Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem

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Abstract

We consider a number of problems that are associated with the 1-Laplace operator $\operatorname{Div}(Du/|Du|)$, the formal limit of the p-Laplace operator for $p \to 1$, by investigating the underlying variational problem. Since corresponding solutions typically belong to BV and not to $\mathcal{W}^{1,1}$, we have to study minimizers of functionals containing the total variation. In particular we look for constrained minimizers subject to a prescribed $\mathcal{L}^1$-norm which can be considered as an eigenvalue problem for the 1-Laplace operator. These variational problems are neither smooth nor convex. We discuss the meaning of Dirichlet boundary conditions and prove existence of minimizers. The lack of smoothness, both of the functional to be minimized and the side constraint, requires special care in the derivation of the associated Euler-Lagrange equation as necessary condition for minimizers. Here the degenerate expression $Du/|Du|$ has to be replaced with a suitable vector field $z \in \mathcal{L}^\infty$ to give meaning to the highly singular 1-Laplace operator. For minimizers of a large class of problems containing the eigenvalue problem, we obtain the surprising and remarkable fact that in general infinitely many Euler-Lagrange equations have to be satisfied.

1 Introduction

Recent developments in degenerate elliptic equations have led to significant new insight in the geometry of the p–Laplace operator $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$ as $p \in (1,\infty)$ tends to ∞, see e.g. [9], [5] and references therein. The limit $p \to 1$ has found less attention, partly because compactness of sequences as $p \to 1$ takes place in weaker norms, see [26, 28, 25]. If one considers the limit problems for $p = 1$ directly, the associated energy is no longer smooth and no longer strictly convex. This rules out a number of standard tricks from the calculus of variations. Moreover it leads to a number of surprising effects.

As model cases we try to give meaning to the eigenvalue problem

$$-\mathrm{Div}\left(\frac{Du}{|Du|}\right) = \lambda \frac{u}{|u|} \tag{1.1}$$

and to the generalized torsion problem

$$-\mathrm{Div}\left(\frac{Du}{|Du|}\right) = f(x) \geq 0 \tag{1.2}$$

in a domain Ω under homogeneous Dirichlet boundary conditions and in particular in points where Du vanishes. When u is positive everywhere in Ω and for $f(x) \equiv \lambda$ equations (1.1) and (1.2) coincide. However, solutions of (1.1) can have large nodal sets and changing sign, and then the situation changes. As noted in [26] the existence of solutions to (1.2) is only possible if f is small compared to λ, with λ defined in Section 2, or if Ω is sufficiently small compared to f, in which case λ is large.

To do so, we study a class of variational problems in $BV(\Omega)$ or in suitable subspaces thereof where the functional to be minimized contains the nonsmooth 1-homogeneous total variation and where the $\mathcal{L}^q$-norm might be prescribed as a constraint for $q \geq 1$. The existence of minimizers is shown in Section 3 where we cover a number of previous results from the literature as special cases. Moreover the (non)uniqueness is discussed in detail for special examples. It should be pointed out that $q = 1$ in the eigenvalue problem which means that also the side constraint is nonsmooth in this case. Therefore the derivation of the Euler-Lagrange equation, which is done in Section 4, is highly nontrivial. In case of (1.2) we have no side constraint but the convex functional is nonsmooth. This situation is much easier to handle. The Euler-Lagrange equation for (1.1) (resp. (1.2)) is given by (4.12) (resp. (4.11)) and contains a vector field $z : \Omega \to B_1(0) \subset \mathbb{R}^n$, which can be identified with $Du/|Du|$ if $|Du|$ is nonzero and well-defined. Otherwise z is a suitable substitute for $Du/|Du|$. Furthermore $u/|u|$ has to be replaced at points where $u(x) = 0$ by some value in $[-1, 1]$. It is remarkable that for minimizers u of a large class of problems, containing the eigenvalue problem, the Euler-Lagrange equation (4.8) has to be satisfied not only for one but for any measurable selection of the set-valued sign function $\mathrm{Sgn}\,(u(x) - h(x))$ on the right hand side (cf. (1.3) below). In the special case of the eigenvalue problem the Euler equation was recently derived by Demengel for one of these selections (see [15] or [16]). We illustrate this phenomenon with explicit examples in Section 5. Moreover, there we give a geometric interpretation of of the assigned vector field z and relate it to generalized constant mean curvature surfaces in Ω. The proof of our results requires some technical material from duality theory and a very general nonsmooth Lagrange-multiplier rule. For the reader's convenience these are presented in an appendix in a self-contained form.

Notation. For a set A let $\bar{A}$ denote its closure and ∂A its boundary. We define its indicator function I_A and its characteristic function χ_A by

$$I_A(x) := \begin{cases} 0 & \text{for } x \in A\,, \\ \infty & \text{otherwise}\,, \end{cases} \qquad \text{and} \quad \chi_A(x) := \begin{cases} 1 & \text{for } x \in A\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

respectively. $d_A(x)$ stands for the distance of a point x to the set A. For a function u we denote by $u_{|\Omega}$ the restriction to the set Ω, by $\operatorname{Div} u$ the divergence in the distributional sense, and by $u^{\partial\Omega}$ the trace on $\partial\Omega$. While $\operatorname{sgn}\alpha$ is the usual sign function on $\mathbb{R}$ we also define the set-valued sign function

$$\operatorname{Sgn}\alpha := \begin{cases} 1 & \text{if } \alpha > 0, \\ [-1,1] & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha < 0. \end{cases} \tag{1.3}$$

The space of p-integrable functions on Ω is denoted by $\mathcal{L}^p(\Omega)$ and its dual by $\mathcal{L}^{p'}(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. The Sobolev space $\mathcal{W}^{1,p}(\Omega)$ contains all p-integrable functions having p-integrable weak derivatives. $\mathcal{C}_0^\infty(\Omega)$ are the infinitely often differentiable functions with compact support. $BV(\Omega)$ stands for the space of functions of bounded variation and $|Du|$ is the total variation measure for these functions. The k-dimensional Hausdorff measure is denoted by $\mathcal{H}^k$, the measure $|\mu|$ is the total variation of the measure μ, and $\mu\lfloor A$ is the restriction of the measure μ to the set A. For a Banach space X its dual is X^* and $\langle\cdot,\cdot\rangle$ is the duality form on $X^* \times X$. By $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ we denote the weak and the weak* convergence. $\partial F(u)$ stands for the convex subdifferential if F is convex and for Clarke's generalized gradient if F is locally Lipschitz continuous.

2 Variational problems

We study a general class of variational problems that contains a number of problems studied in the literature as special case. In particular the eigenvalue problem (1.1) and the torsion problem (1.2) are included. While it appears to be natural to consider these problems in $BV(\Omega)$, they are studied in suitable subspaces $BV(\Omega) \cap \mathcal{L}^p(\Omega)$. This way we cover previous results from the literature, where mostly the case $p = 2$ is investigated, and we verify the existence of solutions with more regularity than merely belonging to $BV(\Omega)$. Nevertheless the important case of minimizers in all of $BV(\Omega)$ (which corresponds to $p = n/(n-1)$) is explicitly contained in Theorem 3.2 below. Let $\Omega \subset \mathbb{R}^n$ always be an open bounded domain with Lipschitz boundary.

For the energy functional

$$E(u) := \int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega} - u_0^{\partial\Omega}|\, d\mathcal{H}^{n-1} - \int_\Omega fu\,dx + \alpha \int_\Omega |u-g|^r\,dx$$

we consider the variational problem

$$E(u) \;\to\; \text{Min!}\,, \quad u \in BV(\Omega) \cap \mathcal{L}^p(\Omega), \tag{2.1}$$

$$\int_\Omega |u-h|^q \, dx = 1\,. \tag{2.2}$$

We assume that

$$u_0^{\partial\Omega} \in \mathcal{L}^1(\partial\Omega)\,, \quad f \in \mathcal{L}^n(\Omega)\,, \quad g \in \mathcal{L}^r(\Omega)\,, \quad h \in \mathcal{L}^q(\Omega)\,, \tag{2.3}$$

$$\frac{n}{n-1} \le p < \infty\,, \quad 1 \le r < \infty\,, \quad 1 \le q \le p\,, \quad \alpha \ge 0 \tag{2.4}$$

are given. Recall that $BV(\Omega)$ is continuously imbedded into $\mathcal{L}^{\frac{n}{n-1}}(\Omega)$ (cf. [2, p. 152]) and that $\mathcal{L}^n(\Omega)$ is the dual of $\mathcal{L}^{\frac{n}{n-1}}(\Omega)$. Thus $\mathcal{L}^n(\Omega) \subset \mathcal{L}^{p'}(\Omega)$ and all expressions in the previous problem are well-defined while the last integral in the energy E might even become infinite. For $p = \frac{n}{n-1}$ we obviously can omit the intersection with $\mathcal{L}^p(\Omega)$ in (2.1). Notice that minimizers of functionals with linear growth typically belong to $BV(\Omega)$ and not necessarily to $\mathcal{W}^{1,1}(\Omega)$, i.e., it is natural to study the previous problem in $BV(\Omega)$ instead of $\mathcal{W}^{1,1}(\Omega)$.

Usually a variational problem contains boundary conditions. Since the trace operator $u \mapsto u^{\partial\Omega}$ on $BV(\Omega)$ has merely quite weak continuity properties (cf. [2, p. 180/181]), the usual prescription of the trace $u^{\partial\Omega}$ is too restrictive if we work in $BV(\Omega)$. Let us show that the surface integral in (2.1) implies boundary conditions in a weaker sense. For this reason we choose an open ball $B \subset \mathbb{R}^n$ with $\bar\Omega \subset B$ and assign to each $u \in BV(\Omega)$ its extension

$$\bar u(x) := \begin{cases} u(x) & \text{on } \Omega, \\ 0 & \text{on } B \setminus \Omega. \end{cases} \tag{2.5}$$

Then $\bar u \in BV(B)$ and

$$D\bar u = Du - u^{\partial\Omega}\nu\mathcal{H}^{n-1}\lfloor\partial\Omega\,, \quad |D\bar u|(B) = |Du|(\Omega) + \int_{\partial\Omega} |u^{\partial\Omega}| \, d\mathcal{H}^{n-1}$$

where ν denotes the outer unit normal of Ω (cf. [2, p. 180/181]). Clearly $\bar u \in \mathcal{L}^p(B)$ as long as $u \in \mathcal{L}^p(\Omega)$. Since the trace operator maps $BV(B \setminus \bar\Omega)$ onto $\mathcal{L}^1(\partial B \cup \partial\Omega)$ (cf. [2, p. 181]), we find $u_0 \in BV(B \setminus \bar\Omega)$ having the given function $u_0^{\partial\Omega}$ as trace on $\partial\Omega$. Therefore the extension

$$\underline{u}_0(x) := \begin{cases} 0 & \text{on } \Omega, \\ u_0(x) & \text{on } B \setminus \Omega \end{cases} \tag{2.6}$$

belongs to $BV(B)$ with $D\underline{u}_0 = Du_0 + u_0^{\partial\Omega}\nu\mathcal{H}^{n-1}\lfloor\partial\Omega$ and

$$|D(\underline{u}_0 + \bar u)|(B) = |Du|(\Omega) + |Du_0|(B \setminus \bar\Omega) + \int_{\partial\Omega} |u^{\partial\Omega} - u_0^{\partial\Omega}| \, d\mathcal{H}^{n-1}\,.$$

We define

$$F(v) := \int_B d|Dv| - \int_\Omega fv \, dx + \alpha \int_\Omega |v-g|^r \, dx$$

for all $v \in BV(B) \cap \mathcal{L}^p(B|_\Omega)$ where

$$\mathcal{L}^p(B|_\Omega) := \{v \in \mathcal{L}^1(B) |\, v_{|\Omega} \in \mathcal{L}^p(\Omega)\}\,.$$

Then we readily see that

$$F(\underline{u}_0 + \bar u) = E(u) + |Du_0|(B \setminus \bar\Omega) \quad \text{for all } \; u \in BV(\Omega) \cap \mathcal{L}^p(\Omega)\,. \tag{2.7}$$

Consequently, (2.1) (2.2) is equivalent to the problem

$$F(v) \;\to\; \text{Min!}, \quad v \in BV(B) \cap \mathcal{L}^p(B|_\Omega)\,, \tag{2.8}$$

$$\int_\Omega |v-h|^q\,dx = 1\,, \tag{2.9}$$

$$v = u_0 \;\text{ a.e. on }\; B \setminus \Omega \tag{2.10}$$

where we can interpret (2.10) as boundary condition in a weak form. We will exploit this equivalence in our analysis below.

3 Existence of minimizers

In this section we provide general existence results for the variational problem stated in Section 2 which generalize a variety of previous results. In particular we extend most of the previous existence results to nonhomogeneous Dirichlet boundary conditions. With the formulation of a number of special cases, that have recently found considerable attention in the literature, we demonstrate the richness of our general variational problem. Finally we show by means of a special example that in general uniqueness cannot be expected for minimizers.

3.1 General existence result

Before we formulate the existence results we state a version of Poincaré's inequality that follows from [20, p. 189]. There is a constant $c > 0$ such that

$$\|v\|_{\mathcal{L}^{\frac{n}{n-1}}(B)} \le c\Big(|Dv|(B) + \int_{\partial B} |v^{\partial B}|\,d\mathcal{H}^{n-1}\Big) \tag{3.1}$$

for all $v \in BV(B)$ where c is the optimal constant from the Gagliardo-Nirenberg-Sobolev inequality (cf. [20, p. 138]). If v satisfies the boundary condition (2.10), then we can take $u_0^{\partial B}$ instead of $v^{\partial B}$ in the surface integral.

Theorem 3.2 (Existence) *Let* (2.3), (2.4) *be satisfied and let either* $c\|f\|_{\mathcal{L}^n} < 1$ *with* $c > 0$ *from* (3.1) *or let* $\alpha > 0$, $r > \frac{n}{n-1}$.

(1) *If* $q < \frac{n}{n-1} = p$*, then Problem* (2.1), (2.2) *has a solution* $u \in BV(\Omega)$.

(2) *If* $\frac{n}{n-1} \le p \le r$, $q < r$, $\alpha > 0$*, then Problem* (2.1), (2.2) *has a solution* $u \in BV(\Omega) \cap \mathcal{L}^p(\Omega)$.

(3) *If* $\frac{n}{n-1} = p$*, then Problem* (2.1) *(without side condition* (2.2)*) has a solution* $u \in BV(\Omega)$.

(4) *If* $p \le r$, $\alpha > 0$*, then Problem* (2.1) *has a solution* $u \in BV(\Omega) \cap \mathcal{L}^p(\Omega)$.

Proof. Let us first consider problem (2.8) - (2.10). We can approximate h by a step function $\tilde h \in \mathcal{L}^\infty(\Omega)$ that is nonzero on finitely many cubes such that

$$0 < \int_\Omega |\tilde h - h|^q\,dx < 1\,.$$

Obviously $\tilde h \in BV(\Omega)$. Furthermore we find some $\tilde t \in \mathbb{R}$ such that $\int_\Omega |u_1 - h|^q\,dx = 1$ for $u_1 := \tilde t \tilde h$. Certainly $\underline{u}_0 + \bar u_1 \in BV(B) \cap \mathcal{L}^p(B|_\Omega)$ (cf. (2.5), (2.6)) and $\underline{u}_0 + \bar u_1$ is admissible for our variational problem with $F(\underline{u}_0 + \bar{u}_1) < \infty$ (choose $u_1 \equiv 0$ in cases (3), (4)). Thus we find a minimizing sequence $\{v_n\} \subset BV(B) \cap \mathcal{L}^p(B|_\Omega)$ of (2.8) - (2.10) with $\lim_{n\to\infty} F(v_n) < \infty$. Using (3.1) we can estimate that

$$\begin{aligned} F(v_n) &\geq |Dv_n|(B) - \|f\|_{\mathcal{L}^n} \|v_n\|_{\mathcal{L}^{\frac{n}{n-1}}} + \alpha \int_\Omega |v_n - g|^r \, dx \\ &\geq (1 - c\|f\|_{\mathcal{L}^n})|Dv_n|(B) - \tilde{c} + \alpha \|v_n - g\|_{\mathcal{L}^r(\Omega)} \end{aligned} \tag{3.3}$$

for a constant $\tilde{c} > 0$. If $1 - c\|f\|_{\mathcal{L}^n} > 0$, then $|Dv_n|(B)$ is bounded and, by (3.1), v_n is bounded in $\mathcal{L}^{\frac{n}{n-1}}(B)$ and thus also in $\mathcal{L}^1(B)$. Hence v_n is bounded in $BV(B)$. In the case where $r > \frac{n}{n-1}$, $\alpha > 0$, we have that

$$\|v_n\|_{\mathcal{L}^{\frac{n}{n-1}}} \leq c_1 \|v_n\|_{\mathcal{L}^r} \leq c_1 \|v_n - g\|_{\mathcal{L}^r} + c_1 \|g\|_{\mathcal{L}^r} .$$

for some constant $c_1 > 0$ and, hence,

$$F(v_n) \geq |Dv_n|(B) + \|v_n - g\|_{\mathcal{L}^r} (\alpha \|v_n - g\|_{\mathcal{L}^r}^{r-1} - c_1 \|f\|_{\mathcal{L}^n}) - c_1 \|f\|_{\mathcal{L}^n} \|g\|_{\mathcal{L}^r} .$$

Also in that case we readily conclude that v_n is bounded in $BV(B)$. Hence there is some $v \in BV(B)$ such that, possibly for a subsequence,

$$v_n \to v \ \text{ in } \ \mathcal{L}^1(B) . \tag{3.4}$$

(cf. [20, p. 176]). Without loss of generality we can assume that

$$v_n(x) \to v(x) \quad \text{a.e. on } \ B .$$

Therefore v satisifies (2.10). In the case where $q < r$, $\alpha > 0$ we have that $v_{n|\Omega}$ is bounded in $\mathcal{L}^r(\Omega)$ by (3.3). Thus $v_{n|\Omega} \rightharpoonup v_{|\Omega}$ in $\mathcal{L}^r(\Omega)$. By $p \leq r$ this implies that $v \in \mathcal{L}^p(\Omega)$. By Hölder's inequality we obtain that

$$\begin{aligned} \int_\Omega |v_n - v|^q \, dx &= \int_\Omega |v_n - v|^{\frac{r-q}{r-1}} |v_n - v|^{\frac{r(q-1)}{r-1}} \, dx \\ &\leq \Big(\int_\Omega |v_n - v| \, dx \Big)^{\frac{r-q}{r-1}} \Big(\int_\Omega |v_n - v|^r \, dx \Big)^{\frac{q-1}{r-1}} \end{aligned}$$

Using (3.4) we see that $v_{n|\Omega}$ converges to $v_{|\Omega}$ in $\mathcal{L}^q(\Omega)$ and, thus, v satisfies (2.9) in that case. If $q < \frac{n}{n-1}$, then we use that v_n is bounded in $BV(B)$ and, thus, it is also bounded in $\mathcal{L}^{\frac{n}{n-1}}(B)$ by (3.1). Now we can argue as above with $\frac{n}{n-1}$ instead of r to get that v satisfies (2.9) and that $v_{|\Omega} \in \mathcal{L}^q(\Omega)$ also in this case, i.e., v is admissible for the variational problem.

Recalling that v_n is bounded in $\mathcal{L}^{\frac{n}{n-1}}(B)$ we can assume that $v_n \rightharpoonup v$ in $\mathcal{L}^{\frac{n}{n-1}}(B)$. Therefore

$$\int_\Omega f v_n \, dx \to \int_\Omega f v \, dx .$$

For $\alpha > 0$ we have that $v_{n|\Omega}$ is bounded in $\mathcal{L}^r(\Omega)$ by (3.3). If $r > 1$, then $v_{n|\Omega} \rightharpoonup v_{|\Omega}$ in $\mathcal{L}^r(\Omega)$ and the weak lower semicontinuity of the norm implies that

$$\|v - g\|_{\mathcal{L}^r} \leq \liminf_{n\to\infty} \|v_n - g\|_{\mathcal{L}^r} .$$

If $r = 1$, then we get $\|v_n - g\|_{\mathcal{L}^r} \to \|v - g\|_{\mathcal{L}^r}$ by (3.4). The lower semicontinuity of the total variation in $BV(B)$ then implies that

$$F(v) \le \liminf_{n\to\infty} F(v_n)\,.$$

Since $\{v_n\}$ was a minimizing sequence, v solves (2.8) - (2.10). By the identity (2.7) the restriction $u := v_{|\Omega}$ is a solution of (2.1), (2.2) as long as the assumptions of cases (1), (2) are met. Cases (3) and (4) easily follow by the same arguments. $\diamondsuit$

3.2 Important special cases

Let us now demonstrate by several special cases that the previous general variational problem covers a rich collection of problems enjoying broad interest in the literature.

Eigenvalue problem. In the literature the problem

$$\int_\Omega |Du|^p\,dx \to \text{Min!}\,, \quad u \in \mathcal{W}^{1,p}(\Omega)\,,$$

$$\int_\Omega |u|^p\,dx = 1, \quad u = 0 \text{ on } \partial\Omega$$

is studied comprehensively for $1 < p < \infty$. Using a Lagrange multiplier rule we are led to the corresponding Euler-Lagrange equation

$$-\text{div}\,\big(|Du|^{p-2}Du\big) = \lambda |u|^{p-2}u \quad \text{on } \Omega\,, \quad u = 0 \text{ on } \partial\Omega \tag{3.5}$$

which is called the eigenvalue problem for the p-Laplace operator. As already observed in [30, p.445], the case $p = 1$ is highly singular. It has recently been analyzed by studying the limit $p \to 1$ (cf. Kawohl & Fridman [28]), see also Demengel [15] for a slightly different approximation. We have to realize that solutions of the variational problem for $p = 1$ belong to $BV(\Omega)$ but not to $\mathcal{W}^{1,1}(\Omega)$. This way we also encounter the difficulties with the Dirichlet boundary conditions as discussed before. Moreover, the eigenvalue problem (3.5) is not well defined for $p = 1$ if we have in mind that solutions might be piecewise constant. Thus it seems to be reasonable to consider the variational problem

$$\int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\,d\mathcal{H}^{n-1} \quad \to \quad \text{Min!}\,, \quad u \in BV(\Omega), \tag{3.6}$$

$$\int_\Omega |u|\,dx = 1 \tag{3.7}$$

in the case $p = 1$ where the surface integral compensates the Dirichlet data in a generalized way. The existence of a solution, that has been verified already in [28, Theorem 8] and [15, p. 888], is recovered by Theorem 3.2. In the next section we will provide a substitute for equation (3.5) as necessary condition for minimizers which can be considered as eigenvalue problem for the 1-Laplace operator.

Generalized torsion problem. In linear elasticity the torsion of an infinitely long elastic bar with cross section Ω can be described in terms of the solutions to $-\Delta u = 1$ in Ω, $u = 0$ on $\partial\Omega$.

Here $|Du(x)|$ respresents the magnitude of stress. Similar problems were studied for nonlinear materials with the Laplacian operator replaced by the p-Laplacian in [26]. The limit $p \to 1$ remained somewhat miraculous. Depending on the size and shape of Ω the family of corresponding solutions u_p could converge to zero, to a characteristic function of Ω or diverge to ∞. Formally the limit equation for $-\Delta_p = f(x)$ is given by (1.2). So now we further pursue an approach from [26] and study the variational problem

$$\int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} - \int_\Omega f u\, dx \quad \to \quad \text{Min!}, \quad u \in BV(\Omega) \tag{3.8}$$

which is a special case of (2.1) without side condition (2.2). Theorem 3.2 provides the existence of a solution if

$$\|f\|_{\mathcal{L}^n} < \frac{1}{c} \tag{3.9}$$

with c from (3.1). This result can already be found in [11]. According to [11] condition (3.9) can also be replaced with a bound $1/\tilde{c}$ on f in the Lorentz space $\mathcal{L}^{n,\infty}(\Omega)$ where $\tilde{c}$ is the optimal imbedding constant of $\mathcal{L}^{\frac{n}{n-1},\infty}(\Omega)$ into $W_0^{1,1}(\Omega)$. In certain special cases we can supplement these existence results by a precise description of the solution set as follows.

Proposition 3.10 (Special solutions in the generalized torsion problem) *Let $\tilde{u}$ be a minimizer of* (3.6), (3.7) *and set* $\tilde{\lambda} := \int_\Omega d|D\tilde{u}| + \int_{\partial\Omega} |\tilde{u}^{\partial\Omega}|\, d\mathcal{H}^{n-1}$.

(1) *If* $\|f\|_{\mathcal{L}^\infty(\Omega)} < \tilde{\lambda}$*, then the trivial solution* $0 \equiv u \in BV(\Omega)$ *is the unique solution of problem* (3.8).

(2) *If* $f(x) = \tilde{\lambda} \operatorname{sgn} \tilde{u}(x)$ *a.e. on* Ω*, then* $u \in BV(\Omega)$ *is a minimizer of* (3.8) *if and only if* u *solves* (3.6), (3.7) *and* $\operatorname{sgn} u = \operatorname{sgn} \tilde{u}$ *a.e. on* Ω*. In particular any positive multiple of* $\tilde{u}$ *is a minimizer.*

From the example corresponding to Figure 2 below we see that in case (2) there can also be solutions that are not a multiple of $\tilde{u}$.

Proof. In the first case the $\mathcal{L}^\infty$-bound for f readily implies that

$$\int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} \geq \tilde{\lambda} \int_\Omega |u|\, dx > \int_\Omega f u\, dx \tag{3.11}$$

for all $u \neq 0$. Hence the energy in (3.8) is positive for all $u \in BV(\Omega)$ with the exception of $u \equiv 0$ which verifies the assertion. In the second case we observe that the energy in (3.8) is nonnegative and that any minimizer u has to have zero energy. In this case all three terms in (3.11) must be equal. But this is only possible if u solves (3.6), (3.7) and $\operatorname{sgn} u = \operatorname{sgn} \tilde{u}$ a.e. on Ω. On the other hand these conditions are sufficient, since they imply that the energy is zero. $\diamondsuit$

In [32] the more general problem

$$\int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} - \int_\Omega f u\, dx + \int_\Omega I_{[-1,1]}(u)\, dx \quad \to \quad \text{Min!}, \quad u \in BV(\Omega)$$

is studied in the 1-dimensional case where $I_{[-1,1]}$ denotes the indicator function of the interval $[-1,1]$. In other words we still have the side constraint $\|u\|_{\mathcal{L}^\infty(\Omega)} \leq 1$ in the generalized torsion problem. Arguing as in the proof of Theorem 3.2 we readily get the existence of a solution without a bound on f in this case.

Image processing. Another special example is provided by variational problems of the type

$$\int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} + \alpha \int_\Omega |u-g|^r\, dx \quad \rightarrow \quad \text{Min!}, \qquad u \in BV(\Omega),$$

which play a major role in mathematical image processing. Here g is a given (blurred) image whose contures one wants to sharpen. For $r = 2$ we refer for instance to [4, p. 7]. In [10] the existence of a minimizer is shown for $r = 1$ and a comparison of $r = 2$ with $r = 1$ is given. Our Theorem 3.2 ensures the existence of a solution in $BV(\Omega)$ for any $r \geq 1$, $g \in \mathcal{L}^r(\Omega)$.

Let us give some heuristic explanation for the previous variational problem. Interpreting the last integral (with α as a multiplier) as a penalty term, one wants to minimize the total variation of a function, while staying close to g. An alternative approach to this task would be to study the associated so called total variation flow equation

$$u_t - \operatorname{Div}\left(\frac{Du}{|Du|}\right) = 0 \tag{3.12}$$

under initial data $u(0,x) = g(x)$. Let us demonstrate how this degenerate parabolic equation gives rise to a different variational problem containing the total variation. Suppose $g(x) \geq 0$. The naive ansatz $u(t,x) = T(t)v(x)$ satisfies (3.12) if and only if

$$T'(t)v(x) = \operatorname{Div}\left(\frac{Dv}{|Dv|}\right),$$

i.e. if $T'(t)$ is (a negative) constant, say $-\lambda$, and if v satisfies the equation

$$\operatorname{Div}\left(\frac{Dv}{|Dv|}\right) + \lambda v = 0. \tag{3.13}$$

If there is a solution v of (3.13), then we obtain a separable solution of (3.12) by

$$u(t,x) = -\lambda(t - t_0)v(x)$$

that decays to zero in finite time t_0 with speed λ. Notice that a search for solutions of (3.13) gives rise to the variational problem (2.1), (2.2) with $f \equiv 0$, $\alpha = 0$, $q = 2$, $p = \frac{n}{n-1}$ or to (2.1) alone with $f \equiv 0$, $\alpha = \lambda$, $p = \frac{n}{n-1}$, $r = 2$. While Theorem 3.2 provides a solution only for $n = 2$ in the first case, it provides a solution for any $n \in \mathbb{N}$ in the second case. The first examples that we are aware of appear in a study of Dibos and Koepfler [17]. Indeed, if $g(x)$ is the characteristic function of a disk of radius R, then $\lambda = 2/R$ and a solution of (3.12) is given by $u(x,t) = (1 - \frac{2}{R}t)\chi_{B_R}$. A more systematic search for solutions of (3.12) and (3.13) can be found in the papers [6] and [7] of Belletini, Caselles and Novaga.

3.3 Uniqueness of minimizers

What can we say about uniqueness or the sign of minimizers of our general variational problem (2.1), (2.2)? Let us discuss this question for the special case of the eigenvalue problem (3.6), (3.7). The first eigenfunction of the p-Laplacian operator with $p \in (1, \infty)$ is known to be nonzero in Ω and unique (modulo sign change and scaling), for a variational proof see [8]. The minimizers of (3.6), (3.7), however, are in general of changing sign, have large nullsets, and are not unique. Let us demonstrate this with some simple examples that are already mentioned in [26] and [28]. We call a set $C \subset \Omega$ Cheeger set of Ω, if it minimizes the ratio $|\partial D|/|D|$ (of $(n-1)$–dimensional perimeter of D over n–dimensional volume of D) among all subsets D of Ω. To give an example, if Ω is a square in $\mathbb{R}^2$, then its (unique) Cheeger set is a "rounded square", i.e. the Minkowski sum of a smaller square S, centered in Ω and with area π/λ^2, and a disk of radius $1/\lambda$, see Figure 1.

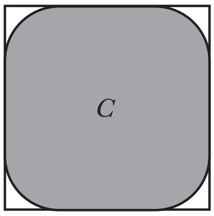

Figure 1: A square and its Cheeger set.

In [28] it was shown that (suitable multiples of) characteristic functions of Cheeger sets are minimizers of (3.6), (3.7), and [29] contains examples of nonunique Cheeger sets.

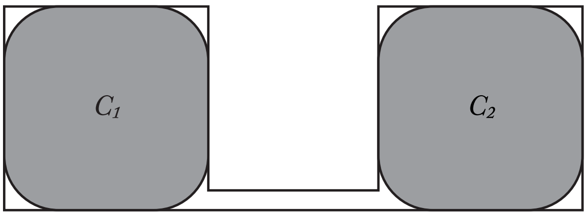

Figure 2: A (nonconvex) barbell-type domain and its Cheeger sets.

Figure 2 displays a situation in which C_1, C_2 and $C_1 \cup C_2$ are all Cheeger sets. Any function $u(x) = c_1\chi_{C_1}(x) + c_2\chi_{C_2}(x)$ with $\sum_1^2 |c_i| \; |C_i| = 1$ is for this domain Ω a minimizer of (3.6), (3.7). Therefore minimizers are neither unique nor of one sign. Finally we should mention that, although the minimizers $u(x)$ are discontinuous, almost all of their level sets are piecewise smooth and Lipschitz-regular. In two dimensions this follows from [31], and in higher dimensions one can first derive the result that almost all level sets of minimizers are Cheeger sets (see [28]) and then refer to results from [23] to verify that the boundary of Cheeger sets is piecewise smooth.

4 Necessary Conditions for Minimizers

In this section we first derive the Euler-Lagrange equation as a necessary condition for minimizers of (2.1), (2.2). Then we show that infinitely many Euler-Lagrange equations have to be satisfied for minimizers of a large subclass of variational problems containing the eigenvalue problem. Here we restrict our attention to energy functionals

$$E(u) := \int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} - \int_\Omega fu\, dx + \alpha \int_\Omega |u-g|^r\, dx \tag{4.1}$$

and consider the variational problem

$$E(u) \quad \to \quad \text{Min!}\,, \quad u \in BV(\Omega) \cap \mathcal{L}^p(\Omega), \tag{4.2}$$

$$\int_\Omega |u-h|^q\, dx = 1\,. \tag{4.3}$$

Instead of (2.3), (2.4) we now assume that

$$f \in \mathcal{L}^{p'}(\Omega)\,, \quad g \in \mathcal{L}^r(\Omega)\,, \quad h \in BV(\Omega) \cap \mathcal{L}^p(\Omega)\,, \tag{4.4}$$

$$\frac{n}{n-1} \le p < \infty\,, \quad 1 \le q, r \le p\,, \quad \alpha \in \mathbb{R}\,. \tag{4.5}$$

Theorem 4.6 *Let $u \in BV(\Omega) \cap \mathcal{L}^p(\Omega)$ be a minimizer of* (4.2), (4.3) *with E as in* (4.1) *and let* (4.4), (4.5) *be satisfied. Then there is some $\lambda \in \mathbb{R}$ and some $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ with*

$$\|z\|_{\mathcal{L}^\infty} \le 1\,, \quad \operatorname{Div} z \in \mathcal{L}^{p'}(\Omega)\,, \quad \int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} = -\int_\Omega u \operatorname{Div} z\, dx\,, \tag{4.7}$$

where $\|z\|_{\mathcal{L}^\infty} = 1$ if $u \ne 0$, such that:

(i) *if $q = r = 1$, then (with the notation of* (1.3)*)*

$$0 \;\in\; -\operatorname{Div} z - f + \alpha \operatorname{Sgn}(u-g) - \lambda \operatorname{Sgn}(u-h) \quad \textit{a.e. on } \Omega, \tag{4.8}$$

(ii) *if $r > 1$ or $q > 1$, then we have* (4.8) *with $|u-g|^{r-2}(u-g)$ or $|u-h|^{q-2}(u-h)$ instead of* $\operatorname{Sgn}(u-g)$ *or* $\operatorname{Sgn}(u-h)$*, respectively.*

(iii) *If $\alpha = 0$ and $h = 0$, then $\lambda = E(u)$ in* (i) *and* (ii).

Notice that (4.8) becomes an equation (i.e., the right hand side is a singelton) for $q, r > 1$. The case where we neglect the side condition (4.3) in our variational problem is a simple consequence of the proof of the previous theorem.

Corollary 4.9 *Let $u \in BV(\Omega) \cap \mathcal{L}^p(\Omega)$ be a minimizer of* (4.2) *with E as in* (4.1) *and let* (4.4), (4.5) *be satisfied. Then there is some $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ satisfying* (4.7)*, where $\|z\|_{\mathcal{L}^\infty} = 1$ if $u \ne 0$, such that*

$$0 \;\in\; -\operatorname{Div} z - f + \alpha \operatorname{Sgn}(u-g) \quad \textit{a.e. on } \Omega \tag{4.10}$$

for $r = 1$. For $r > 1$ we have (4.10) *with $|u-g|^{r-2}(u-g)$ instead of* $\operatorname{Sgn}(u-g)$.

Before we present the proof of Theorem 4.6 at the end of this section we discuss some applications. Let us start with the *generalized torsion problem.* The necessary condition for minimizers of the unconstrained problem (3.8) is provided by Corollary 4.9. It says that there exists some $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ satisfying (4.7), where $\|z\|_{\mathcal{L}^\infty} = 1$ if $u \neq 0$, such that

$$-\mathrm{Div}\, z = f \quad \text{a.e. on } \Omega. \tag{4.11}$$

It is in this sense that one has to understand (1.2). Note that the Euler equation for this problem was neither discussed in [26] nor in [11]. In [15] it is indicated how convex variational problems leading to a right hand side $f(x, u)$ might be treated.

Let us now apply Theorem 4.6 to the *eigenvalue problem.* For the special case where $f, h = 0$, $p = \frac{n}{n-1}$, $q = 1$, $\alpha = 0$ in (4.2), (4.3) we obtain problem (3.6), (3.7). In this case Theorem 4.6 provides the necessary condition that

$$-\mathrm{Div}\, z \in \lambda\, \mathrm{Sgn}\, u \quad \text{a.e. on } \Omega, \qquad \lambda = E(u) > 0 \tag{4.12}$$

for some $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ with

$$\|z\|_{\mathcal{L}^\infty} = 1\,, \quad \mathrm{Div}\, z \in \mathcal{L}^n(\Omega)\,, \quad E(u) = -\int_\Omega u \mathrm{Div}\, z\, dx\,. \tag{4.13}$$

Here the inclusion in (4.12) means that there is a measurable selection s of $\mathrm{Sgn}\, u$, i.e., $s(x) \in \mathrm{Sgn}\,(u(x))$ a.e. on Ω, such that

$$-\mathrm{Div}\, z = \lambda s \qquad \text{a.e. on } \Omega\,. \tag{4.14}$$

Notice that this relation is a generalization of the formal eigenvalue problem (1.1) for the 1-Laplace operator

$$-\mathrm{Div}\, \frac{Du}{|Du|} = \lambda \frac{u}{|u|}$$

where z and s replace the possibly undetermined expressions $Du/|Du|$ and $u/|u|$, respectively. We consider (4.12) as *eigenvalue problem* for the 1-Laplace operator and call λ *eigenvalue* and u *eigensolution.* Occasionally also the unconstrained problem

$$\int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} - \lambda \int_\Omega |u|\, dx \;\; \to \;\; \text{Min!}\,, \quad u \in BV(\Omega)\,,$$

is studied for the smallest eigenvalue λ. The functional considered here is obviously nonnegative and all eigensolutions u corresponding to λ are minimizers. Notice that Corollary 4.9 again implies (4.12) as necessary condition for solutions of this problem. Incidently the dependence of λ on Ω was recently studied in [24].

Unfortunately Theorem 4.6 does not provide further information about the selection s entering (4.14). Thus the result of Demengel [15] for nonnegative minimizers u, that (4.14) always has to be satisfied for the special selection $s \equiv 1$, appeared to be much more precise. But the situation was not completely clear, since we also constructed some explicit solution in a special case with some selection $s \not\equiv 1$. The following result, that is not restricted to the eigenvalue problem, clarifies this situation by stating that for *any* measurable selection s there is some z such that (4.14) is satisfied.

It turns out that the necessary condition (4.8) can be significantly sharpened in the case $\alpha \geq 0$ as long as $q = 1$ and g, h are small.

Theorem 4.15 *Let $u \in BV(\Omega) \cap \mathcal{L}^p(\Omega)$ be a minimizer of* (4.2), (4.3) *with E as in* (4.1) *and let* (4.4), (4.5) *be satisfied with $\alpha \geq 0$, $q = 1$, $\alpha \|g\|^r_{\mathcal{L}^r} < E(u)$, $\|h\|_{\mathcal{L}^1} < 1$. Then for any measurable selection $s(x) \in \operatorname{Sgn}(u(x) - h(x))$ a.e. on Ω there is some $\lambda > 0$ and some $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ with*

$$\|z\|_{\mathcal{L}^\infty} = 1\,, \quad \operatorname{Div} z \in \mathcal{L}^{p'}(\Omega)\,, \quad \int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} = -\int_\Omega u \operatorname{Div} z\, dx\,, \tag{4.16}$$

such that,

(i) *if $r = 1$, then*

$$\lambda s \;=\; -\operatorname{Div} z - f + \alpha\, \tilde{s}(x) \quad \text{a.e. on } \Omega, \tag{4.17}$$

for some suitable measurable selection $\tilde{s}(x) \in \operatorname{Sgn}(u(x) - g(x))$ a.e. on Ω.

(ii) *if $r > 1$, then we have* (4.17) *with $|u - g|^{r-2}(u - g)$ instead of $\tilde{s}$.*

(iii) *If $g = h = 0$ and $r = 1$, then always $\lambda = E(u)$.*

Notice that, in contrast to (4.17), condition (4.8) provides only one equation for a suitable selection $s(x)$ of the set-valued map $\operatorname{Sgn}(u(x) - h(x))$ as necessary condition for a minimizer u. If we take into account that minimizers u may be zero on a set with positive measure (cf. the examples in the previous section), then the previous result has a new quality by saying that infinitely many equations have to be satisfied. Formally one could derive a similar result for $1 < q \leq p$. However, since the convex subdifferential of the function in the side condition (4.3) is a singelton in this case, we would not get more than in Theorem 4.6 (cf. Proposition 4.23 below).

Obviously the previous theorem covers the eigenvalue problem and we obtain the next corollary as a special case.

Corollary 4.18 *Let $u \in BV(\Omega)$ be a minimizer of the eigenvalue problem* (3.6), (3.7)*. Then for any measurable selection $s(x) \in \operatorname{Sgn}(u(x))$ a.e. on Ω there is some $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ with*

$$\|z\|_{\mathcal{L}^\infty} = 1\,, \quad \operatorname{Div} z \in \mathcal{L}^n(\Omega)\,, \quad \int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} = -\int_\Omega u \operatorname{Div} z\, dx\,, \tag{4.19}$$

such that for $\lambda := E(u)$

$$-\operatorname{Div} z = \lambda\, s \quad \text{a.e. on } \Omega. \tag{4.20}$$

In the next section we provide some simple examples showing how the vector fields z can be constructed for different selections s.

The derivation of a necessary condition for minimizers of a constrained problem like (4.2), (4.3) usually uses a Lagrange multiplier rule. A fundamental difficulty now is the lack in differentiability of the energy and also of the functional in the side condition for $q = 1$. Since the energy function E is convex, a necessary condition for a minimizer u should employ the convex subdifferential $\partial E(u)$. If we want to characterize that subdifferential, which is a subset of the dual space

$(BV(\Omega) \cap \mathcal{L}^p(\Omega))^*$, we are confronted with the further difficulty that almost nothing is known about the structure of the space $BV(\Omega)^*$ (cf. Ambrosio, Fusco & Pallara [2]). Therefore we are not able to evaluate the structure of the elements in $\partial E(u)$ as elements of $(BV(\Omega) \cap \mathcal{L}^p(\Omega))^*$. But it turns out that the structure of the elements of $\partial E(u)$ can be derived if $BV(\Omega) \cap \mathcal{L}^p(\Omega)$ is considered as a subspace of $\mathcal{L}^p(\Omega)$ and if E is extended on $\mathcal{L}^p(\Omega)$ in a trivial way, cf. Andreu-Vaillo, Casseles & Mazón [4] and references therein. These methods would suffice to derive an equation for minimizers of (4.2), but we still have to handle the side condition (4.3). Since (4.3) destroys the convexity of the problem, we have to realize that convex analysis is insufficient for our needs. Moereover the nonsmooth calculus of Clarke's generalized gradients, that is mostly worked out for locally Lipschitz continuous functionals, lacks the generality necessary for our problem (cf. Clarke [12]). We will employ some nonsmooth Lagrange multiplier rule that is based on Degiovanni's weak slope and that can handle nonconvex problems with a merely lower semicontinuous energy (cf. Appendix).

As preparation for the proof of Theorem 4.6 we reformulate problem (4.2), (4.3) as an equivalent problem on the space

$$X := \mathcal{L}^p(\Omega)\,.$$

For this reason we define functionals on X by

$$E_1(u) := \begin{cases} \int_\Omega d|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\, d\mathcal{H}^{n-1} & \text{for } u \in BV(\Omega) \cap \mathcal{L}^p(\Omega)\,, \\ \infty & \text{for } u \in \mathcal{L}^p(\Omega) \setminus BV(\Omega)\,, \end{cases} \tag{4.21}$$

$$E_2(u) := -\int_\Omega f u\, dx\,, \quad E_3(u) := \alpha \int_\Omega |u-g|^r\, dx \quad \text{for } u \in \mathcal{L}^p(\Omega)\,,$$

$$G(u) := \int_\Omega |u-h|^q\, dx \quad \text{for } u \in \mathcal{L}^p(\Omega)\,.$$

Notice that all functionals are well-defined on X as long as (4.4), (4.5) are satisfied. Without danger of confusion we identify the energy E with its extension

$$E(u) := E_1(u) + E_2(u) + E_3(u) \quad \text{for all } u \in X\,. \tag{4.22}$$

Then a minimizer $u \in BV(\Omega) \cap \mathcal{L}^p(\Omega)$ of (4.2), (4.3) is also a minimizer of the modified problem

$$E(u) \;\to\; \text{Min!}\,, \quad u \in X\,, \quad G(u) = 1\,.$$

In order to apply the nonsmooth Lagrange multiplier rule stated in Proposition 6.3 in the Appendix we have to determine the needed derivatives and subdifferentials.

Proposition 4.23 *Let* (4.4), (4.5) *be satisfied. Then:*

(1) *The functional E_1 is convex, lower semicontinuous, and positively homogeneous of degree 1 on X. Moreover $u^* \in \partial E_1(u)$ for $u \in X$ if and only if there is some $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ with*

$$\|z\|_{\mathcal{L}^\infty} \le 1\,, \quad u^* = -\mathrm{Div}\, z \in X^* = \mathcal{L}^{p'}(\Omega)\,, \tag{4.24}$$

$$E_1(u) = \langle u^*, u\rangle = -\int_\Omega u \mathrm{Div}\, z\, dx\,. \tag{4.25}$$

If $E_1(u) > 0$, then $\|z\|_{\mathcal{L}^\infty} = 1$ in (4.24).

(2) *The functional E_2 is continuously differentiable on X with*

$$E_2'(u) = -f \quad \textit{for all } u \in X\,.$$

(3) *For $q > 1$ the functional G is convex, locally Lipschitz continuous, and Gâteaux differentiable on X with*

$$G'(u) = |u-h|^{q-2}(u-h) \quad \textit{for all } u \in X\,.$$

The convex subdifferential is given by

$$\partial G(u) = \{|u-h|^{q-2}(u-h)\} \quad \textit{for all } u \in X\,. \tag{4.26}$$

(4) *For $q = 1$ the functional G is convex and Lipschitz continuous on X. Moreover we have that $u^* \in \partial G(u) \subset \mathcal{L}^{p'}(\Omega)$ for $u \in X$ if and only if*

$$u^*(x) \in \operatorname{Sgn}(u(x) - h(x)) \quad \textit{a.e. on } \Omega\,. \tag{4.27}$$

Notice that the treatment of E_3 is covered by (3) and (4) if we take into account that $\partial\alpha G(u) = \alpha\partial G(u)$ for all $\alpha \in \mathbb{R}$ (cf. Clarke [12]). A result similar to (1) can be found in [4].

Proof. We start with the verification of (1). We define

$$M^* := \{v^* \in \mathcal{L}^{p'}(\Omega) |\, v^* = -\operatorname{Div} z \ \text{ for some } \ z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n),\ \|z\|_{\mathcal{L}^\infty(\Omega)} \le 1\}$$

and, to show that M^* is closed, we consider a sequence $\{v_n^*\} \subset M^*$ with $v_n^* \to v^*$ in $\mathcal{L}^{p'}(\Omega)$. We find $z_n \in \mathcal{L}^\infty(\Omega)$ such that $\|z_n\|_{\mathcal{L}^\infty(\Omega)} \le 1$ and $v_n^* = -\operatorname{Div} z_n$. Hence

$$\int_\Omega v_n^* \varphi\, dx = \int_\Omega z_n D\varphi\, dx \quad \text{for all } \ \varphi \in \mathcal{C}_0^\infty(\Omega),\ \ n \in \mathbb{N}\,. \tag{4.28}$$

Since the sequence $\{z_n\}$ is bounded in $\mathcal{L}^\infty(\Omega)$, we have that

$$z_n \overset{*}{\rightharpoonup} z \quad \text{in } \ \mathcal{L}^\infty(\Omega)$$

at least for a subsequence. Taking the limit in (4.28) we get that

$$\int_\Omega v^* \varphi\, dx = \int_\Omega z D\varphi\, dx \quad \text{for all } \ \varphi \in \mathcal{C}_0^\infty(\Omega)\,,$$

i.e., $v^* = -\operatorname{Div} z \in \mathcal{L}^{p'}(\Omega)$. By

$$\|z\|_{\mathcal{L}^\infty} \le \liminf_{n\to\infty} \|z_n\|_{\mathcal{L}^\infty} \le 1$$

we obtain that $v^* \in M^*$, i.e., M^* is closed.

The conjugate function of the indicator function I_{M^*} of M^* is given by

$$I_{M^*}^*(v) = \sup_{v^* \in \mathcal{L}^{p'}(\Omega)} (\langle v^*, v\rangle - I_{M^*}(v^*)) = \sup_{v^* \in M^*} \langle v^*, v\rangle \quad \text{for all } \ v \in \mathcal{L}^p(\Omega)\,.$$

Using Proposition 6.6 and Proposition 6.12 from the Appendix, we can estimate for any $v^* \in M^*$, $v \in \mathcal{L}^p(\Omega)$

$$\begin{aligned}\langle v^*, v\rangle &= \int_\Omega v^* v\, dx = -\int_\Omega v \mathrm{Div}\, z\, dx \\ &= \int_\Omega d(z, Dv) - \int_{\partial\Omega} [z,\nu] v^{\partial\Omega}\, d\mathcal{H}^{n-1} \\ &\le \|z\|_{\mathcal{L}^\infty(\Omega)} \Big(\int_\Omega d|Dv| + \int_{\partial\Omega} |v^{\partial\Omega}|\, d\mathcal{H}^{n-1} \Big) \\ &\le E_1(v)\,. \end{aligned} \tag{4.29}$$

Hence

$$I^*_{M^*}(v) \le E_1(v) \quad \text{for all } \ v \in \mathcal{L}^p(\Omega)\,.$$

Now, for $v \in \mathcal{L}^p(\Omega)$ we define $\bar{v}$ according to (2.5) and, using the definition of the total variation of $\bar{v}$ in B, we get that

$$\begin{aligned} E_1(v) &= \int_B d|D\bar{v}| \\ &= \sup\Big\{ \int_B \bar{v}\mathrm{Div}\, z\, dx \,\Big|\, z \in \mathcal{C}_0^\infty(B, \mathbb{R}^n),\ \|z\|_{\mathcal{L}^\infty(B)} \le 1 \Big\} \\ &= \sup\Big\{ \int_\Omega v\mathrm{Div}\, z\, dx \,\Big|\, z \in \mathcal{C}_0^\infty(B, \mathbb{R}^n),\ \|z\|_{\mathcal{L}^\infty(B)} \le 1 \Big\} \\ &\le \sup\Big\{ \int_\Omega v\mathrm{Div}\, z\, dx \,\Big|\, z \in \mathcal{L}^\infty(\Omega),\ \|z\|_{\mathcal{L}^\infty(\Omega)} \le 1,\ \mathrm{Div}\, z \in \mathcal{L}^{p'}(\Omega) \Big\} \\ &= \sup\Big\{ \int_\Omega v^* v\, dx \,\Big|\, v^* \in M \Big\} \;=\; I^*_{M^*}(v) \quad \text{for all } \ v \in \mathcal{L}^p(\Omega)\,. \end{aligned}$$

We conclude that

$$I^*_{M^*}(v) = E_1(v) \quad \text{for all } \ v \in \mathcal{L}^p(\Omega)\,.$$

Since M^* is closed and convex, I_{M^*} is convex and lower semicontinuous. Therefore,

$$I_{M^*} = (I^*_{M^*})^* = E_1^*$$

(cf. [18, Prop. 3.1, 4.1]). Consequently, $v^* \in \partial E_1(v)$ if and only if

$$E_1(v) + E_1^*(v^*) = E_1(v) + I_{M^*} = \langle v^*, v\rangle$$

(cf. [18, Prop. 5.1]). We readily conclude that

$$v^* \in \partial E_1(v) \quad \text{if and only if} \quad E_1(v) = \langle v^*, v\rangle\,, \ \ v^* \in M^*\,.$$

In particular, $\partial E_1(0) = M^*$ and, by the estimates yielding (4.29), $\|z\|_{\mathcal{L}^\infty(\Omega)} = 1$ for $v \ne 0$. But this verifies (1).

For statement (2) we observe that E_2 is a linear continuous functional on X. Then the assertion follows easily.

Let us now show (3). The functional G is obviously convex. Since $1 < q \le p$, it is also locally Lipschitz continuous on X. Moreover straightforward arguments yield that G is Gâteaux differentiable with

$$E_2'(u) = |u-h|^{q-2}(u-h) \quad \text{for all} \quad u \in \mathcal{L}^p(\Omega)\,.$$

Thus the convex subdifferential is given as in (4.26).

In assertion (4) we have that $q = 1$ and G is convex also in this case. The continuous imbedding of $\mathcal{L}^p(\Omega)$ into $\mathcal{L}^1(\Omega)$ implies that G is Lipschitz continuous on X. For the characterization of the convex subdifferential we recall that $u^* \in \partial G(u)$ if and only if

$$\int_\Omega u^*(v-u)\,dx \le \int_\Omega |v-h|\,dx - \int_\Omega |u-h|\,dx \quad \text{for all } \; v \in X\,. \tag{4.30}$$

Since

$$\int_\Omega u^*(v-u)\,dx = \int_\Omega u^*((v-h)-(u-h))\,dx\,,$$

we readily see that $u^* \in \partial G(u)$ as long as (4.27) is satisfied. Thus it remains to show the opposite. Let us assume that $u^* \in \partial G(u)$, i.e., (4.30) is satisfied. If we choose

$$v = \begin{cases} 2u-h & \text{on } \tilde\Omega\,, \\ u & \text{on } \Omega\setminus\tilde\Omega\,, \end{cases} \qquad \text{and} \qquad v = \begin{cases} h & \text{on } \tilde\Omega\,, \\ u & \text{on } \Omega\setminus\tilde\Omega\,, \end{cases}$$

for any open $\tilde\Omega \subset \Omega$ in (4.30), then we get that

$$\int_{\tilde\Omega} u^*(u-h)\,dx = \int_{\tilde\Omega} |u-h|\,dx \; \text{ for all open } \; \tilde\Omega\subset\Omega\,.$$

Consequently

$$u^*(u-h) = |u-h| \quad \text{a.e. on } \; \Omega \tag{4.31}$$

and, by (4.30),

$$\int_\Omega u^*(v-h)\,dx \le \int_\Omega |v-h|\,dx \quad \text{for all } \; v\in X\,.$$

The last condition implies that $|u^*| \le 1$ a.e. on Ω. Hence we derive (4.27) from (4.31) which completes the proof. $\diamondsuit$

Proof of Theorem 4.6. Let $u \in BV(\Omega)\cap\mathcal{L}^p(\Omega)$ be a minimizer of (4.2), (4.3). Then u is also a minimizer of the extended problem

$$E(v) \;\; \to \;\; \text{Min!}\,, \quad v\in X\,,$$
$$G(v) = \int_\Omega |v-h|^q\,dx = 1$$

with E according to (4.22). As in the proof of Theorem 3.2 we find an admissible v with $E(v) < \infty$ and, thus, $E(u) < \infty$. Therefore $E(u_\pm) < \infty$ for $u_+ := 2u-h$, $u_- := h$, since $h \in BV(\Omega)\cap\mathcal{L}^p(\Omega)$. Using the characterization of $\partial G(u)$ according to Proposition 4.23 we obtain that

$$G'(u;u-u_+) = G'(u;u_- - u) = G'(u;h-u) = \max_{u^*\in\partial G(u)} -\langle u^*, u-h\rangle = -G(u) = -1$$

where $G'(u;v)$ denotes the directional derivative of G at u in direction v. By Proposition 4.23 we can apply the Lagrange multiplier rule stated in Proposition 6.3 in the Appendix. If we still use the sum rule for generalized gradients saying that $\partial(E_2+E_3)(u) = E_2'(u) + \partial E_3(u)$ (cf. [12]), then we obtain the existence of $\tilde\lambda\in\mathbb{R}$, $u_1^* \in \partial E_1(u)$, $u_3^*\in\partial E_3(u)$, $u_G^* \in \partial G(u)$ such that

$$u_1^* - f + u_3^* + \tilde\lambda u_G^* = 0\,. \tag{4.32}$$

Now assertions (i), (ii) are direct consequences of Proposition 4.23 with $\lambda := -\tilde{\lambda}$.

If $\alpha = 0$ and $h = 0$, then (4.32) and Proposition 4.23 imply that

$$0 = \langle u_1^*, u\rangle - \langle f, u\rangle - \lambda\langle u_G^*, u\rangle = E(u) - \lambda G(u) = E(u) - \lambda\,.$$

This verifies the first part of (iii). ♢

Proof of Corollary 4.9. We argue as in the previous proof where we omit the steps concerning the functional G and we apply the second part of Proposition 6.3. ♢

Proof of Theorem 4.15. As in the previous proof of Theorem 4.6 we extend the problem on $X = \mathcal{L}^p(\Omega)$ and use the notation E and G as before. Obviously u is also a minimizer of this extended problem which we want to treat with Proposition 6.4.

Obviously E is convex and G is convex and continuous on X. With $\tilde{u} := -u$ we have that

$$E(u+\tilde{u}) = E(0) = \alpha\|g\|_{\mathcal{L}^r}^r < E(u)\,, \quad G(u+\tilde{u}) = G(0) = \|h\|_{\mathcal{L}^1} < 1\,,$$

and, by $u \in \mathcal{L}^r(\Omega)$,

$$E(u-\tilde{u}) = E(2u) < \infty\,.$$

By Proposition 6.4 we obtain that, for any $g^* \in \partial G(u)$ there is $\tilde{\lambda} \geq 0$ and $e^* \in \partial E(u)$ such that $g^* = \tilde{\lambda} e^*$. Obviously $u \neq 0$ and, thus, $g^* \neq 0$ by Proposition 4.23. Hence

$$\lambda g^* = e^* \quad \text{for} \quad \lambda := 1/\tilde{\lambda} > 0\,.$$

By the continuity of E_2, E_3 (cf. (4.22)) we have that

$$\partial E(u) = \partial E_1(u) + \partial E_2(u) + \partial E_3(u)\,.$$

Therefore

$$e^* = e_1^* + e_2^* + e_3^* \quad \text{for suitable} \quad e_i^* \in \partial E_i(u)\,, \;\; i = 1,2,3\,.$$

Taking into account the structure of the convex subdifferentials according to Proposition 4.23 and the arbitrariness of g^*, we obtain the assertions (i) and (ii). For (iii) we multiply (4.17) with u and integrate it to obtain that $\lambda = E(u)$ for any choice of g^*. ♢

5 Geometric Interpretation

In this section we give geometric interpretations and explicit constructions of the vector field z occuring in Theorem 4.6 and Theorem 4.15 for the special case of the eigenvalue problem (3.6), (3.7), i.e., we choose f, h, α equal to zero and $q = 1$ in (4.2), (4.3).

Let us first consider the case of a square Ω. It is known that a suitable multiple of the characteristic function $u = \chi_C$ of the Cheeger set C of Ω is a minimizer of the eigenvalue problem with $\lambda = |\partial C|/|C|$ (cf. Figure 1). Notice that the curved part of the boundary ∂C has curvature

λ. Inside C the function u is positive and, thus, we have to find a corresponding vector field $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ satisfying (4.13) and

$$-\operatorname{Div} z = \lambda \qquad \text{a.e. on } C\,.$$

It turns out that there is a classical solution w of the precribed mean curvature equation

$$-\operatorname{div}\left(\frac{Dw}{\sqrt{1+|Dw|^2}}\right) = \lambda \qquad \text{in } C \tag{5.1}$$

which satisfies the boundary condition $\partial w/\partial \nu = -\infty$ on ∂C, see [21]. Here ν is the normal vector of C, pointing outward. Notice that the flux $\frac{Dw}{\sqrt{1+|Dw|^2}}$ equals $-\nu$ on ∂C. If we set

$$z(x) = \begin{cases} \frac{Dw}{\sqrt{1+|Dw|^2}} & \text{if } x \in C, \\ -Dd_C(x) & \text{if } x \in (\Omega \setminus C) \end{cases} \tag{5.2}$$

(where $Dd_C(x)$ denotes the gradient of the distance function d_C), then $z : \Omega \to B_1 \subset \mathbb{R}^2$ is bounded in $L^\infty(\Omega)$ by 1 and satisfies $-\operatorname{Div} z = \lambda$ in C by (5.1) and $-\operatorname{Div} z \in (0, \lambda)$ in the complement of $\overline{C}$, because the curvature of level sets of $d_C(x)$ does not exceed λ. Thus (4.12) holds, i.e. $-\operatorname{Div} z \in \lambda \operatorname{Sgn} u$ a.e. on Ω. Moreover, z is even continuous in Ω, in particular across $\partial C \cap \Omega$.

We should point out that our construction does not yield a vector field for which

$$-\operatorname{Div} z = \lambda \quad \text{in } \mathcal{D}'(\Omega), \tag{5.3}$$

but the existence of such a field was derived by Demengel as a necessary condition in [15]. This equation does not contradict (4.12), since $s(x) = 1 \in \operatorname{Sgn}(u(x))$ almost everywhere on Ω. Moreover (5.3) also follows from the stronger assertion in Theorem 4.15. What about an explicit solution for the more specific equation (5.3)? It can be constructed by modifying z from (5.2) on $\Omega \setminus C$ as follows. By shifting the circular arcs which form $\partial C \cap \Omega$ diagonally outward, each point in $\Omega \setminus C$ lies on a circular arc of radius $1/\lambda$. The exterior normal field $\nu(x)$ to this foliation satisfies $\operatorname{Div} \nu = \lambda$. Therefore

$$\breve{z}(x) := \begin{cases} \frac{Dw}{\sqrt{1+|Dw|^2}} & \text{if } x \in C, \\ -\nu(x) & \text{if } x \in (\Omega \setminus C), \end{cases} \tag{5.4}$$

constitutes indeed an explicit (and continuous) solution of (5.3), if Ω is the square from Figure 1. Both z and $\tilde{z}$ are associated with the same eigenfunction $u(x) = \chi_C(x)$.

Let us now turn to a more complicated situation and assume that u is a minimizer for the nonconvex domain in Figure 2 and u is positive in C_1 and negative in C_2. If for $i = 1, 2$ the function w_i denotes the solution of (5.1) in C_i, we may set

$$\tilde{z}(x) = \begin{cases} \frac{Dw_1}{\sqrt{1+|Dw_|1^2}} & \text{if } x \in C_1, \\ \frac{-Dw_2}{\sqrt{1+|Dw_2|^2}} & \text{if } x \in C_2, \\ -Dd_{C_1}(x) & \text{if } x \in (\Omega \setminus (C_1 \cup C_2)) \text{ and } d_{C_1}(x) < d_{C_2}(x), \\ Dd_{C_2}(x) & \text{if } x \in (\Omega \setminus (C_1 \cup C_2)) \text{ and } d_{C_1}(x) \geq d_{C_2}(x), \end{cases} \tag{5.5}$$

then again $\tilde{z}$ satisfies (4.8), except on the line segment that cuts the domain vertically into two halfs. Notice that $\tilde{z} = -\nu$ on ∂C_1, while $\tilde{z} = \nu$ on ∂C_2 and that $\tilde{z}$ is now discontinous on the set where $d_{C_1}(x) = d_{C_2}(x)$. However, only the vertical component $\tilde{z}_2(x)$ of $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$ is discontinuous there. This problen can be overcome as follows. Suppose the barbell domain from Figure 2 is symmetric with respect to $\{x_1 = 0\}$. Let $\eta(x_1)$ be a smooth function that vanishes near zero, is one outside a neighbourhood of zero and satisfies $0 \le \eta \le 1$ on $\mathbb{R}$. Now set $z_1(x) = \tilde{z}_1(x)$ and $z_2(x) = \eta(x_1)\tilde{z}_2(x)$ with $\tilde{z}_i$ defined by (5.5). Then $z(x)$ is continuous and, if we observe that $\frac{\partial \tilde{z}_1}{\partial x_1}$ and $\frac{\partial \tilde{z}_2}{\partial x_2}$ are both negative (resp. positive) in the left (resp. right) half-plane, we realize that

$$-\mathrm{Div}\, z(x) = -\frac{\partial \tilde{z}_1}{\partial x_1}(x) - \eta(x_1)\frac{\partial \tilde{z}_2}{\partial x_2}(x) \in \lambda\ \mathrm{Sgn}\, u \quad \text{a.e. on } \Omega \tag{5.6}$$

and, thus, (4.12) holds as desired.

6 Appendix

6.1 Results from nonsmooth analysis

Let X be a real Banach space, let X^* be its dual space, and let $\langle \cdot, \cdot \rangle$ be the corresponding duality form. Moreover let $F_1 : X \mapsto \mathbb{R} \cup \{\infty\}$ be lower semicontinuous and convex, and let $F_2, G : X \mapsto \mathbb{R}$ be locally Lipschitz continuous. We consider the minimization problem

$$F(u) := F_1(u) + F_2(u) \quad \to \quad \text{Min!}\,, \qquad u \in X\,, \tag{6.1}$$

$$G(u) = 0\,. \tag{6.2}$$

The functions F, G are not necessarily differentiable and neither convex analysis nor Clarke's calculus of generalized gradients provide a Lagrange multiplier rule applicable to this problem. But notice that a minimizer u of (6.1), (6.2) is also an unconstrained minimizer of the function $F + I$ on X with the indicator function

$$I(v) := \begin{cases} 0 & \text{if } G(v) = 0\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

As a simple consequence of its definition, Degiovanni's weak slope $|d(F+I)|(u)$ equals zero for a minimizer u and, thus, u is a critical point of the function $F + I$ (cf. Degiovanni [13] for details about the weak slope). This fact allows the derivation of a nonsmooth Lagrange multiplier rule for problem (6.1), (6.2). The first part of the next proposition provides a specialization of Degiovanni & Schuricht [14, Corollary 3.6]. For the proof of the second part we have to use that the weak slope $|dF|(u)$ is zero for a minimzer u of (6.1) and we have to adapt the proof of [14, Theorem 3.5] to this case (cf. also [14, Remark 3.2]). By $G^0(u; v)$ and $\partial G(u)$ we denote Clarke's generalized directional derivative and Clarke's generalized gradient for the locally Lipschitz continuous function G, respectively. If G is convex these notions coincide with the usual directional derivative and with the convex subdifferential, respectively (cf. Clarke [12]).

Proposition 6.3

(1) *Let* $u \in X$ *be a minimizer of* (6.1), (6.2) *with* $F(u) < \infty$ *and assume that there exist* $u_\pm \in X$ *with* $F(u_\pm) < \infty$ *such that*

$$G^0(u; u_- - u) < 0\,, \quad G^0(u; u - u_+) < 0\,.$$

Then $\partial F_1(u) \neq \emptyset$ *and there exist* $\lambda \in \mathbb{R}$, $f_1^* \in \partial F_1(u)$, $f_2^* \in \partial F_2(u)$, $g^* \in \partial G(u)$ *such that*

$$f_1^* + f_2^* + \lambda g^* = 0\,.$$

(2) *Let* $u \in X$ *be a minimizer of* (6.1) *with* $F(u) < \infty$. *Then* $\partial F_1(u) \neq \emptyset$ *and there exist* $f_1^* \in \partial F_1(u)$, $f_2^* \in \partial F_2(u)$ *such that*

$$f_1^* + f_2^* = 0\,.$$

In the case of convex functionals, i.e., if also F_2 and G are convex, the second assertion is trivial and the first one can be derived much easier by tools of convex analysis (e.g. similar to the proof of Proposition 6.4 below). The next proposition shows that the necessary condition for a minimizer can be essentially sharpened in special situations.

Proposition 6.4 *Let* X *be a Banach space, let* $F : X \mapsto \mathbb{R} \cup \{\infty\}$ *be convex, let* $G : X \mapsto \mathbb{R}$ *be convex and continuous, and let*

$$E(u) = \min_{G(v)=0} E(v) < \infty\,.$$

Moreover, let there exist $\tilde{u} \in X$ *such that*

$$E(u + \tilde{u}) < E(u)\,, \;\; G(u + \tilde{u}) < 0\,, \;\; E(u - \tilde{u}) < \infty\,. \tag{6.5}$$

Then

$$\partial G(u) \;\subset\; \bigcup_{t \geq 0} t \partial E(u)\,.$$

Before we prove this proposition we wish to thank the referee for pointing out a simpler proof in the special situation where u solves

$$E(u) = \min_{G(v)=1} E(v) < \infty$$

for functionals E, G that are convex, lower semicontinuous, and 1-homogeneous with $E(u) > 0$ and with $G(v) > 0$ for $v \neq 0$ (observe that (6.5) is satisfied with $\tilde{u} = -u$ in that case). Indeed, due to homogeneity, the minimality of u implies that $E(\frac{v}{G(v)}) \geq E(u)$ for all $v \neq 0$. Thus, for each $g^* \in \partial G(u)$, $v \in X$,

$$E(v) \geq G(v)E(u) \geq E(u) + E(u)\langle g^*, v - u\rangle\,.$$

Hence $tg^* \in \partial E(u)$ for $t = E(u) > 0$. Note that $\partial G(u)$ might be empty if G is not continuous.

Proof of Proposition 6.4. We define the convex level sets

$$A := \{v \in X |\, E(v) \leq E(u)\}\,, \qquad C := \{v \in X |\, G(v) \leq 0\}\,.$$

We have that

$$A \subset C\,,$$

since $u+\tilde{u} \in A \cap \operatorname{int} C$ and since u minimizes E on the boundary ∂C. We now fix any $u^* \in \partial G(u)$. Since the assertion is trivial for $u^* = 0$, we consider the case $u^* \neq 0$ and set

$$X^- := \{v \in X |\ \langle u^*, v\rangle \leq 0\}\,, \qquad X_0 := \{v \in X |\ \langle u^*, v\rangle = 0\}\,.$$

The definition of the subdifferential readily implies that $C \subset u + X^-$ and $\tilde{u} \in X^- \setminus X_0$. Since $E(v) \geq E(u)$ for all v on the boundary ∂C, we obtain

$$E(v) \geq E(u) \quad \text{for all } \ v \in u + X_0\,.$$

Thus the directional derivative of E satisfies

$$E'(u; v) \geq 0 \quad \text{for all } \ v \in X_0\,.$$

Since $\tilde{u} \in \operatorname{int} X^-$, for all $v, w \in X_0$ we obtain that

$$0 \leq E'(u; v + w) \leq E'(u; v + \tilde{v}) + E'(u; w - \tilde{v})$$

and, thus,

$$\sup_{v \in X_0} -E'(u; v + \tilde{v}) \leq \inf_{w \in X_0} E'(u; w - \tilde{v}) \leq E'(u; -\tilde{v}) < \infty\,.$$

Hence there is some $c \in \mathbb{R}$ with

$$\sup_{v \in X_0} -E'(u; v + \tilde{v}) \leq c \leq \inf_{w \in X_0} E'(u; w - \tilde{v})$$

and we can define a linear functional $\tilde{u}^* \in X^*$ by

$$\langle \tilde{u}^*, v + \tau\tilde{v}\rangle := -\tau c \quad \text{for all } \ v \in X_0, \ \ \tau \in \mathbb{R}\,.$$

For $\tau < 0$, $v \in X_0$ we have that

$$\langle \tilde{u}^*, v + \tau\tilde{v}\rangle = -\tau c \leq -\tau E'(u; -\tfrac{1}{\tau}v - \tilde{v}) = E'(u; v + \tau\tilde{v})$$

and, for $\tau > 0$, $v \in X_0$,

$$\langle \tilde{u}^*, v + \tau\tilde{v}\rangle = -\tau c \leq \tau E'(u; \tfrac{1}{\tau}v + \tilde{v}) = E'(u; v + \tau\tilde{v})\,.$$

Thus $\langle \tilde{u}^*, w\rangle \leq E'(u; w)$ for all $w \in X$ and, therefore, $\tilde{u}^* \in \partial E(u)$. Since u^* and $\tilde{u}^*$ have the same null space X_0, there is $\lambda \in \mathbb{R}$ with $\tilde{u}^* = \lambda u^*$. Since $\tilde{u}^* \neq 0$ by $E(u + \tilde{u}) < E(u)$, we obtain that $\lambda \neq 0$. The arbitrariness of u^* implies the assertion. $\diamondsuit$

6.2 Traces and pairings

Here we summarize some results from Anzellotti [3] as necessary for our analysis (cf. also [4, Appendix]). For the convenience of the reader we present a slightly modified self-contained version with shorter proofs.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let ν be its outward unit normal on $\partial\Omega$. Recall that

$$\mathcal{W}^{1,1}(\Omega) \subset BV(\Omega) \subset \mathcal{L}^p(\Omega) \quad \text{for } 1 \le p \le \tfrac{n}{n-1}\,.$$

By $u^{\partial\Omega}$ we denote the trace of $u \in BV(\Omega)$ or $u \in \mathcal{W}^{1,1}(\Omega)$ that belongs to $\mathcal{L}^1(\partial\Omega)$. Furthermore we set

$$\mathcal{L}^\infty_q(\Omega) := \{z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n) |\, \mathrm{Div}\, z \in \mathcal{L}^q(\Omega)\} \quad \text{for } q \ge 1\,.$$

Proposition 6.6 *For each $z \in \mathcal{L}^\infty_1(\Omega)$ there is a function $[z,\nu] \in \mathcal{L}^\infty(\partial\Omega)$, called* normal trace *of z, such that*

$$\|[z,\nu]\|_{\mathcal{L}^\infty(\partial\Omega)} \le \|z\|_{\mathcal{L}^\infty(\Omega)}\,, \tag{6.7}$$

$$\int_\Omega u \mathrm{Div}\, z\, dx + \int_\Omega z \cdot Du\, dx = \int_{\partial\Omega} [z,\nu] u\, d\mathcal{H}^{n-1} \quad \textit{for all } u \in \mathcal{C}^\infty(\bar\Omega)\,. \tag{6.8}$$

If even $z \in \mathcal{L}^\infty_q(\Omega)$, $1 \le q \le \infty$, then

$$\int_\Omega u \mathrm{Div}\, z\, dx + \int_\Omega z \cdot Du\, dx = \int_{\partial\Omega} [z,\nu] u^{\partial\Omega}\, d\mathcal{H}^{n-1} \quad \textit{for all } u \in \mathcal{W}^{1,1} \cap \mathcal{L}^{q'}(\Omega)\,. \tag{6.9}$$

Proof. For fixed $z \in \mathcal{L}^\infty_1(\Omega)$ we define the linear mapping

$$\alpha(u) := \int_\Omega u \mathrm{Div}\, z\, dx + \int_\Omega z Du\, dx \quad \text{for all } u \in \mathcal{C}^\infty(\bar\Omega)\,. \tag{6.10}$$

Let $\varrho \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ denote the standard mollifier and let $\varrho_m(x) := m^n \varrho(mx)$. Moreover let Ω_ε denote the ε-neighborhood of Ω for some $\varepsilon > 0$ and define the mollifications $z_m := \varrho_m * z$. Then

$$\begin{aligned}
\int_\Omega u(x)\, \mathrm{Div}\, z_m(x)\, dx &= \int_\Omega u(x) \Big(\int_{\Omega_\varepsilon} z(y) \cdot D\varrho_m(x-y)\, dy \Big)\, dx \\
&= \int_\Omega u(x) \Big(\int_{\Omega_\varepsilon} \varrho_m(x-y)\, \mathrm{Div}\, z(y)\, dy \Big)\, dx \\
&= \int_{\Omega_\varepsilon} \mathrm{Div}\, z(y) \Big(\int_\Omega u(x) \varrho_m(x-y)\, dx \Big)\, dy \\
&\overset{m\to\infty}{\to} \int_\Omega u(x)\, \mathrm{Div}\, z(x)\, dx\,.
\end{aligned}$$

for any $u \in \mathcal{C}^\infty(\bar\Omega)$. Consequently,

$$\begin{aligned}
|\alpha(u)| &= \lim_{m\to\infty} \Big| \int_\Omega u \mathrm{Div}\, z_m\, dx + \int_\Omega z_m Du\, dx \Big| = \lim_{m\to\infty} \Big| \int_{\partial\Omega} z_m \cdot \nu\, u\, d\mathcal{H}^{n-1} \Big| \\
&\le \|z\|_{\mathcal{L}^\infty(\Omega)} \|u^{\partial\Omega}\|_{\mathcal{L}^1(\partial\Omega)} \quad \text{for all } u \in \mathcal{C}^\infty(\bar\Omega)\,.
\end{aligned} \tag{6.11}$$

For $u, v \in \mathcal{C}^\infty(\bar\Omega)$ we readily conclude that

$$\alpha(u-v) = 0 \quad \text{if} \quad u^{\partial\Omega} = v^{\partial\Omega} \;\; \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega\,,$$

i.e., $\alpha(u)$ depends only on $u^{\partial\Omega}$. Thus a linear function $\beta : Y \mapsto \mathbb{R}$ on a subspace $Y \subset \mathcal{L}^1(\partial\Omega)$ is defined by

$$\beta(u^{\partial\Omega}) := \alpha(u) \quad \text{for all } \; u \in \mathcal{C}^\infty(\bar{\Omega})\,.$$

Since β is continuous by (6.11), the Hahn-Banach theorem provides a norm preserving extension of β on $\mathcal{L}^1(\partial\Omega)$ that can be represented by a function $[z,\nu] \in \mathcal{L}^\infty(\partial\Omega)$ such that

$$\alpha(u) = \beta(u^{\partial\Omega}) = \int_{\partial\Omega} [z,\nu] u^{\partial\Omega}\, d\mathcal{H}^{n-1} \quad \text{for all } \; u \in \mathcal{C}^\infty(\bar{\Omega})\,.$$

Now the assertions (6.7), (6.8) are immediate consequences of (6.10) and (6.11).

Assume now that $z \in \mathcal{L}^\infty_q(\Omega)$ and $u \in \mathcal{W}^{1,1}(\Omega) \cap \mathcal{L}^{q'}(\Omega)$ for $1 < q \le \infty$. Then we can approximate u by a sequence $u_m \in \mathcal{C}^\infty(\bar{\Omega})$ such that

$$u_m \to u \quad \text{in } \; \mathcal{W}^{1,1}(\Omega)\,, \quad u_m \to u \quad \text{in } \; \mathcal{L}^{q'}(\Omega)$$

(cf. the proof of [19, Theorem 3, p. 252]). Since the trace $u^{\partial\Omega}$ is continuous on $\mathcal{W}^{1,1}(\Omega)$, we obtain (6.9) by taking the limit of (6.8) with u_m. If $q = 1$, then we find a sequence $u_m \in \mathcal{C}^\infty(\bar{\Omega})$ with $u_m \to u$ in $\mathcal{W}^{1,1}(\Omega)$. In addition we can assume that $u_m(x) \to u(x)$ a.e. on Ω. Then we can go to the limit in (6.8) by majorized convergence. $\diamondsuit$

In order to extend the Gauss-Green formula (6.8) or (6.9) to $u \in BV(\Omega)$ we have to clarify the meaning of the second term in this case. As Šilhavý observed, the special case of $\Omega = \mathbb{R}^n$ and $p = 1$ was already covered by Whitney where his cap product agrees with the pairing (z, Du) (cf. Whitney [34], Šilhavý [33]).

Proposition 6.12 *For any $u \in BV(\Omega) \cap \mathcal{L}^p(\Omega)$, $1 < p < \infty$, and $z \in \mathcal{L}^\infty_{p'}(\Omega)$ there exist a Radon measure on Ω denoted by (z, Du) such that*

$$\int_\Omega u \mathrm{Div}\, z\, dx + \int_\Omega d(z, Du) = \int_{\partial\Omega} [z,\nu] u^{\partial\Omega}\, d\mathcal{H}^{n-1} \tag{6.13}$$

and

$$\langle (z, Du), \varphi\rangle = - \int_\Omega u\varphi \mathrm{Div}\, z\, dx - \int_\Omega u z \cdot D\varphi\, dx \quad \textit{for all } \; \varphi \in \mathcal{C}^\infty_0(\Omega)\,. \tag{6.14}$$

The measures (z, Du) and $|(z, Du)|$ are absolutely continuous with repsect to $|Du|$ and, for any open $\tilde{\Omega} \subset \Omega$,

$$|\langle (z, Du), \varphi\rangle| \le \|\varphi\|_\infty \, \|z\|_{\mathcal{L}^\infty(\tilde{\Omega})} \int_{\tilde{\Omega}} d|Du| \quad \textit{for all } \; \varphi \in \mathcal{C}^\infty_0(\tilde{\Omega})\,, \tag{6.15}$$

$$\Big| \int_{\breve{\Omega}} d(z, Du) \Big| \le \int_{\breve{\Omega}} d|(z, Du)| \le \|z\|_{\mathcal{L}^\infty(\tilde{\Omega})} \int_{\breve{\Omega}} d|Du| \quad \textit{for all Borel sets } \; \breve{\Omega} \subset \tilde{\Omega}\,. \tag{6.16}$$

Proof. We fix $u \in BV(\Omega) \cap \mathcal{L}^p(\Omega)$, $z \in \mathcal{L}^\infty_{p'}(\Omega)$ and define a linear mapping $(z, Du) : \mathcal{C}^\infty_0(\Omega) \mapsto \mathbb{R}$ according to (6.14). Moreover let us fix any open set $\tilde{\Omega} \subset \Omega$ and observe that the restriction $u_{|\tilde{\Omega}} \in BV(\tilde{\Omega})$. We can approximate $u_{|\tilde{\Omega}}$ by a sequence $u_m \in \mathcal{C}^\infty(\tilde{\Omega}) \cap BV(\tilde{\Omega})$ such that

$$u_m \to u \quad \text{in } \; \mathcal{L}^1(\tilde{\Omega})\,, \quad |Du_m|(\tilde{\Omega}) \to |Du|(\tilde{\Omega})\,. \tag{6.17}$$

(cf. [20, Theorem 2, p. 172]). Using partial integration we obtain that

$$\begin{aligned}
&\Big|\int_{\tilde\Omega} u_m\varphi \mathrm{Div}\, z\,dx + \int_{\tilde\Omega} u_m z\cdot D\varphi\,dx\Big| \\
&\qquad = \Big|\int_{\tilde\Omega} \varphi z\cdot Du_m\,dx\Big| \;\le\; \|\varphi\|_\infty\,\|z\|_{\mathcal{L}^\infty(\tilde\Omega)}\int_{\tilde\Omega} d|Du_m| \quad \text{for } \varphi\in\mathcal{C}_0^\infty(\tilde\Omega)\,.
\end{aligned}$$

By (6.14), (6.17) the limit in the previous inequality gives (6.15). Consequently we can identify (z,Du) with a Radon measure on Ω. (6.15) implies (6.16) by standard arguments of measure theory and we obtain that (z,Du) and $|(z,Du)|$ are absolutely continuous with respect to $|Du|$.

It remains to show (6.13). For any $\varepsilon>0$ we find an open $\tilde\Omega\subset\subset\Omega$ and $\varphi\in\mathcal{C}_0^\infty(\Omega)$ such that

$$|Du|(\partial\tilde\Omega)=0\,,\quad |Du|(\Omega\setminus\tilde\Omega)<\varepsilon\,,\quad 0\le\varphi(x)\le 1 \text{ on } \Omega\,,\quad \varphi(x)=1 \text{ on } \tilde\Omega\,. \tag{6.18}$$

Consulting the proof of [20, Theorem 2, p. 172] we see that there is a sequence $u_m\in\mathcal{C}^\infty(\Omega)\cap BV(\Omega)\cap\mathcal{L}^p(\Omega)$ approximating $u\in BV(\Omega)\cap\mathcal{L}^p(\Omega)$ such that

$$u_m\to u \quad\text{in } \mathcal{L}^p(\Omega)\,,\quad |Du_m|(\Omega)\to|Du|(\Omega)\,,\quad |Du_m|(\Omega\setminus\tilde\Omega)\le 3|Du|(\Omega\setminus\tilde\Omega)\,. \tag{6.19}$$

Consequently, by (6.16) and (6.19),

$$\begin{aligned}
&\Big|\int_\Omega d(z,Du_m)-\int_\Omega d(z,Du)\Big| \\
&\qquad\le\; |\langle(z,Du_m-Du),\varphi\rangle| + \int_\Omega(1-\varphi)\,d|(z,Du_m)| + \int_\Omega(1-\varphi)\,d|(z,Du)| \\
&\qquad\le\; |\langle(z,Du_m-Du),\varphi\rangle| + 4\|z\|_{\mathcal{L}^\infty(\Omega)}\int_{\Omega\setminus\tilde\Omega} d|Du|\,.
\end{aligned}\tag{6.20}$$

The first term on the right hand side converges to zero. By (6.18) and the arbitrariness of $\varepsilon>0$ we conclude that

$$\lim_{m\to\infty}\int_\Omega z\cdot Du_m\,dx=\int_\Omega d(z,Du)\,.$$

Notice that $u_m\in\mathcal{W}^{1,1}(\Omega)$ and that $u_m^{\partial\Omega}\to u^{\partial\Omega}$ in $\mathcal{L}^1(\partial\Omega)$ by (6.19) (cf. [2, Theorem 3.88]). Thus

$$\begin{aligned}
\int_{\partial\Omega}[z,\nu]u^{\partial\Omega}\,d\mathcal{H}^{n-1} &= \lim_{m\to\infty}\int_{\partial\Omega}[z,\nu]u_m^{\partial\Omega}\,d\mathcal{H}^{n-1} \\
&\overset{(6.9)}{=} \lim_{m\to\infty}\int_\Omega u_m\mathrm{Div}\,z\,dx+\int_\Omega z\cdot Du_m\,dx \\
&= \int_\Omega u\mathrm{Div}\,z\,dx+\int_\Omega d(z,Du)
\end{aligned}\tag{6.21}$$

which verifies (6.13). $\diamondsuit$

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