Contact between nonlinearly elastic bodies

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Abstract

We study the contact between nonlinearly elastic bodies by variational methods. After the formulation of the mechanical problem we provide existence results based on polyconvexity and on quasiconvexity. Then we derive the Euler-Lagrange equation as a necessary condition for minimizers. Here Clarke's generalized gradients are the essential tool to treat the nonsmooth obstacle condition.

1 Introduction

The deformation of a body in nature is always restricted by the presence of other bodies and by the presence of itself, since matter cannot interpenetrate. Problems that focus on the associated phenomenon of touching are called contact problems. In the literature mostly the contact between an elastic body and a rigid obstacle is considered while the possibility of self-contact is usually neglected. Within the general framework of nonlinear elasticity Ciarlet & Nečas [7] treated elastic contact almost 20 years ago based on strong regularity assumptions for the deformation such that the methods of smooth analysis became applicable. Here we study the contact between two nonlinearly elastic bodies with a completely different approach which allows much weaker regularity conditions. Our results can also be considered as some contribution to a general treatment of elastic self-contact, a problem contained in J. Ball's collection of important open problems (cf. [5, Problem 7]).

Contact problems are typically highly nonsmooth due to the general unilateral restriction for deformations. The geometric simplifications employed in linear elasticity cause that admissible deformations usually form a convex set and that the investigation of corresponding contact problems is closely related to the study of variational inequalities - a tool of convex analysis. These methods cannot be transferred to general nonlinear problems but it turned out that the nonsmooth calculus of generalized gradients developed by F. Clarke (see [8]) is suitable for the successful treatment of the nonsmoothness inherent in contact problems within the fully nonlinear theory. This is carried out for the (nonlinearly) elastic contact with a rigid obstacle in Schuricht [16], [17] and for the elastic self-contact of rods in Schuricht & v.d. Mosel [18]. In the present paper we extend these investigations to the contact of two elastic bodies. Here we are confronted with a number of new technical difficulties, since we have basically no information about the regularity of the boundary of the deformed bodies. This has to be balanced by additional assumptions compared with the case of rigid obstacles.

In Section 2 we formulate a general variational problem describing the contact between two nonlinearly elastic bodies. We prevent interpenetration of the bodies by an inequality constraint based on a signed distance function instead of an abstract set inclusion. The existence of solutions for polyconvex and quasiconvex energies is shown by standard methods in Section 3. Here the contact constraint can be treated similarly to the case of a rigid obstacle (cf. Schuricht [17]). In Section 4 we derive the Euler-Lagrange equation as necessary condition for local minimizers of the energy which is equivalent to the mechanical equilibrium condition in integral form. The main difficulty in the proof is caused by the fact that the functional entering the inequality constraint has bad regularity properties. In order to ensure that it is locally Lipschitz continuous we need additional conditions (as finite dilatation) ensuring that deformations correspond to open mappings. Instead of a usual smooth Lagrange multiplier rule, we then apply a nonsmooth one from Clarke's calculus of generalized gradients. In order to be able to evaluate the structure of the Lagrange multiplier corresponding to the contact constraint we have to handle functions that are the pointwise maximum of a class of functions. Here the key is a nonsmooth calculus rule that we apply twice and that typically has no smooth analogue. Furthermore the characterization of certain generalized gradients only succeeds by means of two new calculus rules provided in Section 5. We end up with a multiplier that describes the contact forces between the elastic bodies where the direction of the forces belongs to certain convex cones that might be interpreted as normal to the contact surface. However, a precise description of these cones seems to be very difficult, since in fact no information about the regularity of the contact surface and the influence of small perturbations on it is available. In Section 5 we first provide a brief summary of basic properties of Clarke's generalized gradients sufficient for our investigations. Then we extend a chain rule from Schuricht [16] to non-reflexive spaces and we derive a characterization of the generalized gradient $\partial f(u)$ by the gradients $\partial f(v)$ where $f(v) \neq f(u)$.

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Notation. By A^c, $\mathrm{cl}\, A$ or $\bar{A}$, $\mathrm{int}\, A$, ∂A, $\overline{\mathrm{conv}} A$, $\overline{\mathrm{conv}}^* A$, and $|A|$ we denote the complement, the closure, the interior, the boundary, the closed convex hull, the weak* closed convex hull, and the Lebesgue measure of the set A. The distance of a point to the set A is given by $\mathrm{dist}_A(\cdot)$. sign α is the usual sign function for real numbers and $|a|$ stands for the Euclidean norm in $\mathbb{R}^n$. For a matrix $F \in \mathbb{R}^{3\times 3}$ we express by $|F|$, $\det F$, and $\mathrm{adj}\, F$ any fixed norm, the determinant, and the adjugate (i.e., $F\,\mathrm{adj}\, F = \det F \;\mathrm{id}$). If X is a Banach space, then X^* stands for its dual space, $\langle \cdot, \cdot \rangle$ for the duality form on $X^* \times X$, $u_n \to u$ for the strong, $u_n \rightharpoonup u$ for the weak, and $u_n \overset{*}{\rightharpoonup} u$ for the weak* convergence. $B_\varepsilon(x)$ is the open ball of radius ε around x. By $\mathcal{C}^k(\Omega)$ we denote the usual space of $k-$times continuously differentiable functions, by $\mathcal{C}_0^k(\Omega)$ its subspace of all functions with compact support, by $\mathcal{L}^p(\Omega)$ the Lebesgue space of $p-$integrable functions, and by $\mathcal{W}^{1,p}(\Omega)$ the Sobolev space of p-integrable functions having p-integrable weak derivatives. The space of (positive) Radon measures on Ω is identified by $R[\Omega]$ while $R_1[\Omega]$ and $R_{\leq 1}[\Omega]$ refer to the subset of probability measures and measures of total mass less than 1, respectively. (Without danger of confusion we also write $R[\tilde{\Omega}]$ for a measure on Ω whose support is contained in $\tilde{\Omega} \subset \Omega$.) For a locally Lipschitz continuous function $f : X \to \mathbb{R}$ Clarke's generalized gradient is denoted by $\partial f(u)$ and the generalized directional derivative by $f^\circ(u; v)$.

2 Formulation of the mechanical problem

We consider two elastic bodies that occupy the open bounded domains $\Omega_i \subset \mathbb{R}^3$, $i = 1, 2$, in their reference configuration where we assume that the Ω_i have disjoint closure and Lipschitz boundary. The deformations of these elastic bodies are described by functions $u \in \tilde{X} := \mathcal{W}^{1,p}(\Omega; \mathbb{R}^3)$ with $\Omega := \Omega_1 \cup \Omega_2$ where we usually assume that $p > 3$ to focus on continuous deformations. Using the notation $u_i := u|_{\Omega_i}$ $(i = 1, 2)$ we characterize the deformation of one of the bodies.

The material of the bodies is assumed to be hyperelastic, i.e., there is a density function $W : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ such that the stored energy of the deformed configuration $u \in \mathcal{W}^{1,p}(\Omega)$ is given by

$$E_s(u) := \int_\Omega W(x, Du(x))\, dx\ .$$

As usual we suppose that $W(x, \cdot)$ is rank-1-convex and continuously differentiable on $\mathbb{R}^{3\times 3}$ for all $x \in \Omega$ and that $W(\cdot, F)$ is measurable on Ω for all $F \in \mathbb{R}^{3\times 3}$. All external forces acting on the bodies may be described by a vector valued measure $f \in R[\bar{\Omega}]$ corresponding to the potential energy

$$E_p(u) := -\int_{\bar{\Omega}} u(x)\, df(x)\ .$$

Hence

$$E(u) := E_s(u) + E_p(u) = \int_\Omega W(x, Du(x))\, dx - \int_{\bar{\Omega}} u(x)\, df(x)$$

is the total energy assigned to a configuration u.

Let us now discuss restrictions that we may impose on admissible deformations. The mechanical requirement that deformations u should be locally invertible and orientation preserving enters the theory by the condition that

$$\det Du > 0 \quad \text{a.e. on} \quad \Omega \tag{2.1}$$

which can be ensured by

$$W(x, F) = \infty \quad \text{if} \quad \det F \le 0\,. \tag{2.2}$$

Even for smooth deformations u condition (2.1) implies local but not global invertibility of u, i.e., it does not prevent interpenetration of the material. An analytical condition excluding interpenetration but allowing self-contact is given by

$$\int_\Omega \det Du(x)\, dx \le |u(\Omega)| \tag{2.3}$$

and was introduced by Ciarlet & Nečas [7]. It turns out that conditions like (2.2) and (2.3) can be taken into account to verify the existence of energy minimizing configurations, but the derivation of the corresponding Euler-Lagrange equation as necessary condition succeeds only under strong regularity assumptions on u. The treatment of (2.3) in [7] is based on deformations $u \in \mathcal{C}^1(\bar{\Omega})$ which, in particular, prevents "local" self-contact (i.e., touching of points that are also arbitrarily close to each other in the reference configuration). In such cases the problem of self-contact can be reduced to the investigation of the contact between two elastic bodies.

In this paper we treat the problem that different elastic bodies can touch but should not penetrate each other while we neglect the possibility of self-penetration. According to our previous discussion this can be also considered as a partial problem for the treatment of self-contact. Since constraint (2.3) seems not to be accessible to direct regularity arguments taking into account deformations $u \in \mathcal{W}^{1,p}(\Omega)$, we use a different approach. To prevent interpenetration of the different bodies we have to demand that

$$\operatorname{int} u_1(\bar{\Omega}_1) \cap \operatorname{int} u_2(\bar{\Omega}_2) = \emptyset$$

or

$$u_1(x) \in \overline{\mathbb{R}^3 \setminus u_2(\bar{\Omega}_2)} \quad \text{for all } \ x \in \bar{\Omega}_1\,. \tag{2.4}$$

The investigation of the contact between an elastic body and a rigid obstacle in Schuricht [17] has shown that it is useful to replace an abstract set inclusion like (2.4) by an analytically more tractable inequality condition based on a suitable distance function. Therefore we introduce the signed distance function $d : \mathbb{R}^3 \times \mathcal{W}^{1,p}(\Omega) \to \mathbb{R}$ as

$$d(q, v)\Big(= d(q, v_2)\Big) := \begin{cases} \ \ \operatorname{dist}_{v(\partial\Omega_2)}(q) & \text{if} \quad q \in \operatorname{int} v(\Omega_2), \\ -\operatorname{dist}_{v(\partial\Omega_2)}(q) & \text{if} \quad q \notin \operatorname{int} v(\Omega_2) \end{cases} \tag{2.5}$$

(though $d(q,\cdot)$ is defined on $\mathcal{W}^{1,p}(\Omega)$ we sometimes use the notation given in parentheses to indicate that the function in fact only depends on v_2). Then we can replace (2.4) with the condition

$$g(u) := \max_{x\in\bar{\Omega}_1} d(u_1(x), u_2) \le 0 \tag{2.6}$$

where g is a real function on $\tilde{X}$ and we set

$$\Omega_c^1(u) := \{x \in \bar{\Omega}_1 |\; d(u_1(x), u_2) = 0\}$$

and

$$\Omega_c^2(u) := \{y \in \partial\Omega_2 |\; u_2(y) = u_1(x) \;\text{ for some }\; x \in \bar{\Omega}_1\}\,.$$

We impose Dirichlet boundary conditions

$$u(x) = u_D(x) \quad \text{on } \Gamma_D$$

where $\Gamma_D := \Gamma_D^1 \cup \Gamma_D^2$, $\Gamma_D^i \subset \partial\Omega_i$, $\Gamma_D^i \neq \emptyset$, $i = 1, 2$, and $u_D \in \mathcal{W}^{1,p}(\Omega)$ is a given function satisfying

$$u_D(\Gamma_D^2) \cap u_D(\Gamma_D^1) = \emptyset\,. \tag{2.7}$$

Taking into account the discussed constraints we study the variational problem

$$\begin{aligned} E(u) &\;\to\; \text{Min!}\,, \quad u \in \mathcal{W}^{1,p}(\Omega)\,, && (2.8)\\ u &\;=\; u_D \quad \text{on } \Gamma_D\,, && (2.9)\\ g(u) &\;\le\; 0 && (2.10) \end{aligned}$$

providing equilibrium configurations of pairs of non-penetrating elastic bodies. Though we are mainly interested in the case $p > 3$ that ensures continuous deformations, we may allow $p \le 3$ in the existence results. In that case, however, $u \in \mathcal{W}^{1,p}(\Omega)$ might not be continuous and we have to be more careful with the formulation of the side conditions. While (2.9) has to be understood in the sense of trace, we choose the precise representative of u in (2.4). This means that (2.4) can merely be demanded a.e. on $\bar{\Omega}_1$ and, accordingly, we have to take the essential supremum in (2.6).

Usually one has the imagination that deformations should correspond to open mappings u. In this case one has that $\partial u(\Omega) \subset u(\partial\Omega)$ and that $\Omega_c^1(u)$ identifies the points of contact. This, however, is not true for all $u \in \mathcal{W}^{1,p}(\Omega)$. In general $u(\partial\Omega)$ might contain interior points of $u(\Omega)$ and the preimage of $\partial u(\bar{\Omega})$ must not be a subset of $\partial\Omega$. This can cause $d(\cdot, v)$ to be discontinuous and there might be elements in $\Omega_c^1(u)$ that do not correspond to contact points. Nevertheless it turns out that (2.10) can be used to verify the existence of equilibrium configurations exhibiting contact. But for the derivation of the corresponding Euler-Lagrange equation we have to restrict our attention to deformations corresponding to open mappings.

Note. Alternatively to (2.5) we could define $d(\cdot,\cdot)$ based on $\pm\text{dist}_{\partial v(\Omega_2)}(q)$. While d according to (2.5) is locally Lipschitz continuous on a suitable set of deformations (cf. Lemma 4.4 below), $v \to d(q, v)$ has much less regularity in the other case and is discontinuous even under the restriction to open mappings v. To see this we consider an open deformation u_2 with self-contact, i.e., $u_2(\Gamma') = u_2(\Gamma'')$ for disjoint parts Γ', Γ'' of the boundary $\partial\Omega_2$. Then we choose a (relatively) interior point q of that contact surface which is simultaneously an interior point of $u_2(\Omega_2)$.

Hence $d(q,u) > 0$. On the other hand we find arbitrarily small (open) perturbations v of u_2 that "open" the contact and, thus, $d(q,v) \leq 0$.

Let us now look for a suitable class of Sobolev functions u such that u_2 is open on Ω_2. It is known that $u_2 \in \mathcal{W}^{1,p}(\Omega_2)$ with $p \geq 3$ is open if it has integrable dilatation, i.e.,

$$\gamma(x) := \frac{|Du_2(x)|^3}{\det Du_2(x)} \in \mathcal{L}^{\tilde{p}}(\Omega_2) \quad \text{for some } \tilde{p} > 2 \tag{2.11}$$

where we agree that (2.11) is violated in the limit case $u_2 = const.$ (cf. Villamor & Manfredi [19]). In [12] Manfredi & Villamor provide an example of a polyconvex energy functional with

$$W(x,F) \to \infty \quad \text{as} \quad \det F \to 0 \tag{2.12}$$

such that (2.11) is satisfied for all configurations u having finite elastic energy $E_s(u)$. On the other hand it seems to be reasonable to demand that

$$W(x,F) \to \infty \quad \text{as} \quad \frac{\det F}{|F|^3} \to 0 \tag{2.13}$$

instead of (2.12), since $|Du(x)|^3/\det Du(x)$ somehow measures the local distortion of the body (note that (2.13) is also compatible with (2.2)). The previous arguments justify a condition like (2.11) in elasticity. However, for our analysis in Section 4, we need that not merely the minimizing solution u is open but also small perturbations of it. Hence we have to restrict our considerations there to solutions u with $u_2 \in \mathcal{W}^{1,\infty}(\Omega_2)$ and

$$\det Du_2(x) \geq \tilde{\beta} \quad \text{a.e. on } \Omega_2 \tag{2.14}$$

for some $\tilde{\beta} > 0$. In this case we get that the dilatation of u_2 and of small perturbations of it in $\mathcal{W}^{1,\infty}(\Omega_2)$ is finite, i.e., (2.11) holds with $\tilde{p} = \infty$ (cf. also Reshetnyak [15] or Fonseca & Gangbo [11, p. 151]).

3 Existence of minimizers

In this section we study the existence of solutions of the above formulated variational problem. Results of that kind in nonlinear elasticity are based either on quasiconvexity (cf. Morrey [13]) or on polyconvexity (cf. Ball [4]) of W while conditions like (2.2) can be considered merely in the latter case. Corresponding existence results taking into account the global injectivity of the elastic deformation and the restriction by a rigid obstacle can be found in Ciarlet & Nečas [6], [7], Baiocchi et al. [3], Schuricht [17]. For our variational problem we basically have to adopt the standard arguments to the treatment of the side condition (2.10) which, however, provides no serious difficulties.

For the first result we consider the following hypotheses:

($A0$) $\Omega_i \subset \mathbb{R}^3$, $i = 1,2$, are bounded domains with disjoint closure and Lipschitz Boundary.

($A1$) *Polyconvexity:* There exists $h : \Omega \times \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that

- (a) $h(x,\cdot,\cdot,\cdot)$ is continuous and convex for all $x \in \Omega$,
- (b) $h(\cdot, F, \operatorname{adj} F, \det F)$ is measurable for all $F \in \mathbb{R}^{3\times3}$,
- (c) $W(x,F) = h(x, F, \operatorname{adj} F, \det F)$ for all $(x,F) \in \Omega \times \mathbb{R}^{3\times3}$.

($A2$) *Coercivity:* There are constants $\alpha > 0$, $2 \le p < \infty$, $q \ge \frac{p}{p-1}$ such that
(a) $W(x,F) \ge \alpha\,(|F|^p + |\mathrm{adj}\,F|^q)$ for all $(x,F) \in \Omega \times \mathbb{R}^{3\times 3}$,
(b) $W(x,F) = \infty$ if and only if $\det F \le 0$.

($A3$) If $p \le 3$, then E_p is continuous on $\mathcal{L}^{p^*}(\Omega)$ with $p^* = \frac{3p}{3-p}$ for $p < 3$, $p^* = +\infty$ for $p = 3$ (here we implicitly assume that f is of that kind such that E_p is well defined on $\mathcal{L}^{p^*}(\Omega)$).

($A4$) $\Gamma_D^i \subset \partial\Omega_i$, $\Gamma_D^i \ne \emptyset$, $\left|\Gamma_D^i\right| > 0$ for $p \le 3$, $i = 1,2$, $u_D \in \mathcal{W}^{1,p}(\Omega;\mathbb{R}^3)$ is given ($|\cdot|$ denotes the two-dimensional Haussdorff measure).

Theorem 3.1 *Let* ($A0$) $-$ ($A4$) *be fulfilled and assume that there exists* $\tilde{u} \in \mathcal{W}^{1,p}(\Omega)$ *satisfying (2.9), (2.10), and* $E(\tilde{u}) < \infty$. *Then the variational problem (2.8)-(2.10) has a solution* $u \in \mathcal{W}^{1,p}(\Omega)$ *with* $\det Du > 0$ *a.e. on* Ω.

The proof basically proceeds by standard arguments. We merely have to check that condition (2.10) is stable under weak convergence which is carried out at the end of this section. If we drop condition (b) in (A2), then the theorem remains true for $p \ne 3$ up to the statement that $\det Du > 0$ a.e. on Ω.

Polyconvexity was introduced by Ball [4] to handle energies with property (2.2) in existence theory but it is still open to derive the Euler-Lagrange equation for corresponding solutions without hypothesizing further regularity even in the case without contact and global invertibility constraints. Thus we cannot expect to obtain the Euler-Lagrange equation for solutions of our problem. Hence it is reasonable to neglect (2.2) and to look for existence results with a less restrictive convexity constraint. For this reason we replace ($A1$), ($A2$) with the following hypotheses:

($A1'$) *Quasiconvexity:* Let $W : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ be such that
(a) $W(x,\cdot)$ is continuous for all $x \in \Omega$,
(b) $W(\cdot,F)$ is measurable for all $F \in \mathbb{R}^{3\times 3}$,
(c) for all open $\tilde{\Omega} \subset \Omega$, $F \in \mathbb{R}^{3\times 3}$, $\varphi \in \mathcal{C}_0^1(\tilde{\Omega})$, a.e. $\tilde{x} \in \Omega$

$$\int_{\tilde{\Omega}} W(\tilde{x},F)\,dx \le \int_{\tilde{\Omega}} W(\tilde{x},F + D\varphi)\,dx\,.$$

($A2'$) *Coercivity:* The are constants α, β, $\gamma > 0$, $1 < p < \infty$ such that

$$\alpha|F|^p \le W(x,F) \le \beta + \gamma|F|^p \quad \text{for all } (x,F) \in \Omega \times \mathbb{R}^{3\times 3}.$$

Theorem 3.2 *Let* ($A0$), ($A1'$), ($A2'$), ($A3$), ($A4$) *be satisfied and let (2.9), (2.10) be fulfilled by some* $\tilde{u} \in \mathcal{W}^{1,p}(\Omega)$. *Then the variational problem (2.8)-(2.10) has a solution in* $\mathcal{W}^{1,p}(\Omega)$.

Proof of Theorem 3.1. Despite side condition (2.10) the proof proceeds like that of Ball [4] (cf. also Dacorogna [9], Müller [14] for some extensions). For the convenience of the reader we sketch these arguments.

Let $\{u_n\}$ be a minimizing sequence of the variational problem. By (A2) and since $\|Dv\|_{\mathcal{L}^p}$ is an equivalent norm on the space $\{v \in \mathcal{W}^{1,p}(\Omega) | v = u_D \text{ on } \Gamma_D\}$ according to (A0), (A4), there exists a subsequence (denoted the same way) with

$$u_n \rightharpoonup u \text{ in } \mathcal{W}^{1,p}(\Omega), \quad \mathrm{adj}\,Du_n \rightharpoonup A \text{ in } \mathcal{L}^q\left(\Omega,\mathbb{R}^{3\times 3}\right). \tag{3.1}$$

We have $A = \operatorname{adj} Du$ and $\det Du_n \rightharpoonup \det Du$ in the distributional sense (cf. Dacorogna [9, Ch. 4, Th. 2.6]). Moreover, $u_n \to u$ in $\mathcal{L}^{p^*}(\Omega)$ if $p \leq 3$ and in $\mathcal{C}(\bar{\Omega})$ if $p > 3$.

For $p > 3$ we have that $\det Du_n$, $\det Du \in \mathcal{L}^{p/3}(\Omega)$ by Hölder's inequality and, hence, $\det Du_n \rightharpoonup \det Du$ in $\mathcal{L}^{p/3}(\Omega)$. If $2 \leq p < 3$, then $\det Du_n \rightharpoonup \det Du$ in $\mathcal{L}^r(\Omega)$ with $r = \frac{2q}{3} > 1$ by Müller [14]. In both cases we can thus apply standard lower semicontinuity results for convex integrands to E_s (see [9, Ch. 3, Th. 3.4]) and, since E_p is weakly continuous by the arguments following (3.1) and by (A3), we obtain that

$$E(u) \leq \liminf_{n\to\infty} E(u_n)\,,$$

i.e., u is a solution of the variational problem if it respects (2.10).

For the remaining case $p = 3$ we use that $\det Du_n \geq 0$ a.e. on Ω by $(A2)$ to prove that $\det Du_n \rightharpoonup \det Du$ in $\mathcal{L}^1(K)$ for all compact $K \subset \Omega$. By the same semicontinuity result as above and by $W \geq 0$ we conclude that

$$\int_K W(x, Du)\,dx \leq \liminf_{n\to\infty} \int_K W(x, Du_n)\,dx \leq \liminf_{n\to\infty} \int_\Omega W(x, Du_n)\,dx$$

for all compact $K \subset \Omega$. Choosing an increasing sequence of compact $K_m \subset \Omega$ exhausting Ω we again obtain that u minimizes E (cf. Müller [14]).

Let us now investigate whether u satisfies (2.10). Obviously $d(q, v)$ is lower semicontinuous on $\mathbb{R}^3 \times \mathcal{C}(\bar{\Omega})$. For $p > 3$ we readily verify that (2.10) is met for u, since $u_n \to u$ in $\mathcal{C}(\bar{\Omega})$. Now we consider the case $p \leq 3$ where $u_n \to u$ in $\mathcal{L}^p(\Omega)$ and, at least for a subsequence, also $u_n(x) \to u(x)$ a.e. on Ω. Let us assume that

$$d(u_1(x), u_2) > 0 \quad \text{on a set } \ \Omega_+ \subset \bar{\Omega}_1 \ \text{ with } \ |\Omega_+| > 0\,. \tag{3.2}$$

Since not all of the pairwise disjoint sets $\left\{x \in \bar{\Omega}_1 \middle|\ \frac{1}{n} \leq d(u_1(x), u_2) < \frac{1}{n-1}\right\}$, $n \in \mathbb{N}$ (take $\frac{1}{0} = \infty$), can have measure zero, we can suppose that $d(u_1(x), u_2) \geq \varepsilon$ on Ω_+ for some $\varepsilon > 0$. By Egorov's theorem, $u_n \to u$ uniformly on a subset $\Omega_0 \subset \bar{\Omega}_1$ with $|\bar{\Omega}_1 \backslash \Omega_0| \leq |\Omega_+|/2$. By construction, $|\Omega_{0+}| > 0$ for $\Omega_{0+} := \Omega_0 \cap \Omega_+$ and $d(u_{1,n}(x), u_{2,n}) \geq \varepsilon/2$ on Ω_{0+} for sufficiently large n. But this disagrees with $g(u_n) \leq 0$. Consequently (3.2) must be wrong and u satisfies (2.10). $\diamondsuit$

Proof of Theorem 3.2. Let $\{u_n\}$ denote a minimizing sequence of the variational problem. Similarly to the previous proof we obtain that $u_n \rightharpoonup u$ in $\mathcal{W}^{1,p}(\Omega)$ by (A0), (A2'), (A4) (at least for a subsequence) and that u is admissible. Using (A1') and (A2') we get the lower semicontinuity of E_s by a result of Acerbi and Fusco [1]. Since E_p is weakly continuous on $\mathcal{W}^{1,p}(\Omega)$ by (A3), u minimizes E. $\diamondsuit$

4 Euler-Lagrange equation

4.1 Formulation of the results

In this section we formulate the Euler-Lagrange equation in the weak form for minimizers of the variational problem (2.8)-(2.10). We invoke the following hypotheses:

(B1) $\Omega_i \subset \mathbb{R}^3$, $i = 1, 2$, are bounded domains with disjoint closure and Lipschitz boundary.

(B2) $W(x, \cdot)$ is continuously differentiable on $\mathbb{R}^{3\times 3}$ for all $x \in \Omega$,
$W(\cdot, F)$ is measurable on Ω for all $F \in \mathbb{R}^{3\times 3}$.

(B3) There are $c > 0$, $\gamma \in \mathcal{L}^1(\Omega)$ such that

$$|DW(x, F)| \le c|F|^p + \gamma(x) \quad \text{for all } (x, F) \in \Omega \times \mathbb{R}^{3\times 3} \tag{4.1}$$

(DW denotes the partial derivative with respect to F).

(B4) The Dirichlet boundary conditions are such that $u_D(\Gamma_D^1) \cap u_D(\Gamma_D^2) = \emptyset$.

The standard growth condition (B3) ensures differentiability of the elastic energy E_s in $\mathcal{W}^{1,p}(\Omega)$.

As space of variations we choose the Banach space $X := \mathcal{W}^{1,\infty}(\Omega)$ and its subspace

$$X_0 := \left\{ v \in \mathcal{W}^{1,\infty}(\Omega) |\ v = 0 \text{ on } \Gamma_D \right\} . \tag{4.2}$$

According to our discussion in Section 2 we here consider minimizing solutions u satisfying (2.14) and $u_2 \in \mathcal{W}^{1,\infty}(\Omega_2)$. This way we ensure that u and small perturbations $u + v$ with $v \in X$ are always open on Ω_2.

Since the real function g in (2.10) is not smooth on X we have to employ a nonsmooth Lagrange multiplier rule to derive the Euler-Lagrange equation which should contain a term describing the contact forces in the case of touching. Our experience tells us that the contact forces should be directed normally to the surfaces of the deformed bodies, but we do not know whether these surfaces are smooth and possess a normal. Our analysis rather yields a cone at each contact point containing the direction of the contact force. For this reason we introduce the following sets where $q \in \mathbb{R}^3$. For $w \in \mathcal{W}^{1,p}(\Omega)$ we define

$$\Gamma(q, w) := \{ y \in \partial\Omega_2 \mid |w_2(y) - q| = d(q, w_2) \text{ or } |w_2(y) - q| = -d(q, w_2) \} . \tag{4.3}$$

Notice that for $u \in \mathcal{W}^{1,p}(\Omega)$ with $g(u) = 0$

$$\Gamma(u_1(x), x) \subset \Omega_c^2(u) \quad \text{for all} \quad x \in \Omega_c^1(u) .$$

For $w \in \mathcal{W}^{1,p}(\Omega)$ with $q \notin w_2(\partial\Omega_2)$ (i.e., $d(q, w) \ne 0$) let

$$d^*(q, y, w)\Big(= d^*(q, y, w_2)\Big) := \operatorname{sign} d(q, w)\, \frac{w(y) - q}{|w(y) - q|} \quad \text{for } y \in \Gamma(q, w) . \tag{4.4}$$

For $u \in \mathcal{W}^{1,p}(\Omega)$ with $q \in u_2(\partial\Omega_2)$ (i.e., $d(q, u) = 0$) we define

$$\mathcal{D}^*(q, u)\Big(= \mathcal{D}^*(q, u_2)\Big) := \bigcap_{\sigma > 0} \overline{\operatorname{conv}} \Big(\bigcup_{\substack{\|v\|_X \le \sigma \\ d(q, u+v) \ne 0}} \ \bigcup_{y \in \Gamma(q, u+v)} d^*(q, y, u + v) \Big) \tag{4.5}$$

where we identify $d^*(q, y, u + v)$ with the set consisting of this one element. To avoid formal difficulties we set $\mathcal{D}^*(q, u) := 0$ for $q \notin u_2(\partial\Omega_2)$.

Let us now formulate the Euler-Lagrange equation as the main theorem of the present paper.

Theorem 4.1 *Let $(B1)-(B4)$ be fulfilled and let $u \in \mathcal{W}^{1,p}(\Omega)$, $p > 3$, be a local minimizer of the variational problem (2.8)-(2.10) such that $u_2 \in \mathcal{W}^{1,\infty}(\Omega_2)$ and that (2.14) is satisfied for some $\tilde{\beta} > 0$. Then there exist a constant $\lambda \geq 0$, a Radon measure $\mu_c \in R[\bar{\Omega}_1]$ supported on $\Omega_c^1(u)$, not both zero, Radon measures $\mu_x \in R[\bar{\Omega}_2]$ supported on $\Gamma(u_1(x), u)$, $x \in \bar{\Omega}_1$, and a mapping*

$$(x, y) \to d_c^*(x, y) \in \mathcal{D}^*(u_1(x), u) \subset \mathbb{R}^3 \quad on \quad \bar{\Omega}_1 \times \bar{\Omega}_2 \tag{4.6}$$

such that the weak form of the Euler-Lagrange equation

$$\begin{aligned} 0 \;\; = \;\; & \lambda \int_\Omega DW(x, Du(x))D\varphi(x)\, dx - \lambda \int_{\bar{\Omega}} \varphi(x)\, df(x) \\ & + \int_{\Omega_c^1(u)} \int_{\Omega_c^2(u)} \langle d_c^*(x, y), \varphi_2(y) - \varphi_1(x) \rangle \, d\mu_x(y)\, d\mu_c(x) \end{aligned} \tag{4.7}$$

is satisfied for all $\varphi \in X_0$ where all occurring integrals exist. In particular, $\mu_c = 0$ if $g(u) < 0$, i.e., if $\Omega_c^1(u) = \emptyset$.

We can choose $\lambda = 1$ in (4.7) if for all $\tilde{x} \in \Omega_c^1(u)$ there is an open neighborhood $U(\tilde{x})$ such that

$$0 \notin \overline{\mathrm{conv}} \bigcup_{x \in \Omega_c^1(u) \cap U(\tilde{x})} \mathcal{D}^*(u_1(x), u)\,. \tag{4.8}$$

Remark 4.2

1) The last term in (4.7) is the Lagrange multiplier corresponding to the contact condition and describes the contact forces between the bodies. In the case where both u_1 and u_2 are globally injective on $\bar{\Omega}_1$ and $\bar{\Omega}_2$, respectively, the vector $d_c^*(x, y)$ provides the contact force between the touching points $u_1(x) = u_2(y)$, the measure μ_x is concentrated on the single point $u_2^{-1}(u_1(x))$, and the measure μ gives somehow the distribution of the contact force on the contact set $\Omega_c^1(u) \subset \bar{\Omega}_1$.

2) The vectors in the set $\mathcal{D}^*(u_1(x), u)$ might be considered as certain normal directions for the contact surface $u_2(\partial\Omega_2)$ at the point $u_1(x)$. In the case of a smooth contact surface it contains the normal direction but it is still open whether the set is possibly larger (cf. Corollary 4.3 below). This question is closely related with the (geometric) regularity of the boundary $u_2(\partial\Omega_2)$ and its behavior under small perturbations.

3) Condition (4.6) provides merely a convex set for the direction of the contact force. This does not mean that the direction $d_c^*(x, y)$ is undetermined, but we have to realize that the precise direction cannot be obtained from the shape of the contact surface. Notice that the definition of $\mathcal{D}^*(u_1(x), u)$ is based on the shape of $u_2(\partial\Omega_2)$. Since we can apply the theorem again after interchanging the role of Ω_1 and Ω_2, it might happen that one case gives a better, i.e., a more restrictive, condition for the direction $d_c^*(x, y)$. Observe that the "best case" must not correspond to the same Ω_i for different "contact pairs" (x, y).

4) If self-contact occurs for an elastic body, then we can basically distinguish between a local case where self-touching occurs for points that are arbitrarily close in some (e.g., stress-free) reference configuration and a nonlocal case where parts of the body touch each other that are far away from each other in the reference configuration. If, in the last case, we consider suitable neighborhoods of these parts as separate bodies, then we could prescribe Dirichlet conditions on the "cutted surfaces" and apply the previous theorem. In this sense our result allows a partial treatment of self-contact.

As already mentioned it is hard to give a precise description of the sets $\mathcal{D}^*(u_1(x), u)$. The next corollary provides a set which is always contained in $\mathcal{D}^*(u_1(x), u)$ and which can be characterized much easier. For $\mathcal{O} := u_2(\partial\Omega_2)$ we introduce the signed distance function

$$d_{\mathcal{O}}(q) := \begin{cases} \ \ \mathrm{dist}_{\mathcal{O}}(q) & \text{if} \quad q \in u_2(\Omega_2), \\ -\mathrm{dist}_{\mathcal{O}}(q) & \text{if} \quad q \notin u_2(\Omega_2), \end{cases}$$

and $\partial d_{\mathcal{O}}(q) \subset \mathbb{R}^3$ denotes its generalized gradient.

Corollary 4.3 *Let the assumptions of Theorem 4.1 be satisfied. Then*

$$-\partial d_{\mathcal{O}}(u_1(x)) \subset \mathcal{D}^*(u_1(x), u) \quad \text{for all } x \in \Omega_c^1(u)\,. \tag{4.9}$$

In particular, we conclude that $\mathcal{D}^*(u_1(x), u)$ contains the normal direction of $\mathcal{O}$ if the surface $\mathcal{O} = u_2(\partial\Omega_2)$ is smooth near $u_1(x)$, since $d_{\mathcal{O}}(\cdot)$ is smooth near $u_1(x)$ in that case. On the other hand it is open whether $\mathcal{D}^*(u_1(x), u)$ might be larger even in that case.

4.2 Proofs

Let us start with some notational convention for this section. In addition to the restrictions v_1, v_2 of v introduced at the beginning of Section 2 we consider sequences $\{v_n\}$. To avoid confusion we agree that the explicit indices "1" and "2" never refer to elements of a sequence. By $v_{i,n}$ $(i = 1, 2)$ we denote the restrictions of v_n.

Proof of Theorem 4.1. We present the proof in several steps where $u \in \mathcal{W}^{1,p}(\Omega)$ denotes a local minimizer according to the theorem.

(a) *Modified problem.* Since we study $\mathcal{W}^{1,\infty}$ perturbations of the solution u, it is convenient to introduce the functions

$$\hat{E}(v) := E(u + v)\,, \quad \hat{g}(v) := g(u + v)\,, \quad \hat{W}(x, F) := W(x, Du(x) + F)\,.$$

Analogously we define $\hat{E}_p$ and $\hat{E}_s$. With X, X_0 according to (4.2) we consider the modified variational problem

$$\hat{E}(v) \ \to \ \text{Min!}, \quad v \in X\,, \tag{4.10}$$

$$v \ \in \ X_0\,, \tag{4.11}$$

$$\hat{g}(v) \ \leq \ 0\,. \tag{4.12}$$

Obviously $v = 0$ is a local minimizer of this variational problem by the continuous imbedding $\mathcal{W}^{1,\infty}(\Omega) \hookrightarrow \mathcal{W}^{1,p}(\Omega)$. It seems to be a little artificial to include (4.11) as a side condition instead of just replacing X with X_0. But it is technically advantageous for the investigation of the structure of the generalized gradient of $\hat{g}$ to consider rigid translations of the body $u_2(\Omega_2)$. This would not be possible within X_0 where we had to bother ourselves with some localization and corresponding technical cut off functions.

(b) *Differentiation of the energy $\hat{E}$ in X.* $\hat{E}_p$ is a linear continuous functional on the space of continuous functions. Thus $\hat{E}_p$ is continuously differentiable on X and we easily get that

$$\langle \hat{E}_p'(v), \varphi \rangle = -\int_{\bar{\Omega}} \varphi(x)\, df(x) \quad \text{for all } \varphi \in X\,. \tag{4.13}$$

By standard arguments using $(B3)$ we get that $\hat{E}_s$ is Gâteaux differentiable on X with

$$\langle \hat{E}_s'(v), \varphi\rangle = \int_\Omega D\hat{W}(x, Dv(x))\ D\varphi(x)\ d(x) \quad \text{for all } \varphi \in X\,. \tag{4.14}$$

Let $v_n \to v$ in X. Then there exist $\varphi_n \in X$, $\|\varphi_n\| \le 1$ with

$$\begin{aligned}
\|\hat{E}_s'(v) - \hat{E}_s'(v_n)\| &= \sup_{\varphi\in X,\ \|\varphi\|\le 1} |\langle \hat{E}_s'(v) - \hat{E}_s'(v_n), \varphi\rangle| \\
&\le \left| \int_\Omega \Big(D\hat{W}(x, Dv(x)) - D\hat{W}(x, Dv_n(x))\Big) D\varphi_n(x)\, dx \right| + \frac{1}{n} \\
&\le \int_\Omega |D\hat{W}(x, Dv(x)) - D\hat{W}(x, Dv_n(x))|\, dx + \frac{1}{n}\,.
\end{aligned}$$

Using $(B3)$, the continuity of $DW(x,\cdot)$, and the dominated convergence theorem, we obtain that $\hat{E}_s'(v_n) \to \hat{E}_s'(v)$ for $n \to \infty$ in X^*, i.e., $\hat{E}_s'(\cdot)$ is continuous on X^*. Hence $\hat{E} = \hat{E}_s + \hat{E}_p$ is continuously differentiable and, consequently,

$$\{\hat{E}'(0)\} = \partial\hat{E}'(0) = \partial\hat{E}_s'(0) + \partial\hat{E}_p'(0) = \{\hat{E}_s'(0)\} + \{\hat{E}_p'(0)\}$$

(cf. Proposition 5.1).

(c) *Generalized gradient* $\partial\hat{g}(0)$. Using the auxiliary function $p : \bar{\Omega}_1 \times X \to \mathbb{R}^3 \times \mathcal{W}^{1,p}(\Omega_2)$ given by

$$p(x, v) := (u_1(x) + v_1(x), u_2 + v_2)$$

we can rewrite $\hat{g}$ as

$$\hat{g}(v) = \max_{x\in\bar{\Omega}_1} d(p(x, v))\,.$$

Obviously, $p(\cdot,\cdot)$ is continuous. The mapping $v \to p(x, v) - (u_1(x), u_2)$ is linear and bounded and, therefore, differentiable for each x with

$$\begin{aligned}
p_v(x, v)\ \tilde{v} &= \lim_{t\searrow 0} \frac{1}{t}\Big(p(x, v + t\tilde{v}) - p(x, v)\Big) \\
&= \lim_{t\searrow 0} \frac{1}{t}\Big((u_1(x) + v_1(x) + t\tilde{v}_1(x), u_2 + v_2 + t\tilde{v}_2) - (u_1(x) + v_1(x), u_2 + v_2)\Big) \\
&= (\tilde{v}_1(x), \tilde{v}_2) \quad \text{for all } \tilde{v} \in X\,.
\end{aligned}$$

If $(x_n, v_n) \to (x, v)$ in $\bar{\Omega}_1 \times X$, then there are $\tilde{v}_n \in X$ such that

$$\begin{aligned}
\|p_v(x_n, v_n) - p_v(x, v)\| &= \sup_{\tilde{v}\in X,\ \|\tilde{v}\|\le 1} \|(\, p_v(x_n, v_n) - p_v(x, v)\,)\,\tilde{v}\| \\
&= \sup_{\tilde{v}\in X,\ \|\tilde{v}\|\le 1} \|(\tilde{v}_1(x_n), \tilde{v}_2) - (\tilde{v}_1(x), \tilde{v}_2)\| \\
&\le |\tilde{v}_{1,n}(x_n) - \tilde{v}_{1,n}(x)| + \tfrac{1}{n} \le |x_n - x| + \tfrac{1}{n} \overset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

(note that $\tilde{v}_1$ is Lipschitz continuous), i.e., $p_v(\cdot,\cdot)$ is continuous.

Since we need Lipschitz continuity of the function d near the minimizer u we introduce the function $\hat{d} : \mathbb{R}^3 \times X \to \mathbb{R}$ by

$$\hat{d}(q, v) := d(q, u + v)\,. \tag{4.15}$$

Later in this section we verify

Lemma 4.4 *$\hat{d} : \mathbb{R}^3 \times X \to \mathbb{R}$ is Lipschitz continuous on $\mathbb{R}^3 \times B_\varepsilon(0)$ for some $\varepsilon > 0$.*

By Proposition 5.2 below we obtain that $\hat{g}$ is locally Lipschitz continuous on $B_\varepsilon(0) \subset X$ and that for each $g^* \in \partial\hat{g}(0)$ there is a Radon measure $\mu \in R[\bar{\Omega}_1]$ supported on $\Omega_c^1(u)$ and a μ-integrable function

$$x \to \hat{d}^*(x) \in \partial d(u_1(x), u_2) \subset (\mathbb{R}^3 \times X)^* \quad \text{on } \bar{\Omega}_1$$

such that

$$\begin{aligned} \langle g^*, \varphi\rangle &= \int_{\Omega_c^1(u)} \langle \hat{d}^*(x), p_v(x,u)\,\varphi\rangle\, d\mu(x) \\ &= \int_{\Omega_c^1(u)} \langle \hat{d}^*(x), (\varphi_1(x), \varphi_2)\rangle\, d\mu(x) \quad \text{for all } \varphi \in X \end{aligned} \tag{4.16}$$

(notice that $\varphi_i := \varphi|_{\Omega_i}$).

We now define $d_q(v)\big(:= d_q(v_2)\big) := \hat{d}(q, v)$ on X. The following lemma, which we will prove later, puts elements of $\partial\hat{d}(u_1(x), 0) \subset (\mathbb{R}^3 \times X)^*$ in relation to those of $\partial d_{u_1(x)}(0) \subset X^*$ (see Section 5 for the basic notions). Note that we occasionally identify $\tilde{q} \in \mathbb{R}^3$ with the constant function $w(x) = \tilde{q}$.

Lemma 4.5 *Let $q \in \mathbb{R}^3$ and $v \in X$ be fixed. Then:*

(i) $\hat{d}^\circ((q,v); (\tilde{q}, \tilde{v})) = d_q^\circ(v; \tilde{v} - \tilde{q})$ for all $\tilde{q} \in \mathbb{R}^3$, $\tilde{v} \in X$.

(ii) For every $\hat{d}^* = (q^*, d^*) \in \partial\hat{d}(q,v)$ we get $d^* \in \partial d_q(v)$ and $\langle q^*, \tilde{q}\rangle = -\langle d^*, \tilde{q}\rangle$ for all $\tilde{q} \in \mathbb{R}^3$. Otherwise, every $d^* \in \partial d_q(v)$ provides an element $\hat{d}^* \in \partial\hat{d}(q,v)$ by $\langle \hat{d}^*, (\tilde{q}, \tilde{v})\rangle = \langle d^*, \tilde{v} - \tilde{q}\rangle$ for all $\tilde{q} \in \mathbb{R}^3$, $\tilde{v} \in X$.

The lemma allows to rewrite (4.16) as

$$\langle g^*, \varphi\rangle = \int_{\Omega_c^1(u)} \langle d^*(x), \varphi_2 - \varphi_1(x)\rangle\, d\mu(x) \quad \text{for all } \varphi \in X$$

where $d^*(x) \in \partial d_{u_1(x)}(0) \subset X^*$ for all $x \in \Omega_c^1(u)$.

A characterization of $\partial d_q(0)$ is given in the next lemma which we will prove later.

Lemma 4.6 *Let $q = u_1(x_1) = u_2(\tilde{x})$ with $x_1 \in \partial\Omega_1$, $\tilde{x} \in \partial\Omega_2$, i.e., $\hat{d}(q,0) = d(q,u) = 0$. Then we have that*

$$\partial d_q(0) \subset \Big\{ \int_{\partial\Omega_2} \mathcal{D}^*(q,u)\, d\mu(y) \Big|\; \mu \in R_{\le 1}[\Gamma(q,u)] \Big\}, \tag{4.17}$$

i.e., for each $d^* \in \partial d_q(0)$ there is a Radon measure μ supported on $\Gamma(q,u)$ and a function $y \to \delta^*(y) \in \mathcal{D}^*(q,u) \subset \mathbb{R}^3$ such that

$$\langle d^*, \psi\rangle = \int_{\partial\Omega_2} \langle \delta^*(y), \psi_2(y)\rangle\, d\mu(y) \quad \textit{for all } \psi \in X$$

where the integrand is always μ-integrable. If $0 \notin \mathcal{D}^*(q,u)$, then $\mu \neq 0$.

We thus obtain for every $g^* \in \partial\hat{g}(0)$ a measure $\mu \in R[\Omega_c^1(u)]$, measures $\mu_x \in R[\Gamma(u_1(x),u)]$ and a mapping $(x,y) \to d^*(x,y) \in \mathcal{D}^*(u_1(x),u) \subset \mathbb{R}^3$ on $\bar{\Omega}_1 \times \bar{\Omega}_2$ such that

$$\langle g^*, \varphi\rangle = \int_{\Omega_c^1(u)} \int_{\Omega_c^2(u)} \langle d^*(x,y), \varphi_2(y) - \varphi_1(x)\rangle \, d\mu_x(y)\, d\mu(x) \quad \text{for all } \varphi \in X\,. \tag{4.18}$$

(d) *Normal cone of* X_0. Note that X_0 is a closed (convex) subspace of X and, thus, the normal cone $N_{X_0}(0)$ (in the sense of convex analysis) is obviously given by

$$N_{X_0}(0) = \{b^* \in X^* |\ \langle b^*, v\rangle = 0 \ \text{ for all } \ v \in X_0\}\,. \tag{4.19}$$

(e) *Nonsmooth Lagrange multiplier rule.* Applying the Lagrange multiplier rule stated in Proposition 5.1 below to the modified problem (4.10) - (4.12) we find $\lambda, \tilde{\lambda} \geq 0$, not both zero, $g^* \in \partial\hat{g}(0)$, and $b^* \in N_{X_0}(0)$ such that

$$0 = \lambda \hat{E}'(0) + \tilde{\lambda} g^* + b^*, \quad \tilde{\lambda}\hat{g}(0) = \tilde{\lambda} g(u) = 0\,. \tag{4.20}$$

(4.13), (4.14), (4.18), and (4.19) readily imply the Euler-Lagrange equation (4.7) by $d_c^*(x,y) = d^*(x,y)$ and $\mu_c = \tilde{\lambda}\mu$. If $g(u) < 0$, then $\tilde{\lambda} = 0$ by (4.20) and, thus, $\mu_c = 0$.

Let us now assume that $\lambda = 0$ while (4.8) is satisfied. Then $\mu_c \neq 0$, i.e., there is $\tilde{x} \in \Omega_c^1(u)$ such that $\mu_c(B_\varepsilon(\tilde{x})) > 0$ for all $\varepsilon > 0$. We choose an open neighborhood $U(\tilde{x})$ such that (4.8) is satisfied and a smooth function $\alpha : \bar{\Omega}_1 \to [0,1]$ with $\alpha(\tilde{x}) = 0$ and $\alpha(x) = 1$ on $\bar{\Omega}_1 \setminus U(\tilde{x})$. By a separation argument we find some $b \in \mathbb{R}^3$ and $\beta \in \mathbb{R}$ with

$$0 < \beta \leq \langle b, d\rangle \quad \text{for all } \ d \in M$$

where M denotes the set given in (4.8). Recall that $\mu_x \neq 0$ for all $x \in U(\tilde{x})$ by Lemma 4.6. Equation (4.7) with the constant function $\varphi_2(y) = b$ and with $\varphi_1(x) = b\alpha(x)$ now gives the contradiction that

$$\begin{aligned}
0 &= \int_{\Omega_c^1(u)} \int_{\Omega_c^2(u)} \langle d^*(x,y), b - b\alpha(x)\rangle \, d\mu_x(y)\, d\mu_c(x) \\
&= \int_{U(\tilde{x})} \int_{\Omega_c^2(u)} \langle d^*(x,y), b - b\alpha(x)\rangle \, d\mu_x(y)\, d\mu_c(x) \\
&\geq \int_{U(\tilde{x})} \int_{\Omega_c^2(u)} \beta(1-\alpha(x)) d\mu_x(y)\, d\mu_c(x) \\
&= \int_{U(\tilde{x})} \beta(1-\alpha(x))\mu_x(\bar{\Omega}_2)\, d\mu_c(x) \ > \ 0\,.
\end{aligned} \tag{4.21}$$

Consequently $\lambda > 0$ and, by scaling, we can always obtain that $\lambda = 1$ which completes the proof of Theorem 4.1. $\diamondsuit$

Proof of Lemma 4.4. By our assumptions the solution u has finite dilatation on Ω_2 and there is a small $\varepsilon > 0$ such that this property is preserved for all perturbations $w = u + v$ with $v \in B_\varepsilon(0) \subset X$. Hence all these perturbations w are open mappings on Ω_2 and, therefore, they satisfy $\partial w(\bar{\Omega}_2) \subset w(\partial\Omega_2)$ (cf. Section 2).

It is sufficient to prove the Lipschitz continuity of $\hat{d}$ separately in each variable, since always

$$|\hat{d}(q,v) - \hat{d}(q',v')| \le |\hat{d}(q,v) - \hat{d}(q',v)| + |\hat{d}(q',v) - \hat{d}(q',v')| \,.$$

Note that $\hat{d}(\cdot, v)$ is a (signed) distance function on $\mathbb{R}^3$ having Lipschitz constant 1, i.e., for fixed $v \in X$,

$$|\hat{d}(q,v) - \hat{d}(q',v)| \le |q - q'| \quad \text{for all } \ q, q' \in \mathbb{R}^3 \,.$$

It remains to study $\hat{d}(q, \cdot)$ for fixed $q \in \mathbb{R}^3$. For $v, v' \in B_\varepsilon(0)$ we set $w = u + v$, $w' = u + v'$. Then the following cases are possible:

(a) $q \in w(\bar{\Omega}_2) \cap w'(\bar{\Omega}_2)$,

(b) $q \notin w(\bar{\Omega}_2) \cup w'(\bar{\Omega}_2)$,

(c) $q \in w(\bar{\Omega}_2) \cap w'(\bar{\Omega}_2)^c$ or, symmetrically, $q \in w(\bar{\Omega}_2)^c \cap w'(\Omega_2)$.

Let us now successively treat the different cases.

Case (a). Without loss of generality we can assume that

$$d(q,w) = |q - w(x)| \ge |q' - w'(x')| = d(q,w') \quad \text{for some } \ x, x' \in \partial\Omega_2 \,.$$

Then we have

$$\begin{aligned} |\hat{d}(q,v) - \hat{d}(q,v')| &= |d(q,w) - d(q,w')| \;=\; |q - w(x)| - |q - w'(x')| \\ &\le \big||q - w(x')| - |q - w'(x')|\big| \;\le\; |w(x') - w'(x')| \\ &= |v(x') - v'(x')| \;\le\; \|v - v'\|_{\mathcal{W}^{1,\infty}} \,. \end{aligned}$$

Case (b). This can be treated analogously to (a).

Case (c). We consider the case where $q \notin w'(\bar{\Omega}_2)$ and $q = w(\tilde{x})$ for some $\tilde{x} \in \bar{\Omega}_2$. Then

$$|d(q,w) - d(q,w')| = \mathrm{dist}_{w(\partial\Omega_2)}(q) + \mathrm{dist}_{w'(\partial\Omega_2)}(q) = |q - w(x)| + |q - w'(x')|$$

for suitable $x, x' \in \partial\Omega_2$.

Assume now that $\varrho := \mathrm{dist}_{w'(\partial\Omega_2)}(q) > \|w - w'\|_{\mathcal{L}^\infty}$. Thus $w(\bar{\Omega}_2) \subset B_\varrho(w'(\bar{\Omega}_2))$. Since w' is open, $\partial w'(\bar{\Omega}_2) \subset w'(\partial\Omega_2)$. But this yields the contradiction that $q = w(\tilde{x}) \notin B_\varrho(w'(\bar{\Omega}_2))$. Hence

$$\mathrm{dist}_{w'(\partial\Omega_2)}(q) \le \|w - w'\|_{\mathcal{L}^\infty} \,.$$

Consequently,

$$\mathrm{dist}_{w(\partial\Omega_2)}(q) = |q - w(x)| \le |q - w(x')| \le |q - w'(x')| + |w'(x') - w(x')| \le 2\|w - w'\|_{\mathcal{L}^\infty} \,.$$

Thus we finally get that

$$|\hat{d}(q,v) - \hat{d}(q,v')| = |d(q,w) - d(q,w')| \le 3\|w - w'\|_{\mathcal{L}^\infty} \le 3\|v - v'\|_{\mathcal{W}^{1,\infty}}$$

which completes the proof of the lemma. $\diamondsuit$

Proof of Lemma 4.5. Obviously

$$\hat{d}(q,v) = \hat{d}(q', v + q' - q) \quad \text{for any } \ q, q' \in \mathbb{R}^3 \,.$$

Thus, for $q, \tilde{q} \in \mathbb{R}^3$, $v, \tilde{v} \in X$,

$$\begin{aligned}
\hat{d}^\circ((q,v);(\tilde{q},\tilde{v})) &= \limsup_{(q',v')\to(q,v),t\searrow 0} \frac{\hat{d}(q'+t\tilde{q}, v'+t\tilde{v}) - \hat{d}(q',v')}{t} \\
&= \limsup_{(q',v')\to(q,v),t\searrow 0} \frac{\hat{d}(q, v'+(q-q')+t(\tilde{v}-\tilde{q})) - \hat{d}(q, v'+q-q')}{t} \\
&= \limsup_{w'\to v,t\searrow 0} \frac{\hat{d}(q, w'+t(\tilde{v}-\tilde{q})) - \hat{d}(q,w')}{t} \\
&= d_q^\circ(v;\tilde{v}-\tilde{q})) \qquad (4.22)
\end{aligned}$$

which verifies (i).

Let us now prove (ii). First we assume that $\hat{d}^* = (q^*, d^*) \in \partial\hat{d}(q,v) \subset \mathbb{R}^3 \times X^*$. For all $(\tilde{q},\tilde{v}) \in \mathbb{R}^3 \times X$ we have that (cf. (5.1))

$$\langle \hat{d}^*, (\tilde{q},\tilde{v})\rangle = \langle q^*, \tilde{q}\rangle + \langle d^*, \tilde{v}\rangle \le \hat{d}^\circ((q,v);(\tilde{q},\tilde{v})) = d_q^\circ(v;\tilde{v}-\tilde{q})\,.$$

Inserting first the constant function $\tilde{v} = \tilde{q}$ and then replacing $(\tilde{v},\tilde{q}) = (\tilde{q},\tilde{q})$ with $(-\tilde{q},-\tilde{q})$, we get that

$$\langle q^*, \tilde{q}\rangle = -\langle d^*, \tilde{q}\rangle \quad \text{for all } \tilde{q} \in \mathbb{R}^3$$

by $d_q^\circ(v;0) = 0$. Therefore,

$$\langle d^*, \tilde{v}-\tilde{q}\rangle \le d_q^\circ(v;\tilde{v}-\tilde{q}) \quad \text{for all } (\tilde{q},\tilde{v}) \in \mathbb{R}^3 \times X$$

and, hence,

$$\langle d^*, \tilde{w}\rangle \le d_q^\circ(v;\tilde{w}) \quad \text{for all } \tilde{w} \in X$$

which proves that $d^* \in \partial d_q(v)$.

Now suppose that $d^* \in \partial d_q(v)$. For all $(\tilde{q},\tilde{v}) \in \mathbb{R}^3 \times X$ we have that

$$\langle d^*, \tilde{v}\rangle - \langle d^*, \tilde{q}\rangle = \langle d^*, \tilde{v}-\tilde{q}\rangle \le d_q^\circ(v;\tilde{v}-\tilde{q}) = \hat{d}^\circ((q,v);(\tilde{q},\tilde{v}))\,.$$

If we define $\hat{d}^* := (q^*, d^*) \in (\mathbb{R}^3 \times X)^*$ by

$$\langle \hat{d}^*, (\tilde{q},\tilde{v})\rangle = \langle (q^*,d^*), (\tilde{q},\tilde{v})\rangle = \langle q^*, \tilde{q}\rangle + \langle d^*, \tilde{v}\rangle = -\langle d^*, \tilde{q}\rangle + \langle d^*, \tilde{v}\rangle\,,$$

then we obtain $\hat{d}^* \in \partial\hat{d}(q,v)$ and (ii) is verified. $\diamondsuit$

Proof of Lemma 4.6. Since u_2 has finite dilatation, it is an open mapping on Ω_2. Thus every neighborhood of q contains inner and exterior points of $u(\bar{\Omega}_2)$. Hence there are arbitrarily small constant functions $v(y) = \tilde{q}$ with $d_q(v) \ne 0$, i.e., $\operatorname{int} d_q(0)^{-1} = \emptyset$. Therefore

$$\partial d_q(0) = \overline{\operatorname{conv}}^*\Big\{d^* \in X^* |\ d^* \in \operatorname{clust}^*(d_j^*),\ d_j^* \in \partial d_q(v_j),\ v_j \to 0,\ d_q(v_j) \ne 0\Big\}\,. \qquad (4.23)$$

by Proposition 5.3. Let now $\{d_j^*\}$, $\{v_j\}$ be sequences as in (4.23) and set $w_j := u + v_j$. To characterize the gradients $\partial d_q(v_j)$ we choose an open neighborhood $U(0) \subset X$ and define

$$\alpha_y(v) := -|(u+v)(y) - q| \quad \text{for } v \in U(0),\ y \in \partial\Omega_2\,.$$

Then

$$d_q(v) = \operatorname{sign} d(q, u+v) \min_{y\in\partial\Omega_2} -\alpha_y(v) = -\operatorname{sign} d(q, u+v) \max_{y\in\partial\Omega_2} \alpha_y(v)\,.$$

Obviously $\partial\Omega_2$ is a compact set, $y \to \alpha_y(v)$ is continuous on $\partial\Omega_2$ for all $v \in U(0)$, and $v \to \alpha_y(v)$ is Lipschitz continuous on $U(0)$ with Lipschitz constant 1 for all $y \in \partial\Omega_2$. Since $q \notin w_j(\partial\Omega_2)$ by $d_q(v_j) \neq 0$, we readily verify that $\alpha_y(\cdot)$ is continuously differentiable at v_j for all $y \in \Gamma(q, w_j)$, $j \in \mathbb{N}$, with

$$\langle \alpha_y'(v_j), v\rangle = \frac{q - w_j(x)}{|q - w_j(x)|} \cdot v(x)\,.$$

Thus

$$\partial\alpha_y(v_j) = \left\{ -\frac{w_j(y) - q}{|w_j(y) - q|} \right\} = \left\{\alpha_y'(v_j)\right\} \tag{4.24}$$

(cf. the arguments following (5.1)). Thus we can apply Clarke [8, Theorem 2.8.2] to obtain that

$$\partial d_q(v_j) \subset \left\{ \int_{\partial\Omega_2} -\operatorname{sign} d(q, u+v)\ \partial\alpha_y(v_j)\ d\mu_j(y) \;\middle|\; \mu_j \in R_1[\Gamma(q, w_j)] \right\} \tag{4.25}$$

with $\Gamma(q, w_j)$ as defined in (4.3). Using the notation from (4.4) we deduce that for every $d_j^* \in \partial d_q(v_j)$ there is $\mu_j \in R_1[\Gamma(q, w_j)]$ such that

$$\langle d_j^*, v\rangle = \int_{\partial\Omega_2} v(y) \cdot d^*(q, y, w_j)\, d\mu_j(y) \quad \text{for all } v \in X \tag{4.26}$$

where all integrals exist. By the unique polar decomposition the positive real measure μ_j, $j \in \mathbb{N}$, coincides with the total variation $|\omega_j|$ of the vector measure $\omega_j := d^*(p, \cdot, w_j)\mu_j$, since all $d^*(p, y, w_j)$ are unit vectors (cf. Ambrosio et al. [2, p.14]). By the boundedness of the $d^*(q, \cdot, w_j)$ and μ_j we find a (positive) real measure ν and a vector measure ω such that, at least for a subsequence,

$$\omega_j \overset{*}{\rightharpoonup} \omega\,, \qquad \mu_j \overset{*}{\rightharpoonup} \nu\,, \qquad \nu \geq |\omega| \tag{4.27}$$

in the sense of measures (cf.[2, p. 26, 28]). Using the polar decomposition $\omega = \bar{\delta}^*(\cdot)|\omega|$ (i.e., $|\omega|$ is the total variation of ω and $\bar{\delta}^*(\cdot)$ is a $|\omega|$-integrable function on $\partial\Omega_2$ with $|\bar{\delta}^*(y)| = 1$ for all y) we obtain that

$$\langle d_j^*, v\rangle = \int_{\partial\Omega_2} v(y) \cdot d\omega_j(y) \to \int_{\partial\Omega_2} v(y) \cdot d\omega(y) = \int_{\partial\Omega_2} v(y) \cdot \bar{\delta}^*(y)\, d|\omega|(y) \quad \text{for all } v \in X\,. \tag{4.28}$$

Summarizing we can say that each sequence $\{d_j^*\}$ according to the right hand side in (4.23) has a weak* convergent subsequence where the limit d^* corresponds to a positive real measure $|\omega|$ and a mapping $\bar{\delta}^*(\cdot)$ such that

$$\langle d^*, v\rangle = \int_{\partial\Omega_2} v(y) \cdot \bar{\delta}^*(y)\, d|\omega|(y) \quad \text{for all } v \in X\,. \tag{4.29}$$

In addititon we can assign a measure ν to d^* by (4.27). We have that

$$\lim_{\varepsilon\searrow 0} \frac{\nu(B_\varepsilon(y))}{|\omega|(B_\varepsilon(y))} =: \eta(y) \geq 1 \quad \text{exists for } |\omega|\text{-a.e. } y \in \operatorname{supp}|\omega|$$

and that $\eta(\cdot)$ is $|\omega|$-integrable (cf. [2, p. 54]). If we set

$$\delta^*(y) := \frac{\bar{\delta}^*(y)}{\eta(y)}, \qquad \mu := \eta|\omega|,$$

then (4.29) becomes

$$\langle d^*, v\rangle = \int_{\partial\Omega_2} v(y)\cdot\delta^*(y)\, d\mu(y) \quad \text{for all } v \in X. \tag{4.30}$$

Moreover we have that

$$|\omega|(\tilde{\Gamma}) \le \nu(\tilde{\Gamma}) \le \liminf_{j\to\infty} \mu_j(\tilde{\Gamma}) \quad \text{for all (relatively) open sets } \tilde{\Gamma} \subset \partial\Omega_2. \tag{4.31}$$

Notice that μ equals the absolutely continuous part of ν and, thus, $\mu \in R_{\le 1}[\partial\Omega_2]$ (cf. Evans & Gariepy [10, p. 42]. We claim that

$$\operatorname{supp}\mu = \operatorname{supp}|\omega| \subset \Gamma(q,u) \tag{4.32}$$

where the equality is obvious and for the inclusion we use the next lemma.

Lemma 4.7 *For every $\varepsilon > 0$ there exists $j_0 = j_0(\varepsilon) \in \mathbb{N}$ such that for all $j \ge j_0$ we have*

$$\Gamma(q, w_j) \subset B_\varepsilon(\Gamma(q,u)).$$

Proof of Lemma 4.7. Suppose there are $\tilde{\varepsilon} > 0$ and a sequence $\{x_j\}$ of points $x_j \in \Gamma(q, w_j)$ such that $\operatorname{dist}(x_j, \Gamma(q,u)) \ge \tilde{\varepsilon}$ for all $j \in \mathbb{N}$. Without loss of generality we can assume that $x_j \to \bar{x} \in \partial\Omega_2 \setminus \Gamma(q,u)$. This implies that $\varepsilon_1 := |u(\bar{x}) - q| > 0$. By the uniform convergence of the sequence v_j we find some $j_1 \in \mathbb{N}$ such that $|w_j(x) - u(x)| \le \varepsilon_1/4$ for all $x \in \partial\Omega_2$, $j \ge j_1$. Since u is continuous there exists $\delta > 0$ with $u(B_\delta(\bar{x}) \cap \bar{\Omega}_2) \subset B_{\varepsilon_1/4}(u(\bar{x}))$. Moreover, we can choose $j_2 \ge j_1$ in order to get $x_j \in \partial\Omega_2 \cap B_\delta(\bar{x})$ for all $j \ge j_2$. Then we obtain that

$$|w_j(x_j) - u(\bar{x})| \le |w_j(x_j) - u(x_j)| + |u(x_j) - u(\bar{x})| \le \varepsilon_1/4 + \varepsilon_1/4 = \varepsilon_1/2 \tag{4.33}$$

and, hence,

$$|w_j(x_j) - q| \ge |u(\bar{x}) - q| - |w_j(x_j) - u(\bar{x})| \ge \varepsilon_1 - \varepsilon_1/2 = \varepsilon_1/2 \tag{4.34}$$

for all $j \ge j_2$. On the other hand, for $j \ge j_2$ we know that

$$|w_j(\tilde{x}) - q| = |w_j(\tilde{x}) - u(\tilde{x})| \le \varepsilon_1/4 \quad \text{for all } \tilde{x} \in \Gamma(q,u). \tag{4.35}$$

Using (4.34), this implies that $x_j \notin \Gamma(q, w_j)$ which completes the proof. $\diamondsuit$

Let us now continue with the proof of Lemma 4.6 by supposing that (4.32) is wrong. Then there exists $y_0 \in \partial\Omega_2 \setminus \Gamma(q,u)$ such that $|\omega|(B_\varepsilon(y_0) \cap \partial\Omega_2) > 0$ for all $\varepsilon > 0$. Choose $\breve{\Gamma} := B_{\varepsilon_0}(y_0) \cap \partial\Omega_2$ with $\varepsilon_0 := \operatorname{dist}(x_0, \Gamma(q,u))/2 > 0$. By Lemma 4.7 and (4.31) we get the contradiction that

$$0 < |\omega|(\breve{\Gamma}) \le \lim_{j\to\infty} \mu_j(\breve{\Gamma}) = 0 \tag{4.36}$$

which verifies (4.32).

Let $G_{\rm clust} \subset X^*$ denote the set on the right hand side in (4.23). By $G_{\rm lim} \subset X^*$ we denote the set defined as $G_{\rm clust}$ but where $d^* \in {\rm clust}^*(d_j^*)$ is replacecd with $d_j^* \overset{*}{\rightharpoonup} d^*$. Clearly $G_{\rm lim} \subset G_{\rm clust}$. Let us assume that $G_{\rm lim} \neq G_{\rm clust}$. Since these weak* closed convex sets are uniquely determined by their support functions on X (cf. Clarke [8, p. 29]), we find some $\tilde{v} \in X$ and $d_0^* \in G_{\rm clust}$ such that

$$\langle d^*, \tilde{v}\rangle < \langle d_0^*, \tilde{v}\rangle \quad \text{for all } \; d^* \in G_{\rm lim}\,. \tag{4.37}$$

Let $d_0^* \in {\rm clust}^*(d_j^*)$ according to the definition of $G_{\rm clust}$. Then, for some subsequence, $\langle d_{j'}^*, \tilde{v}\rangle \to \langle d_0^*, \tilde{v}\rangle$. By our previous investigations we find again a subsequence such that $d_{j''}^* \overset{*}{\rightharpoonup} d_1^* \in G_{\rm lim}$ which contradicts (4.37). Hence $G_{\rm lim} = G_{\rm clust}$, i.e., all elements $d^* \in \partial d_q(0)$ have a structure as given in (4.29) or, equivalently, in (4.30).

We now fix some $d^* \in \partial d_q(0)$. According to our previous arguments we assign $\delta^*(\cdot)$, μ, ν, $\eta(\cdot)$, and sequences w_j, μ_j such that (4.26), (4.28), and (4.30) are valid. We claim that

$$\delta^*(y) \in \mathcal{D}^*(q,u) \quad \text{for } \mu\text{-a.e. } \; y \in \Gamma(q,u)\,. \tag{4.38}$$

Let us suppose the opposite, i.e, there exists a Lebesgue point y_0 of δ^* such that $\delta^*(y_0) \notin \mathcal{D}^*(q,u)$. Without loss of generality we can assume that y_0 is also Lebesgue point of the μ-integrable function η. Using the notation

$$\mathcal{M}_\sigma := \overline{\rm conv} \bigcup_{\substack{\|v\|_X \le \sigma \\ d(q,u+v)\neq 0}} \; \bigcup_{y \in \Gamma(q,u+v)} d^*(q,y,u+v)\,, \quad \sigma > 0\,, \tag{4.39}$$

we thus find $\sigma_0 > 0$ such that $\delta^*(y_0) \notin \mathcal{M}_{\sigma_0}$. Since $\mathcal{M}_{\sigma_0}$ is closed and convex, there are $b \in \mathbb{R}^3$ and $\beta > 0$ such that

$$\langle b, \delta^*(y_0)\rangle > \langle b, d^*\rangle + \beta \quad \text{for all } \; d^* \in \mathcal{M}_{\sigma_0} \tag{4.40}$$

by a separation argument. By $w_j = u + v_j \to u$ there is some $j_0 \in \mathbb{N}$ with $\|v_j\| < \sigma_0$ for all $j > j_0$. Since $\mu_j/\mu_j(B_\varepsilon(y_0))$ is a probability measure on $B_\varepsilon(y_0)$,

$$\int_{B_\varepsilon(y_0)} d^*(q,y,w_j)\, \frac{d\mu_j(y)}{\mu_j(B_\varepsilon(y_0))} \in \mathcal{M}_{\sigma_0} \quad \text{for all } \; \varepsilon > 0,\; j > j_0,$$

and, hence,

$$\frac{1}{\mu_j(B_\varepsilon(y_0))} \int_{B_\varepsilon(y_0)} \langle b, d^*(q,y,w_j)\rangle d\mu_j(y) < \langle b, \delta^*(y_0)\rangle - \beta \quad \text{for all } \; \varepsilon > 0,\; j > j_0\,. \tag{4.41}$$

Since y_0 is a Lebesgue point of δ^* and η, we have that

$$\frac{1}{\mu(B_\varepsilon(y_0))} \int_{B_\varepsilon(y_0))} \langle b, \delta^*(y)\rangle \, d\mu(y) \overset{\varepsilon\to 0}{\longrightarrow} \langle b, \delta^*(y_0)\rangle \tag{4.42}$$

and

$$\frac{\mu(B_\varepsilon(y_0))}{\nu(B_\varepsilon(y_0))} = \frac{\mu(B_\varepsilon(y_0))}{|\omega|(B_\varepsilon(y_0))} \frac{|\omega|(B_\varepsilon(y_0))}{\nu(B_\varepsilon(y_0))} = \frac{|\omega|(B_\varepsilon(y_0))}{\nu(B_\varepsilon(y_0))} \int_{B_\varepsilon(y_0)} \eta(y)\, d|\omega|(y) \overset{\varepsilon\to 0}{\longrightarrow} 1\,. \tag{4.43}$$

Certainly $\nu(\partial B_\varepsilon(y_0)) = 0$ for a.e. $\varepsilon > 0$ and, therefore,

$$\frac{1}{\mu_j(B_\varepsilon(y_0))} \int_{B_\varepsilon(y_0)} \langle b, d^*(q,y,w_j)\rangle \, d\mu_j(y) \overset{j\to\infty}{\longrightarrow} \frac{1}{\nu(B_\varepsilon(y_0))} \int_{B_\varepsilon(y_0)} \langle b, \delta^*(y)\rangle \, d\mu(y) \tag{4.44}$$

by (4.27) for a.e. $\varepsilon > 0$ (cf. [2, p. 28]). By (4.42) and (4.43) there is $\varepsilon_0 > 0$ such that

$$\frac{1}{\nu(B_\varepsilon(y_0))} \int_{B_\varepsilon(y_0)} \langle b, \delta^*(y)\rangle \, d\mu(y) \geq \langle b, \delta^*(y_0)\rangle - \frac{\beta}{2}$$

for a.e. $0 < \varepsilon < \varepsilon_0$. But this contradicts (4.41), (4.44) and verifies (4.38) and (4.17).

We now assume that $0 \notin \mathcal{D}^*(q,u)$ and argue similarly as in the verification of (4.38). Since $\mathcal{D}^*(q,u) \subset \mathbb{R}^3$ is a closed convex set, we find $b \in \mathbb{R}^3$ and $\beta \in \mathbb{R}$ such that

$$0 < \beta \leq \langle b, d^*\rangle \quad \text{for all } \; d^* \in \mathcal{D}^*(q,u)\,.$$

By (4.28), (4.30) with the constant function $\tilde{v}(y) = b$, and by the fact that all μ_j are probability measures, we obtain that

$$0 < \beta \leq b \cdot \int_{\partial\Omega_2} d^*(q,y,w_j)\, d\mu_j(y) = \langle d_j^*, \tilde{v}\rangle \to \langle d^*, \tilde{v}\rangle = b \cdot \int_{\partial\Omega_2} \delta^*(y)\, d\mu(y)$$

which excludes that $\mu = 0$ and finishes the proof. $\diamondsuit$

Proof of Corollary 4.3. The function $d_{\mathcal{O}}(\cdot)$ is Lipschitz continuous on $\mathbb{R}^3$. If $M_d \subset \mathbb{R}^3$ denotes the set of all points where the gradient $Dd_{\mathcal{O}}(q)$ exists, then $\mathbb{R}^3 \setminus M_d$ has measure zero by Rademacher's theorem. For the set $\mathcal{O} = u_2(\partial\Omega_2)$ having measure zero we obtain that

$$\partial d_{\mathcal{O}}(q) = \operatorname{conv}\big\{ \lim_{i\to\infty} Dd_{\mathcal{O}}(q_i) |\; q_i \to q, q_i \in (M_d \setminus \mathcal{O})\big\} \tag{4.45}$$

by Clarke [8, Theorem 2.5.1]. The next lemma characterizes the gradients $Dd_{\mathcal{O}}(q)$.

Lemma 4.8 *Let $d_{\mathcal{O}}$ be differentiable in $\tilde{q} \in \mathbb{R}^3 \setminus u_2(\partial\Omega_2)$. Then there exists an uniquely defined vector $q' \in u_2(\partial\Omega_2)$ such that $d_{\mathcal{O}}(\tilde{q}) = \operatorname{sign}\big(d_{\mathcal{O}}(\tilde{q})\big)\, |\tilde{q} - q'|$. Moreover*

$$Dd_{\mathcal{O}}(\tilde{q}) = \operatorname{sign}\big(d_{\mathcal{O}}(\tilde{q})\big) \frac{\tilde{q} - q'}{|\tilde{q} - q'|}\,. \tag{4.46}$$

Proof of Lemma 4.8. It is sufficient to consider the case $d_{\mathcal{O}}(\tilde{q}) > 0$. By the compactness of $\mathcal{O}$ and the continuity of $d_{\mathcal{O}}$ there is some q' with $d := d_{\mathcal{O}}(\tilde{q}) = |\tilde{q} - q'|$. We define $z(t) := q' + b\,t$ with $b := (\tilde{q} - q')/d$ for $t \in [0,d]$. Then $d_{\mathcal{O}}(z(t)) = |z(t) - q'|$ for all $t \in [0,d]$ and, hence, $\langle Dd_{\mathcal{O}}(\tilde{q}), b\rangle = 1$. Since b is a unit vector and since $|Dd_{\mathcal{O}}(\tilde{q})| \leq 1$ by the Lipschitz continuity of $d_{\mathcal{O}}$ with Lipschitz constant 1, we obtain that $Dd_{\mathcal{O}}(\tilde{q}) = b$ which implies the uniqueness of q' and (4.46). $\diamondsuit$

Let us continue with the proof of Corollary 4.3 by verifying (4.9). We fix any $x \in \Omega_c^1(u)$ and set $q := u_1(x)$. Now we choose sequences $\{q_j\} \subset (M_d \setminus \mathcal{O})$ and $\{q_j'\} \subset \mathcal{O} = u_2(\partial\Omega_2)$ such that

$$q_j \to q\,, \quad d_{\mathcal{O}}(q_j) = \operatorname{sign}\big(d_{\mathcal{O}}(q_j)\big)\, |q_j - q_j'|\,, \quad \operatorname{sign}\big(d_{\mathcal{O}}(q_j)\big) \frac{q_j - q_j'}{|q_j - q_j'|} \to \bar{d}\,. \tag{4.47}$$

Note that $\bar{d} \in \partial d_{\mathcal{O}}(q)$ by (4.45) and Lemma 4.8. We claim that

$$-\bar{d} \in \mathcal{D}^*(q,u) \tag{4.48}$$

which would imply (4.9) by (4.45), since $\mathcal{D}^*(q,u)$ is a closed convex set.

Let us consider the sequence of constant functions $v_j(x) := q - q_j$ that belongs to X and, obviously, $v_j \to 0$ in X by (4.47). Furthermore we find $y_j \in \partial\Omega_2$ with $q_j' = u_2(y_j)$. For simplicity of presentation we assume that $q_j \in u_2(\Omega_2)$ for all $j \in \mathbb{N}$, i.e., $d_{\mathcal{O}}(q_j) > 0$ (otherwise we have to multiply with "-1" occasionally). Using the translated sets $\mathcal{O}_j := \mathcal{O} + q - q_j$ we find that

$$d(q, u + v_j) = \mathrm{dist}_{\mathcal{O}_j}(q) = d_{\mathcal{O}}(q_j) = |q_j - q_j'| = |q_j - u_2(y_j)| = \left|q - \big(u_2(y_j) + v_j(y_j)\big)\right|.$$

Hence $y_j \in \Gamma(q, u + v_j)$ and, by (4.4), (4.47),

$$d^*(q, y_j, u + v_j) = \frac{u_2(y_j) + v_j(y_j) - q}{|u_2(y_j) + v_j(y_j) - q|} = \frac{q_j' - q_j}{|q_j' - q_j|} \to -\bar{d}.$$

This implies (4.48) by (4.5) and finishes the proof. $\diamondsuit$

5 Clarke's generalized gradient

In this section we briefly summarize basic properties of Clarke's generalized gradients for locally Lipschitz continuous functions and we prove some auxiliary results we had used in our previous analysis. For a more comprehensive presentation the reader is referred to Clarke [8].

Let X be a Banach space and $f : X \to \mathbb{R}$ a locally Lipschitz continuous function. The *generalized directional derivative* $f^\circ(u;h)$ of f at u in the direction v is defined as

$$f^\circ(u;v) := \limsup_{w \to u,\, t \searrow 0} \frac{f(w + tv) - f(w)}{t}.$$

The mapping $v \to f^\circ(u;v)$ is positively homogeneous, subadditive and satisfies $|f^\circ(u;v)| \le l_f \|v\|$ on X where l_f is the Lipschitz constant of f near u.

The *generalized gradient* $\partial f(u)$ of f at u is given by

$$\partial f(u) := \{f^* \in X^* |\ \langle f^*, v\rangle \le f^\circ(u;v) \quad \text{for all } \ v \in X\}. \tag{5.1}$$

$\partial f(u)$ is a nonempty, convex and weak* compact subset of X^* and it is bounded by the Lipschitz constant l_f. If f is continuously differentiable, then $\partial f(u)$ is the singleton $\{f'(u)\}$, whereas for a convex f the set $\partial f(u)$ agrees with the subdifferential of convex analysis. The next theorem summarizes additional properties as necessary for our investigations. Here $\mathrm{clust}^*(f_i^*) \subset X^*$ denotes the set of all weak* cluster points f^* of the sequence $\{f_i^*\} \subset X^*$, i.e., each neighborhood of f^* in the weak* topology contains infinitely many elements of the sequence. Recall that each bounded sequence $\{f_i^*\} \subset X^*$ has a weak* cluster point.

Proposition 5.1 *Let $f, g, g_i : X \to \mathbb{R}$, $i = 1, \ldots, n$, be Lipschitz continuous near $u \in X$. Then:*

(i) $\partial(\alpha f)(u) = \alpha \partial f(u)$ *for all* $\alpha \in \mathbb{R}$.

(ii) $\partial \sum_{i=1}^n g_i(u) \subset \sum_{i=1}^n \partial g_i(u)$.

(iii) If $\{u_i\} \subset X$, $\{f_i^*\} \subset X^*$ *are sequences with* $f_i^* \in \partial f(u_i)$, $u_i \to u$ *and if* $f^* \in \mathrm{clust}^*(f_i^*)$, *then* $f^* \in \partial f(u)$.

(iv) *(Chain Rule) Let Y be a Banach space, $F : X \to Y$ continuously differentiable at $u \in X$, and $g : Y \to \mathbb{R}$ Lipschitz continuous near $F(u)$. Then $f := g \circ F$ is Lipschitz continuous near u and*

$$\partial f(u) \subset \partial g(F(u)) \circ F'(u) \,,$$

i.e., for $f^* \in \partial f(u)$ there exists $g^* \in \partial g(F(u))$ such that

$$\langle f^*, v\rangle = \langle g^* \circ F'(u), v\rangle = \langle g^*, F'(u)v\rangle_{Y^* \times Y} \quad \textit{for all } v \in X \,.$$

(v) *(Lebourgs Mean Value Theorem) Let f be Lipschitz continuous on an open neighborhood of the line segment $[u, v]$ for $u, v \in X$. Then there is $w \in (u, v)$ and $f^* \in \partial f(w)$ such that*

$$f(v) - f(u) = \langle f^*, w\rangle.$$

(vi) *(Lagrange Multiplier Rule) If u is a local minimizer of f subject to the restrictions $g(v) \leq 0$, $g_i(v) = 0$, $i = 1, \ldots, n$, and $u \in C$ for a closed subset $C \subset X$, then there exist constants λ_f, $\lambda \geq 0$, and $\lambda_i \in \mathbb{R}$, not all zero, such that*

$$0 \in \lambda_f \partial f(u) + \lambda \partial g(u) + \sum_{i=1}^{n} \lambda_i \partial g_i(u) + N_C(u) \quad \textit{and} \quad \lambda g(u) = 0$$

where

$$N_C(u) := \{v^* \in X^* | \ \langle v^*, v\rangle \leq 0 \ \textit{for all } v \textit{ with } \operatorname{dist}^\circ_C(u; v) = 0\}$$

is the normal cone of C at u (it agrees with the normal cone of convex analysis if C is convex).

Let us now characterize the generalized gradient of functions of the type

$$g(v) := \max_{x \in \Omega} d(p(x, v)) \,. \tag{5.2}$$

We assume that

(a) X, Y are Banach spaces, Ω is a metrizable sequentially compact topological space, $U \subset X$, $V \subset Y$ are open,

(b) $p : \Omega \times U \to V$ is continuous and $v \to p(x, v)$ is differentiable for all $x \in \Omega$ such that the derivative $p_v(\cdot, \cdot)$ is continuous on $\Omega \times U$,

(c) $d : V \to \mathbb{R}$ is Lipschitz continuous.

The function g is well defined by the compactness of Ω and we introduce the nonempty closed set

$$\Omega(v) := \{x \in \Omega | \ g(v) = d(p(x, v))\} \,.$$

Proposition 5.2 *Assume that (a)-(c) hold. Then the function g given by (5.2) is locally Lipschitz continuous on U and*

$$\partial g(v) \subset \left\{ \int_\Omega \partial d(p(x, v)) \circ p_v(x, v) \, d\rho(x) \ \Big| \ \rho \in R[\Omega(v)] \right\} \quad \textit{for all } v \in U \tag{5.3}$$

where the set on the right hand side consists of all elements $g^* \in X^*$ that correspond to a mapping $d^* : \Omega \to Y^*$ with $d^*(x) \in \partial d(p(x,v))$ and a measure $\rho \in R[\Omega]$ supported on $\Omega(v)$ such that

$$x \to \langle d^*(x) \circ p_v(x,v), w\rangle = \langle d^*(x), p_v(x,v)w\rangle$$

is ρ-integrable for all $w \in X$ and

$$\langle g^*, w\rangle = \int_\Omega \langle d^*(x), p_v(x,v)w\rangle \, d\rho(x) \quad \textit{for all } w \in X\,.$$

The previous result generalizes Schuricht [16, Proposition 6.10] so far that the reflexivity of Y is dropped. Before we carry out the proof we still formulate a characterization of $\partial f(u)$ by means of the sets $\partial f(v)$ for v with $f(v) \neq f(u)$.

Proposition 5.3 *Let $f : X \to \mathbb{R}$ be locally Lipschitz continuous with $f(u) = 0$.*

(a) If $u \notin \overline{\operatorname{int}(f^{-1}(0))}$, then

$$\partial f(u) = \overline{\operatorname{conv}}^* \Big\{ u^* \in X^* \,|\, u^* \in \operatorname{clust}^*(u_j^*),\ u_j^* \in \partial f(u_j),\ u_j \to u,\ f(u_j) \neq 0 \Big\}\,.$$

(b) If $u \in \overline{\operatorname{int}(f^{-1}(0))}$, then

$$\partial f(u) = \overline{\operatorname{conv}}^* \Big(\{0\} \,\cup\, \Big\{ u^* \in X^* \,|\, u^* \in \operatorname{clust}^*(u_j^*),\ u_j^* \in \partial f(u_j),\ u_j \to u,\ f(u_j) \neq 0 \Big\} \Big)\,.$$

Proof of Proposition 5.2. We consider the functionals

$$f_x : X \to \mathbb{R} \quad \text{defined by} \quad f_x(v) := d(p(x,v))\,, \ \ v \in U\,,$$

depending on the parameter $x \in \Omega$. As in Clarke [8, Chapter 2.8] we define a generalized gradient taking into account the parameters by

$$\partial_{[\Omega]} f_x(v) := \overline{\operatorname{conv}}^* \{ f^* \in X^* |\ f^* \in \operatorname{clust}^*(f_i^*),\ f_i^* \in \partial f_{x_i}(v_i),\ v_i \to v, x_i \to x, x_i \in \Omega\}\,. \tag{5.4}$$

Since $f_x(\cdot)$ is locally Lipschitz continuous on U for all $x \in \Omega$, this gradient is well defined and we readily get that

$$\partial f_x(v) \subset \partial_{[\Omega]} f_x(v) \quad \text{for all } v \in U\,.$$

Clarke [8, Theorem 2.8.2] implies that

$$\partial g(v) \subset \Big\{ \int_\Omega \partial_{[\Omega]} f_x(v)\, d\rho(x) \Big|\ \rho \in R[\Omega(v)] \Big\}$$

for all $v \in U$. We claim to show that

$$\partial_{[\Omega]} f_x(v) \subset \partial d(p(x,v)) \circ p_v(x,v) \tag{5.5}$$

which would complete the proof of the proposition.

By definition $\partial_{[\Omega]} f_x(v)$ is the weak* closed convex hull of the set F^* consisting of all $f^* \in X^*$ such that

$$f^* \in \operatorname{clust}^*(f_k^*) \ \text{ where } \ f_k^* \in \partial f_{x_k}(v_k),\ v_k \to v,\ x_k \to x,\ x_k \in \Omega\,. \tag{5.6}$$

Since the generalized gradient $\partial d(p(x,v)) \subset Y^*$ is convex and weak* compact, the set $\partial d(p(x,v)) \circ p_v(x,v) \subset X^*$ is convex and weak* closed. Hence (5.5) is verified if

$$F^* \subset \partial d(p(x,v)) \circ p_v(x,v) \tag{5.7}$$

can be shown. For this reason let us assume that there is $f^* \in F^*$ with $f^* \notin \partial d(p(x,v)) \circ p_v(x,v)$. Using a convex separation argument we find some $w \in X$ such that

$$\langle f^*, w\rangle < \langle d^* \circ p_v(x,v), w\rangle \quad \text{for all } \ d^* \in \partial d(p(x,v))\,. \tag{5.8}$$

Since f^* is a weak* cluster point of a sequence $\{f_k^*\}$ based on sequences $\{v_k\}$, $\{x_k\}$ according to (5.6), we have that (possibly for a subsequence)

$$\lim_{k\to\infty} \langle f_k^*, w\rangle = \langle f^*, w\rangle\,.$$

The chain rule in Proposition 5.1 implies that

$$\partial f_x(v) \subset \partial d(p(x,v)) \circ p_v(x,v) \quad \text{for all } \ v \in U, \ \ x \in \Omega\,.$$

Hence there are $d_k^* \in \partial d(p(x_k, v_k))$ such that

$$f_k^* = d_k^* \circ p_v(x_k, v_k)\,, \quad k \in \mathbb{N}\,.$$

The sets $\partial d(p(x_k,v_k)) \subset Y^*$ are uniformly bounded, since $d(\cdot)$ is Lipschitz continuous by assumption (c). Hence $\{d_k^*\}$ is bounded in Y^* and has a weak* cluster point $d^* \in Y^*$. Therefore we can find a subsequence (denoted the same way) such that

$$\lim_{k\to\infty} \langle d_k^*, p_v(x,v)w\rangle = \langle d^*, p_v(x,v)w\rangle\,.$$

Using the continuity of $p_v(\cdot,\cdot)$ and the boundedness of $\{d_k^*\}$ we obtain that

$$\begin{aligned} \langle f^*, w\rangle &= \lim_{k\to\infty} \langle f_k^*, w\rangle = \lim_{k\to\infty} \langle d_k^*, p_v(x_k,v_k)w\rangle \\ &= \lim_{k\to\infty} \Big(\langle d_k^*, p_v(x,v)w\rangle + \langle d_k^*, (p_v(x_k,v_k) - p_v(x,v))w\rangle \Big) \\ &= \langle d^*, p_v(x,v)w\rangle\,. \end{aligned} \tag{5.9}$$

By Proposition 5.1, (iii) and by the construction of d^* we know that $d^* \in \partial d(p(x,v))$ contradicting (5.8). But this verifies (5.7) and finishes the proof. ◊

Proof of Proposition 5.3.

(a) We set

$$S(u) = \overline{\text{conv}}^*\Big\{ u^* \in X^* |\ u^* \in \text{clust}^*(u_j^*),\ u_j^* \in \partial f(u_j),\ u_j \to u,\ f(u_j) \neq 0 \Big\}$$

and, obviously, $S(u) \subset \partial f(u)$ by Proposition 5.1, (iii). For the opposite inclusion it is enough to show that for each $v \in X$ there is some $u^* \in S(u)$ with

$$f^\circ(u;v) \le \langle u^*, v\rangle\,. \tag{5.10}$$

For that we fix $v \in X$ and we choose corresponding sequences $u_j \to u$, $t_j \searrow 0$ such that $t_{j+1} < t_j$, $j \in \mathbb{N}$, and

$$\lim_{j\to\infty} \frac{f(u_j + t_j v) - f(u_j)}{t_j} = f^\circ(u;v)\,.$$

Below we distinguish a finite number of cases for the signs of $f(u_j)$ and $f(u_j + t_j v)$, i.e., at least one of the cases has to be met by infinitely many elements. Thus we argue for a corresponding subsequence in each case.

(a_1) Let $f(u_j) \geq 0$, $f(u_j + t_j v) > 0$. By τ_j we denote the minimal $\tau \in [0, t_j)$ such that $f(u_j + sv) > 0$ for any $s \in (\tau, t_j]$. By the mean value theorem (cf. Proposition 5.1) we find $s_j \in (\tau_j, t)$ and $u_j^* \in \partial f(u_j + s_j v)$ such that

$$\frac{f(u_j + t_j v) - f(u_j)}{t_j} \leq \frac{f(u_j + t_j v) - f(u_j + \tau_j v)}{t_j - \tau_j} = \langle u_j^*, v\rangle\,.$$

The sequence $\{u_j^*\}$ is bounded by the Lipschitz continuity of f near u and, therefore, it has a weak* cluster point u^* that clearly belongs to $S(u)$. Then (5.10) follows in this case.

(a_2) Let $f(u_j) < 0$, $f(u_j + t_j v) \leq 0$. Similar arguments as in (a_1) apply.

(a_3) Let $f(u_j) < 0 < f(u_j + t_j v)$. There exists $\tau_j \in (0, t_j)$ such that $f(u_j + \tau_j v) = 0$. We have either

$$\frac{f(u_j + t_j v) - f(u_j)}{t_j} \leq \frac{f(u_j + t_j v) - f(u_j + \tau_j v)}{t_j - \tau_j}$$

or

$$\frac{f(u_j + t_j v) - f(u_j)}{t_j} \leq \frac{f(u_j + \tau_j v) - f(u_j)}{\tau_j}\,.$$

In the first case we are reduced to (a_1) and in the second case to (a_2).

(a_4) Let $f(u_j) = f(u_j + t_j v) = 0$. We choose a sequence $v_j = v + w_j$ such that $w_j \to 0$ and $f(u_j + \frac{1}{2} t_j v_j) \neq 0$. This is possible, because $u \notin \overline{\operatorname{int}(f^{-1}(0))}$. If $f(u_j + \frac{1}{2} t_j v_j) > 0$ we find, as in (a_1), some $u_j^* \in \partial f(u_j + s_j v_j)$ with $f(u_j + s_j v_j) \neq 0$, $s_j \in (0, t_j)$ such that

$$\frac{f(u_j + t_j v) - f(u_j)}{t_j} \leq \frac{f(u_j + \frac{1}{2} t_j v_j) - f(u_j)}{\frac{1}{2} t_j} = \langle u_j^*, v + w_j\rangle\,.$$

Otherwise, if $f(u_j + \frac{1}{2} t_j v_j) < 0$, we argue as in ($a_2$) to find some $u_j^* \in \partial f(u_j + s_j v_j)$ with $f(u_j + s_j v_j) \neq 0$, $s_j \in (0, t_j)$ such that

$$\frac{f(u_j + t_j v) - f(u_j)}{t_j} \leq \frac{f(u_j + t_j v) - f(u_j + \frac{1}{2} t_j v_j)}{\frac{1}{2} t_j} = \langle u_j^*, v - w_j\rangle\,.$$

Now (5.10) follows as in (a_1).

(a_5) Let $f(u_j) > 0 \geq f(u_j + t_j v)$ or $f(u_j) \geq 0 > f(u_j + t_j v)$. By $\underline{\tau}_j$ and $\overline{\tau}_j$ we denote the minimal and the maximal $\tau \in [0, t_j]$ such that $f(u_j + \tau v) = 0$. If $\underline{\tau}_j < \overline{\tau}_j$ at least for a subsequence, then we have

$$\frac{f(u_j + t_j v) - f(u_j)}{t_j} \leq \frac{f(u_j + \overline{\tau}_j v) - f(u_j + \underline{\tau}_j v)}{\overline{\tau}_j - \underline{\tau}_j}$$

and we are reduced to case (a_4). Otherwise let $\tau_j := \underline{\tau}_j = \overline{\tau}_j$ for almost all elements.

If $0 < \tau_j < t_j$, we find $\underline{s}_j \in (0, \tau_j)$, $v_j^* \in \partial f(u_j + \underline{s}_j v)$, $\overline{s}_j \in (\tau_j, t_j)$ and $w_j^* \in \partial f(u_j + \overline{s}_j v)$ such that

$$\begin{aligned}\frac{f(u_j + t_j v) - f(u_j)}{t_j} &= \frac{f(u_j + t_j v) - f(u_j + \tau_j v)}{t_j} + \frac{f(u_j + \tau_j v) - f(u_j)}{t_j} = \\ &= \frac{t_j - \tau_j}{t_j} \langle v_j^*, v \rangle + \frac{\tau_j}{t_j} \langle w_j^*, v \rangle .\end{aligned}$$

If $\tau_j = 0$ or $\tau_j = t_j$, the argument is similar with some obvious simplifications. As before we find weak* cluster points v^*, w^* of $\{v_j^*\}$, $\{w_j^*\}$, respectively, and, possibly for a subsequence, $\frac{\tau_j}{t_j} \to \lambda \in [0,1]$ and $\frac{t_j - \tau_j}{t_j} \to 1 - \lambda$. Since $(1-\lambda)v^* + \lambda w^* \in S(u)$, (5.10) follows and completes the proof of (a).

(b) Since 0 belongs to the right hand side, it is enough to consider the case

$$\lim_j \frac{f(u_j + t_j v) - f(u_j)}{t_j} > 0 .$$

Here we can apply cases (a_1), (a_2), (a_3), where the assumption $u \notin \overline{\mathrm{int}(f^{-1}(0))}$ is not needed. $\diamondsuit$

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