z)$ to be infinite. When x, y and z are non-collinear (and thus distinct) there is a unique circle passing through them and

$$R(x,y,z) = \frac{|x-y|}{|2\sin[\sphericalangle(x-z,y-z)]|} = \frac{|x-y|}{2\left|\frac{x-z}{|x-z|} \wedge \frac{y-z}{|y-z|}\right|}. \tag{2}$$

If two points coincide, however, say $x = z$ or $y = z$, then there are many circles through the three points and we take $R(x, y, z)$ to be the smallest possible radius namely the distance $|x-y|/2$. We should point out that with this choice the function $R(x, y, z)$ fails to be continuous at points, where at least two of the arguments x, y, z, coincide. Notice nevertheless that, by definition, $R(x, y, z)$ is symmetric in its arguments. In [13] it was shown that $\kappa_G[\gamma](s)$ is always positive for closed curves. In the case of smooth curves that have no or only transversal crossings, our definition of κ_G agrees with the inverse of the *global radius of curvature* defined in [7], but it is different for curves with double covered regions, see [13] and the discussion in [8]. We define the *global curvature of* γ by

$$\mathcal{K}[\gamma] := \sup_{s \in S_L} \kappa_G[\gamma](s). \tag{3}$$

The curve γ is said to be *simple* if its arc length parameterization $\Gamma : S_L \to \mathbb{R}^3$ is injective. Otherwise there exist $s, t \in S_L$ $(s \neq t)$ for which $\Gamma(s) = \Gamma(t)$. Any such pair will be called a *double point* of Γ.

The condition that the global curvature $\mathcal{K}[\gamma]$ is finite identifies curves with a $C^{1,1}$-arc length parameterization without double points which was shown in [13, Theorem 1 (iii)]:

Proposition 2.1. *Let $\gamma \in \mathcal{L}$ with arc length parameterization $\Gamma : S_L \to \mathbb{R}^3$. Then $\mathcal{K}[\gamma] < \infty$ if and only if γ is simple and $\Gamma \in C^{1,1}([0, L], \mathbb{R}^3)$.*

In particular, if $\mathcal{K}[\gamma]$ is finite, then

$$|\Gamma'(s_1) - \Gamma'(s_2)| \leq \mathcal{K}[\gamma]|s_1 - s_2| \quad \forall s_1, s_2 \in S_L, \tag{4}$$

i.e., Γ' has Lipschitz constant $\mathcal{K}[\gamma]$.

Remark. The necessary condition for $\mathcal{K}[\gamma]$ to be finite implies that the second derivative $\Gamma''(s)$ of Γ exists for a.e. $s \in S_L$, since $C^{1,1}([0, L], \mathbb{R}^3)$ is isomorphic to the Sobolev space $W^{2,\infty}([0, L], \mathbb{R}^3)$. Thus (4) actually implies

$$\|\Gamma''\|_{L^\infty} \leq \mathcal{K}[\gamma]\,. \tag{5}$$

The next result (cf. [13, Theorem 2 (ii)]) clarifies how the supremum in (1) is realized on a closed curve. Later this will lead to the definition of generalized local curvature for rectifiable loops that are not necessarily differentiable, and also to an alternative characterization of global curvature in terms of the radius of a circle uniquely determined by two points and one tangent.

Proposition 2.2. *Let $\gamma \in \mathcal{L}$ with $\mathcal{K}[\gamma] < \infty$, and let $s \in S_L$. Then there exists a sequence $(\sigma_j, \tau_j) \to (\sigma, \tau)$ in $S_L \times S_L$ with $s \neq \sigma_j \neq \tau_j \neq s$ for all $j \in \mathbb{N}$ satisfying*

$$\kappa_G[\gamma](s) \quad = \quad \lim_{j\to\infty} \frac{1}{R(\Gamma(s), \Gamma(\sigma_j), \Gamma(\tau_j))}, \tag{6}$$

such that either (a) $s = \sigma = \tau$, or (b) $s \neq \sigma = \tau$.

Remarks. 1. The different options in Proposition 2.2 can occur simultaneously, as can, e.g., be seen for the planar circle. Here, at every point the local curvature and the global curvature coincide, i.e., the different limit cases hold simultaneously at every point due to the high degree of symmetry of the circle. Another interesting example is that of a *stadium curve*, which consists of two parallel straight line segments of equal length and distance d connected by two planar half circles of radius $d/2$. These curves have constant local curvature $\kappa = 0$ along the line segments and $\kappa = 2/d$ along the half circles, i.e., curvature jumps. The global curvature $\mathcal{K}[\gamma]$, however, is equal to $2/d$ and one can show that $\mathcal{K}[\gamma] = \kappa_G[\gamma](s)$ for all $s \in S_L$, see [13, Theorem 2 (i)]. Stadium curves appear also in the examples of ideal $C^{1,1}$-links considered in [3].

2. The previous example has shown that a jump in local curvature must not necessarily be accompanied by a discontinuity of $\kappa_G[\gamma](\cdot)$. Nevertheless $\kappa_G[\gamma](\cdot)$ may fail to be continuous in the case where $\mathcal{K}[\gamma] < \infty$. Indeed, one can construct an example of a closed curve γ of class $C^{1,1}$ with oscillations in local curvature such that for some s the global curvature $\kappa_G[\gamma](s)$ is strictly less than the limit of $\kappa_G[\gamma](s_i)$ where $s_i \to s$ and $\kappa_G[\gamma](s_i)$ is realized be local curvature.

3. The statement of Proposition 2.2 says, roughly speaking, that the supremum in the definition of $\kappa_G[\gamma](s)$ cannot exclusively be achieved by two distinct parameters $\sigma, \tau \in S_L$. Moreover, due to the symmetry of $R(.,.,.)$ a third feasible option (c), namely $s = \sigma \neq \tau$, does not occur exclusively, that

is, without option (b) at the same time. Observe, however, that R might not be continuous at the limit point (s, σ, τ).

4. Notice that the assumption $\mathcal{K}[\gamma] < \infty$ is equivalent to demanding that γ possesses an injective arc length parameterization of class $C^{1,1}$ according to Proposition 2.1.

In case (a) of Proposition 2.2 the value $\kappa_G[\gamma](s)$ expresses a local property of the curve at $s \in S_L$, which coincides with the classical local curvature of $\gamma \in \mathcal{L}$ at s if γ is *smooth*.[1] For curves $\gamma \in \mathcal{L}$ which are only continuous and rectifiable in general, this observation motivates the following definition of the *generalized local curvature* $\kappa[\gamma](s)$ *of* $\gamma \in \mathcal{L}$ *at* $s \in S_L$ as

$$\kappa[\gamma](s) := \limsup_{\substack{(\tau_j,\sigma_j)\to(s,s)\\ s\neq\tau_j\neq\sigma_j\neq s}} \frac{1}{R(\Gamma(s),\Gamma(\tau_j),\Gamma(\sigma_j))}\,. \tag{7}$$

Note that for $\gamma \in \mathcal{L}$ the generalized local curvature $\kappa[\gamma]$ can take values in $[0, \infty]$, and that

$$\kappa[\gamma](s) \leq \kappa_G[\gamma](s) \ \text{ for all } \ s \in S_L. \tag{8}$$

For curves $\gamma \in \mathcal{L}$ with a $C^{1,1}$-arc length parameterization Γ we proved in [13, Proposition 1] how the generalized local curvature is related to the second derivative of Γ :

Proposition 2.3. *Let* $\Gamma : S_L \to \mathbb{R}^3$ *be the arc length parameterization of a simple curve* $\gamma \in \mathcal{L}$ *with* $\Gamma \in C^{1,1}([0, L], \mathbb{R}^3)$*. Then*

(i) $\kappa[\gamma](s) \leq \operatorname{ap\,lim\,sup}_{\sigma\to s} |\Gamma''(\sigma)|$ *for all* $s \in S_L$.

(ii) $\kappa[\gamma](s) = |\Gamma''(s)|$ *for a.e.* $s \in S_L$.

Here, ap lim sup denotes the approximate limes superior as defined, e.g. in [5, p. 47]. Suppose we have the local bound $|\Gamma''(\sigma)| \leq \kappa_0$ for a.e. σ in some open subinterval $J \subset S_L$, then, according to (i), $\kappa[\gamma](\sigma) \leq \kappa_0$ for all $\sigma \in J$. The essence of part (ii) is that for curves $\gamma \in \mathcal{L}$ with finite global curvature (hence γ simple and $\Gamma \in W^{2,\infty}$ by Proposition 2.1), we can identify $\kappa[\gamma]$ with $|\Gamma''|$ a.e. on $[0, L]$, and from (5) we infer

$$\|\kappa[\gamma]\|_{L^\infty} = \|\Gamma''\|_{L^\infty} \leq \mathcal{K}[\gamma] \ \text{ for all } \ \gamma \in \mathcal{L} \ \text{ with } \ \mathcal{K}[\gamma] < \infty. \tag{9}$$

[1] This can easily be verified by expanding Γ about s when calculating the term $R(\Gamma(s), \Gamma(\sigma_j), \Gamma(\tau_j))$ in (6) in case (a) for j large, see also [13, Lemma 7].

In the light of this inequality, we say for curves γ with $\mathcal{K}[\gamma] < \infty$, that *the global curvature* $\mathcal{K}[\gamma]$ *is locally not attained* if and only if

$$\|\Gamma''\|_{L^\infty} < \mathcal{K}[\gamma]. \tag{10}$$

Curves with this property are considered in Proposition 2.5 below. They play an essential role in [14], where the Euler-Lagrange equations for energy minimizing elastic rods are derived. In fact, it seems that this property is needed for the variational techniques we apply in the proof of Theorem 3.1.

For $\gamma \in \mathcal{L}$ with finite global curvature the alternative (b) in part (ii) of Proposition 2.2 expresses a nonlocal property of the curve. It motivates a different characterization of $\mathcal{K}[\gamma]$ which is analytically more tractable. Let $x, y, z \in \mathbb{R}^3$ be such that the vectors $x-y$ and z are linearly independent. By P we denote the plane spanned by $x-y$ and z. Then there is a unique circle contained in P through x and y, and tangent to z in the point y. We denote the radius of that circle by $r(x, y, z)$ and set $r(x, y, z) := \infty$, if $x-y$ and z are collinear. Using elementary geometric arguments r can be computed as

$$r(x, y, z) = \frac{|x-y|}{2\left|\frac{x-y}{|x-y|} \wedge \frac{z}{|z|}\right|}, \tag{11}$$

which shows that $r(x, y, z)$ is continuous on the set of triples (x, y, z) with the property, that $x-y$ and z are linearly independent. But it fails to be continuous at points, where, e.g., x and y coincide. Recall that for curves γ with $\mathcal{K}[\gamma] < \infty$ Proposition 2.1 says, that the corresponding arc length parameterization Γ possesses a Lipschitz continuous unit tangent field Γ' on $[0, L]$. Hence, for every pair $(s, \sigma) \in S_L \times S_L$, the radius $r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma))$ is well defined, and we obtain the following identities for κ_G and $\mathcal{K}$ (cf. [13, Lemma 2]):

Lemma 2.4. *Let $\gamma \in \mathcal{L}$ be such that $\mathcal{K}[\gamma] < \infty$, then at least one of the following statements* (A),(B) *is true:*

$$\kappa_G[\gamma](s) = \sup_{\substack{\sigma \in S_L \\ \sigma \neq s}} \frac{1}{r(\Gamma(s), \Gamma(\sigma), \Gamma'(\sigma))}, \tag{A}$$

$$\kappa_G[\gamma](s) = \sup_{\substack{\sigma \in S_L \\ \sigma \neq s}} \frac{1}{r(\Gamma(\sigma), \Gamma(s), \Gamma'(s))} = \kappa[\gamma](s). \tag{B}$$

If for $s \in S_L$ part (b) *of Proposition 2.2 holds, then alternative* (A) *above is true.*

In addition,

$$\mathcal{K}[\gamma] \quad = \quad \sup_{\substack{s,\sigma\in S_L\\ s\neq\sigma}} \frac{1}{r(\Gamma(s),\Gamma(\sigma),\Gamma'(\sigma))}\,. \tag{12}$$

Because of the representation of $\mathcal{K}[\gamma]$ as a supremum in (12) the following set $A[\gamma]$, where the global curvature $\mathcal{K}[\gamma]$ is attained, is of particular interest.

$$A[\gamma] := \{(s,\sigma) \in [0,L] \times [0,L] : \mathcal{K}[\gamma] = \frac{1}{r(\Gamma(s),\Gamma(\sigma),\Gamma'(\sigma))}\}. \tag{13}$$

Note that also $\Gamma(s) \neq \Gamma(\sigma)$ for $(s,\sigma) \in A[\gamma]$, if $\mathcal{K}[\gamma]$ is finite. The set $A[\gamma]$ can be empty, e.g., in the case when γ parameterizes a regular ellipse, where the local curvature $\kappa[\gamma]$ is maximal and equal to $\mathcal{K}[\gamma]$ at exactly the two vertices. In other words, for an ellipse, $\mathcal{K}[\gamma]$ is attained *exclusively* locally. On the other hand, if γ describes a circle, one has $A[\gamma] = [0,L] \times [0,L]\setminus$ diagonal.

We conclude this section with a continuity property of the global curvature $\kappa_G[\gamma](.)$ on the set of parameters with sufficiently large values for $\kappa_G[\gamma](s)$ (notice that this condition is not met in the example mentioned in the second remark following Proposition 2.2).

Proposition 2.5. *Let $\gamma \in \mathcal{L}$ satisfy* (10) *with $\mathcal{K}[\gamma] < \infty$. Then $\kappa_G[\gamma](.) : S_L \to (0,\infty)$ is continuous on the set*

$$\tilde{S} := \{s \in S_L : \kappa_G[\gamma](s) > \|\Gamma''\|_{L^\infty}\}.$$

Proof. Let $s \in \tilde{S}$. According to Proposition 2.2 we can find a sequence $(\sigma_j,\tau_j) \to (\sigma,\tau)$ in $S_L \times S_L$ with $s \neq \sigma_j \neq \tau_j \neq s$ for all $j \in \mathbb{N}$ with

$$\kappa_G[\gamma](s) = \lim_{j\to\infty} \frac{1}{R(\Gamma(s),\Gamma(\sigma_j),\Gamma(\tau_j))}\,,$$

such that either (a) $s = \sigma = \tau$, or (b) $s \neq \sigma = \tau$.

Case (a) is not possible since by (7), Proposition 2.3 (i), and by our assumption that $s \in \tilde{S}$ we would obtain

$$\kappa_G[\gamma](s) \underset{(7)}{\leq} \kappa[\gamma](s) \underset{\text{Prop.2.3}(i)}{\leq} \|\Gamma''\|_{L^\infty} \underset{\text{Def. of } \tilde{S}}{<} \kappa_G[\gamma](s),$$

which is absurd. Hence we are in the situation (b) which allows us to go to the limit $j \to \infty$ in the formula (2), since γ is simple and $\Gamma \in C^{1,1}([0,L],\mathbb{R}^3)$ by Proposition 2.1 and $\mathcal{K}[\gamma] < \infty$. We obtain

$$\kappa_G[\gamma](s) = \frac{1}{r(\Gamma(s),\Gamma(\tau),\Gamma'(\tau))}\,, \tag{14}$$

where the function $r(.,.,.)$ is given by (11). By continuity of $r(x,y,z)$ in the vicinity of non-collinear vectors $x-y$ and z we can find for

$$\varepsilon \in (0, \kappa_G[\gamma](s) - \|\Gamma''\|_{L^\infty})$$

some number $\delta = \delta(s) > 0$ such that for all $s', \tau' \in S_L$ with $|s-s'| \le \delta$ and $|\tau - \tau'| \le \delta$

$$\frac{1}{r(\Gamma(s'),\Gamma(\tau'),\Gamma'(\tau'))} \ge \frac{1}{r(\Gamma(s),\Gamma(\tau),\Gamma'(\tau))} - \varepsilon/2.$$

Consequently, by (1), and for $|\tau'' - \tau'|$ sufficiently small by (2) and (11)

$$\begin{aligned}\kappa_G[\gamma](s') \;&\underset{(1)}{\ge}\; \frac{1}{R(\Gamma(s'),\Gamma(\tau'),\Gamma(\tau''))} \ge \frac{1}{r(\Gamma(s'),\Gamma(\tau'),\Gamma'(\tau'))} - \varepsilon/2\\ &\ge\; \frac{1}{r(\Gamma(s),\Gamma(\tau),\Gamma'(\tau))} - \varepsilon = \kappa_G[\gamma](s) - \varepsilon > \|\Gamma''\|_{L^\infty}.\end{aligned}$$

Hence $s' \in \tilde{S}$ as well and as before we obtain

$$\kappa_G[\gamma](s') = \frac{1}{r(\Gamma(s'),\Gamma(\bar\tau),\Gamma'(\bar\tau))} \tag{15}$$

for some $\bar\tau \ne s'$. Therefore, by Lemma 2.4

$$\begin{aligned}\kappa_G[\gamma](s) \;&\underset{\text{Lemma 2.4}}{\ge}\; \frac{1}{r(\Gamma(s),\Gamma(\bar\tau),\Gamma'(\bar\tau))}\\ &=\; \frac{1}{r(\Gamma(s'),\Gamma(\bar\tau),\Gamma'(\bar\tau))} + \frac{1}{r(\Gamma(s),\Gamma(\bar\tau),\Gamma'(\bar\tau))}\\ &\qquad -\frac{1}{r(\Gamma(s'),\Gamma(\bar\tau),\Gamma'(\bar\tau))}\\ &=\; \kappa_G[\gamma](s') + \frac{1}{r(\Gamma(s),\Gamma(\bar\tau),\Gamma'(\bar\tau))} - \frac{1}{r(\Gamma(s'),\Gamma(\bar\tau),\Gamma'(\bar\tau))}\,.\end{aligned}$$

Hence we arrive at

$$\kappa_G[\gamma](s') - \kappa_G[\gamma](s) \le \frac{1}{r(\Gamma(s'), \Gamma(\bar\tau), \Gamma'(\bar\tau))} - \frac{1}{r(\Gamma(s), \Gamma(\bar\tau), \Gamma'(\bar\tau))},$$

and exchanging the roles of s and s' in the previous considerations also

$$\kappa_G[\gamma](s) - \kappa_G[\gamma](s') \le \frac{1}{r(\Gamma(s), \Gamma(\tau), \Gamma'(\tau))} - \frac{1}{r(\Gamma(s'), \Gamma(\tau), \Gamma'(\tau))}.$$

Since the right-hand side of each inequality (for fixed τ, and $\bar\tau$) tends to zero as $s' \to s$ we have proven the claim. $\square$

Corollary 2.6. *Let $\gamma \in \mathcal{L}$ satisfy* (10) *with $\mathcal{K}[\gamma] < \infty$. Then the set*

$$\tilde S_\rho := \{s \in S_L : \kappa_G[\gamma](s) \ge \|\Gamma''\|_{L^\infty} + \rho\}$$

is closed for each $\rho \in (0, \mathcal{K}[\gamma] - \|\Gamma''\|_{L^\infty})$.

Proof. Let $s_i \to s_0$ as $i \to \infty$, where $s_i \in \tilde S_\rho$ for all $i \in \mathbb{N}$. Then, since $s_i \in \tilde S$ for all $i \in \mathbb{N}$, where $\tilde S$ is defined in Proposition 2.5, we obtain as in (14)

$$\kappa_G[\gamma](s_i) = \frac{1}{r(\Gamma(s_i), \Gamma(\tau_i), \Gamma'(\tau_i))}, \tag{16}$$

where $s_i \ne \tau_i$ for all $i \in \mathbb{N}$. A straightforward argument involving Taylor's expansion implies that for all $i \in \mathbb{N}$

$$|s_i - \tau_i| > \eta := \frac{1 - \frac{\|\Gamma''\|_{L^\infty}}{\|\Gamma''\|_{L^\infty} - \rho}}{\|\Gamma''\|_{L^\infty}}, \tag{17}$$

since $\|\Gamma''\|_{L^\infty} + \rho < \kappa_G[\gamma](s_i)$ (see [13, Lemma 8] for details). Without loss of generality we can assume that $\tau_i \to \tau_0 \in S_L$. Hence $|s_0 - \tau_0| > \eta$ and $\Gamma(s_0) \ne \Gamma(\tau_0)$. Using the continuity of the function $1/r(.,.,.)$ (cf. (11)) we obtain for all i sufficiently large

$$\frac{1}{r(\Gamma(s_0), \Gamma(\tau_i), \Gamma'(\tau_i))} \ge \frac{1}{r(\Gamma(s_i), \Gamma(\tau_i), \Gamma'(\tau_i))} - \frac{\rho}{4}. \tag{18}$$

Using (1) one deduces for $\tau_i' \in [0,1]$ with $|\tau_i' - \tau_i|$ sufficiently small by (2) and (11)

$$\kappa_G[\gamma](s_0) \geq \frac{1}{R(\Gamma(s_0),\Gamma(\tau_i),\Gamma(\tau_i'))} \geq \frac{1}{r(\Gamma(s_0),\Gamma(\tau_i),\Gamma'(\tau_i))} - \frac{\rho}{4}. \tag{19}$$

Thus, by (16)–(19)

$$\kappa_G[\gamma](s_0) \geq \kappa_G[\gamma](s_i) - \rho/2 \geq \|\Gamma''\|_{L^\infty} + \rho/2,$$

since $s_i \in \tilde{S}_\rho$. Hence $s_0 \in \tilde{S}$ defined in Proposition 2.5, i.e. $s_i \to s_0$ in $\tilde{S}$. Now we can use the continuity of $\kappa_G[\gamma](.)$ on $\tilde{S}$ to obtain

$$\kappa_G[\gamma](s_0) = \lim_{i\to\infty} \kappa_G[\gamma](s_i) \geq \|\Gamma''\|_{L^\infty} + \rho\ ,$$

hence $s_0 \in \tilde{S}_\rho$. $\square$

Remark. If we apply estimate (17) to (s,τ) in (14), then we even obtain Lipschitz continuity of $\kappa_G[\gamma](.)$ on $\tilde{S}_\rho$ with a Lipschitz constant depending on ρ.

3 Ideal knots

An ideal knot in the sense of [2], [9] and [11] is a non-self-intersecting tube of fixed radius $\theta > 0$ and prescribed knot type with a centreline curve γ of minimal length. To be more precise, an ideal knot is a solution of the variational problem

$$L(\gamma) = \int_I |\gamma'(\sigma)|\, d\sigma \to \text{Min!}, \quad \gamma \in W^{1,q}, \quad q \in (1,\infty), \tag{20}$$

subject to the conditions

$$\gamma(0) = \gamma(1), \quad \mathcal{K}[\gamma] \leq \theta^{-1} \quad \text{and} \quad \gamma(\bar{I}) \simeq \tilde{\gamma}(\bar{I}). \tag{21}$$

Here $\theta > 0$ is a constant representing the prescribed thickness of the curves in competition. In fact, the upper bound on the global curvature corresponds to a uniform lower bound on the global radius of curvature (cf. [13]), and it provides by [8, Lemma 3] or [13, Proposition 3] a geometrically exact model for the self-contact or excluded volume constraint imposed on the curve by a tubular neighbourhood of fixed radius θ.

The fixed curve $\tilde{\gamma} \in W^{1,q} \cap \mathcal{L}$ represents the prescribed knot or *isotopy*[2] type and are assumed to satisfy $\mathcal{K}[\tilde{\gamma}] \leq \theta^{-1}$.

The existence of a curve of class $W^{1,q}(I, \mathbb{R}^3)$ solving the minimization problem (20),(21) is established in [8, Theorem 4] for each knot type $\tilde{\gamma}$ and the corresponding arc length parameterization Γ belongs to $C^{1,1}([0, L], \mathbb{R}^3)$ where L denotes the length of the minimizing curve. Using the simple reparameterization $\gamma(\tau) := \Gamma(L \cdot \tau)$ for $\tau \in \bar{I} = [0, 1]$ we obtain the same curve parameterized on $[0, 1]$ with constant velocity L. Obviously γ is again a solution of (20),(21) and $\gamma \in C^{1,1}([0, 1], \mathbb{R}^3)$.

Theorem 3.1. *Let $\gamma \in W^{1,q}(I, \mathbb{R}^3)$ be a solution of problem* (20),(21) *such that γ is the parameterization with constant speed (i.e., $\gamma \in C^{1,1}([0, 1], \mathbb{R}^3)$, and the global curvature $\mathcal{K}[\gamma]$ be locally not attained. Then $\mathcal{K}[\gamma] = \theta^{-1}$, and for all $s \in [0, 1]$ there exists a (nonnegative) Radon measure μ_s on $[0, 1]$ supported on*

$$I_s := \{\tau \in [0, 1] : (s, \tau) \in A[\gamma] \ \textit{or} \ (\tau, s) \in A[\gamma]\}$$

(i.e., μ_s is the zero measure if $I_s = \emptyset$) such that

$$\gamma''(s) = \int_0^1 [\gamma(\tau) - \gamma(s)] \, d\mu_s(\tau) \quad \textit{for a.e. } s \in [0, 1]. \tag{22}$$

Remarks.

1. The condition that $\mathcal{K}[\gamma]$ be locally not attained excludes the circle of radius θ, for which $\mathcal{K}[\gamma]$ is attained locally and globally at all points. This circle is the obvious ideal knot in the trivial isotopy class of unknots, as one can easily deduce from [8, Lemma 3 (i)].

2. A simple scaling argument (cf. Section 5) shows that $\mathcal{K}[\gamma] = \theta^{-1}$ for any ideal knot, i.e., the global curvature $\mathcal{K}[\gamma]$ is automatically attained for an ideal configuration.

[2] Two continuous closed curves $K_1, K_2 \subset \mathbb{R}^3$ are isotopic, denoted as $K_1 \simeq K_2$, if there are open neighbourhoods N_1 of K_1, N_2 of K_2, and a continuous mapping $\Phi : N_1 \times [0, 1] \to \mathbb{R}^3$ such that $\Phi(N_1, \tau)$ is homeomorphic to N_1 for all $\tau \in [0, 1]$, $\Phi(x, 0) = x$ for all $x \in N_1$, $\Phi(N_1, 1) = N_2$, and $\Phi(K_1, 1) = K_2$. Roughly speaking, two curves are in the same isotopy class if one can be continuously deformed onto the other.

3. Notice that

$$\gamma''(s) \in \overline{\text{conv}}\,\{\,\rho\left(\gamma(\tau)-\gamma(s)\right)|\,\rho \geq 0,\ \tau \in I_s\} \quad \text{for a.e. } s \in [0,1]$$

by (22) and the definition of the integral ($\overline{\text{conv}}$ - closed convex hull).

Let us define

$$I_c := \{s \in [0,1] : I_s \neq \emptyset\}$$

Then we have the following consequences of the previous theorem:

Corollary 3.2. *Assume that the assumptions of the previous theorem are satisfied. Then:*

(i) *I_c is closed.*

(ii) *If $\kappa_G[\gamma](s) < \theta^{-1}$ then there is $\delta(s) > 0$ such that $\gamma''(\tau) = 0$ for a.e. $\tau \in [0,1]$ with $|\tau - s| < \delta(s)$, i.e., γ is straight on a neighbourhood of s.*

(iii) *Let $\tilde{I} \subset [0,1]$ be an open interval and let $I_c \cap \tilde{I}$ have (Lebesgue) measure zero. Then γ is straight on $\tilde{I}$.*

(iv) *Let γ be curved on an open interval $\tilde{I} \subset [0,1]$, i.e, $\gamma''(s) \neq 0$ a.e. on $\tilde{I}$, then $\tilde{I} \subset I_c$ and $\kappa_G[\gamma](s) = \theta^{-1}$ on $\tilde{I}$.*

Remark.
1. If s_0 is an isolated point in I_c (or also an accumulation point of isolated points s_n in I_c), then γ has be be straight on a neighborhood of s_0 by (iii).
2. Notice that the previous results can be transferred to *ideal links*, i.e., ideal configurations composed of several closed knotted and linked curves having minimal length under a uniform upper bound on the global curvature.

4 Framed curves

In this section we provide a special representation of curves $\gamma \in C^{1,1}([0,1],\mathbb{R}^3)$ having constant speed parameterization as key ingredient for the proof of Theorem 3.1. This will enable us to apply the methods developed in [14].

We consider pairs (γ, D), called *framed curves*, where $\gamma : \bar{I} \to \mathbb{R}^3$ is a curve of class $W^{2,\infty}(I,\mathbb{R}^3)$ parameterized by constant speed equipped with a frame field $D : \bar{I} \to SO(3)$ of class $W^{1,\infty}(I, \mathbb{R}^3)$. Notice that $D(s) = (d_1(s)|d_2(s)|d_3(s))$ consists of three orthonormal column-vectors $d_i(s)$ ($i = 1, 2, 3$) for each $s \in \bar{I} = [0, 1]$. Thus a framed curve can be considered as curve with an orthonormal frame attached to each point $\gamma(s)$. Let us mention that our smoothness assumptions for (γ, D) constitute a proper subset of more general framed curves treated, e.g., in [8] and [13]. We call (γ, D) a *closed framed curve* if

$$\gamma(0) = \gamma(1) \quad \text{and} \quad d_3(0) = d_3(1). \tag{23}$$

Proposition 4.1 below states that a framed curve $(\gamma, D) \in W^{2,\infty}(I, \mathbb{R}^3) \times W^{1,\infty}(I, \mathbb{R}^{3\times 3})$ may be uniquely determined from shape and placement variables

$$w = (u, v, \gamma_0, D_0) \in X_0 := L^\infty(I, \mathbb{R}^3) \times \mathbb{R}_+ \times \mathbb{R}^3 \times SO(3)$$

with $u = (u^1, u^2, u^3)$ via the equations

$$\begin{aligned} d_k'(s) &= \Big[\sum_{i=1}^{3} u^i(s) d_i(s) \Big] \wedge d_k(s) \qquad \text{for a.e. } s \in I, \quad k = 1, 2, 3, \\ \gamma'(s) &= v d_3(s) \qquad \text{for a.e. } s \in I, \\ \gamma(0) &= \gamma_0, \quad D(0) = D_0, \end{aligned} \tag{24}$$

Notice that X_0 is a proper subset of the corresponding Banach space

$$X := L^p(I, \mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}.$$

The triples (u^1, u^2, u^3) and $(0, 0, v)$ may be identified as the coordinates, in the moving frame $\{d_i\}$, of the Darboux vector for the frame field $D(s)$ and the tangent vector $\gamma'(s)$, respectively. Notice that $u \in L^\infty(I, \mathbb{R}^3)$ and v describe the shape of a framed curve, whereas γ_0 and D_0 describe its spatial placement. The constant $v > 0$ in particular is the speed of the parameterization of γ.

The following result is a special case of [8, Lemma 6].

Proposition 4.1. *To each framed curve $(\gamma, D) \in W^{2,\infty} \times W^{1,\infty}$ we can associate a unique $w = w(\gamma, D) \in X_0$ determined by* (24)*. Conversely, to each $w \in X_0$ we can associate a unique framed curve $(\gamma, D) = (\gamma[w], D[w]) \in W^{2,\infty} \times W^{1,\infty}$ such that* (24) *holds.*

As already mentioned we claim to apply the variational techniques for framed curves developed in [14] in the proof of our main results. Here the key observation is that for each curve $\gamma \in W^{2,\infty}$ having constant speed parameterization we can find a frame field $D \in W^{1,\infty}$ such that (γ, D) is a framed curve.

Proposition 4.2. *For each curve $\gamma \in W^{2,\infty}(I, \mathbb{R}^3)$ with $|\gamma'(\sigma)| = v > 0$ there are $u^1, u^2 \in L^\infty(I)$, $\gamma_0 \in \mathbb{R}^3$, and $D_0 \in SO(3)$, such that for any $u^3 \in L^\infty(I)$ equations (24) provide a frame field $D : \bar{I} \to SO(3)$ of class $W^{1,\infty}(I, \mathbb{R}^{3\times 3})$ such that (γ, D) is a framed curve.*

Proof. Given a curve $\gamma \in W^{2,\infty}(I, \mathbb{R}^3)$ with $|\gamma'(\sigma)| = const. > 0$ we set $d_3 := \gamma'/|\gamma'|$ which is of class $W^{1,\infty}(I, \mathbb{R}^3)$ and has unit length. Moreover we define $v := |\gamma'| > 0$. To obtain a complete frame field $D = (d_1|d_2|d_3) : \bar{I} \to SO(3)$ of class $W^{1,\infty}(I, \mathbb{R}^3)$ we observe that the length of the curve $\gamma'(\bar{I}) \subset S^2$ is bounded since $\gamma \in W^{2,\infty}(I, \mathbb{R}^3)$. Hence the two-dimensional Hausdorff-measure $\mathcal{H}^2(\gamma'(\bar{I}))$ of the image curve $\gamma'(\bar{I})$ of the tangent is zero, in particular we find a fixed vector $\tilde{d} \in S^2 \backslash d_3(\bar{I})$ and define $d_1 := \tilde{d} \wedge d_3$ and $d_2 := d_3 \wedge d_1$ on $\bar{I}$. Now we apply the first part of Proposition 4.1 to the framed curve $(\gamma, D) \in W^{2,\infty}(I, \mathbb{R}^3) \times W^{1,\infty}(I, \mathbb{R}^{3\times 3})$ and obtain $u^i \in L^\infty(I)$, $i = 1, 2, 3$, and $\gamma_0 := \gamma(0)$, $D_0 := D(0)$, such that (24) holds. Finally we note that we can change the function $u^3 \in L^\infty(I)$ arbitrarily without changing the differential equation and the initial condition for d_3 in (24). Thus, by the second part of Proposition 4.1, we obtain a framed curve (γ, D) for any $u^3 \in L^\infty(I)$. □

Remarks.

1. The preceding proposition allows to represent any curve of class $W^{2,\infty}(I, \mathbb{R}^3)$ by an element $w \in X_0$. Thus, instead to consider perturbations of an ideal knot in $W^{2,\infty}$, we can study corresponding variations inX_0. This has the essential advantage that the freedom in parameterization of competing curves is removed.
2. Below we assign the length of γ to framed curves (γ, D) as some kind of energy. For a mechanical interpretation one may view these objects as unshearable elastic strings which are able to develop tensile stresses but without resistance towards bending and torsion.

The previous investigations allow us to reformulate the variational problem stated in Section 3 in terms of $w \in X_0$. This way we restrict our attention to constant speed parameterizations on the fixed parameter interval $I = [0,1]$ as representatives for curves in $W^{2,\infty}$. To each element $w = (u, v, \gamma_0, D_0)$ in $X_0 = L^\infty(I, \mathbb{R}^3) \times \mathbb{R}_+ \times \mathbb{R}^3 \times SO(3)$ we assign the "energy"

$$E(w) := \int_0^1 v d\sigma = v = L(\gamma). \tag{25}$$

Then we are looking for minimizers of the problem

$$E(w) \quad \to \quad \text{Min!}\,, \quad w \in X_0, \tag{26}$$

$$\begin{aligned} \gamma[w](1) &= \gamma_0, && (27)\\ d_3[w](1) &= d_3[w](0), && (28)\\ \mathcal{K}[w] &\le \theta^{-1}, && (29)\\ \gamma[w] &\simeq \tilde{\gamma}(\bar{I}). && (30) \end{aligned}$$

Here and from now on we use the short notation $\mathcal{K}[w]$, $A[w]$ etc. for $\mathcal{K}[\gamma[w]]$, $A[\gamma[w]]$, etc. Note that (28)is automatically satisfied for a constant speed ideal knot, since γ' is continuous and $\gamma' = d_3$. Consequently, if γ is a (reparameterized) solution of (20), (21) and $w = w[\gamma] \in X_0$ is the element assigned according to Proposition 4.2 (with, e.g., $u^3 \equiv 0$), then w solves the variational problem (26)–(30).

Remark. The minimizing curves cannot expected to be unique, there are concrete examples of nonunique minimizing links constructed by Cantarella et al. in [3]. Hence when we speak of *the* minimizer we mean a particular suitably reparameterized minimizing curve whose existence was established independently in [3] and [8].

5 Proofs

We claim to prove Theorem 3.1. Similarly as in [14] we proceed in several steps first modifying and reducing the variational problem and then deriving variational formulas for the quantities involved. Throughout this section we assume that $w = (u, v, \gamma_0, D_0) \in X_0$ is a solution of the variational problem (26)–(30) such that the global curvature $\mathcal{K}[w]$ is locally not attained.

Modified variational problem. We follow the approach in [14, Section 5] to linearize the space X_0 by using the correspondence of small variations of the identity in $SO(3)$ with small variations in $\mathbb{R}^3$ near the origin by means of the rotation vector. To be more precise, small perturbations of $D_0 \in SO(3)$ have the form $D_0 U(\overset{\triangle}{\alpha})$ with a continuous and invertible mapping $U : B_{\delta_1}(0) \subset \mathbb{R}^3 \to SO(3)$ for sufficiently small $\delta_1 > 0$, where $U(0) = Id \in SO(3)$. We can identify each slightly perturbed configuration

$$(u+ \overset{\triangle}{u}, v+ \overset{\triangle}{v}, \gamma_0+ \overset{\triangle}{\gamma}_0, D_0 \overset{\triangle}{D}) \in X_0$$

with an element

$$\overset{\triangle}{w}= (\overset{\triangle}{u}, \overset{\triangle}{v}, \overset{\triangle}{\gamma}_0, \overset{\triangle}{\alpha}) \in L^\infty(I, \mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 =: Y \tag{31}$$

by $\overset{\triangle}{D}= U(\overset{\triangle}{\alpha})$. With the norm

$$\| \overset{\triangle}{w} \|_Y := \| \overset{\triangle}{u} \|_{L^\infty} + | \overset{\triangle}{v} | + | \overset{\triangle}{\gamma}_0 | + | \overset{\triangle}{\boldsymbol{\alpha}} | \tag{32}$$

for $\overset{\triangle}{w}= (\overset{\triangle}{u}, \overset{\triangle}{v}, \overset{\triangle}{\gamma}_0, \overset{\triangle}{\alpha}) \in Y$ we obtain $(Y, \|.\|_Y)$ as a Banach space, whereas the original set X_0 is not a linear space.

For notational convenience we introduce the modified energy function

$$\breve{E}(\overset{\triangle}{w}) := E(v+ \overset{\triangle}{v}) \quad \text{for} \quad \overset{\triangle}{w}\in B_\delta(0) \subset Y, \tag{33}$$

where $B_\delta(0)$ is a small neighbourhood of $0 \in Y$ with $\delta > 0$ not fixed yet but sufficiently small. Analogously, we define $\breve{\gamma}[\overset{\triangle}{w}]$, $\breve{D}[\overset{\triangle}{w}]$, $\breve{\mathcal{K}}[\overset{\triangle}{w}]$, etc. Note that $\breve{\gamma}[0] = \gamma[w]$, $\breve{D}[0] = D[w]$, etc.

Now we consider the *modified variational problem*

$$\breve{E}(\overset{\triangle}{w}) \quad \longrightarrow \quad \text{Min!}, \quad \overset{\triangle}{w}\in Y, \tag{34}$$

$$\begin{aligned}
\breve{\gamma}[\overset{\triangle}{w}](1) &= \gamma_0+ \overset{\triangle}{\gamma}_0, && (35)\\
\breve{d}_3[\overset{\triangle}{w}](1) &= \breve{d}_3[\overset{\triangle}{w}](0), && (36)\\
\breve{\mathcal{K}}[\overset{\triangle}{w}] &\leq \theta^{-1}, && (37)\\
\breve{\gamma}[\overset{\triangle}{w}](\bar{I}) &\simeq \tilde{\gamma}(\bar{I}). && (38)
\end{aligned}$$

Since $w \in X_0$ was assumed to be a solution of (26)-(30),

$$\overset{\triangle}{w}= 0 \quad \text{is a local minimizer of (34)–(38).} \tag{39}$$

Reduction of the modified problem. Some of the constraints of the modified variational problem are redundant and need to be reformulated to obtain a clear-cut situation. Furthermore we will replace $\mathcal{K}[\gamma]$ by its square to obtain better regularity properties for the side condition (37).

We recall from [14, Lemma 5.3] that small variations

$$\overset{\triangle}{w}_1 := (\overset{\triangle}{u}, 0, \overset{\triangle}{\gamma}_0, \overset{\triangle}{\alpha}) \in Y$$

do not violate the constraint of a given knot class. Variations of the form $\overset{\triangle}{w}_2 := (0, \overset{\triangle}{v}, 0, 0) \in Y$ on the other hand, correspond to merely changing the length of the curve within the same knot class. Since general variations $\overset{\triangle}{w} \in Y$ may be decomposed as $\overset{\triangle}{w} = \overset{\triangle}{w}_1 + \overset{\triangle}{w}_2$ we conclude that the topological condition of a prescribed knot class can be removed without affecting the fact that $\overset{\triangle}{w} = 0$ is a local minimizer of the modified variational problem:

Lemma 5.1. *Let $w = (u, v, \gamma_0, D_0) \in X_0$ be a solution of the variational problem* (26)–(30). *Then $\breve{\gamma}[\overset{\triangle}{w}] \simeq \gamma[w]$ for all $\| \overset{\triangle}{w} \|_Y$ sufficiently small.*

In order to replace (37) by an equivalent condition we introduce the functions

$$P[\overset{\triangle}{w}](s,\sigma) \ := \ (\breve{\gamma}[\overset{\triangle}{w}](s), \breve{\gamma}[\overset{\triangle}{w}](\sigma), \breve{\gamma}'[\overset{\triangle}{w}](\sigma)), \tag{40}$$

$$H(x,y,t) \ := \ \frac{4|(x-y)\wedge t|^2}{|x-y|^4|t|^2}, \ \text{ for } \ x,y,t \in \mathbb{R}^3, x \neq y, t \neq 0, \tag{41}$$

and note that according to (11),(12) we may write

$$\breve{\mathcal{K}}[\overset{\triangle}{w}]^2 = \sup_{\substack{s,\sigma \in S_L \\ s \neq \sigma}} H(P[\overset{\triangle}{w}](s,\sigma)). \tag{42}$$

Now we can replace (37) with

$$g(\overset{\triangle}{w}) := \breve{\mathcal{K}}[\overset{\triangle}{w}]^2 - \theta^{-2} \ \leq \ 0. \tag{43}$$

To remove the redundancy in the boundary condition (36) we are going to replace the three scalar conditions (36) by just two scalar equations, see (47), (48) below. (Note that $d_3[\overset{\triangle}{w}](s)$ is always a unit vector for each $\overset{\triangle}{w} \in Y$, thus it has only two degrees of freedom.)

This way we get the *reduced variational problem*

$$\breve{E}(\overset{\triangle}{w}) \ \to \ \text{Min!}\,, \quad \overset{\triangle}{w} \in Y, \tag{44}$$

$$g(\overset{\triangle}{w}) \leq 0, \tag{45}$$

$$g_0(\overset{\triangle}{w}) := \breve{\gamma}[\overset{\triangle}{w}](1) - (\gamma_0 + \overset{\triangle}{\gamma}_0) = 0, \tag{46}$$

$$g_1(\overset{\triangle}{w}) := \breve{d}_3[\overset{\triangle}{w}](1) \cdot (D_0 U(\overset{\triangle}{\alpha}))_1 = 0, \tag{47}$$

$$g_2(\overset{\triangle}{w}) := \breve{d}_3[\overset{\triangle}{w}](1) \cdot (D_0 U(\overset{\triangle}{\alpha}))_2 = 0, \tag{48}$$

where, for $M \in \mathbb{R}^{3\times 3}$, we denote the k-th column vector by $(M)_k$, $k = 1, 2, 3$.

Lemma 5.2. *The reduced variational problem* (44)–(48) *has a local minimizer at* $\overset{\triangle}{w} = 0$.

Proof. Taking into account the previous arguments one merely needs to show that (47), (48) imply (36) for $\| \overset{\triangle}{w} \|_Y$ sufficiently small. Note that $\breve{d}_3[\overset{\triangle}{w}](1)$ is collinear with $\breve{d}_3[\overset{\triangle}{w}](0)$ by (47), (48). [8, Lemma 8] implies that $\breve{d}_3[\overset{\triangle}{w}](s) \to \breve{d}_3[0](s)$ for all $s \in [0, 1]$ as $\| \overset{\triangle}{w} \|_Y \to 0$. By $\breve{d}_3[0](1) = d_3[w](1) = d_3[w](0) = \breve{d}_3[0](0)$ (cf. (28)) we conclude that $\breve{d}_3[\overset{\triangle}{w}](1) = \breve{d}_3[\overset{\triangle}{w}](0)$ for $\| \overset{\triangle}{w} \|_Y$ sufficiently small. □

Computation of derivatives. The mapping

$$(\overset{\triangle}{w}, s) \mapsto (\breve{\gamma}[\overset{\triangle}{w}](s), \breve{D}[\overset{\triangle}{w}](s))$$

from $B_\delta(0) \times [0, 1]$ into $\mathbb{R}^3 \times \mathbb{R}^{3\times 3}$ is of class $C^1(B_\delta(0) \times [0, L], \mathbb{R}^3 \times \mathbb{R}^{3\times 3})$ for some sufficiently small $\delta > 0$ (depending on w), which can be shown as in [14, Lemma 5.5]. In [14, Lemma A.1] we gave an explicit formula for the Fréchet derivative of the mapping $\overset{\triangle}{w} \mapsto \breve{d}_k[\overset{\triangle}{w}](s)$, $k = 1, 2, 3$, at the origin $0 \in Y$, which we denote by $\partial_w \breve{d}_k[0](s)$. (Note that $\breve{d}_k[\overset{\triangle}{w}](s)$ is independent of variations $\overset{\triangle}{v}$ which are not considered in [14].) In fact, we have

$$\partial_w \breve{d}_k[0](s)\, \overset{\triangle}{w} = z(s) \wedge d_k[w](s), \quad k = 1, 2, 3,\ s \in [0, 1]. \tag{49}$$

Here, $z = z[\overset{\triangle}{u}, \overset{\triangle}{\alpha}]$ is a special characterization of elements $\overset{\triangle}{u} \in L^\infty([0, 1], \mathbb{R}^3)$, and $\overset{\triangle}{\alpha} \in \mathbb{R}^3$ by the uniquely assigned function

$$z(s) = z(0) + \int_0^s \sum_{i=1}^{3} \overset{\triangle}{u}{}^i(\tau) d_i[w](\tau)\, d\tau \tag{50}$$

with

$$z(0) \wedge d_k[w](0) = (D_0 U'(0) \overset{\triangle}{\alpha})_k, \quad k = 1, 2, 3, \tag{51}$$

where U' denotes the derivative of U with respect to α at $0 \in \mathbb{R}^3$, which is invertible as pointed out in [14, Appendix]. Note that $z \in W^{1,\infty}([0,1], \mathbb{R}^3)$. In particular,

$$z(0) = 0 \quad \text{ for } \quad \overset{\triangle}{w} = (\overset{\triangle}{u}, \overset{\triangle}{v}, \overset{\triangle}{\gamma}_0, 0) \in Y. \tag{52}$$

Thus, by (24)

$$\partial_w \breve{\gamma}[0](s)\, \overset{\triangle}{w} = \overset{\triangle}{\gamma}_0 + v \int_0^s z(\tau) \wedge d_3[w](\tau)\, d\tau + \overset{\triangle}{v} \int_0^s d_3[w](\tau)\, d\tau, \tag{53}$$

for all $s \in [0, 1]$, $\overset{\triangle}{w} = (\overset{\triangle}{u}, \overset{\triangle}{v}, \overset{\triangle}{\gamma}_0, \overset{\triangle}{\alpha}) \in Y$.

The energy functional $\breve{E} : B_\delta(0) \subset Y \longrightarrow \mathbb{R}$ is obviously differentiable satisfying

$$\breve{E}'(0)\, \overset{\triangle}{w} \; = \int_0^1 \overset{\triangle}{v}\; d\tau \; = \overset{\triangle}{v} \quad \text{ for } \overset{\triangle}{w} \in Y. \tag{54}$$

Similarly as in [14, Lemma 5.7] one verifies that the constraint functions g_0, g_1, and g_2 are differentiable on $B_\delta(0) \subset Y$ for $\delta > 0$ sufficiently small, and we obtain

$$\begin{aligned} g_0'(0)\, \overset{\triangle}{w} \;&=\; v z(0) \wedge \int_0^1 d_3[w](t)\, dt + \overset{\triangle}{v} \int_0^1 d_3[w](t)\, dt \\ &\quad + v \int_0^1 z'(t) \wedge \int_t^1 d_3[w](\tau)\, d\tau\, dt \\ &\underset{(27)}{=} v \int_0^1 z'(t) \wedge \int_t^1 d_3[w](\tau)\, d\tau\, dt, \end{aligned} \tag{55}$$

$$g_1'(0)\, \overset{\triangle}{w} \;=\; \int_0^1 z'(t) \cdot d_{02}\, dt \tag{56}$$

$$g_2'(0)\, \overset{\triangle}{w} \;=\; -\int_0^1 z'(t) \cdot d_{01}\, dt \tag{57}$$

where $z \in W^{1,\infty}([0,1], \mathbb{R}^3)$ is given by (50),(51).

Notice that g is not smooth, but one can show (cf. [14, Lemma 5.10]) that g is Lipschitz continuous on $B_\delta(0) \subset Y$ for some $\delta > 0$ sufficiently small. Thus we can compute its generalized gradient in the sense of Clarke [4]. As a preliminary tool we show

Lemma 5.3. *Let w be a minimizing configuration for* (26)–(30)*, such that $\mathcal{K}[w]$ is locally not attained. Then there are constants $\delta > 0$ and $\tilde{\eta} \in (0, L/2\pi)$ (both depending on the minimizer w) such that*

$$g(\overset{\triangle}{w}) = \max_{(s,\sigma)\in\tilde{\mathcal{Q}}} H(P[\overset{\triangle}{w}](s,\sigma)) - \theta^{-2} \quad \text{for all} \quad \overset{\triangle}{w} \in B_\delta(0) \subset Y, \tag{58}$$

where

$$\tilde{\mathcal{Q}} := \{(s,\sigma) \in [0,1] \times [0,1] : 1 - \tilde{\eta} \geq s - \sigma \geq \tilde{\eta}\}. \tag{59}$$

In particular, $A[\breve{\gamma}[\overset{\triangle}{w}]] \subset \tilde{\mathcal{Q}}$ for all $\overset{\triangle}{w} \in B_\delta(0)$, where $A[\breve{\gamma}[\overset{\triangle}{w}]]$ is the set defined in (13).

Proof. It suffices to prove the continuity of the mapping $\breve{\mathcal{K}}[.]$ near the origin of Y, i.e.,

$$\breve{\mathcal{K}}[\overset{\triangle}{w}] \to \breve{\mathcal{K}}[0] = \mathcal{K}[w] \ \text{ as } \ \| \overset{\triangle}{w} \|_Y \to 0. \tag{60}$$

Then we can argue exactly as in the proof of [14, Lemma 5.9] to prove the claim.

For the proof of (60) we notice that according to (40)–(42) we have

$$\breve{\mathcal{K}}[\overset{\triangle}{w}] = v(v + \overset{\triangle}{v})^{-1}\breve{\mathcal{K}}[\overset{\triangle}{w}_0], \tag{61}$$

where

$$\overset{\triangle}{w}_0 := (\overset{\triangle}{u}, 0, \overset{\triangle}{\gamma}_0, \overset{\triangle}{\alpha}), \tag{62}$$

since $\breve{d}_3[\overset{\triangle}{w}] = \breve{d}_3[\overset{\triangle}{w}_0]$ by (24). Variations of the special form (62) keep the length fixed according to (24), and [14, Lemma 3.2] implies that $\breve{\mathcal{K}}[.]$ is continuous with respect to these particular variations. Hence we conclude with (61) that

$$\begin{aligned} |\breve{\mathcal{K}}[\overset{\triangle}{w}] - \breve{\mathcal{K}}[0]| &\leq |\breve{\mathcal{K}}[\overset{\triangle}{w}] - \breve{\mathcal{K}}[\overset{\triangle}{w}_0]| + |\breve{\mathcal{K}}[\overset{\triangle}{w}_0] - \breve{\mathcal{K}}[0]| \\ &\to 0 \ \text{ as } \ \overset{\triangle}{w} \to 0, \end{aligned}$$

because $\overset{\triangle}{w} \to 0$ implies $\overset{\triangle}{w}_0 \to 0$ and $\overset{\triangle}{v} \to 0$. □

Due to the characterization (58) of g we can apply the nonsmooth chain rule proved in [14, Appendix] to get an explicit formula for the generalized gradient $\partial g(0)$.

Lemma 5.4. *Let w be a minimizer of (26)–(30), such that $\mathcal{K}[w]$ is locally not attained. Then the function g as defined in (43) is Lipschitz continuous on $B_\delta(0) \subset Y$ for some small $\delta > 0$ depending on w, that is, the generalized gradient $\partial g(0) \subset X^*$ exists. Furthermore, for any $g^* \in \partial g(0)$ there is a Radon measure μ^* on $[0,1] \times [0,1]$ with nonempty support on $A[w]$, see (13), such that*

$$\begin{aligned} \langle g^*, \overset{\triangle}{w} \rangle_{Y^* \times Y} &= -\int_0^1 z'(t) \cdot v \int_t^1 d_3(\tau) \wedge f_c^*(\tau)\, d\tau\, dt \\ &\quad - z(0) \cdot v \int_0^1 d_3(t) \wedge f_c^*(t)\, dt \\ &\quad - \overset{\triangle}{v} \int_0^1 d_3[w] \cdot f_c^*(t)\, dt, \end{aligned} \tag{63}$$

where

$$f_c^*(\tau) := \int_{\mathcal{Q}_\tau} [\gamma[w](s) - \gamma[w](\sigma)]\, d\mu^*(s,\sigma), \tag{64}$$

$$\mathcal{Q}_\tau := \{(s,\sigma) \in [0,1] \times [0,1] : \sigma \le \tau \le s\} \;\; \textit{for} \;\; \tau \in [0,1]. \tag{65}$$

Proof. We can proceed as in the proof of [14, Lemma 5.10] the only difference being the evaluation of the Fréchet derivative $P_w[0](s,\sigma)$ of the function P defined in (40). In contrast to [14, formula (131)] we obtain by (49), (53)

$$P_w[0](s,\sigma)\, \overset{\triangle}{w} = \begin{pmatrix} \overset{\triangle}{\gamma}_0 + v \int_0^s z(t) \wedge d_3[w](t)\, dt + \overset{\triangle}{v} \int_0^s d_3[w](t)\, dt \\ \overset{\triangle}{\gamma}_0 + v \int_0^\sigma z(t) \wedge d_3[w](t)\, dt + \overset{\triangle}{v} \int_0^\sigma d_3[w](t)\, dt \\ z(\sigma) \wedge d_3[w](\sigma)\big) \end{pmatrix}.$$

Thus for any $g^* \in \partial g(0)$ there is a probability Radon measure $\bar{\mu}$ on $\tilde{\mathcal{Q}}$ supported on $A[w]$ such that

$$\begin{aligned} \langle g^*, \overset{\triangle}{w} \rangle_{Y^* \times Y} &= -\mathcal{K}[w]^4 \int_{\tilde{\mathcal{Q}}} [\gamma[w](s) - \gamma[w](\sigma)] \cdot \\ &\qquad \Big[v \int_\sigma^s z(t) \wedge d_3[w](t)\, dt + \overset{\triangle}{v} \int_\sigma^s d_3[w](t)\, dt \Big]\, d\bar{\mu}(s,\sigma) \end{aligned}$$

for all $\overset{\triangle}{w} \in Y$. After extending the measure $\bar{\mu}$ from $\tilde{\mathcal{Q}}$ to the triangle

$$\bar{\mathcal{Q}} := \{(s,\sigma) \in [0,1] \times [0,1] : s \ge \sigma\} \supset \tilde{\mathcal{Q}}$$

by zero (again denoted by $\bar{\mu}$), we proceed as in [14] (essentially by Fubini's Theorem after integrating by parts) to obtain for $\mu^* := \mathcal{K}[w]^4\bar{\mu}$ and f_c^* as in (64)

$$\begin{aligned} \langle g^*, \overset{\triangle}{w} \rangle_{Y^* \times Y} &= -\int_0^1 z'(t) \cdot v \int_t^1 d_3[w](\tau) \wedge f_c^*(\tau)\, d\tau\, dt \\ &\quad -z(0) \cdot v \int_0^1 d_3[w](t) \wedge f_c^*(t)\, dt \\ &\quad - \overset{\triangle}{v} \int_0^1 d_3[w](t) \cdot f_c^*(t)\, dt. \end{aligned} \tag{66}$$

□

Lagrange multiplier rule. By Lemma 5.2 we know that $\overset{\triangle}{w}= 0$ is a local minimizer for the reduced variational problem (44)–(48). We are now in the position to apply the nonsmooth Lagrange multiplier rule of Clarke (cf. [4, Chapter 6]) to this variational problem, since the energy function $\breve{E}$ and the constraints $g, g_0, g_i, i = 1, 2$, are Lipschitz continuous near $0 \in Y$. Hence there exist multipliers $\lambda_E, \lambda \geq 0$, $\lambda_0 \in \mathbb{R}^3$, $\lambda_1, \lambda_2 \in \mathbb{R}$, not all zero, such that

$$0 \in \lambda_E \breve{E}'(0) + \lambda \partial g(0) + \lambda_0 \cdot g_0'(0) + \sum_{i=1}^{2} \lambda_i g_i'(0) \tag{67}$$

with

$$\lambda g(0) = 0. \tag{68}$$

In other words, there exists $g^* \in \partial g(0) \subset Y^*$, such that

$$0 = \left\langle \lambda_E \breve{E}'(0) + \lambda_0 \cdot g_0'(0) + \sum_{i=1}^{2} \lambda_i g_i'(0), \overset{\triangle}{w} \right\rangle_{Y^* \times Y} + \lambda \langle g^*, \overset{\triangle}{w} \rangle_{Y^* \times Y} \tag{69}$$

for all $\overset{\triangle}{w}\in Y$.

Inserting the expressions (54), (55)–(57) and (64) into (69) and setting

$m_0 := \lambda_1 d_{02} - \lambda_2 d_{01}$ we thus arrive at

$$
\begin{aligned}
0 \;=\; & \lambda_E \overset{\triangle}{v} + v \int_0^1 z'(t) \cdot \left(\int_t^1 d_3[w](\tau) \wedge \lambda_0 \, d\tau \right) dt \\
& + \int_0^1 z'(t) \cdot m_0 \, dt - \lambda v \int_0^1 z'(t) \cdot \int_t^1 d_3[w](\tau) \wedge f_c^*(\tau) \, d\tau \, dt \qquad (70) \\
& - \lambda \overset{\triangle}{v} \int_0^1 d_3[w](t) \cdot f_c^*(t) \, dt - \lambda z(0) \cdot v \int_0^1 d_3[w](t) \wedge f_c^*(t) \, dt
\end{aligned}
$$

for all $\overset{\triangle}{w} \in Y$. Notice that the equation is independent of the variations $\overset{\triangle}{\gamma}_0$. (Recall that $\overset{\triangle}{u}$ uniquely determines z' by (50), and that z' can be any function in $L^\infty([0,1], \mathbb{R}^3)$ by a suitable choice of $\overset{\triangle}{u} \in L^\infty([0,1], \mathbb{R}^3)$. Analogously $z(0)$ can be any vector in $\mathbb{R}^3$ by a suitable choice of $\overset{\triangle}{\alpha} \in \mathbb{R}^3$, see (51).)

Taking special variations of the form $\overset{\triangle}{w} = (0, \overset{\triangle}{v}, 0, 0) \in Y$ we obtain from (61) that $\breve{\mathcal{K}}[\overset{\triangle}{w}] = v(v + \overset{\triangle}{v})^{-1} \breve{\mathcal{K}}[0] = v(v + \overset{\triangle}{v})^{-1} \mathcal{K}[w]$. Assume that $\mathcal{K}[w] < \theta^{-1}$. Then, for $\| \overset{\triangle}{w} \|_Y$ sufficiently small, one has $\breve{\mathcal{K}}[\overset{\triangle}{w}] < \theta^{-1}$ and $\overset{\triangle}{w}$ satisfies all side conditions (45)–(48). For $\overset{\triangle}{v} < 0$, however, $\breve{E}(\overset{\triangle}{w}) = v + \overset{\triangle}{v} < v = \breve{E}(0) = E(w)$ according to (25). Thus $w = 0$ could not have been a local minimizer. Hence $\mathcal{K}[w] = \mathcal{K}[\gamma] = \theta^{-1}$.

Next we show that $\lambda \neq 0$ in (70). Indeed, $\lambda = 0$ in (70) implies

$$
0 = \lambda_E \overset{\triangle}{v} + v \int_0^1 z'(t) \cdot \left(\int_t^1 d_3[w](\tau) \wedge \lambda_0 \, d\tau \right) dt + \int_0^1 z'(t) \cdot m_0 \, dt.
$$

The choice $\overset{\triangle}{v} \neq 0$ and $\overset{\triangle}{u} = 0$ giving $z' \equiv 0$ by (50) leads to $\lambda_E = 0$. Arbitrary variations $\overset{\triangle}{u} \in L^\infty(I, \mathbb{R}^3)$ correspond to arbitrary $z' \in L^\infty(I, \mathbb{R}^3)$ by (50), hence we can apply the Fundamental Lemma in the calculus of variations to obtain

$$
0 = v \int_t^1 d_3[w](\tau) \wedge \lambda_0 \, d\tau + m_0 \qquad (71)
$$

for a.e. $t \in [0,1]$, and then by continuity for all $t \in [0,1]$. Differentiation with respect to t gives

$$
d_3[w](t) \wedge \lambda_0 = 0 \ \text{ for all } \ t \in [0,1], \qquad (72)
$$

again by continuity of $d_3[w] \in C^{0,1}([0,1],\mathbb{R}^3)$. Inserting (72) into (71) we arrive at $m_0 = 0$, i.e., $\lambda_1 = \lambda_2 = 0$ by definition of m_0. If $\lambda_0 = 0$ then all Lagrange multipliers in (67) would vanish contradicting the Lagrange multiplier rule, hence $\lambda_0 \neq 0$. But then by (72) the vectors $d_3[w](t) = \gamma'[w](t)$ and λ_0 would be collinear for all $t \in [0,1]$ contradicting the fact that $\gamma[w]$ is a closed curve. Hence $\lambda \neq 0$, i.e., $\lambda > 0$ (since $\lambda \geq 0$ by the Lagrange multiplier rule).

Evaluating (70) for $\overset{\triangle}{v}= 0$ and setting $f_c(t) := \lambda v f_c^*(t)$ for $t \in [0,1]$, and $f_0 := v\lambda_0$ we obtain

$$0 \;=\; \int_0^1 z'(t) \cdot \left(\int_t^1 d_3[w](\tau) \wedge [f_0 - f_c(\tau)]\, d\tau + m_0 \right) dt,$$

which implies by the Fundamental Lemma in the calculus of variations

$$0 = \int_t^1 d_3[w](\tau) \wedge [f_0 - f_c(\tau)]\, d\tau + m_0 \tag{73}$$

for all $t \in [0,1]$ by continuity. Differentiating this with respect to t we obtain

$$d_3[w](t) \wedge [f_0 - f_c(t)] = 0 \;\text{ for a.e. } t \in [0,1],$$

which by (73) leads to $m_0 = 0$. Hence we can write

$$f_c(t) = b(t) d_3[w](t) + f_0 \;\text{ for a.e. } t \in [0,1] \tag{74}$$

for some function b. Notice that [14, Corollary 4.2] applies to f_c, which is a constant multiple of f_c^* defined in (64). Hence $f_c \cdot d_3[w] \in W^{1,\infty}([0,1])$ and, by (74), $b \in W^{1,\infty}([0,1])$ and $f_c \in W^{1,\infty}([0,1],\mathbb{R}^3)$. Taking derivatives in (74) we get

$$f_c'(t) = b'(t) d_3[w](t) + b(t) d_3'[w](t) \quad \text{a.e. on } [0,1].$$

Since $d_3'[w] \cdot d_3[w] = 0$ on $[0,1]$ and, by [14, Corollary 4.2], $f_c' \cdot d_3[w] = 0$ a.e. on $[0,1]$, there is a constant $b_0 \in \mathbb{R}$ with

$$b(t) = b_0 \;\text{ for all }\; t \in [0,1].$$

Consequently,

$$f_c'(t) = b_0 d_3'[w](t) \;\text{ for a.e. } t \in [0,1]. \tag{75}$$

If $b_0 = 0$, then $f_c(t) = f_0$ for all $t \in [0,1]$ by (74). With [14, Lemma 5.11] we conclude that
$$\lambda v \mu^* = \lambda v \mathcal{K}[w]^4 \bar{\mu} = 0.$$
Hence $\bar{\mu} = 0$ by $\lambda > 0$. This contradicts the fact that $\bar{\mu}$ is a probability Radon measure according to the proof of Lemma 5.4. Therefore $b_0 \neq 0$. By the identity $\gamma'[w](t) = v d_3[w](t)$ according to (24) we can rewrite (75) as

$$f_c'(t) = (b_0/v)\gamma''[w](t) \;\text{ for a.e. } t \in [0,1]. \tag{76}$$

To determine the sign of the constant b_0 we take variations

$$\overset{\triangle}{w} = (0, \overset{\triangle}{v}, 0, 0) \in Y \;\text{ with }\; \overset{\triangle}{v} \neq 0$$

in (70) to obtain

$$\begin{aligned} 0 &= \lambda_E - \lambda \int_0^1 d_3[w](t) \cdot f_c^*(t)\, dt \\ &= \lambda_E - v^{-1} \int_0^1 d_3[w](t) \cdot [f_c(t) - f_0]\, dt \\ &\underset{(74)}{=} \lambda_E - b_0/v, \end{aligned}$$

where we used that $\int_0^1 d_3[w](t) \cdot f_0\, dt = 0$ by (24) and (27). Consequently, $\lambda_E \neq 0$, i.e., $\lambda_E > 0$ by the Lagrange multiplier rule, hence $b_0 > 0$ as well. (The condition $\lambda_E > 0$ expresses *normality* in the Lagrange multiplier rule.)

Recall that $f_c = \lambda v f_c^*$ with f_c^* according to (64) and $\lambda, v > 0$. Now we can argue as in the proof of [14, Corollary 4.2 (iii)], which rests on [1, Theorem 2.28]. Hence there are (nonegative) Radon measures μ_s for $s \in [0,1]$ such that
$$\operatorname{supp} \mu_s \subset I_s \quad (\text{i.e., } \mu_s = 0 \text{ if } I_s = \emptyset.)$$
and
$$\gamma''(s) = \frac{v}{b_0} f_c'(s) = \int_0^1 [\gamma(\tau) - \gamma(s)]\, d\mu_s(\tau) \quad \text{a.e. on } [0,1].$$
This verifies Theorem 3.1. $\square$

Proof of Corollary 3.2. (i) We can rewrite the set I_c as

$$I_c = \{s \in [0,1] : \kappa_G[\gamma](s) = \theta^{-1}\} \tag{77}$$

by definition of $A[\gamma]$ in (13), together with Lemma 2.4 and by the fact that $\mathcal{K}[\gamma] = \theta^{-1}$ according to Theorem 3.1. Thus (i) follows from Corollary 2.6.

(ii) If $\kappa_G[\gamma](s) < \theta^{-1}$ we find $\delta = \delta(s)$ such that $\kappa_G[\gamma](\tau) < \theta^{-1}$ for all $\tau \in [0,1]$ with $|\tau - s| < \delta$. Otherwise we could find a sequence $s_i \to s$ with $\kappa_G[\gamma](s_i) = \theta^{-1}$ for all $i \in \mathbb{N}$, but then $\kappa_G[\gamma](s) = \theta^{-1}$ according to assertion (i). Thus $\tau \notin I_c$ for $|s - \tau| < \delta$. Hence $I_\tau = \emptyset$ and $\mu_\tau = 0$ for all $|\tau - s| < \delta$. Therefore $\gamma''(\tau) = 0$ for a.e. $|\tau - s| < \delta$, which implies the assertion.

(iii) Obviously $\mu_s = 0$ for a.e. $s \in \tilde{I}$. Thus $\gamma''(s) = 0$ a.e. on $\tilde{I}$, i.e., γ is straight on $\tilde{I}$ by $\gamma \in W^{2,\infty}$.

(iv) Since $\gamma''(s) \neq 0$ a.e. on $\tilde{I}$, $\mu_s \neq 0$ for a.e. $s \in \tilde{I}$. Thus there is a set $I_0 \subset \tilde{I}$ of (Lebesgue) measure zero such that $\tilde{I} \setminus I_0 \in I_c$. Since the closure of $\tilde{I} \setminus I_0$ relatively to $\tilde{I}$ is $\tilde{I}$ and since I_c is closed, we readily get $\tilde{I} \subset I_c$, i.e., $\kappa_G[\gamma](s) = \theta^{-1}$ on $\tilde{I}$ by (77). □

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