An alternative derivation of the eigenvalue equation for the 1-Laplace operator

Friedemann Schuricht
Mathematisches Institut, Universität zu Köln
50923 Köln, Germany

Abstract

Minimizers of the total variation subject to a prescribed $\mathcal{L}^1$-norm are considered as eigensolutions of the 1-Laplace operator. The derivation of the corresponding eigenvalue equation, which requires particular care due to the lack of smoothness, is carried out in a previous paper by using a special Lagrange multiplier rule based on Degiovanni's weak slope. The present paper provides a simpler proof that exploits the special structure of the problem and does not go beyond convex analysis.

1 Introduction

Minimizing problems for the total variation of BV-functions formally lead to Euler-Lagrange equations containing the 1-Laplace operator Div $\frac{Du}{|Du|}$, which is the formal limit of the p-Laplace operator div $(|Du|^{p-2}Du)$ as $p \to 1$. The investigation of such problems enjoys increasing interest during the last years while the degeneracy of the problems is a major challange to the underlying analysis. Since the formal 1-Laplace operator is not well-defined for minimizers in general, it was a particular question to give meaning to the highly singular operator. One way to approach this question is to approximate the degenerate problem by a sequence a regular ones and to study the limit (cf. [4, 6, 7, 8]). The drawback of this approach is that the result might depend on the special smoothing choosen. Therefore a direct treatment of the singular problem is desirable. An essential step into that direction was the computation of the convex subdifferential of the total variation in a suitable space (cf. [1] and references therein). The presence of a nonsmooth side condition as in the eigenvalue problem

$$\int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} \quad \to \quad \text{Min!}\,, \quad u \in BV(\Omega),$$

$$\int_\Omega |u|\, dx = 1$$

makes the problem nonconvex and leads to additional difficulties. With a special nonconvex Lagrange multiplier rule from Degiovanni & Schuricht [3], a necessary condition for a minimizer u of the previous variation problem has been derived by Schuricht & Kawohl [9]. More precisley, for any measurable selection $s(x)$ of the set-valued sign function $\operatorname{Sgn}(u(x))$ we find some vector field $z \in \mathcal{L}^\infty(\Omega)$ satisfying some compatibility condition for u (cf. (2.5)) such that

$$-\operatorname{Div} z(x) = \lambda s(x) \quad \text{a.e. on } \Omega\,, \quad \lambda = E(u)\,. \tag{1.1}$$

Here the vector field z can be considered as a substitute of $Du/|Du|$ in the formal 1-Laplace operator and s substitutes the formal multiplier $u/|u|$ of the $\mathcal{L}^1$-norm. A surprising aspect of this result is that, in general, infinitely many equations have to be satisfied by the minimizer. The corresponding proof in [9] is based on Degiovanni's weak slope (cf. [2]) and goes far beyond convex analysis. The purpose of the present paper is to provide an alternative simpler derivation of the eigenvalue equation (1.1) for the 1-Laplace operator by merely using nonsmooth tools from convex analysis. The new proof exploits the special structure of the problem and rests on the key observation that the identity $su = |u|$, that is satisfied for any selection s occuring in (1.1), allows to replace the nonconvex equality constraint in the variational problem with a suitable convex inequality constraint.

Notation. Let $\operatorname{Div} u$ denote the divergence of u in the distributional sense and let $u^{\partial\Omega}$ be the trace of u on $\partial\Omega$. While $\operatorname{sgn}\alpha$ is the usual sign function on $\mathbb{R}$ we also define the set-valued sign function

$$\operatorname{Sgn}\alpha = \begin{cases} 1 & \text{if } \alpha > 0, \\ [-1,1] & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha < 0. \end{cases} \tag{1.2}$$

The space of p-integrable functions on Ω is denoted by $\mathcal{L}^p(\Omega)$ and its dual by $\mathcal{L}^{p'}(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. We write $BV(\Omega)$ for the space of functions of bounded variation and $|Du|$ is the total variation measure for these functions. The k-dimensional Hausdorff measure is denoted by $\mathcal{H}^k$. Let X^* denote the dual of a Banach space X. We define the indicator function I_A of a set A by

$$I_A(x) := \begin{cases} 0 & \text{for } x \in A\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

For the convex subdifferential of a function F we write $\partial F(u)$.

2 Variational problems

We assume that $\Omega \subset \mathbb{R}^n$ is an open bounded domain with Lipschitz boundary and consider the variational problem

$$E(u) := \int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} \quad \to \quad \text{Min!}\,, \quad u \in BV(\Omega) \cap \mathcal{L}^p(\Omega), \tag{2.1}$$

$$G(u) := \int_\Omega |u|\, dx = 1\,. \tag{2.2}$$

The surface integral in the energy E is a replacement for boundary conditions and ensures homogeneous Dirichlet data in a weak sense appropriate for BV-functions (cf. [9]). The formal Euler-Lagrange equation of this variational problem, which can be considered as eigenvalue problem for the 1-Laplace operator, is given by

$$-\text{Div}\, \frac{Du}{|Du|} = \lambda\, \frac{u}{|u|} \quad \text{on } \Omega \tag{2.3}$$

for some $\lambda > 0$. It turns out that minimizers of (2.1), (2.2) may vanish on a set having positive maesure (cf. Kawohl & Fridman [8]). For such solutions the expressions in (2.3) are not well-defined and a suitable substitute is needed. The derivation of a necessary condition for minimizers of a variational problem as (2.1), (2.2) with side condition is usually done by using a Lagrange multiplier rule. The difficultity in our special case is the lack of smoothness. The energy E is a convex but merely lower semicontinuous functional while G is Lipschitz continuous. One way to treat such a nonsmoothness is by an approximation of the problem with a sequence of smooth problems. Since the result obtained by such a procedure might depend on the special smoothing chosen, we are interested in a direct treatment of the highly singular problem. In doing so we are confronted with the further difficulty that almost nothing is known about the structure of the dual of $BV(\Omega)$, the space that contains the gradients of the functionals E and G. We can circumvent this last difficulty by a simple extension of the problem on all of $\mathcal{L}^p(\Omega)$. Using this strategy the next theorem was derived in Schuricht & Kawohl [9].

Theorem 2.4 *Let $u \in BV(\Omega) \cap \mathcal{L}^p(\Omega)$ be a minimizer of* (2.1), (2.2) *with $\frac{n}{n-1} \le p < \infty$. Then for each measurable selection $s(x) \in \text{Sgn}\,(u(x))$, $x \in \Omega$, there is some $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ with*

$$\|z\|_{\mathcal{L}^\infty} = 1\,, \quad \text{Div}\, z \in \mathcal{L}^{p'}(\Omega)\,, \quad \int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} = -\int_\Omega u \text{Div}\, z\, dx\,, \tag{2.5}$$

such that

$$-\text{Div}\, z = \lambda s \quad \textit{a.e. on } \Omega\,, \quad \lambda = E(u)\,. \tag{2.6}$$

Notice that (2.6) provides infinitely many Euler-Lagrange equations in general, since it has to be satisfied for *any* measurable selection s. Special examples where the vector field z is constructed for the same minimizer u but different s can be found in [9]. The proof of Theorem (2.4) in [9] essentially uses a very general Lagrange multiplier rule based on Degiovanni's weak slope (cf. Degiovanni [2] for details about the weak slope and Schuricht & Degiovanni [3] for the Lagrange multiplier rule used). In this paper we present an alternative proof of Theorem (2.4) by merely using convex analysis and omitting the technicalities of weak slope. Here the key idea is to exploit the special structure of our problem that allows us to replace the nonsmooth and nonconvex equality constraint (2.2) with a smooth convex inequality constraint. More precisely, for each selection s as in the theorem we can replace (2.2) with such an inequality constraint containing s where u is also minimizer of the new problem. Then we derive the Euler-Lagrange equation for the modified problem which provides (2.6) as necessary condition for any selectionn s. Notice that a similar replacement of (2.2) with the special selection $s \equiv 1$ was used in Demengel [4].

Proof of Theorem 2.4. We extend E on the space $X := \mathcal{L}^p(\Omega)$ by

$$E(v) := \begin{cases} \int_\Omega d|Dv| + \int_{\partial\Omega} |v|\, d\mathcal{H}^{n-1} & \text{for } v \in BV(\Omega) \cap \mathcal{L}^p(\Omega)\,, \\ \infty & \text{for } v \in \mathcal{L}^p(\Omega) \setminus BV(\Omega)\,, \end{cases}$$

and $G(v) = \int_\Omega |v|\, dx$ is naturally defined on all of X. Now we fix an arbitrary measurable selection $s \in \mathcal{L}^\infty(\Omega)$ of $\mathrm{Sgn}\,(u)$, i.e.,

$$s(x) \in \mathrm{Sgn}\,(u(x)) \quad \text{a.e. on } \Omega\,. \tag{2.7}$$

Notice that

$$\int_\Omega sv\, dx \le \int_\Omega |v|\, dx \quad \text{for all } v \in X\,, \qquad \int_\Omega |u|\, dx = \int_\Omega su\, dx\,.$$

Since E, G are both convex and 1-homogeneous, u is also a solution of the modified problem

$$E(v) \;\; \to \;\; \text{Min!}\,, \quad v \in C$$

with

$$C := \left\{ v \in X \,\Big|\, \int_\Omega sv\, dx \ge 1 \right\}.$$

Observe that any solution of this problem must belong to the boundary of C. Using the indicator function I_C of C, the function u is also an unconstrained minimizer of

$$E(v) + I_C(v) \;\; \to \;\; \text{Min!}\,, \quad v \in X\,.$$

Since s can be identified with a linear continuous functional on X, the set C is a closed convex half-space in X with nonempty interior. Hence I_C is convex and it is continuous at some point of X. Thus u minimizes the convex function $E + I_C$ on X and we therefore have that

$$0 \in \partial(E + I_C)(u)\,.$$

The sum rule of convex analysis implies that there is $E^* \in \partial E(u)$, $I^* \in \partial I_C(u)$ such that

$$E^* + I^* = 0\,. \tag{2.8}$$

According to [9, Prop. 3.18] we have that $E^* \in \partial E(u) \subset \mathcal{L}^{p'}(\Omega)$ if and only if there is some $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ with

$$\|z\|_{\mathcal{L}^\infty} = 1\,, \quad E^* = -\mathrm{Div}\, z\,, \quad E(u) = -\int_\Omega u \mathrm{Div}\, z\, dx\,. \tag{2.9}$$

Since $v \to \int_\Omega sv\, dx$ is a continuous linear functional on X and since u belongs to the boundary of C, the normal cone $N_C(u)$ of C at u is given by

$$N_C(u) = \{ts |\, t \le 0\} \subset X^*\,.$$

We readily verify that $N_C(u)$ coincides with the subdifferential $\partial I_C(u)$. By (2.8) we thus find $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ satisfying (2.9) such that

$$-\mathrm{Div}\, z = \lambda s \quad \text{a.e. on } \Omega$$

for some $\lambda > 0$. Multiplying this equation with u and integrating over Ω we obtain that $\lambda = E(u)$ by (2.9) and $\int_\Omega |u|\, dx = 1$. The arbitrariness of the selection s according to (2.7) completes the asertion. $\diamondsuit$

Remark. The previous proof derives necessary conditions for the minimizer u by using a special construction. But, in contrast to the approach in [9] where an inclusion for the subdifferentials of the total variation and of the $\mathcal{L}^1$-norm is evaluated, we cannot be completely sure in the previous proof that we really catched all necessary conditions of first order. Only by using the fact that the subdifferential $\partial G(u)$ consists of all functions s satisfying (2.7) (cf. [9]) we know that our analysis here is complete.

The proof of Theorem 2.4 can certainly be extended to any convex functional E that is *nondecreasing on rays starting at the origin*, i.e., for any $v \in X$ the real function $t \to E(tv)$ is nondecreasing for $t \ge 0$. We readily obtain the following result.

Corollary 2.10 *Let $E : X \to \mathbb{R}$ be convex and nondecreasing on rays starting at the origin and let $u \in \mathcal{L}^p(\Omega)$, $1 \le p < \infty$, be a minimizer of*

$$E(u) \;\to\; \text{Min!}\,, \quad u \in \mathcal{L}^p(\Omega)\,,$$

$$\int_\Omega |u|\, dx = 1\,.$$

Then for each measurable function s with $s(x) \in \mathrm{Sgn}\,(u(x))$ for all $x \in \Omega$ there is some $e^* \in \partial E(u) \subset \mathcal{L}^{p'}(\Omega)$ and some $\lambda > 0$ such that

$$e^*(x) = \lambda s(x) \quad \textit{a.e. on } \Omega\,.$$

Notice that the multiplier λ might depend on the special selection s in this corollary.

References

[1] F. Andreu-Vaillo, V. Caselles, and J.M. Mazón. *Parabolic quasilinear Equations Minimizing Linear Growth Functionals.* Birkhäuser, Basel, 2004

[2] M. Degiovanni and M. Marzocchi. A critical point theory for nonsmooth functionals. *Ann. Mat. Pura Appl. (4)* 167 (1994) 73-100

[3] M. Degiovanni and F. Schuricht. Buckling of nonlinearly elastic rods in the presence of obstacles treated by nonsmooth critical point theory. *Math. Ann.* 311 (1998) 675–728

[4] F. Demengel. Functions locally almost 1-harmonic. *Appl. Anal.* 83 (2004) 865–896

[5] F. Demengel. Some existence results for noncoercive “1-Laplacian” operator. *Asymptot. Anal.* 43 (2005) 287–322

[6] P. Juutinen. Absolutely minimizing functions. *Indiana Univ. Math. J.* 54 (2005) 1015–1029

[7] B. Kawohl. On a family of torsional creep problems. *J. Reine Angew. Math.* 410 (1990) 1–22

[8] B. Kawohl and V. Fridman. Isoperimetric estimates for the first eigenvalue of the p–Laplace operator and the Cheeger constant. *Comment. Math. Univ. Carolinae* 44 (2003) 659–667

[9] B. Kawohl and F. Schuricht. Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem. *Manuscript*