A new mathematical foundation for contact interactions in continuum physics

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This paper is dedicated to Eberhard Zeidler with gratitude on the occasion of his 65th birthday

Abstract

The investigation of contact interactions, such as traction and heat flux, that are exerted by contiguous bodies across the common boundary is a fundamental issue in continuum physics. However, the traditional theory of stress established by Cauchy and extended by Noll and his successors is insufficient for handling the lack of regularity in continuum physics due to shocks, corner singularities, and fracture. This paper provides a new mathematical foundation for the treatment of contact interactions. Based on mild physically motivated postulates, which differ essentially from those used before, the existence of a corresponding interaction tensor is established. While in earlier treatments contact interactions are basically defined on surfaces, here contact interactions are rigorously considered as maps on pairs of subbodies. This allows the action exerted on a subbody to be defined not only, as usual, for sets with a sufficiently regular boundary, but also for Borel sets (which include all open and all closed sets). In addition to the classical representation of such interactions by means of integrals on smooth surfaces, a general representation using the distributional divergence of the tensor is derived. In the case where concentrations occur, this new approach allows a more precise description of contact phenomena than before.

Contents

1 Introduction

> *It is a widespread belief even today that classical mechanics is a dead subject, that its foundations were made clear long ago, and that all that remains to be done is to solve special problems. This is not so.*
>
> Walter Noll (1959, [37])

In continuum physics the underlying laws have to be satisfied not only for an entire body but also for all of its parts. Typically, these laws account for interactions between contiguous parts exerted across the common boundary involving contact forces, heat flux, entropy flux, and electromagnetic fields. It turns out that all these phenomena have the same nature. Cauchy discovered in 1823 that such *contact interactions* depend linearly on the normal field of the surface on which they act. More precisely, there is a tensor field τ independent of the normal field such that the interaction $f(S)$ between two parts of the body having the common boundary S can be represented by

$$f(S) = \int_S \tau \cdot \nu \, da \tag{1.1}$$

where ν denotes the unit normal to S and a is a suitable surface measure. This famous observation had been fundamental not only for the understanding of contact interactions but also for the whole theory of continuum physics (cf. [8, 9]). Contact phenomena as mentioned above are called contact interactions in our presentation to preserve generality while it is sufficient to consider $f(S)$ just as a surface traction in continuum mechanics or as a heat flux in thermodynamics. For accounts of Cauchy's proof based on his celebrated tetrahedron argument see Antman [2], Ciarlet [14], Gurtin [22], Truesdell [60], and Zeidler [65].

Fifty years ago Noll began his seminal effort to establish a theoretical foundation for continuum physics that is based on simple physically motivated but mathematically precise postulates. His sophisticated characterization of contact phenomena had to resolve two essential questions:

(1) *How can the nature of contact interactions be effectively described with simple physically motivated hypotheses?*

(2) *What is a body and what is an appropriate class of subbodies?*

Since the advantages of Cauchy's powerful result should not be given up, the answers to these questions have to resolve the conflict between simple postulates and the analytical requirements that are necessary for the derivation of the tensorial structure according to (1.1). Here regularity assumptions on the interaction f and on the boundaries of the subbodies, reflecting the development of analytical methods, play a central role. As it is pointed out below, these regularity assumptions have been weakened more and more during the last decades. Nevertheless, we have to realize that the traditional theory of stress established by Cauchy and his successors is insufficient for present needs in modern continuum physics where one has to handle singularities that are naturally present in shocks, corners, contact of bodies, and fracture. To be able to describe these phenomena in further detail we not only have to weaken restrictions of regularity, but we also need a richer structure within the theory. According to Noll the decision under what conditions the tensorial structure of contact interactions can be derived is "perhaps the hardest problem of the mathematical foundations of continuum physics" [40, p. 27]. Here we provide a new approach to this task: We not only propose a completely new set of postulates characterizing contact interactions but we also lay down the foundation for a much richer theory than before, a theory better able to deal with singular behavior in continuum physics.

Foundations of the present theory. Let us call the parts of the entire body under consideration subbodies. It would seem to be natural that an interaction between two subbodies in contact that is exerted across the common boundary should be described by a mapping f defined on a suitable class $\mathcal{S}$ of surfaces. While question (1) is related to the properties of such a mapping f, question (2) is related to the domain on which it should be defined, since $\mathcal{S}$ is understood to be a collection of parts of the boundaries of subbodies. The basic assumption that f should be additive with respect to disjoint decompositions of its argument, i.e., $f(S_1 \cup S_2) = f(S_1) + f(S_2)$ whenever $S_1 \cap S_2 = \emptyset$, is eminently natural by virtue of its simplicity and its agreement with experience. Consequently, it is both conceptually and analytically reasonable to suppose that the collection $\mathcal{B}$ of subbodies should be something like an algebra of sets which always contains finite unions, finite intersections, and the complement of its elements. These simple structural conditions are usually supplemented with requirements on the regularity of f and the boundaries of the elements of $\mathcal{B}$. During the past half century the regularity assumptions have been weakened more and more. A further substantial structural condition is contained only in Cauchy's famous theorem itself with the additional postulate that contact forces in continuum mechanics should depend on the shape of a surface merely through its normal field. While this condition caused some speculations whether a realistic theory should take into account further properties of the shape of the surfaces, in 1959 Noll [37, Theorem IV] derived Cauchy's postulate essentially as a consequence of the balance of linear momentum.

The regularity assumptions imposed on the mapping f concern essentially the representation or estimate of $f(S)$, $S \in \mathcal{S}$, by means of surface densities (possibly combined with smoothness requirements) and suitable balance laws. This should ensure the existence of a tensor field τ such that the interaction $f(S)$ between two subbodies having the common boundary $S \in \mathcal{S}$ can be represented as in (1.1). The desire to have such a formula enforces the limitation of the domain of f to sufficiently regular surfaces S that possess a normal field and, thus, $\mathcal{B}$ should contain only subbodies with a correspondingly regular boundary.

The selection of a suitable class of subbodies raises an apparent conceptual difficulty. At first glance it seems reasonable that a subbody $B \in \mathcal{B}$ should contain its boundary ∂B, since otherwise the common surface S of two contiguous bodies, which appears to be *the* fundamental object for the contact interaction, just does not exist. On the other hand, we have already argued that $\mathcal{B}$ should be something like an algebra of sets and, as complements of "closed" subbodies, we would get "open" subbodies. Then we had to decide whether "open" subbodies with common boundaries can exert a contact interaction on each other and whether it might be the same as that for the corresponding "closed" subbodies. If we say that it does not matter whether the boundary belongs to B or not, then we have to clarify subsequent questions as, e.g., whether the union of two "open" subbodies is the same as the union of the corresponding "closed" subbodies. Notice that these difficulties are manifestations of the tacitly assumed symmetry that the role of the two touching subbodies should be interchangeable. Furthermore, they imply that the class $\mathcal{B}$ of subbodies can merely assumed to be a Boolean algebra (cf. SIKORSKI [53]), which is like an algebra of sets but where, roughly speaking, the union and the intersection are more general operators that do not precisely account for boundary points.

Historical development. Let us now outline the development begun by Noll fifty years ago. While the relevant results were formulated partially in terms of continuum mechanics and of thermodynamics, the essential ideas apply to all cases where contact interactions occur. The original idea of NOLL [37] in 1959 is that subbodies of a given body correspond to sets with a piecewise smooth boundary, that contact forces correspond to vector-valued surface measures on the boundaries of subbodies with bounded integrable surface densities, and that all other forces are so-called body forces corresponding to measures with bounded volume densities. Furthermore, it was assumed that forces should be balanced, i.e., they satisfy a balance law as, e.g., the balance of linear momentum. This implies that the resultant contact force exerted on subbodies by the surrounding material is a measure that is also assumed to possess an integrable volume density.

In 1966 NOLL [38] extended his axiomatic treatment and proposed that a system of subbodies should have the structure of a Boolean algebra for the reasons mentioned above. Certainly a usual algebra of sets as, e.g., the set of all Borel sets (that contains countable unions and intersections of open and closed sets) or the set of all subsets would have this structure. But in view of the conceptual difficulty concerning the boundary of subbodies mentioned above, these algebras had been considered to be unreasonable for the treatment of contact forces. Taking account of the additional analytical desideratum of having a divergence theorem available, he assumed that subbodies correspond to closures of open sets with a piecewise smooth boundary. Unfortunately, such a system is not closed under intersections. Moreover, neither the regular regions of KELLOGG [30] nor the standard domains of WHITNEY [62] have this closedness property, which is needed in the analysis. Since the problem of defining classes of subbodies that are closed under set-theoretic operations could not be solved for many years, a concrete example for a system of subbodies satisfying all of Noll's axioms had long been unavailable (cf. also NOLL [39, 40]).

Based on his specific postulates for subbodies, NOLL [38] also proposed that a force f should be defined on pairs of (disjoint) subbodies where $f(B, A)$ stands for the resultant force exerted on B by A. Intuitively, f should be additive in both arguments at least with respect to finite disjoint decompositions. NOLL [39] called such an f an interaction to cover phenomena such as heat and entropy flux in thermodynamics. As a fundamental hypothesis it was assumed that only two kinds of interactions can occur: contact interactions f_{c} and body interactions f_{b}. Here $f_{\mathrm{c}}(B, A)$ is supposed to depend merely on the common surface S of A, B and to be bounded by a multiple of the surface area of S. For f_{b} the second argument is in fact considered to be a fixed external body E and $f_{\mathrm{b}}(\cdot, E)$ is supposed to be an absolutely continuous volume measure on the whole body. Under these conditions, the representation of a contact force by means of a tensor as in (1.1) could be demonstrated and, analogously, the tensorial structure of the heat and the entropy flux in thermodynamics could be derived by GURTIN & WILLIAMS in [26, 63, 27]. Some continuity requirement on the densities of the traction field that was necessary in the treatment of Cauchy appeared to be artificial and had often been discussed in the literature (cf. NOLL [37, 39]). This continuity condition could be replaced by a much more reasonable integrability condition by Gurtin, Mizel & Williams [25]. A presentation of the development at this stage and discussions of open questions can be found in Noll [40], Gurtin [23], and in the book of Truesdell [60].

It became increasingly clear that the understanding of contact interactions is one of the crucial challenges in a mathematically precise approach to continuum physics. In this light the treatment of Gurtin & Martins [24] in 1976 can be considered as an extraction of the essential ingredients. They circumvented the difficulties with a system of subbodies by the restriction to planar polygonal surfaces and they introduced the notion of a Cauchy flux as an additive mapping f on surfaces such that, with a constant $c > 0$,

$$|f(S)| \leq c\, \mathrm{area}(S)\,, \quad |f(\partial B)| \leq c\, \mathrm{volume}(B) \tag{1.2}$$

for all surfaces S and all subbodies B. In this way the total force $f(S)$ is considered as the basic concept (rather than its density) and the classical representation formula (1.1) could be verified for each Cauchy flux.

The next major step in the development was the introduction of sets of finite perimeter as a system of subbodies by Banfi & Fabrizio [4, 5] in 1979 and Ziemer [66] in 1983. These sets seem to be an optimal choice for the treatment of contact interactions, since they possess a normal almost everywhere on their (measure-theoretic) boundary and, thus, they allow representations as in (1.1) for all surfaces that are parts of the (measure-theoretic) boundaries of sets of finite perimeter. Furthermore Gurtin, Williams & Ziemer [28] showed that the subclass of normalized sets of finite perimeter (i.e., sets that coincide with their measure-theoretic interior) form a Boolean algebra and, therefore, satisfy all of Noll's axioms for subbodies. On the other hand, it turned out in [24] that it is sufficient for the derivation of the tensorial structure of a contact interaction as in (1.1) to consider the quite small class of subbodies having piecewise planar polygonal boundaries. Noll & Virga [42] proposed the class of fit regions for subbodies (bounded regularly open sets with finite perimeter and negligible boundary) which lies somewhere in between the previously mentioned classes. How rich a system of subbodies should be has been a constantly discussed question and, according to Noll & Virga [42, p. 2], it should "... include all that can possibly be imagined by an engineer but exclude those that can be dreamt up *only* by an ingenious mathematician". Certainly there is no objective answer to this question. From A. Einstein we learn that: "Everything should be made as simple as possible, but not simpler." The mathematical analysis tells us that a contact interaction considered on a very small class of subbodies is already sufficient to derive its tensorial structure according to (1.1). But then the representation formula (1.1) can be extended to a very rich class of subbodies, and one might even ask for the largest class allowing such an extension. This strategy of starting with a small class and ending up with a rich class will be adopted in the new approach to be presented in this paper. It seems to be the only way to get additional information about the nature of contact interactions that might be necessary for the treatment of severe singularities in modern continuum physics. Moreover, it allows a subsequent selection of a smaller class of subbodies that is sufficient for special needs. For a discussion of relevant problems we refer to Williams [64].

The papers of Šilhavý [54, 55] in 1985 and 1991 extended the previous results for Cauchy fluxes f satisfying (1.2) to those for which there are $\mathcal{L}^p$-functions g, h with $p \geq 1$, such that f merely satisfies

$$|f(S)| \le \int_S h\, d(\text{area})\,, \quad |f(\partial B)| \le \int_B g\, d(\text{volume})\,. \tag{1.3}$$

It turned out that the tensor fields τ obtained under these conditions are exactly those whose distributional divergences are integrable. This generality means that the tensorial representation in (1.1) might be well-defined only for "almost all surfaces", a fact that already had been observed by Antman & Osborn [3] (cf. also Ziemer [66]). As a byproduct of his new methods, Šilhavý also generalized the results of [3] about the equivalence between an integral balance law and the principle of virtual power (which corresponds to the weak form of the balance equation).

In many contact problems for elastic bodies concentrated forces occur naturally. Since such forces are usually unknowns of the problem, they have to be assumed to correspond to a measure in general (cf. Schuricht [45, 48, 46, 47]). But also in the presence of corners and shocks, such a generality is necessary for a rigorous treatment. According to Noll [37, p. 281] a theory for continuous bodies should allow the treatment of such concentrations and in 1999, Degiovanni, Marzocchi & Musesti [15] were able to extend the theory to this generality by replacing the absolutely continuous volume measure corresponding to g in (1.3) with an arbitrary Radon measure (cf. also the papers of de Botton, Rodnay, and Segev mentioned below). They in particular showed that a Cauchy flux is already determined by its knowledge on almost all planar rectangular surfaces whose edges are parallel to the axes of a basis. However, the interesting surfaces where concentrations occur had to be disregarded as faces of subbodies. It turned out that the tensor fields τ whose distributional divergence is a measure have to be taken into account for representations as in (1.1). A comprehensive investigation of this class of tensor fields can be found in Chen & Frid [10, 11, 12], Chen & Torres [13], and Šilhavý [56]. Here the existence of a normal trace is studied also for surfaces where concentrations may occur and, to some extent, even tensors τ that are measures are taken into account. In a series of papers, Marzocchi & Musesti have refined the results of [15] where the decomposition of general interactions into a body and a contact part (in analogy to [26, 28]) and corresponding structural properties are investigated in [31], thermodynamical aspects are treated in [32], a more general class of bodies is studied in [34], and the treatment of boundary conditions is considered in [35]. For the investigation of concentrated forces in the framework of linear elasticity we refer to the papers of Boussinesq [6], Flamant [19], Kelvin [59], Sternberg & Eubanks [58], Turteltaub & Sternberg [61], and references therein.

An alternative, but possibly less powerful approach, for the derivation of Cauchy's theorem by using variations of the subbodies was presented by Fosdick & Virga [20]. A completely different treatment relying on the principle of virtual power can be found in a series of papers by de Botton, Rodnay, and Segev [49, 51, 52, 50, 44] and by Degiovanni, Marzocchi, and Musesti [33, 16]. While the usual theory of stress is obtained if the powers are assumed to depend on the first derivative of the velocity field, also powers depending on higher-order derivatives of the velocity field and leading to higher-order stresses are considered. However the interpretation of such higher-order stresses as, e.g., edge-force densities, does not yet seem to be completely clarified. Moreover, the theory is developed on general manifolds within a geometric framework in [49, 51, 52, 50, 44]. The definition of subbodies in relation to fracture and to contact of different bodies is investigated by Noll [41].

Present problems. After this overview let us comment on some problems of the theory at the present stage. Normalized sets of finite perimeter seem to be the class that is generally accepted as system for subbodies today. But in the light of our previous discussion about the selection of a suitable class of subbodies, each normalized set in fact represents a collection of subsets differing only by boundary points and we have not yet resolved the questions raised there. This does not matter as long as no concentrations occur. But let us consider a force exerted at a single point of a body, a situation that is covered by the treatment of [15] to the extent that external forces of that kind are taken into account while surfaces subject to such concentrations are disregarded for defining traction (cf. also [56]). If we cut this body into two pieces just through that point, then there is no criterion to decide whether this point belongs to the left or to the right piece, i.e., the point has either to disappear or we have to count it twice. However both possibilities are inconsistent with the additivity of forces (cf. Figure 1).

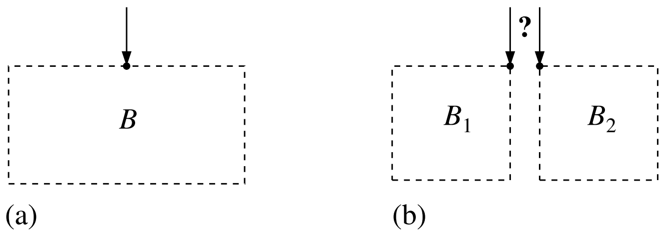

Figure 1: (a) A concentrated force is exerted to a boundary point of the "open" body B. (b) Body B is decomposed into two "open" subbodies B_1, B_2 where the cutting surface "disappears" and one cannot decide on which part the force is exerted.

Hence we should look for a way to "take care" of each point. If we consider the tensor field corresponding to such a singular force, then the divergence has to be a measure with a concentration at the single point. For such a tensor field we can formally compute the traction that is exerted by the surrounding material on small cones with vertices at the points of concentration. Such computations suggest that the material of such cones contributes to the balance of the concentrated force while the contribution depends on the opening angle of the special cone (cf. Podio-Guidugli [43] and Example 2 below). Thus we have to realize that a Cauchy flux defined on surfaces is not capable of describing this situation in detail. For concentrated forces the physical motivation for a condition like the first estimate in (1.3), which is assumed to hold for almost all surfaces, also has to be called into question. Finally, we might ask whether nature "knows" normals, i.e., whether it is natural to define contact interactions only on surfaces that possess a normal field.

God does not care about our mathematical dificulties; He integrates empirically.

Albert Einstein [29]

Ideas of the new approach. Previous treatments of contact interactions had always employed, at least tacitly, the usual concept that a contact interaction should be something that is basically defined on surfaces. Notice that this concept expresses the symmetry that the role of two contiguous interacting bodies should be somehow interchangeable. The essential starting point for our effort to resolve the problems mentioned in the previous paragraph about concentrated contact forces is that we drop this ubiquitous assumption and the corresponding symmetry! We rigorously consider a contact interaction f as defined on pairs of subbodies. In particular we do not assume, as usual, that their intersections contain at most boundary points or that the common "contact surfaces" have to have a normal field. While the disjoint additivity of f in both arguments is natural from the physical point of view, σ-additivity (i.e., countable additivity) would be desirable from the analytical point of view in order to use the powerful machinery of measure and integration theory. However, it turns out that σ-additivity can merely be employed for $f(\cdot, A)$. This manifests the fundamental *asymmetry* of $f(B, A)$ that the role of the subbody A *exerting* an action on B differs from that of the subbody B *resisting* an action from A, i.e., the *action* $f(\cdot, A)$ of A is of a different nature than the *reaction* $f(B, \cdot)$ of B. As a system of subbodies we select the Borel sets or a suitable subalgebra thereof where unions, intersections, and complements are understood in the usual set-theoretic sense. Thus we assign boundary points precisely to subbodies, and the difficulties mentioned above about which of two subbodies in contact owns the common boundary vanish. In the development given here, we have no need for Boolean algebras that are not algebras of sets, and the previous problems appear to be artificial.

The most important question now is to construct a simple and physically natural characterization of contact interactions. Clearly a contact interaction should vanish if the subbodies do not touch (cf. Noll [40]). Furthermore, Noll's "Principle of Local Action" [38, p. 199], formerly called "Principle of Determinism" [36, p. 209], states that the stress at a point should depend only on the response of the material within any arbitrarily small neighborhood of it. While this principle had been formulated as a guiding principle for constitutive laws, we will use it to characterize contact interactions by postulating that the action exerted on a subbody B by a subbody A merely depends on those parts of A outside of B that lie within an arbitrarily small neighborhood of B (cf. Figure 2).

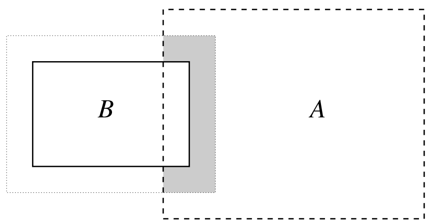

Figure 2: The action exerted on subbody B (solid line) by subbody A (dashed line) merely depends on that material of A which corresponds to the grey region where this region can be chosen to lie within an arbitrarily small (dotted) neighborhood of B.

This simple central condition will only be supplemented by the demand that the material corresponding to a set of measure zero *cannot exert* a nontrivial action and that a nontrivial interaction *can be detected* by the reactions of the surrounding material even if we disregard the material corresponding to a set of measure zero. Both conditions exclude certain singular cases. In particular, we obtain that a single point can resist but cannot exert a nontrivial action, which again expresses the fundamental asymmetry of f. Combined with a natural boundedness condition these quite mild postulates allow the derivation of a corresponding integrable tensor field τ characterizing the contact interaction. For subsets B with a "nice" boundary the action $f(B,A)$ exerted on B by A can be represented in the usual way as in (1.1) as long as no concentration occurs. This classical formula can be extended, e.g., to *all closed* sets B in the sense of a normal trace. But the theory delivers the interaction $f(B,A)$ even for B that are much "worse", since $f(B,A)$ is well-defined for *all* Borel subsets B. For the general case we derive the new fundamental *representation formula*, which can be considered as replacement for Cauchy's classical formula (1.1), that

$$f(B,A) = (\operatorname{div} \tau_A)(B) \qquad \text{with} \quad \tau_A := \begin{cases} \tau \text{ on } A, \\ 0 \text{ otherwise}, \end{cases} \tag{1.4}$$

for all $B \in \mathcal{B}$, $A \in \mathcal{A}$ where $\mathcal{A} \subset \mathcal{B}$ is a suitable subalgebra and where the distributional divergence $\operatorname{div} \tau_A$ has to be understood as a measure. Note that neither surfaces nor normals enter this formula. In the case where a single point resists a nontrivial action, we can exactly identify the contribution of each part of the surrounding material. The foundation for contact interacions in continuum physics presented here differs from earlier theories by giving a more precise and general description that removes certain discrepancies.

In Section 2 different systems of subbodies are introduced, general interactions are defined, and extensions, restrictions, and sums of interactions are discussed. The central notion of a contact interaction is introduced in Section 3 and basic properties are stated. The central assertion that a contact interaction can be represented by means of a tensor field can be found in Section 4. The derived tensor field corresponds to the material or to the spatial description if the underlying contact interaction is related to the reference configuration or to the present state, respectively. The proof rests essentially on measure-theoretic arguments; the major difficulty is to verify some measurability properties that are necessary for the application of disintegration arguments for the construction of the tensor field. Section 5 shows that the distributional divergence of the tensor field and of suitable restrictions of it are measures. This then allows the derivation of not only the usual but also of new representation formulas for $f(B,A)$ for suitable sets B and the equivalence of corresponding balance laws with the principle of virtual power can readily be seen. While in the first sections contact interactions are considered to be defined on a quite small algebra with respect to the second argument, the extension to the much larger algebra of sets of finite perimeter is studied in Section 6. The boundedness condition used for the existence of the corresponding tensor field is discussed in Section 8. Though a slightly weakened boundedness condition ensures a nice mathematical equivalence, both its physical motivation and its physical relevance seem questionable. The Appendix collects basic properties of sets of finite perimeter that are sufficient for understanding the results presented here. Some proofs appearing in the body of this paper, however, use arguments of geometric measure theory going beyond these properties. Moreover, a necessary result about the measurability of real functions is derived.

Notation. For sets A, B let A^c, $A \setminus B$, χ_A, and $\operatorname{dist}_A(x)$ be the complement of A, the points of A not contained in B, the characteristic function of A, and the distance of the point x from A, respectively. $\mathcal{L}^n$ is the n-dimensional Lebesgue outer measure and $\mathcal{H}^k$ is the k-dimensional Hausdorff outer measure on $\mathbb{R}^n$. For $A \subset \mathbb{R}^n$ we write $\operatorname{int} A$, $\operatorname{cl} A$, ∂A for its topological interior, closure, and boundary, respectively, and A_*, $\partial_* A$ for its measure-theoretic interior and boundary, respectively (cf. Appendix). $A \subset \mathbb{R}^n$ is *normalized* if $A_* = A$ and A has *finite perimeter* in the open set $B \subset \mathbb{R}^n$ if $\mathcal{H}^{n-1}(\partial_* A \cap B) < \infty$. Let $\nu_A(x)$ stand for the outer unit normal of A at $x \in \partial_* A$. The open ball of radius $r > 0$ centered at x is denoted by $B_r(x)$. In general, we consider a measure μ to be a signed real measure or a vector-valued measure. Let $\mu\lfloor A$ be the restriction of the measure μ to the set A and let $|\mu|$ be the total variation measure. By spt we denote the support of a function or of a measure. $\mathrm{dens}_A(x)$ denotes the density of the set A at the point x and $\mathrm{ap}\lim_{s\downarrow t} f(s)$ denotes the approximate limit from above of f at t (cf. Appendix). $|x|$ is the Euclidean norm on $\mathbb{R}^n$, and we write

$$\begin{gathered} x = (x_1, \ldots, x_{j-1}, \xi, x_{j+1}, \ldots, x_n) = (x', \xi)_j \in \mathbb{R}^n \\ \text{with } x' \in \mathbb{R}^{n-1}, \;\; \xi = x_j \in \mathbb{R} \end{gathered} \tag{1.5}$$

to distinguish the jth coordinate of $x \in \mathbb{R}^n$ (taken with respect to a fixed orthonormal frame) and let $(x', \xi)_n = (x', \xi)$ for notational convenience. Analogously we define $(B' \times H)_j \subset \mathbb{R}^n$ with $B' \in \mathbb{R}^{n-1}$, $H \subset \mathbb{R}$ as the set of all $(x', \xi)_j$ with $x' \in B'$, $\xi \in H$ and again $(B' \times H)_n = (B' \times H)$. By $y \cdot z$ we mean the scalar product of the vectors y, z and by $Y \cdot y$ the application of the matrix Y to the vector y. The space of (locally) p-integrable functions on C is denoted by $\mathcal{L}^p(C)$ ($\mathcal{L}^p_{\mathrm{loc}}(C)$) where, following Federer [18], Evans & Gariepy [17], et al., we distinguish functions that differ on sets of measure zero: we do not follow the usual practice of identifying such functions as members of the same equivalence class. $\mathcal{C}^\infty(C)$ denotes the set of infinitely differentiable functions on C and $\mathcal{C}^\infty_0(C)$ denotes the subset of all functions having compact support on C. For further notation we refer to Section 2 and, in particular, to the paragraphs following (2.1).

2 Bodies and interactions

Bodies. The simplest way to define subbodies of a body would be to take the system of all subsets or a suitable subsystem of it in the usual set-theoretic sense. The literature suggests that it is a subtle question to define subbodies. If material reactions that are exerted through the common boundary of contiguous subbodies are considered, they are basically defined on surfaces. The problem now is that, if we cut a body into two pieces, there is no obvious way to decide to which part the common surface should belong. To circumvent this difficulty, normalized sets are usually considered as subbodies (cf. [28]). Here boundary points are somehow disregarded and each subbody in fact represents a whole class of sets. This has the effect that in some sense the cutting surface "disappears" if a body is cut into two pieces and that a material surface is "created" if subbodies are glued together (a comparable situation is met in the system proposed by Noll [39, p. 92]). However, such an approach is not compatible with the additivity of forces if concentrated forces occur (cf. Figure 1). Here we present an approach for the description of subbodies that is based on the simple idea of taking subbodies as subsets in the set-theoretic sense. That removes previous discrepancies and there is no need for Boolean algebras as in previous treatments.

We assume that the material points of a body correspond to the points of a set $C \subset \mathbb{R}^N$. In the material description the points of C correspond to the positions of the material points in the reference configuration and in the spatial description they correspond to the positions they occupy at some given time. Note that our further analysis is not influenced by the special choice we make and, thus, it is applicable to both cases. From the physical point of view it seems reasonable to include at least the (relatively) open and closed subsets of C in a suitable collection $\mathcal{B}$ of subbodies of C. Furthermore, we may assume that $\mathcal{B}$ should be an algebra of subbodies of C, also called an *algebra on* C, so that unions, intersections, and complements of finite numbers of sets of $\mathcal{B}$ belong to $\mathcal{B}$:

$$\emptyset,\, C \in \mathcal{B}, \quad B_1 \setminus B_2 \in \mathcal{B}, \quad \bigcup_{k=1}^{m} B_k \in \mathcal{B}, \quad \bigcap_{k=1}^{m} B_k \in \mathcal{B}$$

whenever $B_1, \dots, B_m \in \mathcal{B}$. From the analytical point of view it appears to be convenient to have the powerful tools of measure and integration theory available and, thus, to demand that $\mathcal{B}$ even be a σ-*algebra* on C, i.e., that $\mathcal{B}$ be an algebra on C that also contains countable unions and intersections of sets of $\mathcal{B}$:

$$\bigcup_{k=1}^{\infty} B_k \in \mathcal{B}, \quad \bigcap_{k=1}^{\infty} B_k \in \mathcal{B}$$

whenever $B_1, B_2, \dots \in \mathcal{B}$. Thus, as a natural choice for our treatment, let $C \subset \mathbb{R}^N$ be *a Borel set* and let $\mathcal{B}$ be *the collection of Borel subsets of* C (recall that $\mathcal{B}$ contains any union and any intersection of at most countably many open and closed sets). We will see, however, that the richness of a σ-algebra is inconsistent with certain arguments concerning special forces. This motivates us to consider also suitable subsystems of $\mathcal{B}$ that are merely algebras or just generators of $\mathcal{B}$. Here it is useful to have not only "small" generating systems but also rich algebras. By $\mathcal{A} \subset \mathcal{B}$ we always denote a *subalgebra* of $\mathcal{B}$ which is not yet specified.

Let us now introduce special subsystems of $\mathcal{B}$. By $\mathcal{Q}$ we denote the collection of all (closed) N-*intervals* (N-dimensional rectangular blocks) on C having the form

$$\{(x_1, \dots, x_N) \in C \mid a_i \le x_i \le b_i,\ a_i, b_i \in \mathbb{R},\ i = 1, \dots, N\} \tag{2.1}$$

where the a_i, b_i are called the *coordinates* (corresponding to a fixed orthonormal frame) of the N-interval. The algebra generated by $\mathcal{Q}$ (i.e., the smallest algebra containing $\mathcal{Q}$), which is obviously a subset of $\mathcal{B}$, is denoted by $\mathcal{R}$. The collection of N-intervals merely having coordinates a_i, b_i, $i = 1, \dots, N$, confined to a subset $H \subset \mathbb{R}$ is denoted by $\mathcal{Q}(H)$. We call $\mathcal{Q}^{\mathrm{f}} \subset \mathcal{Q}$ a *full* subsystem of $\mathcal{Q}$ if $\mathcal{Q}^{\mathrm{f}} = \mathcal{Q}(H^{\mathrm{f}})$ for a subset $H^{\mathrm{f}} \subset \mathbb{R}$ such that $\mathcal{L}^1(\mathbb{R} \setminus H^{\mathrm{f}}) = 0$. We say that $\mathcal{Q}^{\mathrm{d}} \subset \mathcal{Q}$ is a *dense* subsystem of $\mathcal{Q}$ if $\mathcal{Q}^{\mathrm{d}} = \mathcal{Q}(H^{\mathrm{d}})$ for a dense subset $H^{\mathrm{d}} \subset \mathbb{R}$. The subsystem $\mathcal{Q}^{\mathrm{c}} \subset \mathcal{Q}$ is called *countable* if $\mathcal{Q}^{\mathrm{c}} = \mathcal{Q}(H^{\mathrm{c}})$ for a countable subset $H^{\mathrm{c}} \subset \mathbb{R}$. We will use the notation $\mathcal{Q}^{\mathrm{cd}} = \mathcal{Q}(H^{\mathrm{cd}})$ for a countable dense subsystem of $\mathcal{Q}$. Note that $\mathcal{B}$ is the smallest σ-algebra containing a dense subsystem of $\mathcal{Q}$. For open C we also consider the system $\mathcal{P}$ of *sets of finite perimeter* in C, i.e.,

$$\mathcal{P} := \{P \cap C \mid P \subset \mathbb{R}^N \text{ has finite perimeter in } C\}$$

which is an algebra on C (cf. Appendix). Obviously, $\mathcal{Q} \subset \mathcal{R} \subset \mathcal{P} \subset \mathcal{B}$.

A *property* is said to *hold for a.e.* $Q \in \mathcal{Q}$ if it is true for a full subsystem $\mathcal{Q}^{\mathrm{f}} \subset \mathcal{Q}$. For $Q \in \mathcal{Q}$ we denote the (closed) ε-*neighborhood* of Q on C with respect to the maximum norm by Q_ε, i.e., if Q has coordinates a_i, b_i, then $Q_\varepsilon \in \mathcal{Q}$ is the (closed) N-interval on C having coordinates $a_i - \varepsilon$, $b_i + \varepsilon$ (cf. (2.1)). Notice that Q_ε depends on the set C on which $\mathcal{Q}$ is considered. Obviously Q_ε is well defined for all $\varepsilon \geq 0$ and, if $\operatorname{int} Q \neq \emptyset$, then also for some $\varepsilon < 0$. For each $Q \in \mathcal{Q}$ and each $x \in C$ there is exactly one $\sigma(x) \in \mathbb{R}$ such that $x \in \partial Q_{\sigma(x)}$. By $\nu_Q(x)$ we denote the outer unit *normal* of $Q_{\sigma(x)}$ at x which is uniquely determined up to the edges of $Q_{\sigma(x)}$.

Sometimes we use the notation $\mathcal{B}_{\tilde{C}}$, $\mathcal{Q}_{\tilde{C}}$, etc. to indicate that the system is a collection of corresponding subsets of $\tilde{C}$ instead of C. But if the index is omitted, then these systems are always taken with respect to the "standard" set C. For an algebra $\mathcal{A}$ on C and a subset $\tilde{C} \subset C$ we define the *restriction* of $\mathcal{A}$ to $\tilde{C}$ by $\mathcal{A}_{|\tilde{C}} := \{A \cap \tilde{C} \mid A \in \mathcal{A}\}$, which obviously is an algebra on $\tilde{C}$ and, if $\tilde{C} \in \mathcal{A}$, then $\mathcal{A}_{|\tilde{C}} \subset \mathcal{A}$ is a subalgebra.

We also call $\mathcal{B}$ a *system of subbodies* of C. Notions of subbody, empty body, disjoint bodies, etc., are taken in the obvious set-theoretic sense. Note that collections of subbodies previously used in the literature are not systems of bodies in our sense. This distinguishes the present approach from former treatments and makes the notion of Boolean algebra superfluous. It turns out naturally in the subsequent analysis that certain arguments can be carried out for all subbodies $B \in \mathcal{B}$ but that other arguments only work for subbodies of an algebra $\mathcal{A}$ that is strictly contained in $\mathcal{B}$. Therefore it does not make sense to ask for *the* best choice of a system of subbodies. We rather have to look for choices appropriate to special aspects of the theory. This way we simultaneously obtain new insights into the nature of contact interactions.

Interactions. In continuum physics, forces are often described by means of (volume, area) densities. However, since a force can only be observed and measured by the interaction between bodies, there is no a priori physical warrant for a general force to have, e.g., an integrable density. It is rather reasonable to describe a force or, more generally, an interaction, by means of a mapping $(B, A) \mapsto f(B, A)$ (that is real-valued for the heat flux and vector-valued for tractions) assigning the resultant action exerted by body A on body B to suitable pairs (B, A). (Notice that the force governing the motion of a falling stone is the force exerted by the earth on the stone.) It would seem physically natural to require that interactions be additive with respect to disjoint unions in each argument and seem mathematically convenient to demand disjoint additivity not only for finite but also for countable unions. However, it turns out that such a requirement for both arguments would be too restrictive, e.g., for contact forces in continuum mechanics or for the heat flux in thermodynamics. Therefore we will demand countable additivity for the first argument, corresponding to the body subject to the action, and merely finite additivity for the second one.

For the analytical description of such quantities we define an *interaction* on C relative to an algebra $\mathcal{A}$ on C as a mapping $f : \mathcal{B} \times \mathcal{A} \mapsto \mathbb{R}^M$ such that

(A1) $f(\emptyset, A) = f(B, \emptyset) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$,

(A2) $f\left(\bigcup_{k=1}^\infty B_k, A \right) = \sum_{k=1}^\infty f(B_k, A)$ for all countable collections of pairwise disjoint sets $B_1, B_2, \ldots$ in $\mathcal{B}$,

(A3) $f\left(B, \bigcup_{k=1}^m A_k \right) = \sum_{k=1}^m f(B, A_k)$ for all finite collections of pairwise disjoint sets $A_1, \ldots, A_m$ in $\mathcal{A}$.

This just means that f is a measure with respect to its first argument, the body subject to the action, and a finitely additive set function with respect to the second argument, the body producing the action. Note that (A3) does not prevent special interactions from being σ-additive with respect to the second argument and thus can be extended to $\mathcal{B} \times \mathcal{B}$. As a simple consequence of measure theory, an interaction f is uniquely determined by its specification on a dense subsystem $\mathcal{Q}^{\mathrm{d}} \subset \mathcal{Q}$ with respect to the first argument and on a generating system of $\mathcal{A}$ with respect to the second argument. We denote the *total variation* of the measure $f(\cdot, A)$, $A \in \mathcal{A}$, which is a real nonnegative measure on $\mathcal{B}$, by $|f|(\cdot, A)$.

Let $C, \breve{C}$ be Borel sets with $C \subset \breve{C}$ and let $\mathcal{A}, \breve{\mathcal{A}}$ be algebras on $C, \breve{C}$, respectively, with $\breve{\mathcal{A}}_{|C} \subset \mathcal{A}$. We define the *zero extension* $\breve{f}$ on $\breve{C}$ relative to $\breve{\mathcal{A}}$ of an interaction f on C relative to $\mathcal{A}$ by

$$\breve{f}(B, A) := f(B \cap C, A \cap C) \quad \text{for all } B \in \mathcal{B}_{\breve{C}},\ A \in \breve{\mathcal{A}}\,.$$

Obviously, $\breve{f}$ is an interaction on $\breve{C}$ and

$$\breve{f}(B, A) = 0 \quad \text{if} \quad B \subset \breve{C} \setminus C \ \text{ or } \ A \subset \breve{C} \setminus C\,. \tag{2.2}$$

If $\mathcal{A} = \mathcal{R}_C$ or $\mathcal{P}_C$ and $\breve{\mathcal{A}} = \mathcal{R}_{\breve{C}}$ or $\mathcal{P}_{\breve{C}}$, respectively, then $\breve{\mathcal{A}}_{|C} = \mathcal{A}$ and the zero extension is always well-defined.

For an interaction f on C relative to $\mathcal{A}$ and a Borel subset $\breve{C} \subset C$ with $\breve{C} \in \mathcal{A}$ we define the *complete restriction* $f_{\breve{C}}$ of f to $\breve{C}$ relative to $\mathcal{A}_{|\breve{C}}$ by

$$f_{\breve{C}}(B, A) := f(B, A) \quad \text{for all } B \in \mathcal{B}_{\breve{C}},\ A \in \mathcal{A}_{|\breve{C}}\,.$$

and the *partial restriction* $f_{(\breve{C})}$ of f to $\breve{C}$ relative to $\mathcal{A}$ by

$$f_{(\breve{C})}(B, A) := f(B, A \cap \breve{C}) \quad \text{for all } B \in \mathcal{B},\ A \in \mathcal{A}\,.$$

Since $\mathcal{A}_{|\breve{C}} \subset \mathcal{A}$ is a subalgebra, the complete restriction $f_{\breve{C}}$ is an interaction on $\breve{C}$ relative to $\mathcal{A}_{|\breve{C}}$ and, since $A \cap \breve{C} \in \mathcal{A}$, the partial restriction $f_{(\breve{C})}$ is an interaction on C relative to $\mathcal{A}$.

Now let f_1, f_2 be interactions on Borel sets C_1, C_2 relative to algebras $\mathcal{A}_1$, $\mathcal{A}_2$, respectively, and let $\mathcal{A}$ be an algebra on $C := C_1 \cup C_2$ such that $\mathcal{A}_{|C_j} \subset \mathcal{A}_j$, $j = 1, 2$. Then we define the *sum* $f_1 + f_2$ on C relative to $\mathcal{A}$ by

$$(f_1 + f_2)(B, A) := f_1(B \cap C_1, A \cap C_1) + f_2(B \cap C_2, A \cap C_2)$$

for all $B \in \mathcal{B}_C$, $A \in \mathcal{A}$, which is obviously an interaction on C. If $\mathcal{A} = \mathcal{R}_C$ or $\mathcal{P}_C$ and $\mathcal{A}_j = \mathcal{R}_{C_j}$ or $\mathcal{P}_{C_j}$ (j=1,2), respectively, then $\mathcal{A}_{|C_j} = \mathcal{A}_j$ for $j = 1, 2$ and the sum is well-defined.

3 Contact interactions

In continuum physics we observe interactions that act between contiguous bodies through the common parts of their boundaries and that are caused by some material response such as forces or heat transfer. In this section we present a new mathematical approach to such interactions as a foundation for continuum physics.

It is very convenient to study interactions exerted across the common part of the boundaries of subbodies $Q \in \mathcal{Q}$, i.e., for N-intervals, due to their simple geometry. This suggests considering such interactions relative to an algebra $\mathcal{A}$ containing $\mathcal{Q}$. On the other hand, it turns out that it is not only convenient but also sufficient for the essential analytical arguments if that algebra merely contains a full subsystem of $\mathcal{Q}$. Therefore we now consider interactions f on a Borel set C relative to an algebra $\mathcal{A} \subset \mathcal{B}$ that contains a full subsystem $\mathcal{Q}^{\mathrm{f}} = \mathcal{Q}(H^{\mathrm{f}}) \subset \mathcal{Q}$ (i.e., $H^{\mathrm{f}} \subset \mathbb{R}$ with $\mathcal{L}^1(\mathbb{R} \setminus H^{\mathrm{f}}) = 0$). Then f is called a *contact interaction* on C if

(C1) $f(Q, A) = f(Q, (Q_\varepsilon \setminus Q) \cap A)$ for all $Q \in \mathcal{Q}^{\mathrm{f}}$, $A \in \mathcal{A}$, $\varepsilon > 0$ with $Q_\varepsilon \in \mathcal{Q}^{\mathrm{f}}$,

(C2) $f(Q, A) = 0$ for all $Q \in \mathcal{Q}^{\mathrm{f}}$, $A \in \mathcal{A}$ with $\mathcal{L}^N(A) = 0$,

(C3) $f(Q, A) = \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(Q_\varepsilon \setminus Z, A \setminus Q_\varepsilon)$ for all $Q \in \mathcal{Q}^{\mathrm{f}}$, $A \in \mathcal{A}$, $Z \in \mathcal{B}$ with $\mathcal{L}^N(Z) = 0$.

Note that these conditions cannot be required for all $Q \in \mathcal{B}$, since then $(Q_\varepsilon \setminus Q) \cap A$ or $A \setminus Q_\varepsilon$ might not belong to $\mathcal{A}$. But $\mathcal{Q}^{\mathrm{f}}$ generates $\mathcal{B}$ and seems to be sufficiently large to provide the essential properties of contact reactions as observed in continuum mechanics or thermodynamics. Furthermore, it is essential to choose closed N-intervals $Q \in \mathcal{Q}^{\mathrm{f}}$ in the first argument, since $f(Q, A)$ and $f(\operatorname{int} Q, A)$ may differ in general. Finally, $Q_\varepsilon \notin \mathcal{Q}^{\mathrm{f}}$ merely for ε on a set of $\mathcal{L}^1$-measure zero.

Observe that we do *not* require, as usual, that the contact interaction $f(B, A)$ be defined only for pairs of *disjoint* subbodies A, B. Instead we merely assume that the material of A outside of B is responsible for the interaction. Now the most important feature of a contact interaction is the *locality* condition (C1) expressing that only this material of a contiguous body can interact with a body which belongs to an arbitrarily small neighborhood of it as observed, e.g., for traction or heat flux (cf. Figure 3). Thus (C1) implements NOLL'S "principle of local action" [38, p. 199] as a characterizing property for contact interactions instead for corresponding constitutive laws. Conditions (C2) and (C3) exclude certain singular cases that seem to be unphysical. (C2) says that a body A having $\mathcal{L}^N$-measure 0 (i.e., having zero volume) *cannot exert* a nonzero contact action, i.e., we need "thick" bodies in the second argument for nontrivial interactions. Note that this *does not prevent* a subbody $B \in \mathcal{B}$ consisting of a single point from *resisting* a nontrivial action exerted by some A. Hence (C2) *does not imply* that the measures $f(\cdot, A)$ have to be absolutely continuous with respect to the $\mathcal{L}^N$-measure. Property (C3) seems to be the most technical one at first glance. It gives a coupling between the measures $f(\cdot, A)$ for different A's and means, roughly speaking, that a possible concentration of the measure $f(\cdot, A)$ is somehow smeared around, i.e., an action exerted from some "thick" A on some "thin" B with $\mathcal{H}^{N-1}$-measure zero has to be "seen" by interactions $f(\tilde{B}, A \setminus B_\varepsilon)$ for $\tilde{B} \subset \partial B_\varepsilon$ with $\mathcal{H}^{N-1}(\tilde{B}) > 0$ and for sufficiently small $\varepsilon > 0$ (cf. Example 1 below). Note that (C3) merely expresses a general limit property of the measure $f(\cdot, A)$ in the case where $Z = \emptyset$, since $f(Q_\varepsilon, A \setminus Q_\varepsilon) = f(Q_\varepsilon, A)$ according to (C1).

We say that the contact interaction f on C relative to $\mathcal{A}$ is *locally bounded* if for each (bounded) $\breve{Q} \in \mathcal{Q}^{\mathrm{f}}$ there is a constant $\gamma_{\breve{Q}} > 0$ such that

$$|f(B, Q \cap \breve{Q})| \le \gamma_{\breve{Q}} \quad \text{for all } B \in \mathcal{B},\ Q \in \mathcal{Q}^{\mathrm{f}}.$$

Using the Hahn decomposition of $f(\cdot, Q \cap \breve{Q})$ we readily see that the total variation $|f|(C, Q \cap \breve{Q})$ for a locally bounded contact interaction f has to be bounded by $2\gamma_{\breve{Q}}$ for all $Q \in \mathcal{Q}^{\mathrm{f}}$.

Let us now provide some simple consequences for contact interactions. We readily derive from (C1) that

$$f(Q,A)=0 \quad \text{for all } Q\in\mathcal{Q}^{\mathrm{f}},\ A\in\mathcal{A} \text{ with } Q\subset(C\setminus \operatorname{cl}A)\,,$$

and, consequently,

$$f(B,A)=0 \quad \text{for all } B\in\mathcal{B},\ A\in\mathcal{A} \text{ with } B\subset(C\setminus \operatorname{cl}A)\,, \tag{3.1}$$

i.e., the support of the measure $f(\cdot,A)$ is contained in the (relatively closed) set $(\operatorname{cl}A)\cap C$. Moreover, (C1) implies that

$$f(Q,A)=0 \quad \text{for all } Q\in\mathcal{Q}^{\mathrm{f}},\ A\in\mathcal{A} \text{ with } A\subset Q\,.$$

For bounded $A\in\mathcal{A}$ we always find $Q\in\mathcal{Q}^{\mathrm{f}}$ with $A\subset Q$ and, taking into account (3.1), we find that

$$f(Q,A)=f(B,A)=0 \quad \text{for all } B\in\mathcal{B},\ A\in\mathcal{A} \text{ with } ((\operatorname{cl}A)\cap C)\subset B,\ A \text{ bounded.}$$

In particular, for $B=C$ we conclude that

$$0=f(C,A)=f(B,A)+f(C\setminus B,A) \quad \text{for all } B\in\mathcal{B},\ A\in\mathcal{A},\ A \text{ bounded,}$$

i.e.,

$$f(B,A)=-f(C\setminus B,A) \quad \text{for all } B\in\mathcal{B},\ A\in\mathcal{A},\ A \text{ bounded.} \tag{3.2}$$

It is an important observation that condition (C1) prevents a contact interaction from being a measure with respect to the second argument. Otherwise the continuity of measures on nested sequences would imply that

$$f(Q,A)\overset{(C1)}{=}f(Q,A\setminus Q)=\lim_{\varepsilon\downarrow 0}f(Q,A\setminus Q_\varepsilon)\overset{(3.1)}{=}0 \tag{3.3}$$

for any $Q\in\mathcal{Q}^{\mathrm{f}}$, $A\in\mathcal{A}$, and $\varepsilon>0$ such that $Q_\varepsilon\in\mathcal{Q}^{\mathrm{f}}$. But this is only possible in the trivial case where f is identically zero. For this reason we require that interactions merely be finitely additive with respect to the second argument.

According to the next result, which is proved at the end of this section, conditions (C1), (C3) are satisfied if they are satisfied for all A of a system generating the algebra $\mathcal{A}$.

Proposition 3.4 *Let f be an interaction on C relative to $\mathcal{A}$ with $\mathcal{Q}^{\mathrm{f}}\subset\mathcal{A}$ and let $\mathcal{A}^g\subset\mathcal{A}$ be a subsystem generating $\mathcal{A}$. If f satisfies* (C1), (C3) *only for all $A\in\mathcal{A}^g$, then f satisfies* (C1), (C3) *also for all $A\in\mathcal{A}$.*

For given contact interactions we now ask to what extent the derived interactions as zero extension, sum, and the restrictions are again contact interactions. Corresponding to the full subsystem $\mathcal{Q}^{\mathrm{f}}=\mathcal{Q}(H^{\mathrm{f}})$ of $\mathcal{Q}=\mathcal{Q}_C$ we here denote by $\mathcal{Q}^{\mathrm{f}}_{\breve{C}}$ the full subsystem of $\mathcal{Q}_{\breve{C}}$ consisting of all closed N-intervals in $\breve{C}$ having coordinates in H^{f}.

Proposition 3.5 (1) *Let f be a contact interaction on C relative to $\mathcal{A}$ with $\mathcal{Q}^{\mathrm{f}} \subset \mathcal{A}$.*

(a) *If $\breve{\mathcal{A}}$ is an algebra on $\breve{C} \supset C$ with $\mathcal{Q}^{\mathrm{f}}_{\breve{C}} \subset \mathcal{A}$ and $\breve{\mathcal{A}}_{|C} \subset \mathcal{A}$, then the zero extension $\breve{f}$ of f on $\breve{C}$ relative to $\breve{\mathcal{A}}$ is a contact interaction on $\breve{C}$.*

(b) *If $\breve{C} \subset C$ with $\breve{C} \in \mathcal{Q}^{\mathrm{f}}$, then the complete restriction $f_{\breve{C}}$ of f on $\breve{C}$ relative to $\mathcal{A}_{|\breve{C}}$ is a contact interaction on $\breve{C}$.*

(c) *If $\breve{C} \subset C$ with $\breve{C} \in \mathcal{A}$, then the partial restriction $f_{(\breve{C})}$ of f on $\breve{C}$ is a contact interaction on C relative to $\mathcal{A}$.*

(2) *Let f_j be contact interactions on C_j relative to $\mathcal{A}_j$ with $\mathcal{Q}^{\mathrm{f}}_{C_j} \subset \mathcal{A}_j$, $j = 1, 2$, and let $\mathcal{A}$ be an algebra on $C := C_1 \cup C_2$ with $\mathcal{Q}^{\mathrm{f}}_C \subset \mathcal{A}$ and $\mathcal{A}_{|C_j} \subset \mathcal{A}_j$, $j = 1, 2$. Then the sum $f_1 + f_2$ is a contact interaction on C relative to $\mathcal{A}$.*

Notice that there might be a difference between the restriction of f on a closed set $\breve{C}$ and on its interior. Before we carry out the proofs of the previous propositions, we illuminate some basic questions by means of typical examples.

Example 1. This example illustrates the kinds of interactions ruled out by condition (C3). Let $C = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$, $\mathcal{Q}^{\mathrm{f}} = \mathcal{Q}(H^{\mathrm{f}}) \subset \mathcal{Q}_C$ with $H^{\mathrm{f}} = \mathbb{R} \setminus \{0\}$, and define the vector-valued measures on C

$$\tau := \begin{pmatrix} \mathcal{H}^1 \lfloor \{(x_1, x_2) \in C \,|\, x_2 = 0\} \\ 0 \end{pmatrix}, \qquad \tau_A := \tau \lfloor A \quad \text{for } A \in \mathcal{Q}^{\mathrm{f}} .$$

With $A = [a_1, b_1] \times [a_2, b_2] \subset \mathcal{Q}^{\mathrm{f}}$ we obtain in the distributional sense that

$$\operatorname{div} \tau = 0 , \quad \operatorname{div} \tau_A = \begin{cases} 0 & \text{if } 0 \notin [a_2, b_2] , \\ \delta_{(a_1,0)} - \delta_{(b_1,0)} & \text{otherwise} \end{cases}$$

where $\delta_{(a,b)}$ denotes the usual δ-distribution concentrated at the point $(a, b) \in C$ (recall that $a_j, b_j \neq 0$). Hence $\operatorname{div} \tau_A$ is a measure for all $A \in \mathcal{Q}^{\mathrm{f}}$ and we set

$$f(B, A) := \operatorname{div} \tau_A(B) \quad \text{for all } B \in \mathcal{B},\ A \in \mathcal{Q}^{\mathrm{f}} . \tag{3.6}$$

The mapping f can readily be extended to an interaction on C relative to the algebra $\mathcal{R}^{\mathrm{f}}$ generated by $\mathcal{Q}^{\mathrm{f}}$. Obviously f satisfies (C1). It is important to notice that (C2) is also satisfied though all measures $f(\cdot, A)$ are concentrated on a set of $\mathcal{L}^2$-measure zero. For the investigation of (C3) we observe that

$$f(\,(-1, \xi] \times (a, b), [\zeta, 1) \times (\alpha, \beta)\,) = \begin{cases} 1 \text{ if } \zeta \leq \xi \text{ and } 0 \in (a, b) \cap (\alpha, \beta) , \\ 0 \text{ otherwise.} \end{cases}$$

Thus, with $Z = \{(x_1, x_2) \in C \,|\, x_2 = 0\}$,

$$\begin{aligned} 1 &= f(\,(-1, \xi] \times (-1, 1), [\xi, 1) \times (-1, 1)\,) \\ &\neq \operatorname*{ap\,lim}_{\varepsilon \downarrow 0} f(\,((-1, \xi + \varepsilon] \times (-1, 1)) \setminus Z, [\xi + \varepsilon, 1) \times (-1, 1)\,) \\ &= 0 \qquad \text{for all } \xi \in H^{\mathrm{f}} , \end{aligned} \tag{3.7}$$

which contradicts (C3). We will see that (C3) prevents interactions that are defined as in (3.6) by means of a measure-valued tensor τ that is singular with respect to $\mathcal{L}^N$.

Example 2. In classical continuum mechanics it is usually assumed that the contact interaction $f(B,A)$ merely depends on the common surface $\partial B \cap \partial A$ for disjoint and sufficiently regular sets A, B and that $A \mapsto f(B,A)$, $A \subset C \setminus B$, defines a measure on ∂B that is absolutely continuous with respect to the $\mathcal{H}^{N-1}$-measure (cf. TRUESDELL [60]). For our setting we have already seen that a contact interaction f cannot be a measure in the second argument, but we could ask whether $A \mapsto f(B,A)$ merely depends on $A \cap \partial B$ and whether it can be extended to a measure on ∂B.

For this reason we consider the example of a planar version of the problems studied by BOUSSINESQ [6] in 1878 (perpendicular point load) and by FLAMANT [19] in 1892 (perpendicular homogeneous line load) for a linearly elastic, isotropic body occupying a half space. More precisely, we consider the tensor field $\tau \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^4)$ given by

$$\tau(x) := \frac{2x_1}{\pi |x|^4} \begin{pmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{pmatrix} \quad \text{for } x_1 \geq 0$$

and extended by zero on the half plane $x_1 < 0$. The divergence of τ exists in the sense of distributions (see also (5.1) below) and is given by the vector-valued measure

$$\operatorname{div} \tau = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta_{(0,0)}$$

where $\delta_{(0,0)}$ denotes the scalar Dirac measure concentrated at $x = (0,0)$. According to our results below we get a contact interaction on $C = \mathbb{R}^2$ relative to the algebra $\mathcal{P}$ by setting

$$f(Q,A) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\partial Q_\sigma \cap A} \tau \cdot \nu_Q \, d\mathcal{H}^{N-1} \, d\sigma$$

for $Q \in \mathcal{Q}$, $A \in \mathcal{P}$ and then extending each $f(\cdot, A)$ to a measure on $\mathcal{B}$. Now we fix $B := [-1,0] \times [-1,1]$ and consider sets $A \subset C \setminus B$ that are sectors of a disk with corner at the origin. Obviously $A \in \mathcal{P}$, and a straightforward computation shows that $f(B,A)$ depends on the opening angle of the sector A at the corner. Moreover $f(B,A)$ even has a non-vanishing x_2-component if A is not symmetric with respect to the x_2-axis (cf. Podio-Guidugli [43] for detailed computations and comprehensive discussions). This fact however shows that we cannot expect in general that $A \mapsto f(B,A)$ depends only on the common surface $\partial A \cap \partial B$ or that it even defines a measure on ∂B.

Example 3. This example demonstrates that a nontrivial contact interaction f can never be extended consistently to the whole σ-algebra $\mathcal{B}$ with respect to the second argument. For this reason let f be a contact interaction on $C := (-1,1) \times (-1,1)$ relative to some algebra $\mathcal{A}$ where we now assume that $\mathcal{A} = \mathcal{B}$. Then the set A given by

$$A := \bigcup_{k \in \mathbb{N}} A_k \quad \text{where} \quad A_k := \left(\tfrac{1}{3k}, \tfrac{1}{3k-1}\right) \times (-1,1)$$

belongs to $\mathcal{A}$. With $Q := (-1,0] \times (-1,1)$ the continuity of a measure on a nested set implies that

$$f(Q,A) = \lim_{\varepsilon \downarrow 0} f(Q_\varepsilon, A) = \lim_{k\to\infty} f(Q_{1/(3k-2)}, A) \overset{(C1)}{=} 0$$

which is not satisfied in general for a nontrivial contact interaction f (cf. Figure 3).

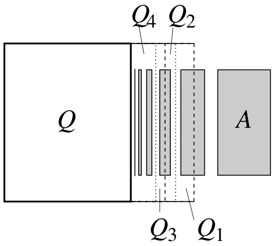

Figure 3: Subbody A corresponds to the grey set of infinitely many strips. The subbodies Q_k approximate Q where the boundary of Q_k on the right is dashed for the Q_{2k+1} and dotted for the Q_{2k}. Since $f(Q_{2k}, A) = 0$ for all k, the sequence $f(Q_k, A)$ oscillates in general and does not converge to $f(Q,A)$.

More generally, let us consider any $Q \in \mathcal{Q}$, $A \in \mathcal{A}$ such that $f(Q,A) \neq 0$. We construct an $\breve{A} \subset A$ with $\breve{A} \in \mathcal{A}$ having a structure analogous to that of A above by removing sufficiently small neighborhoods of the boundaries $\partial Q_{1/k}$, $k \in \mathbb{N}$, from A. Then we obtain a contradiction as before.

This example and (3.3) show that a nontrivial contact interaction f can *neither be σ-additive nor be extended to all of $\mathcal{B}$ with respect to the second argument.*

Proof of Proposition 3.4. Assume that (C1), (C3) are satisfied for A, A_1, $A_2 \in \mathcal{A}^g$. Then, by the identities $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$ and $A_1 \cap A_2 = (A_1^c \cup A_2^c)^c$, it is sufficient to verify (C1), (C3) for $A_1 \cup A_2$ with $A_1 \cap A_2 = \emptyset$ and for $A_2 \setminus A_1$. For $Q \in \mathcal{Q}^f$, $Z \in \mathcal{B}$ with $\mathcal{L}^N(Z) = 0$, and $\varepsilon > 0$ with $Q_\varepsilon \in \mathcal{Q}^f$ we have that

$$\begin{aligned}
f(Q, A_1 \cup A_2) &= f(Q,A_1) + f(Q,A_2) \\
&= f(Q,(Q_\varepsilon \setminus Q) \cap A_1) + f(Q,(Q_\varepsilon \setminus Q) \cap A_2) \\
&= f(Q,(Q_\varepsilon \setminus Q) \cap (A_1 \cup A_2)),
\end{aligned}$$

$$\begin{aligned}
f(Q, A_2 \setminus A_1) &= f(Q,A_2) - f(Q,A_1) \\
&= f(Q,(Q_\varepsilon \setminus Q) \cap A_2) - f(Q,(Q_\varepsilon \setminus Q) \cap A_1) \\
&= f(Q,(Q_\varepsilon \setminus Q) \cap A_2 \setminus A_1),
\end{aligned}$$

$$\begin{aligned}
f(Q, A_1 \cup A_2) &= f(Q,A_1) + f(Q,A_2) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(Q_\varepsilon \setminus Z, A_1 \setminus Q_\varepsilon) + \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(Q_\varepsilon \setminus Z, A_2 \setminus Q_\varepsilon) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(Q_\varepsilon \setminus Z, (A_1 \cup A_2) \setminus Q_\varepsilon),
\end{aligned}$$

$$
\begin{aligned}
f(Q, A_2 \setminus A_1) &= f(Q, A_2) - f(Q, A_1) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(Q_\varepsilon \setminus Z, A_2 \setminus Q_\varepsilon) - \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(Q_\varepsilon \setminus Z, A_1 \setminus Q_\varepsilon) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(Q_\varepsilon \setminus Z, (A_2 \setminus A_1) \setminus Q_\varepsilon)\,,
\end{aligned}
$$

which verifies the assertion. □

Proof of Proposition 3.5. In Section 2 we have seen that the zero extension, the complete and the partial restriction, and the sum are again interactions on the corresponding sets. Thus we still have to check (C1)–(C3) where we tacitly assume that such ε are taken into account such that the neighborhoods Q_ε belong to the corresponding full subsystems $\mathcal{Q}^{\mathrm{f}}$.

For technical convenience we set $\acute{\mathcal{Q}} := \mathcal{Q}_{\mathbb{R}^N}$. Then, for $Q \in \mathcal{Q}^{\mathrm{f}}_C$ we find some $\acute{Q} \in \acute{\mathcal{Q}}$ such that $Q = \acute{Q} \cap C$ and $Q_\varepsilon = \acute{Q}_\varepsilon \cap C$ (notice that Q_ε is the ε-neighborhood of Q in C).

Now we assume that f is a contact interaction on C relative to $\mathcal{A}$. First let $\breve{f}$ be the zero extension of f according to (a). For $\breve{Q} \in \mathcal{Q}^{\mathrm{f}}_{\breve{C}}$ we choose $\acute{Q} \in \acute{\mathcal{Q}}$ such that $\breve{Q} = \acute{Q} \cap \breve{C}$. Obviously, $\breve{A} \cap C \in \breve{\mathcal{A}}_{|C} \subset \mathcal{A}$ for $\breve{A} \in \breve{\mathcal{A}}$ and $\mathcal{L}^N(Z \cap C) = 0$ for $Z \subset \breve{C}$ with $\mathcal{L}^N(Z) = 0$. Then

$$
Q := \breve{Q} \cap C = \acute{Q} \cap \breve{C} \cap C = \acute{Q} \cap C \in \mathcal{Q}^{\mathrm{f}}_C\,.
$$

Hence

$$
\begin{aligned}
\breve{f}(\breve{Q}, \breve{A}) &= f(\breve{Q} \cap C, \breve{A} \cap C) \\
&= f(\breve{Q} \cap C, ((\breve{Q} \cap C)_\varepsilon \setminus (\breve{Q} \cap C)) \cap \breve{A} \cap C) \\
&= f(\breve{Q} \cap C, ((\acute{Q}_\varepsilon \cap C) \setminus (\acute{Q} \cap C)) \cap \breve{A} \cap C) \\
&= f(\breve{Q} \cap C, (\acute{Q}_\varepsilon \setminus \acute{Q}) \cap \breve{C} \cap \breve{A} \cap C) \\
&= f(\breve{Q} \cap C, (\breve{Q}_\varepsilon \setminus \breve{Q}) \cap \breve{A} \cap C) \\
&= \breve{f}(\breve{Q}, (\breve{Q}_\varepsilon \setminus \breve{Q}) \cap \breve{A})\,,
\end{aligned}
$$

$$
\begin{aligned}
\breve{f}(\breve{Q}, \breve{A}) &= f(\breve{Q} \cap C, \breve{A} \cap C) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f((\breve{Q} \cap C)_\varepsilon \setminus (Z \cap C), (\breve{A} \cap C) \setminus (\breve{Q} \cap C)_\varepsilon) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f((\breve{Q}_\varepsilon \cap C) \setminus (Z \cap C), (\breve{A} \cap C) \setminus (\breve{Q}_\varepsilon \cap C)) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f((\breve{Q}_\varepsilon \setminus Z) \cap C, (\breve{A} \setminus \breve{Q}_\varepsilon) \cap C) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} \breve{f}(\breve{Q}_\varepsilon \setminus Z, \breve{A} \setminus \breve{Q}_\varepsilon)\,.
\end{aligned}
$$

Since $\breve{f}(\breve{Q}, \breve{A}) = f(\breve{Q} \cap C, \breve{A} \cap C) = 0$ for $\breve{A} \in \breve{\mathcal{A}}$ with $\mathcal{L}^N(\breve{A}) = 0$, we conclude that $\breve{f}$ is a contact interaction.

Let now $f_{\breve{C}}$ be the complete restriction of f on $\breve{C}$ according to (b). Since $\breve{C} \in \mathcal{Q}^{\mathrm{f}}_C$, there is $\acute{Q}^\star \in \acute{\mathcal{Q}}$ with $\breve{C} = \acute{Q}^\star \cap C$. For $\breve{Q} \in \mathcal{Q}^{\mathrm{f}}_{\breve{C}}$ we choose $\acute{Q} \in \acute{\mathcal{Q}}$ such that $\breve{Q} = \acute{Q} \cap \breve{C}$. Hence, $\breve{Q} = \acute{Q} \cap \acute{Q}^\star \cap C \in \mathcal{Q}^{\mathrm{f}}_C$. We have to distinguish between the neighborhood $\breve{Q}_\varepsilon = \acute{Q}_\varepsilon \cap \breve{C}$ in $\breve{C}$ and the neighborhood $\breve{Q}^C_\varepsilon := (\acute{Q} \cap \acute{Q}^\star)_\varepsilon \cap C = \acute{Q}_\varepsilon \cap \acute{Q}^\star_\varepsilon \cap C$ in C. For $\breve{A} \in \mathcal{A}_{|\breve{C}} \subset \mathcal{A}$ and $\breve{Z} \in \mathcal{B}_{\breve{C}} \subset \mathcal{B}_C$

with $\mathcal{L}^N(\breve{Z}) = 0$ we obtain that

$$\begin{aligned}
f_{\breve{C}}(\breve{Q}, \breve{A}) &= f(\breve{Q}, \breve{A}) \\
&= f(\breve{Q}, (\breve{Q}_\varepsilon^C \setminus \breve{Q}) \cap \breve{A}) \\
&= f(\breve{Q}, ((\acute{Q}_\varepsilon \cap \acute{Q}_\varepsilon^\star \cap C) \setminus (\acute{Q} \cap \acute{Q}^\star \cap C)) \cap \breve{A}) \\
&= f_{\breve{C}}(\breve{Q}, (\breve{Q}_\varepsilon \setminus \breve{Q}) \cap \breve{A}) \qquad (\text{since } \breve{A} \subset \breve{C}),
\end{aligned}$$

$$\begin{aligned}
f_{\breve{C}}(\breve{Q}, \breve{A}) &= f(\breve{Q}, \breve{A}) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(\breve{Q}_\varepsilon^C \setminus \breve{Z}, \breve{A} \setminus \breve{Q}_\varepsilon^C) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f((\acute{Q}_\varepsilon \cap \acute{Q}_\varepsilon^\star \cap C) \setminus \breve{Z}, \breve{A} \setminus (\acute{Q}_\varepsilon \cap \acute{Q}_\varepsilon^\star \cap C)) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f_{\breve{C}}(\breve{Q}_\varepsilon \setminus \breve{Z}, \breve{A} \setminus \breve{Q}_\varepsilon)
\end{aligned}$$

where in the last equality we have used that $\breve{A} \subset \breve{C}$ and $f(C \setminus \breve{C}, \breve{A} \setminus \breve{Q}_\varepsilon) = 0$ by (3.1). For $\breve{A} \in \mathcal{A}_{|\breve{C}} \subset \mathcal{A}$ with $\mathcal{L}^N(\breve{A}) = 0$ we readily find that $f_{\breve{C}}(\breve{Q}, \breve{A}) = f(\breve{Q}, \breve{A}) = 0$ and, thus, $f_{\breve{C}}$ is a contact interaction on $\breve{C}$.

Now let $f_{(\breve{C})}$ be the partial restriction of f to $\breve{C}$. Choosing $Q \in \mathcal{Q}_C^{\mathrm{f}}$, $A \in \mathcal{A}$, and $Z \in \mathcal{B}$ with $\mathcal{L}^N(Z) = 0$ we obtain that $A \cap \breve{C} \in \mathcal{A}$ by $\breve{C} \in \mathcal{A}$ and, therefore,

$$\begin{aligned}
f_{(\breve{C})}(Q, A) &= f(Q, A \cap \breve{C}) \\
&= f(Q, (Q_\varepsilon \setminus Q) \cap A \cap \breve{C}) \\
&= f_{(\breve{C})}(Q, (Q_\varepsilon \setminus Q) \cap A),
\end{aligned}$$

$$\begin{aligned}
f_{(\breve{C})}(Q, A) &= f(Q, A \cap \breve{C}) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(Q_\varepsilon \setminus Z, (A \cap \breve{C}) \setminus Q_\varepsilon) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(Q_\varepsilon \setminus Z, (A \setminus Q_\varepsilon) \cap \breve{C}) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f_{(\breve{C})}(Q_\varepsilon \setminus Z, A \setminus Q_\varepsilon).
\end{aligned}$$

For $A \in \mathcal{A}$ with $\mathcal{L}^N(A) = 0$ we find that $f_{(\breve{C})}(Q, A) = f(Q, A \cap \breve{C}) = 0$ and, consequently, $f_{(\breve{C})}$ is a contact interaction on C.

Finally we assume that f_1, f_2 are contact interactions on C_1, C_2 relative to $\mathcal{A}_1$, $\mathcal{A}_2$, respectively, and we consider the sum $f := f_1 + f_2$ according to (2). Let $Q \in \mathcal{Q}_C^{\mathrm{f}}$ with $Q = \acute{Q} \cap C$ for some $\acute{Q} \in \acute{\mathcal{Q}}$, let $A \in \mathcal{A}$, and let $Z \in \mathcal{B}_C$ be such that $\mathcal{L}^N(Z) = 0$. Then $A_j := A \cap C_j \in \mathcal{A}_j$ by assumption and $Q_j := \acute{Q} \cap C_j \in \mathcal{Q}_{C_j}^{\mathrm{f}}$, $j = 1, 2$. Hence

$$\begin{aligned}
f(Q, A) &= \sum_{j=1}^{2} f_j(Q \cap C_j, A \cap C_j) \\
&= \sum_{j=1}^{2} f_j(\acute{Q} \cap C_j, ((\acute{Q}_\varepsilon \cap C_j) \setminus (\acute{Q} \cap C_j)) \cap A_j)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{2} f_j(\acute{Q} \cap C_j, (\acute{Q}_\varepsilon \setminus \acute{Q}) \cap A \cap C_j) \\
&= f(Q, (Q_\varepsilon \setminus Q) \cap A)\,,
\end{aligned}
$$

$$
\begin{aligned}
f(Q,A) &= \sum_{j=1}^{2} f_j(Q \cap C_j, A \cap C_j) \\
&= \sum_{j=1}^{2} \operatorname{ap}\lim_{\varepsilon \downarrow 0} f_j(Q_{j,\varepsilon} \setminus Z, A_j \setminus Q_{j,\varepsilon}) \\
&= \sum_{j=1}^{2} \operatorname{ap}\lim_{\varepsilon \downarrow 0} f_j((\acute{Q}_\varepsilon \cap C_j) \setminus Z, A_j \setminus (\acute{Q}_\varepsilon \cap C_j)) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} \sum_{j=1}^{2} f_j((\acute{Q}_\varepsilon \setminus Z) \cap C_j, (A \setminus \acute{Q}_\varepsilon) \cap C_j) \\
&= \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(Q_\varepsilon \setminus Z, A \setminus Q_\varepsilon)\,.
\end{aligned}
$$

For $A \in \mathcal{A}$ with $\mathcal{L}^N(A) = 0$ we obtain

$$
f(Q,A) = f_1(Q \cap C_1, A \cap C_1) + f_2(Q \cap C_2, A \cap C_2) = 0\,,
$$

which implies that $f = f_1 + f_2$ is a contact interaction on C. $\square$

4 Interaction tensor

In this section we derive the main result that a contact interaction can be represented by means of a tensor. Notice that, for a contact interaction f relative to $\mathcal{A}$, the algebra $\mathcal{A}$ is assumed to contain a full subsystem $\mathcal{Q}^{\mathrm{f}}$. If $\mathcal{R}^{\mathrm{f}}$ denotes the algebra generated by $\mathcal{Q}^{\mathrm{f}}$, then $\mathcal{R}^{\mathrm{f}} \subset \mathcal{A}$, i.e., f is also a contact interaction relative to the algebra $\mathcal{R}^{\mathrm{f}}$. Therefore, it is not restrictive if we first consider contact interactions relative to $\mathcal{R}^{\mathrm{f}}$. It turns out that the corresponding tensor field is already uniquely determined this way. In Section 6 below we will study how far contact interactions relative to $\mathcal{R}^{\mathrm{f}}$ can be extended to larger algebras $\mathcal{A}$. The proof of the following theorem is carried out later in this section.

Theorem 4.1 (Existence of the interaction tensor). *Let $f : \mathcal{B} \times \mathcal{R}^{\mathrm{f}} \mapsto \mathbb{R}^M$ be a locally bounded contact interaction on the Borel set $C \subset \mathbb{R}^N$ relative to the algebra $\mathcal{R}^{\mathrm{f}}$ that is generated by the full system $\mathcal{Q}^{\mathrm{f}} \subset \mathcal{Q}$. Then there exists an* interaction tensor *$\tau \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R}^{M \times N})$ such that f can be represented by*

$$
\begin{aligned}
f(Q,R) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{(Q_\varepsilon \setminus Q) \cap R} \tau \cdot \nu_Q \, d\mathcal{L}^N \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\partial Q_\sigma \cap R} \tau \cdot \nu_Q \, d\mathcal{H}^{N-1}\, d\sigma
\end{aligned} \tag{4.2}
$$

for all $Q \in \mathcal{Q}^{\mathrm{f}}$, $R \in \mathcal{R}^{\mathrm{f}}$. The tensor τ is uniquely determined up to a set of $\mathcal{L}^N$-measure zero.

Remark 4.3 (1) Note that the surface integrals on the right-hand side in (4.2) are well-defined for a.e. $\sigma \in (0, \varepsilon)$, since Q is bounded and $\tau \in \mathcal{L}^1_{\mathrm{loc}}(C)$.

(2) If f is a locally bounded contact interaction on C relative to an algebra $\mathcal{A}$ larger than $\mathcal{R}^{\mathrm{f}}$, then we assign to f the interaction tensor τ corresponding to the restriction of f on $\mathcal{B} \times \mathcal{R}^{\mathrm{f}}$.

(3) Note that (4.2) is valid in general only for closed N-intervals Q and that we get an analogous formula for open N-intervals but with the limit $\varepsilon \uparrow 0$ instead of $\varepsilon \downarrow 0$ due to (3.2), i.e., roughly speaking, for open N-intervals Q we have to approximate from inside instead from outside.

(4) The condition of local boundedness for f is not very restrictive. It does *not* imply that the tensor τ has to be essentially bounded. In particular concentrations as in Example 2 in the previous section are not ruled out by this condition. Its relevance is discussed in some more detail in Section 7.

The next corollary directly follows from the proof of Theorem 4.1.

Corollary 4.4 *Let $f = (f^i)$ be a contact interaction as in the previous theorem and let $\tau = (\tau^{ij})$ be the corresponding interaction tensor. Then there is a full set $\check{H}^{\mathrm{f}} \subset \mathbb{R}$ such that, for all $\xi \in \check{H}^{\mathrm{f}}$, $i = 1, \dots, M$, $j = 1, \dots, N$, the (real) measures*

$$B' \mapsto f^i((B' \times \{\xi\})_j \cap C, (\mathbb{R}^{N-1} \times [\xi, \infty))_j \cap C)\,,$$

defined on the Borel sets $B' \subset \mathbb{R}^{N-1}$, are absolutely continuous with respect to $\mathcal{H}^{N-1}$ with densities $x' \mapsto \tau^{ij}((x', \xi)_j)$, i.e.,

$$f^i((B' \times \{\xi\})_j \cap C, (\mathbb{R}^{N-1} \times [\xi, \infty))_j \cap C) = \int_{(B' \times \{\xi\})_j \cap C} \tau^{ij}((x', \xi)_j)\, d\mathcal{H}^{N-1}(x')$$

for all Borel sets $B' \subset \mathbb{R}^{N-1}$.

It turns out that a contact interaction as in Theorem 4.1 is already determined if it is known on a dense subsystem $\mathcal{Q}^{\mathrm{d}}$ of $\mathcal{Q}$. While this is clear with respect to the first argument, it is not so obvious with respect to the second argument. The proof of the next corollary can be found at the end of this section.

Corollary 4.5 *Let f be a contact interaction on C as in Theorem* 4.1*. Then f is uniquely determined by its specification on $\mathcal{Q}^{\mathrm{d}} \times \mathcal{Q}^{\mathrm{d}} \subset \mathcal{B} \times \mathcal{R}^{\mathrm{f}}$ where $\mathcal{Q}^{\mathrm{d}} = \mathcal{Q}(H^{\mathrm{d}})$ is a dense subset of $\mathcal{Q}$ with $H^{\mathrm{d}} \subset H^{\mathrm{f}}$.*

Recall that a partial restriction or a zero extension of a contact interaction is again a contact interaction under suitable compatibility conditions according to Proposition 3.5. It turns out that the corresponding interaction tensors can be easily obtained from the interaction tensor of the original contact interaction, as stated in the next corollary, whose proof is postponed to the end of this section.

Corollary 4.6 *Let $f : \mathcal{B} \times \mathcal{A} \mapsto \mathbb{R}^M$ be a locally bounded contact interaction on $C \subset \mathbb{R}^N$ relative to $\mathcal{A}$ with $\mathcal{Q}^{\mathrm{f}} \subset \mathcal{A}$ and let τ be the corresponding interaction tensor.*

(1) *For $A \in \mathcal{A}$ the partial restriction $f_{(A)}$ of f to A is a locally bounded contact interaction on C and $\tau_A(x) = 0$ a.e. on $C \setminus A$ for its interaction tensor $\tau_A \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R}^{M\times N})$. If $A \in \mathcal{R}^{\mathrm{f}}$, then the interaction tensor is given, up to a set of $\mathcal{L}^N$-measure zero, by*

$$\tau_A(x) = \begin{cases} \tau(x) & \text{for } x \in A\,, \\ 0 & \text{for } x \in C \setminus A\,. \end{cases} \tag{4.7}$$

(2) *If $\breve{f}$ is the zero extension of f on the Borel set $\breve{C} \supset C$ relative to an algebra $\breve{\mathcal{A}}$ on $\breve{C}$ with $\mathcal{Q}^{\mathrm{f}}_{\breve{C}} \subset \breve{\mathcal{A}}$ and $\breve{\mathcal{A}}_{|C} \subset \mathcal{A}$, then $\breve{f}$ is a locally bounded contact interaction on $\breve{C}$ and its interaction tensor $\breve{\tau}$ is given, up to a set of $\mathcal{L}^N$-measure zero, by*

$$\breve{\tau}(x) = \begin{cases} \tau(x) & \text{for } x \in C\,, \\ 0 & \text{for } x \in \breve{C} \setminus C\,. \end{cases}$$

Proof of Theorem 4.1. Recall that $\mathcal{Q}^{\mathrm{f}} = \mathcal{Q}(H^{\mathrm{f}})$ for some $H^{\mathrm{f}} \subset \mathbb{R}$ with $\mathcal{L}^1(\mathbb{R} \setminus H^{\mathrm{f}}) = 0$. Since all arguments of the proof work for each component of f separately, we can restrict our attention to the scalar case $M = 1$.

Let us first assume that C is an open bounded cube in $\mathbb{R}^N$ and, without any loss of generality, we choose $C = (0,1)^N$. We take $\mathcal{Q} = \mathcal{Q}_C$ and $\mathcal{Q}^{\mathrm{f}} = \mathcal{Q}^{\mathrm{f}}_C$ with respect to that C where we can assume that $0, 1 \in H^{\mathrm{f}}$, since coordinates less than 0 or larger than 1 can be readily replaced with 0 or 1, respectively. Furthermore we set $C' := (0,1)^{N-1}$. Let $\mathcal{Q}'$ denote the closed $(N-1)$-intervals in C' analogously to (2.1) and define $\mathcal{Q}'^{\mathrm{f}} := \mathcal{Q}'(H^{\mathrm{f}}) \subset \mathcal{Q}'$ in analogy to $\mathcal{Q}^{\mathrm{f}}$. We also fix a countable subset $H^{\mathrm{cd}} \subset (H^{\mathrm{f}} \cap [0,1])$ which is dense in $[0,1]$ and with $0, 1 \in H^{\mathrm{cd}}$. Then $\mathcal{Q}'^{\mathrm{cd}} := \mathcal{Q}'(H^{\mathrm{cd}}) \subset \mathcal{Q}'^{\mathrm{f}}$ denotes the set of all closed $(N-1)$-intervals in C' having coordinates $a_i, b_i \in H^{\mathrm{cd}}$, $i = 1, \dots, N-1$, and $\mathcal{Q}'^{\mathrm{cd}}_0 \subset \mathcal{Q}'^{\mathrm{cd}}$ denotes the set of all closed $(N-1)$-intervals in C' with coordinates in H^{cd} where $a_j = b_j$ for at least one index $1 \le j \le N-1$. Note that $C' \in \mathcal{Q}'^{\mathrm{cd}}$ and recall the notation

$$x = (x_1, \dots, x_N) = (x', \xi) \in \mathbb{R}^N \quad \text{with } x' \in \mathbb{R}^{N-1},\ \xi = x_N \in \mathbb{R}\,. \tag{4.8}$$

(a) We show that there is a subset $H^{\mathrm{z}} \subset ((0,1) \cap H^{\mathrm{f}})$ of $\mathcal{L}^1$-measure zero (possibly depending on H^{cd}) such that

$$f(P' \times \{\xi\}, Q' \times [\xi, 1)) = 0 \quad \text{for all } \xi \in ((0,1) \cap H^{\mathrm{f}}) \setminus H^{\mathrm{z}},\ P' \in \mathcal{Q}'^{\mathrm{cd}}_0,\ Q' \in \mathcal{Q}'^{\mathrm{cd}}\,. \tag{4.9}$$

For this purpose we fix $(N-1)$-intervals $P' \in \mathcal{Q}'^{\mathrm{cd}}_0$ and $Q' \in \mathcal{Q}'^{\mathrm{cd}}$. For any $\xi \in ((0,1) \cap H^{\mathrm{f}})$ we have, by (C3), that

$$\begin{aligned} & f(P' \times (0,\xi], Q' \times [\xi, 1)) \\ &\quad = \operatorname*{ap\,lim}_{\varepsilon \downarrow 0} f((P' \times (0,\xi])_\varepsilon, (Q' \times [\xi,1)) \setminus (P' \times (0,\xi])_\varepsilon) \\ &\quad = \operatorname*{ap\,lim}_{\varepsilon \downarrow 0} f((P' \times (0,\xi])_\varepsilon \setminus (P' \times (0,\xi+\varepsilon]), (Q' \times [\xi,1)) \setminus (P' \times (0,\xi])_\varepsilon), \end{aligned}$$

since $\mathcal{L}^N(P' \times (0, \xi+\varepsilon]) = 0$. By the additivity of f in the first argument the last equality implies that

$$0 = \operatorname{ap}\lim_{\varepsilon\downarrow 0} f(P' \times (0, \xi+\varepsilon], (Q' \times [\xi, 1)) \setminus (P' \times (0, \xi])_\varepsilon)\,.$$

Applying (C1) we get for a.e. $\varepsilon > 0$ that

$$f(P' \times (0, \xi+\varepsilon], (Q' \times [\xi, 1)) \setminus (P' \times (0, \xi])_\varepsilon) = f(P' \times (0, \xi+\varepsilon], Q' \times [\xi+\varepsilon, 1))\,.$$

Hence

$$\begin{aligned} 0 &= \operatorname{ap}\lim_{\zeta\downarrow\xi} f(P' \times (0, \zeta], Q' \times [\zeta, 1)) \\ &= \operatorname{ap}\lim_{\zeta\downarrow\xi} f(P' \times \{\zeta\}, Q' \times [\zeta, 1)) \quad \text{for all } \xi \in ((0,1) \cap H^{\mathrm{f}}) \end{aligned} \tag{4.10}$$

(recall (3.1) for the last equality). Thus we can apply Proposition 8.1 to the real function

$$\varphi(\zeta) := f(P' \times \{\zeta\}, Q' \times [\zeta, 1)) \quad \text{for } \zeta \in ((0,1) \cap H^{\mathrm{f}})\,.$$

We obtain that φ is $\mathcal{L}^1$-measurable and that the set

$$H(P', Q') := \{\zeta \in ((0,1) \cap H^{\mathrm{f}}) \,|\, \varphi(\zeta) \neq 0\}$$

has $\mathcal{L}^1$-measure zero. Since H^{cd} is countable, $\mathcal{Q}'^{\mathrm{cd}}$ and $\mathcal{Q}_0'^{\mathrm{cd}}$ are countable, too. Hence

$$H^{\mathrm{z}} := \bigcup_{P' \in \mathcal{Q}_0'^{\mathrm{cd}},\, Q' \in \mathcal{Q}'^{\mathrm{cd}}} H(P', Q')$$

is a set of $\mathcal{L}^1$-measure zero in $(0,1) \cap H^{\mathrm{f}}$ which verifies (4.9).

(b) We now show that the real function

$$\psi_{P'}(\xi) := f(P' \times \{\xi\}, C' \times [\xi, 1)) \quad \text{is } \mathcal{L}^1\text{-measurable on } (0,1) \cap H^{\mathrm{f}} \tag{4.11}$$

for all $P' \in \mathcal{Q}'^{\mathrm{cd}}$.

For this reason we fix $P' \in \mathcal{Q}'^{\mathrm{cd}}$, $\xi \in ((0,1) \cap H^{\mathrm{f}}) \setminus H^{\mathrm{z}}$, and recall that $C' \in \mathcal{Q}'^{\mathrm{cd}}$. By additivity we have that

$$f(P' \times \{\xi\}, C' \times [\xi, 1)) = f(\partial P' \times \{\xi\}, C' \times [\xi, 1)) + f((\operatorname{int} P') \times \{\xi\}, C' \times [\xi, 1))\,.$$

Since $\partial P'$ is a finite union of elements from $\mathcal{Q}_0'^{\mathrm{cd}}$,

$$f(\partial P' \times \{\xi\}, C' \times [\xi, 1)) = f(\partial P' \times \{\xi\}, P' \times [\xi, 1)) = 0 \tag{4.12}$$

by (4.9). According to (3.1),

$$f((\operatorname{int} P') \times \{\xi\}, (C' \setminus P') \times [\xi, 1)) = 0\,.$$

Hence, by additivity,

$$\begin{aligned} f(P' \times \{\xi\}, C' \times [\xi, 1)) &= f(\partial P' \times \{\xi\}, P' \times [\xi, 1)) + f((\operatorname{int} P') \times \{\xi\}, P' \times [\xi, 1)) \\ &= f(P' \times \{\xi\}, P' \times [\xi, 1))\,. \end{aligned}$$

Applying first (3.1) and then (C1) we conclude that

$$\begin{aligned} f(P' \times \{\xi\}, C' \times [\xi, 1)) &= f(P' \times (0, \xi], P' \times [\xi, 1)) \\ &= f(P' \times (0, \xi], P' \times (0, 1)) \end{aligned} \tag{4.13}$$

for all $P' \in \mathcal{Q}'^{\mathrm{cd}}$, $\xi \in ((0,1) \cap H^{\mathrm{f}}) \setminus H^{\mathrm{z}}$. Since $f(\cdot, P' \times (0,1))$ is a measure, the last right term in (4.13) is a BV-function on $(0,1)$ in the variable ξ and, thus, $\mathcal{L}^1$-measurable. This function agrees with the function $\psi_{P'}(\cdot)$ on $(0,1) \cap H^{\mathrm{f}}$ up to a set of $\mathcal{L}^1$-measure zero. Therefore, $\psi_{P'}(\cdot)$ is $\mathcal{L}^1$-measurable on $(0,1) \cap H^{\mathrm{f}}$ for all $P' \in \mathcal{Q}'^{\mathrm{cd}}$.

(c) For each $\xi \in (0,1)$ we define a finite Radon measure on C' by

$$\mu_\xi(P') := f(P' \times \{\xi\}, C' \times [\xi, 1)) \quad \text{for } P' \in \mathcal{B}_{C'},\ \xi \in ((0,1) \cap H^{\mathrm{f}}) \tag{4.14}$$

and we set $\mu_\xi := 0$ for $\xi \in (0,1) \setminus H^{\mathrm{f}}$. We show that

$$\mu(P) := \int_{(0,1)} \int_{C'} \chi_P(x', \xi)\, d\mu_\xi(x')\, d\xi \quad \text{for all Borel sets } P \in \mathcal{B}_C\,, \tag{4.15}$$

where χ_P denotes the characteristic function of the set P, is a well defined Radon measure on $C = C' \times (0,1)$ and that

$$\int_C g(x', \xi)\, d\mu(x', \xi) = \int_{(0,1)} \int_{C'} g(x', \xi)\, d\mu_\xi(x') d\xi \tag{4.16}$$

holds for any bounded Borel function g on $C' \times (0,1)$.

Let us first verify that the measure-valued mapping $\xi \mapsto \mu_\xi$ is $\mathcal{L}^1$-measurable on $(0,1)$, i.e., that the real function

$$\xi \mapsto \mu_\xi(P') \quad \text{is } \mathcal{L}^1\text{-measurable on } (0,1) \tag{4.17}$$

for any Borel set $P' \subset C'$ (cf. Ambrosio et al. [1, p. 56]). By (4.11) we know that (4.17) is true for all $P' \in \mathcal{Q}'^{\mathrm{cd}}$ and that $\mathcal{Q}'^{\mathrm{cd}}$ obviously generates the Borel sets $\mathcal{B}_{C'}$ (note that $C' \in \mathcal{Q}'^{\mathrm{cd}}$). Let $\mathcal{M}' \subset \mathcal{B}_{C'}$ denote the system of all Borel sets P' satisfying (4.17). Using basic properties of measurable functions we readily see that $C' \setminus M' \in \mathcal{M}'$ if $M' \in \mathcal{M}'$. Furthermore, $M_1' \cap M_2' \in \mathcal{M}'$ if $M_1', M_2', M_1' \cup M_2' \in \mathcal{M}'$. Now consider a sequence

$$M_1' \subset M_2' \subset \dots \quad \text{with} \quad M_j' \in \mathcal{M}', \quad M' := \bigcup_{j \in \mathbb{N}} M_j'\,.$$

Since the contact interaction f is locally bounded, $|\mu_\xi|(C') < \infty$ for all $\xi \in (0,1)$. Thus $|\mu_\xi|(M' \setminus M_j') \to 0$ as $j \to \infty$ and, thus, $\mu_\xi(M_j') \to \mu_\xi(M')$ for all $\xi \in (0,1)$. Consequently, $M' \in \mathcal{M}'$. Since $\mathcal{Q}'^{\mathrm{cd}}$ is closed under finite intersections and since countably many elements of $\mathcal{Q}'^{\mathrm{cd}}$ cover C', we obtain that $\mathcal{M}'$ contains the σ-algebra generated by $\mathcal{Q}'^{\mathrm{cd}}$, i.e., $\mathcal{B}_{C'} \subset \mathcal{M}'$ (cf. Ambrosio et al. [1, Proposition 1.8, Remark 1.9]). Hence (4.17) is satisfied for all Borel sets $P' \subset C'$, i.e., $\xi \mapsto \mu_\xi$ is $\mathcal{L}^1$-measurable and, therefore, $\xi \mapsto |\mu_\xi|$ is also $\mathcal{L}^1$-measurable on $(0,1)$ (cf. Ambrosio et al. [1, p. 56, (2.16)]).

By Ambrosio et al. [1, Prop. 2.26],

$$\xi \mapsto \int_{C'} g(x', \xi)\, d\mu_\xi(x')$$

is $\mathcal{L}^1$-measurable on $(0,1)$ for all bounded functions $g : C' \times (0,1) \mapsto \mathbb{R}$ that are measurable with respect to $\mathcal{B}_{C'} \times \Lambda$ where Λ denotes the $\mathcal{L}^1$-measurable subsets of $(0,1)$. Using the local boundedness of the interaction f we have that the measures $|\mu_\xi|$ are uniformly bounded on C' and thus

$$\int_0^1 |\mu_\xi|(C')\, d\xi < \infty\,.$$

Thus the generalized product measure μ defined in (4.15) is a well defined Radon measure on $C = C' \times (0,1)$, and (4.16) is satisfied for any bounded Borel function g on $C' \times (0,1)$ (cf. Ambrosio et al. [1, p. 57]).

(d) According to the Radon-Nikodym theorem we decompose μ into the absolutely continuous part μ^{ac} and the singular part μ^{s} with respect to $\mathcal{L}^N$ on $C \subset \mathbb{R}^N$ and we show that

$$\mu^{\mathrm{s}} = 0\,, \;\text{ i.e., }\; \mu = \mu^{\mathrm{ac}}\,.$$

For this purpose we also decompose all μ_ξ, $\xi \in (0,1)$, into the absolutely continuous part μ_ξ^{ac} and the singular part μ_ξ^{s} with respect to $\mathcal{L}^{N-1}$ on $C' \subset \mathbb{R}^{N-1}$. Let μ^{s} be concentrated on the set $P^{\mathrm{s}} \in \mathcal{B}_C$ of $\mathcal{L}^N$-measure zero and set $P_\xi^{\mathrm{s}} := \{x' \in C' | (x',\xi) \in P^{\mathrm{s}}\}$. Clearly,

$$\mathcal{L}^N(P^{\mathrm{s}}) = 0\,, \quad \mathcal{L}^{N-1}(P_\xi^{\mathrm{s}}) = 0 \quad \text{for a.e. } \xi \in (0,1)\,. \tag{4.18}$$

Thus, by (4.15), we have for all Borel sets $P \subset P^{\mathrm{s}}$ that

$$\mu(P) = \mu^{\mathrm{s}}(P) = \int_{(0,1)} \int_{C'} \chi_P(x',\xi)\, d\mu_\xi^{\mathrm{s}}(x')\, d\xi\,. \tag{4.19}$$

Since it is not clear whether the singular parts μ_ξ^{s} have to be concentrated in P_ξ^{s}, we consider the measures

$$\check{\mu}_\xi^{\mathrm{s}}(P') := \mu_\xi^{\mathrm{s}}(P' \setminus P_\xi^{\mathrm{s}}) \quad \text{for all } P' \in \mathcal{B}_{C'}\,, \;\; \xi \in (0,1) \tag{4.20}$$

which are either zero or singular with respect to $\mathcal{L}^{N-1}$ on C'. For all $P \in \mathcal{B}_C$ we thus have that

$$\begin{aligned} \mu^{\mathrm{ac}}(P) &= \mu(P \setminus P^{\mathrm{s}}) \\ &= \int_{(0,1)} \int_{C'} \chi_P(x',\xi)\, d\mu_\xi^{\mathrm{ac}}(x')\, d\xi + \int_{(0,1)} \int_{C'} \chi_P(x',\xi)\, d\check{\mu}_\xi^{\mathrm{s}}(x')\, d\xi\,. \end{aligned} \tag{4.21}$$

If $\tau^N \in \mathcal{L}^1(C)$ denotes the integrable density of μ^{ac}, then we define measures on C' by

$$\check{\mu}_\xi^{\mathrm{ac}}(P') := \int_{P'} \tau^N(x',\xi)\, d\mathcal{L}^{N-1}(x') \quad \text{for } P' \in \mathcal{B}_{C'}\,, \;\; \xi \in (0,1)\,, \tag{4.22}$$

which are well defined and absolutely continuous with respect to $\mathcal{L}^{N-1}$ on C' for a.e. $\xi \in (0,1)$. Then, Fubini's theorem implies that

$$\mu^{\mathrm{ac}}(P) = \int_{(0,1)} \int_{C'} \chi_P(x',\xi)\, d\check{\mu}_\xi^{\mathrm{ac}}(x')\, d\xi \quad \text{for all } P \in \mathcal{B}\,. \tag{4.23}$$

For fixed $P' \in \mathcal{Q}'^{\mathrm{cd}}$ we now consider (4.21) and (4.23) with $P = P' \times (a,b)$, $(a,b) \subset (0,1)$. The arbitrariness of (a,b) implies the existence of a Borel set $H(P') \subset (0,1)$ with $\mathcal{L}^1(H(P')) = 0$ such that

$$\int_{P'} d\check{\mu}_\xi^{\mathrm{ac}}(x') = \int_{P'} d(\mu_\xi^{\mathrm{ac}} + \check{\mu}_\xi^{\mathrm{s}})(x') \tag{4.24}$$

for all $\xi \in (0,1) \setminus H(P')$ (we tacitly assume that $\check{\mu}_\xi^{\mathrm{ac}}$ is well-defined on $(0,1) \setminus H(P')$ by (4.22)). The set $H_0 := \bigcup_{P' \in \mathcal{Q}'^{\mathrm{cd}}} H(P')$ has obviously $\mathcal{L}^1$-measure zero and (4.24) has to be true for all $P' \in \mathcal{Q}'^{\mathrm{cd}}$ and all $\xi \in (0,1) \setminus H_0$. Since $\mathcal{Q}'^{\mathrm{cd}}$ generates the Borel subsets of C', we obtain the identity

$$\check{\mu}_\xi^{\mathrm{ac}} = \mu_\xi^{\mathrm{ac}} + \check{\mu}_\xi^{\mathrm{s}} \quad \text{for all } \xi \in (0,1) \setminus H_0 \,. \tag{4.25}$$

Recalling that the measures $\check{\mu}_\xi^{\mathrm{ac}}$, μ_ξ^{ac} are absolutely continuous and that the $\check{\mu}_\xi^{\mathrm{s}}$ are singular or zero, we conclude that $\check{\mu}_\xi^{\mathrm{s}}$ has to vanish for all $\xi \in (0,1) \setminus H_0$, i.e.,

$$\check{\mu}_\xi^{\mathrm{ac}} = \mu_\xi^{\mathrm{ac}} \,, \quad \check{\mu}_\xi^{\mathrm{s}} = 0 \,, \quad \mu_\xi^{\mathrm{s}}(C' \setminus P_\xi^{\mathrm{s}}) = 0 \quad \text{for all } \xi \in (0,1) \setminus H_0. \tag{4.26}$$

If we denote the densities of the μ_ξ^{ac} by $\tau_\xi^N \in \mathcal{L}^1(C')$, $\xi \in (0,1)$, then for a.e. $\xi \in (0,1)$

$$\tau^N(x', \xi) = \tau_\xi^N(x') \quad \text{for a.e. } x' \in C' \tag{4.27}$$

by (4.22). According to (4.21) we obtain that

$$\mu^{\mathrm{ac}}(P) = \int_{(0,1)} \int_{C'} \chi_P(x', \xi) \, d\mu_\xi^{\mathrm{ac}}(x') \, d\xi \quad \text{for all } P \subset \mathcal{B}_C \,.$$

Let $P^{\mathrm{s}} = P^{\mathrm{s}+} \cup P^{\mathrm{s}-}$, $P^{\mathrm{s}+} \cap P^{\mathrm{s}-} = \emptyset$, $P^{\mathrm{s}\pm} \in \mathcal{B}_C$, be a Hahn decomposition of the set P^{s} where the singular measure μ^{s} is concentrated (i.e., the measures $\pm\mu^{\mathrm{s}}$ are nonnegative on the disjoint sets $P^{\mathrm{s}\pm}$, respectively; cf. Ambrosio et al. [1, p. 35]). Then we set

$$P_\xi^{\mathrm{s}\pm} := \{x' \in C' | (x', \xi) \in P^{\mathrm{s}\pm}\} \quad \text{for all } \xi \in (0,1), \text{ i.e., } P_\xi^{\mathrm{s}} = P_\xi^{\mathrm{s}+} \cup P_\xi^{\mathrm{s}-} \,.$$

For fixed $P' \in \mathcal{Q}'^{\mathrm{cd}}$ and any interval $(a,b) \subset (0,1)$ we get

$$\pm\mu^{\mathrm{s}}(\, (P' \times (a,b)) \cap P^{\mathrm{s}\pm} \,) = \pm \int_{(a,b)} \int_{P' \cap P_\xi^{\mathrm{s}\pm}} d\mu_\xi^{\mathrm{s}}(x') \, d\xi \geq 0$$

by (4.19). Thus $\pm\mu_\xi^{\mathrm{s}}(P' \cap P_\xi^{\mathrm{s}\pm}) \geq 0$ for a.e. $\xi \in (0,1)$. Since $\mathcal{Q}'^{\mathrm{cd}}$ is countable, we obtain that

$$\pm\mu_\xi^{\mathrm{s}}(P' \cap P_\xi^{\mathrm{s}\pm}) \geq 0 \quad \text{for all } P' \in \mathcal{Q}'^{\mathrm{cd}}, \;\; \xi \in (0,1) \setminus H_0$$

with some possibly larger set $H_0 \subset (0,1)$ of $\mathcal{L}^1$-measure zero. Using (4.26) we see that

$$P_\xi^{\mathrm{s}\pm} \quad \text{is a Hahn decomposition for } \mu_\xi^{\mathrm{s}} \text{ for all } \xi \in (0,1) \setminus H_0 \tag{4.28}$$

and, by (4.18), we can assume that

$$\mathcal{L}^{N-1}(P_\xi^{\mathrm{s}\pm}) = 0 \quad \text{for all } \xi \in (0,1) \setminus H_0 \,. \tag{4.29}$$

Condition (C3) implies that

$$\begin{aligned} f(C' \times \{\xi\}, C' \times [\xi, 1)) &= \operatorname{ap}\lim_{\zeta \downarrow \xi} f(C' \times [\xi, \zeta], C' \times [\zeta, 1)) \\ &= \operatorname{ap}\lim_{\zeta \downarrow \xi} f(C' \times [\xi, \zeta] \setminus P^{\mathrm{s}\pm}, C' \times [\zeta, 1)) \end{aligned}$$

for all $\xi \in (0,1) \cap H^{\mathrm{f}}$. By additivity,

$$0 = \operatorname{ap}\lim_{\zeta \downarrow \xi} f((C' \times [\xi,\zeta]) \cap P^{\mathrm{s}\pm}, C' \times [\zeta,1)) \quad \text{for all } \xi \in (0,1) \cap H^{\mathrm{f}}\,.$$

Using (C1), (4.14), and (4.29), for $\xi \in (0,1) \cap H^{\mathrm{f}}$ we obtain that

$$\begin{aligned} f((C' \times [\xi,\zeta]) \cap P^{\mathrm{s}\pm}, C' \times [\zeta,1)) &= f((C' \times \{\zeta\}) \cap P^{\mathrm{s}\pm}, C' \times [\zeta,1)) \\ &= f(P^{\mathrm{s}\pm}_\zeta \times \{\zeta\}, C' \times [\zeta,1)) \\ &= \mu_\zeta(P^{\mathrm{s}\pm}_\zeta) \quad \text{for all } \zeta \in (\xi,1) \cap H^{\mathrm{f}}\,. \end{aligned}$$

Thus

$$0 = \operatorname{ap}\lim_{\zeta \downarrow \xi} \mu_\zeta(P^{\mathrm{s}\pm}_\zeta) \quad \text{for all } \xi \in (0,1) \cap H^{\mathrm{f}}$$

and, consequently,

$$0 = \mu_\xi(P^{\mathrm{s}\pm}_\xi) = \mu^{\mathrm{s}}_\xi(P^{\mathrm{s}\pm}_\xi) \quad \text{for a.e. } \xi \in (0,1)$$

by Proposition 8.1 and (4.29). Hence, by (4.28),

$$\mu^{\mathrm{s}}_\xi = 0 \quad \text{for a.e. } \xi \in (0,1). \tag{4.30}$$

Therefore $\mu^{\mathrm{s}} = 0$ by (4.19), i.e., $\mu = \mu^{\mathrm{ac}}$ is absolutely continuous.

(e) We now distinguish $\xi = x_j$, $j = 1,\dots,N-1$, instead of $\xi = x_N$ in (4.8). Using the notation $(x',\xi)_j$ and $(P' \times H)_j$ (cf. (1.5)) we can argue as in the previous steps (a)–(d). In analogy to (4.14) we define measures μ^j_ξ on C' by

$$\mu^j_\xi(P') := f((P' \times \{\xi\})_j, (C' \times [\xi,1))_j) \quad \text{for } P' \in \mathcal{B}_{C'}\,,\ \xi \in ((0,1) \cap H^{\mathrm{f}})\,, \tag{4.31}$$

having the absolutely continuous part $\mu^{j,\mathrm{ac}}_\xi$ with density $\tau^j_\xi \in \mathcal{L}^1(C')$ and the singular part $\mu^{j,\mathrm{s}}_\xi$ with respect to $\mathcal{L}^{N-1}$ for all $\xi \in ((0,1) \cap H^{\mathrm{f}})$, $j = 1,\dots,N$. As in (4.15) we then get Borel measures $\mu^1,\dots,\mu^N$ on C that have to be absolutely continuous with respect to $\mathcal{L}^N$ with corresponding densities $\tau^1,\dots,\tau^N \in \mathcal{L}^1(C)$. Let us verify (4.2) for the interaction tensor

$$\tau := (\tau^1,\dots,\tau^N) \in \mathcal{L}^1(C,\mathbb{R}^N)\,.$$

In analogy to (4.27), for a.e. $\xi \in (0,1)$, $j = 1,\dots,N$, we get that

$$\tau^j((x',\xi)_j) = \tau^j_\xi(x') \quad \text{for a.e. } x' \in C'\,. \tag{4.32}$$

Furthermore, $\mu^{j,\mathrm{s}}_\xi = 0$ for a.e. $\xi \in (0,1)$, $j = 1,\dots,N$, in analogy to (4.30). Setting

$$H := \{\xi \in (0,1) \cap H^{\mathrm{f}} \mid \mu^{j,\mathrm{s}}_\xi = 0\,,\ \tau^j((\cdot,\xi)_j) = \tau^j_\xi(\cdot) \text{ a.e. on } C' \text{ for all } j = 1,\dots,N\}\,, \tag{4.33}$$

we thus have that $\mathcal{L}^1(H) = 1$. Furthermore we can assume that

$$|f|((C' \times \{\xi\})_j, C) = 0 \quad \text{for all } \xi \in H,\ j = 1,\dots,N \tag{4.34}$$

(note that $\mathcal{L}^N((C' \times \{\xi\})_j) = 0$). Let $\mathcal{Q}_H := \mathcal{Q}(H) \subset \mathcal{Q}$ denote the set of all (closed) N-intervals in C with coordinates $a_i, b_i \in H$, $i = 1, \dots, N$ (cf. (2.1)).

Now we fix any $P, Q \in \mathcal{Q}^{\mathrm{f}}$. If a_i, b_i, $i = 1, \dots, N$, are the coordinates of P, then we readily find some set $\breve{H} \subset (0,1)$ (depending on P) with $\mathcal{L}^1(\breve{H}) = 1$ such that $a_i - \varepsilon, b_i + \varepsilon \in H$ for all $\varepsilon \in \breve{H}$, $i = 1, \dots, N$, i.e., $P_\varepsilon \in \mathcal{Q}_H$ for all $\varepsilon \in \breve{H}$. By ∂P_ε we here denote the boundary of P_ε relative to the open set C, i.e., $\partial P_\varepsilon \subset C$. Then, by (C1) and (C3),

$$f(P,Q) = \operatorname{ap}\lim_{\varepsilon \downarrow 0} f(P_\varepsilon \setminus A, Q \setminus P_\varepsilon) = \operatorname{ap}\lim_{\varepsilon \downarrow 0} f((\partial P_\varepsilon \cap Q) \setminus A, Q \setminus P_\varepsilon) \tag{4.35}$$

for any $A \in \mathcal{B}_C$ with $\mathcal{L}^N(A) = 0$. We now compute $f((\partial P_\varepsilon \cap Q) \setminus A, Q \setminus P_\varepsilon)$ for all $\varepsilon \in \breve{H}$ which is sufficient for the evaluation of the limit in (4.35).

The boundary ∂P is the union of (possibly empty) planar lateral faces

$$\begin{aligned} S_{a_j} &:= \{(x',\xi)_j \in P \,|\, \xi = a_j\} \qquad \text{and} \\ S_{b_j} &:= \{(x',\xi)_j \in P \,|\, \xi = b_j\}\,, \quad j = 1, \dots, N\,, \end{aligned}$$

having the form

$$S_{a_j} = (P'_j \times \{a_j\})_j \cap C \quad \text{and} \quad S_{b_j} = (P'_j \times \{b_j\})_j \cap C$$

for suitable $(N-1)$-dimensional intervals $P'_j \in \mathcal{Q}'^{\mathrm{f}}$. Then the corresponding lateral faces of P_ε for $\varepsilon > 0$ are

$$S^\varepsilon_{a_j} := ((P'_j)_\varepsilon \times \{a_j - \varepsilon\})_j \cap C \quad \text{and} \quad S^\varepsilon_{b_j} := ((P'_j)_\varepsilon \times \{b_j + \varepsilon\})_j \cap C\,.$$

For suitable $Q'^\varepsilon_{a_j}$, $Q'^\varepsilon_{b_j} \in \{\mathcal{Q}' \cup \emptyset\}$ we have that

$$\begin{aligned} S^\varepsilon_{a_j,Q} &:= S^\varepsilon_{a_j} \cap Q = (Q'^\varepsilon_{a_j} \times \{a_j - \varepsilon\})_j\,, \\ S^\varepsilon_{b_j,Q} &:= S^\varepsilon_{b_j} \cap Q = (Q'^\varepsilon_{b_j} \times \{b_j + \varepsilon\})_j\,, \end{aligned} \tag{4.36}$$

and, hence,

$$\partial P_\varepsilon \cap Q = \bigcup_{j=1}^N (S^\varepsilon_{a_j,Q} \cup S^\varepsilon_{b_j,Q})\,. \tag{4.37}$$

Let $\partial S^\varepsilon_{a_j,Q}$ denote the boundary and $\operatorname{int} S^\varepsilon_{a_j,Q}$ the interior of $S^\varepsilon_{a_j,Q}$ relative to $(C' \times \{a_j - \varepsilon\})_j$; let $\partial S^\varepsilon_{b_j,Q}$ and $\operatorname{int} S^\varepsilon_{b_j,Q}$ be defined analogously. Using the notation

$$S^\varepsilon_\partial := \bigcup_{j=1}^N \left(\partial S^\varepsilon_{a_j,Q} \cup \partial S^\varepsilon_{b_j,Q}\right), \quad S_\partial := \bigcup_{\varepsilon > 0} S^\varepsilon_\partial\,,$$

we get the decomposition

$$\partial P_\varepsilon \cap Q = S^\varepsilon_\partial \;\cup\; \left(\bigcup_{j=1}^N (\operatorname{int} S^\varepsilon_{a_j,Q} \cup \operatorname{int} S^\varepsilon_{b_j,Q}) \right) \tag{4.38}$$

and, obviously, $\mathcal{L}^N(S_\partial) = 0$. Then (4.35) with $A = S_\partial$ implies that

$$\begin{aligned} f(P,Q) &= \operatorname{ap}\lim_{\varepsilon\downarrow 0} f((\partial P_\varepsilon \cap Q) \setminus S_\partial, Q \setminus P_\varepsilon) \\ &= \operatorname{ap}\lim_{\varepsilon\downarrow 0} f((\partial P_\varepsilon \cap Q) \setminus S^\varepsilon_\partial, Q \setminus P_\varepsilon)\,. \end{aligned} \tag{4.39}$$

Taking into account (4.38) and the additivity of f we have to evaluate $f(\operatorname{int} S^\varepsilon_{a_j,Q}, Q \setminus P_\varepsilon)$ and $f(\operatorname{int} S^\varepsilon_{b_j,Q}, Q \setminus P_\varepsilon)$, and it is sufficient to do this for $\varepsilon \in \breve{H}$. If $\operatorname{int} S^\varepsilon_{a_j,Q} \neq \emptyset$, then $\operatorname{int} Q'^\varepsilon_{a_j} \neq \emptyset$ by (4.36) and

$$f(\operatorname{int} S^\varepsilon_{a_j,Q}, Q \setminus P_\varepsilon) = \lim_{\sigma\uparrow 0} f\Big((\,(Q'^\varepsilon_{a_j}\,)_\sigma \times \{a_j - \varepsilon\})_j, Q \setminus P_\varepsilon \Big) \tag{4.40}$$

as a basic property of measures. Analyzing the relation between the sets we conclude that

$$f\Big((\,(Q'^\varepsilon_{a_j}\,)_\sigma \times \{a_j - \varepsilon\})_j, Q \setminus P_\varepsilon \Big) = f\Big((\,(Q'^\varepsilon_{a_j}\,)_\sigma \times \{a_j - \varepsilon\})_j, (\,C' \times (0, a_j - \varepsilon]\,)_j \Big)$$

for $\sigma < 0$ by (C1). The additivity of f in its second argument, (C2), and (4.34), imply that the right-hand side of this equation equals

$$-f\Big((\,(Q'^\varepsilon_{a_j}\,)_\sigma \times \{a_j - \varepsilon\})_j, (\,C' \times [a_j - \varepsilon, 1))_j \Big)$$

for all $\varepsilon \in \breve{H}$. Hence, by (4.31), (4.33),

$$\begin{aligned} f\Big((\,(Q'^\varepsilon_{a_j}\,)_\sigma \times \{a_j - \varepsilon\})_j, Q \setminus P_\varepsilon \Big) &= -\mu^j_{a_j-\varepsilon}(\,(Q'^\varepsilon_{a_j}\,)_\sigma\,) \\ &= -\mu^{j,\mathrm{ac}}_{a_j-\varepsilon}(\,(Q'^\varepsilon_{a_j}\,)_\sigma\,) \\ &\xrightarrow{\sigma\uparrow 0} -\mu^{j,\mathrm{ac}}_{a_j-\varepsilon}(Q'^\varepsilon_{a_j}) \end{aligned} \tag{4.41}$$

for $\varepsilon \in \breve{H}$, $j = 1, \ldots, N$. Consequently, by (4.33), (4.40),

$$f(\operatorname{int} S^\varepsilon_{a_j,Q}, Q \setminus P_\varepsilon) = -\mu^{j,\mathrm{ac}}_{a_j-\varepsilon}(Q'^\varepsilon_{a_j}) = -\int_{Q'^\varepsilon_{a_j}} \tau^j((x', a_j - \varepsilon)_j)\, dx'$$

and, analogously,

$$f(\operatorname{int} S^\varepsilon_{b_j,Q}, Q \setminus P_\varepsilon) = \mu^{j,\mathrm{ac}}_{b_j+\varepsilon}(Q'^\varepsilon_{b_j}) = \int_{Q'^\varepsilon_{b_j}} \tau^j((x', b_j + \varepsilon)_j)\, dx'$$

for all $\varepsilon \in \breve{H}$, $j = 1, \ldots, N$. Notice that the previous identities are also satisfied if $\operatorname{int} S^\varepsilon_{a_j,Q} = \emptyset$ or $\operatorname{int} S^\varepsilon_{b_j,Q} = \emptyset$. Recalling (4.38), (4.39) we obtain that

$$\begin{aligned} f(P,Q) &= \operatorname{ap}\lim_{\varepsilon\downarrow 0} f((\partial P_\varepsilon \cap Q) \setminus S^\varepsilon_\partial, Q \setminus P_\varepsilon) \\ &= \operatorname{ap}\lim_{\varepsilon\downarrow 0} \sum_{j=1}^N \Big(f(\operatorname{int} S^\varepsilon_{a_j,Q}, Q \setminus P_\varepsilon) + f(\operatorname{int} S^\varepsilon_{b_j,Q}, Q \setminus P_\varepsilon) \Big) \\ &= \operatorname{ap}\lim_{\varepsilon\downarrow 0} \sum_{j=1}^N \Big(\mu^{j,\mathrm{ac}}_{b_j+\varepsilon}(Q'^\varepsilon_{b_j}) - \mu^{j,\mathrm{ac}}_{a_j-\varepsilon}(Q'^\varepsilon_{a_j}) \Big) \end{aligned} \tag{4.42}$$

$$
\begin{aligned}
&= \operatorname{ap}\lim_{\varepsilon\downarrow 0}\sum_{j=1}^{N}\Big(\int_{Q'^{\varepsilon}_{b_j}} \tau^j((x',b_j+\varepsilon)_j)\,dx' - \int_{Q'^{\varepsilon}_{a_j}} \tau^j((x',a_j-\varepsilon)_j)\,dx'\Big)\\
&\overset{(4.36)}{=} \operatorname{ap}\lim_{\varepsilon\downarrow 0}\sum_{j=1}^{N}\Big(\int_{S^{\varepsilon}_{b_j,Q}} \tau^j(x)\,d\mathcal{H}^{N-1}(x) - \int_{S^{\varepsilon}_{a_j,Q}} \tau^j(x)\,d\mathcal{H}^{N-1}(x)\Big)\\
&\overset{(4.37)}{=} \operatorname{ap}\lim_{\varepsilon\downarrow 0}\int_{\partial P_\varepsilon\cap Q} \tau(x)\cdot\nu_P(x)\,d\mathcal{H}^{N-1}(x)\,. \qquad (4.43)
\end{aligned}
$$

The equality of the sum in (4.42) with the integral in (4.43), the definition of the μ^j_ξ, the fact that $\mu^j_\xi = \mu^{j,\mathrm{ac}}_\xi$ for $\xi \in H$, and the local boundedness of f imply that the real-valued function

$$
\psi(\varepsilon) := \int_{\partial P_\varepsilon\cap Q} \tau(x)\cdot\nu_P(x)\,d\mathcal{H}^{N-1}(x)
$$

is essentially bounded on $(0,1)$. Fubini's theorem implies that ψ is $\mathcal{L}^1$-measurable on $(0,1)$. By Proposition 8.1 we thus obtain that

$$
\begin{aligned}
f(P,Q) &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{\varepsilon}\int_{\partial P_\sigma\cap Q} \tau(x)\cdot\nu_P(x)\,d\mathcal{H}^{N-1}(x)\,d\sigma\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{(P_\varepsilon\setminus P)\cap Q} \tau(x)\cdot\nu_P(x)\,dx \qquad (4.44)
\end{aligned}
$$

for any $P, Q \in \mathcal{Q}^{\mathrm{f}}$.

The previous construction readily shows that the components τ^j of the interaction tensor τ are uniquely determined up to a set of $\mathcal{L}^N$-measure zero. Note that $\tau^j \in \mathcal{L}^1_{\mathrm{loc}}(C)$ as densities of the absolutely continuous measures μ^j, $j = 1,\dots,N$. This verifies (4.2) for $R \in \mathcal{Q}^{\mathrm{f}}$.

Let now $R \in \mathcal{R}^{\mathrm{f}}$, $Q \in \mathcal{Q}^{\mathrm{f}}$. Obviously ∂R has $\mathcal{L}^N$-measure zero and, thus, $f(Q,R) = f(Q,\operatorname{cl} R)$ by (C2). On the other hand, $\operatorname{cl} R$ is the union of finitely many $R_k \in \mathcal{Q}^{\mathrm{f}}$ with pairwise disjoint interiors and $f(Q,R_k) = f(Q,\operatorname{int} R_k)$ again by (C2). The additivity of $f(Q,\cdot)$ then readily implies (4.2) for all $R \in \mathcal{R}^{\mathrm{f}}$. Thus the theorem is proved for the case where C is a bounded open cube.

(f) Let us now assume that C is a bounded Borel set. We cover C by an open cube $\breve{C}$. By $\breve{f}$ we denote the zero extension of f on $\breve{C}$ relative to $\mathcal{R}^{\mathrm{f}}_{\breve{C}}$. We know that $\breve{f}$ is a contact interaction on $\breve{C}$ relative to $\mathcal{R}^{\mathrm{f}}_{\breve{C}}$ by Proposition 3.5. By our previous proof there is an interaction tensor $\breve{\tau} \in \mathcal{L}^1(\breve{C})$ for $\breve{f}$ such that (4.2) is satisfied. We now show that $\breve{\tau}$ has to vanish a.e. on $\tilde{C} := \breve{C}\setminus C$.

Suppose that there is some j such that $\breve{\tau}^j \neq 0$ on a Borel subset $\tilde{B} \subset \tilde{C}$ with $\mathcal{L}^N(\tilde{B}) > 0$. Without loss of generality let us assume that $\breve{\tau}^j(x) > 0$ for all $x \in \tilde{B}$ and let H be the set defined in (4.33) corresponding to $\breve{f}$. Then there exists a $\xi \in H$ such that $\tilde{B}'_\xi := \{x' \in \breve{C}' \mid (x',\xi)_j \in \tilde{B}\}$ has positive $\mathcal{H}^{N-1}$-measure. Consequently, by (4.31),

$$
\breve{f}((\tilde{B}'_\xi\times\{\xi\})_j, (\breve{C}'\times[\xi,1))_j) = \int_{\tilde{B}'_\xi} \breve{\tau}^j((x',\xi)_j)\,dx' > 0\,.
$$

On the other hand, $\tilde{B}\cap C = \emptyset$ and, by (2.2), we obtain the contradiction that

$$
\breve{f}((\tilde{B}'_\xi\times\{\xi\})_j, (\breve{C}'\times[\xi,1))_j) = 0\,.
$$

Hence $\breve{\tau}(x) = 0$ a.e. on $\tilde{C}$. Now the assertion is readily proved with $\tau(x) := \breve{\tau}(x)$ for all $x \in C$ and, hence, the theorem is valid for any contact interaction f on a bounded Borel set C.

(g) Let us finally consider the case where C is unbounded. For technical simplicity we assume that $\mathbb{Z} \in H^{\mathrm{f}}$. We cover C with bounded closed unit cubes $C_k \in \mathcal{Q}^{\mathrm{f}}_C$, $k \in \mathbb{N}$, having coordinates $a^k_j, b^k_j = a^k_j + 1 \in \mathbb{Z}$, $j = 1, \dots, N$ (cf. (2.1)), i.e.,

$$C = \bigcup_{k \in \mathbb{N}} C_k \,, \qquad \mathcal{L}^N(C_k \cap C_l) = 0 \quad \text{for } k \neq l \,. \tag{4.45}$$

By f_k we denote the complete restriction of f on $C_k \in \mathcal{Q}^{\mathrm{f}}_C$ which is a contact interaction on the bounded Borel set C_k relative to $\mathcal{R}^{\mathrm{f}}_{C_k}$ by Proposition (3.5) and which is obviously locally bounded. Applying our previous proof to the f_k we find interaction tensors $\tau_k \in \mathcal{L}^1(C_k)$ such that (4.2) is satisfied for f_k on C_k. We set

$$\tau(x) := \tau_k(x) \quad \text{if } x \in C_k \,.$$

Clearly $\tau \in \mathcal{L}^1_{\mathrm{loc}}(C)$. Fixing $Q, R \in \mathcal{Q}^{\mathrm{f}}_C$, we find that $Q \cap C_k \neq \emptyset$ only for finitely many $k = k_1, \dots, k_l$. Thus, taking into account (C2) and (4.45), we obtain

$$f(Q,R) = f(Q, R \cap \tilde{C}) + \sum_{k=k_1,\dots,k_l} f(Q, R \cap C_k) \,, \quad \tilde{C} := \bigcup_{\substack{k \in \mathbb{N} \\ k \neq k_1,\dots,k_l}} C_k \,.$$

Since Q and the C_k are closed with respect to C, we get $Q_\varepsilon \cap \tilde{C} = \emptyset$ for some small $\varepsilon > 0$ and, thus, $f(Q, R \cap \tilde{C}) = 0$ by (C1). On the other hand,

$$f(Q, R \cap C_k) \overset{(3.1)}{=} f(Q \cap C_k, R \cap C_k) = f_k(Q \cap C_k, R \cap C_k)$$

for $k = 1, \dots, k_l$. Thus, by (4.2) for f_k,

$$\begin{aligned} f(Q,R) &= \sum_{k=k_1,\dots,k_l} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{(Q_\varepsilon \setminus Q) \cap R \cap C_k} \tau_k \cdot \nu_Q \, d\mathcal{L}^N \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{(Q_\varepsilon \setminus Q) \cap R} \tau \cdot \nu_Q \, d\mathcal{L}^N \,. \end{aligned}$$

This finishes the proof of the Theorem 4.1. □

Proof of Corollary 4.4. The measures f^i correspond to the μ^i_ξ defined in (4.31). The assertion of the corollary follows, since a corresponding set H like that in (4.33) has to have full $\mathcal{L}^1$-measure. Notice that we have to adopt arguments as in part (f) and (g) of the previous proof if C is not an open cube. □

Proof of Corollary 4.5. By the same arguments as in the proof of Theorem 4.1 we can restrict our attention to the case where $M = 1$ and $C = (0,1)^N$. Furthermore we can choose a countable dense subset $H^{\mathrm{cd}} \subset H^{\mathrm{d}}$. Thus $\mathcal{Q}^{\mathrm{cd}} := \mathcal{Q}(H^{\mathrm{cd}})$ is a countable subset of $\mathcal{Q}^{\mathrm{d}}$.

Let us first consider sets of the form $Q = Q' \times (0,1)$ with $Q' \in \mathcal{Q}'^{\mathrm{cd}} := \mathcal{Q}'(H^{\mathrm{cd}})$. Since $f(\cdot, Q)$ is a measure, the real-valued function

$$\varphi_{Q'}(\xi) := f(Q' \times (0,\xi], Q) \overset{(C1)}{=} f(Q' \times \{\xi\}, Q' \times [\xi, 1))$$

has to be continuous from the right on $(0,1)$. Hence each value $\varphi_{Q'}(\xi)$ is uniquely determined as limit of the known values $\varphi_{Q'}(\zeta)$ with $\zeta \in H^{\mathrm{cd}}$ for all $\xi \in (0,1)$, $Q' \in \mathcal{Q}'^{\mathrm{cd}}$. According to the proof of Theorem 4.1, there is $H \subset (0,1) \cap H^{\mathrm{f}}$ with $\mathcal{L}^1(H) = 1$ such that the measure

$$\mu^N_\xi(P') := f(P' \times \{\xi\}, C' \times [\xi, 1))$$

is absolutely continuous with respect to $\mathcal{L}^{N-1}$ on C' for all $\xi \in H$ (cf. (4.14), (4.33)). On the other hand, we can argue as in part (a) of the proof of Theorem 4.1 and obtain that, for a possibly smaller set H of full $\mathcal{L}^1$-measure,

$$f(P' \times \{\xi\}, R' \times [\xi, 1)) = 0 \quad \text{for all } \xi \in H,\ P' \in \mathcal{Q}'^{\mathrm{cd}}_0,\ R' \in \mathcal{Q}'^{\mathrm{cd}}$$

where $\mathcal{Q}'^{\mathrm{cd}}_0$ denotes all elements of $\mathcal{Q}'^{\mathrm{cd}}$ with $\mathcal{H}^{N-1}$-measure zero. Thus

$$f(\partial Q' \times \{\xi\}, C' \times [\xi, 1)) = f(\partial Q' \times \{\xi\}, Q' \times [\xi, 1)) = 0$$

for all $\xi \in H$, $Q' \in \mathcal{Q}'^{\mathrm{cd}}$. Consequently,

$$\mu^N_\xi(Q') = f(Q' \times \{\xi\}, Q' \times [\xi, 1)) = \varphi_{Q'}(\xi) \quad \text{for all } \xi \in H,\ Q' \in \mathcal{Q}'^{\mathrm{cd}},$$

but these values can already considered to be known. Since $\mathcal{Q}'^{\mathrm{cd}}$ generates the Borel sets $\mathcal{B}_{C'}$, the absolutely continuous measures μ^N_ξ and their densities τ^N_ξ are uniquely determined for all $\xi \in H$ this way. By (4.32) the component τ^N of the interaction tensor τ is uniquely determined (up to a set of $\mathcal{L}^N$-measure zero) by

$$\tau^N(x', \xi) = \tau^N_\xi(x') \quad \text{for a.e. } x' \in C' \tag{4.46}$$

for all $\xi \in H$. We obtain the other components $\tau^1, \dots, \tau^{N-1}$ by analogous arguments. Since f is uniquely determined by its interaction tensor τ according to Theorem 4.1, the assertion is proved. $\square$

Proof of Corollary 4.6. (1) $f_{(A)}$ is a contact interaction by Proposition 3.5 and, obviously, it is locally bounded. Thus $f_{(A)}$ has an interaction tensor $\tau_A \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R}^{M \times N})$ by Theorem 4.1. We argue in the same way as in part (f) of the proof of Theorem (4.1) to obtain that $\tau_A(x) = 0$ a.e. on $C \setminus A$. For $A \in \mathcal{R}^{\mathrm{f}}$ we can first replace R with $R \cap A$ and then τ with τ_A in (4.2) to get the corresponding representation formula for $f_{(A)}$. The uniqueness of the interaction tensor (up to a set of $\mathcal{L}^N$-measure zero) verifies the assertion.

(2) The zero extension $\breve{f}$ is a contact interaction on $\breve{C}$ by Proposition 3.5 and it is clearly locally bounded. Again Theorem 4.1 provides the existence of an interaction tensor $\breve{\tau} \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R}^{M \times N})$. Now we can again argue in the same way as in part (f) of the proof of Theorem (4.1) to see that $\breve{\tau}(x) = 0$ a.e. on $\breve{C} \setminus C$. As in the previous case we obtain that $\breve{\tau}$ has to equal τ a.e. on C. $\square$

5 Tensors with divergence measure

In this section we further analyze the structure of interaction tensors of contact interactions and first introduce an important notion. Let $C \subset \mathbb{R}^N$ be open. We say that $\tau \in \mathcal{L}^1_{\text{loc}}(C, \mathbb{R}^{M \times N})$ has *divergence measure* if for each compact $K \subset C$ there exists $c_K \geq 0$ such that

$$\left| \int_C \tau \cdot D\varphi \, d\mathcal{L}^N \right| \leq c_K \max_K |\varphi| \tag{5.1}$$

for all $\varphi \in \mathcal{C}^\infty_0(C, \mathbb{R})$ with $\operatorname{spt} \varphi \subset K$. By Riesz's representation theorem this condition is equivalent with the existence of a vector-valued Radon measure σ on C such that

$$-\int_C \tau \cdot D\varphi \, d\mathcal{L}^N = \int_C \varphi \, d\sigma \tag{5.2}$$

for all Lipschitz continuous functions $\varphi : C \mapsto \mathbb{R}^M$ having compact support. In the sense of distributions we then have that $\sigma = \operatorname{div} \tau$. Note that σ is independent of a change of τ on a set of $\mathcal{L}^N$-measure zero.

We now ask to what extent interaction tensors have divergence measure. In the preceding section we have verified the existence of an interaction tensor τ for contact interactions f on arbitrary Borel sets $C \in \mathbb{R}^N$, but tensor fields having divergence measure are merely defined on open sets C. On the other hand, Corollary 4.6 justifies that we can identify a contact interaction f with its zero extension on a larger set. Therefore it is not restrictive to limit our further attention to contact interactions that are defined on an open set C. As before $\mathcal{Q}^{\mathrm{f}}$ denotes a full subsystem of $\mathcal{Q}$, and $\mathcal{R}^{\mathrm{f}} \subset \mathcal{R}$ is the algebra generated by $\mathcal{Q}^{\mathrm{f}}$.

Theorem 5.3 *Let $f : \mathcal{B} \times \mathcal{R}^{\mathrm{f}} \mapsto \mathbb{R}^M$ be a locally bounded contact interaction on the open set $C \subset \mathbb{R}^N$ relative to $\mathcal{R}^{\mathrm{f}}$ and let $\tau \in \mathcal{L}^1_{\text{loc}}(C, \mathbb{R}^{M \times N})$ be the corresponding interaction tensor. Then τ has divergence measure and* $\operatorname{div} \tau = f(\cdot, C)$.

If f is a locally bounded contact interaction on C relative to some algebra $\mathcal{A}$, then we know by Corollary 4.6 that the partial restriction $f_{(A)}$ of f on $A \in \mathcal{A}$ is again a locally bounded contact interaction on C corresponding to some interaction tensor $\tau_A \in \mathcal{L}^1_{\text{loc}}(C, \mathbb{R}^{M \times N})$ while τ_A can be given explicitly by (4.7) for $A \in \mathcal{R}^{\mathrm{f}}$. Thus we can apply Theorem 5.3 to all these τ_A.

Corollary 5.4 *Let f be a locally bounded contact interaction on C relative to some algebra $\mathcal{A}$ containing $\mathcal{R}^{\mathrm{f}}$ and let $\tau_A \in \mathcal{L}^1_{\text{loc}}(C, \mathbb{R}^{M \times N})$ denote the interaction tensor of the partial restriction $f_{(A)}$ of f for $A \in \mathcal{A}$. Then τ_A has divergence measure and $f(\cdot, A) = \operatorname{div} \tau_A$.*

Remark 5.5 (1) With f as in Corollary 5.4 and with the corresponding interaction tensor $\tau = \tau_C$ we find that for all $A \in \mathcal{R}^{\mathrm{f}}$ the tensor field

$$\tau_A(x) := \begin{cases} \tau(x) & \text{for } x \in A\,, \\ 0 & \text{for } x \in C \setminus A\,, \end{cases}$$

has divergence measure according to (4.7).

(2) Let τ denote the interaction tensor of a locally bounded contact interaction f on the open set C relative to some algebra $\mathcal{A}$ with $\mathcal{R}^{\mathrm{f}} \subset \mathcal{A}$. With $\breve{\mathcal{A}} := \mathcal{R}^{\mathrm{f}}_{\mathbb{R}^N}$ we have an algebra on $\mathbb{R}^N$ satisfying $\breve{\mathcal{A}}_{|C} = \mathcal{A}$. Hence the zero extension $\breve{f}$ of f on $\mathbb{R}^N$ is a contact interaction by Proposition 3.5 and the corresponding interaction tensor $\breve{\tau}$ is just the extension of τ on $\mathbb{R}^N$ by zero according to Corollary 4.6. With $A = \mathbb{R}^N \in \breve{\mathcal{A}}$ in Corollary 5.4 we obtain that $\breve{f}(\cdot, \mathbb{R}^N) = \operatorname{div} \breve{\tau}$. By (2.2) we readily imply that $|\operatorname{div} \breve{\tau}|(\partial C) = 0$ since $\partial C \subset \mathbb{R}^N \setminus C$.

Proof of Theorem 5.3. Except for the first part, the proof basically coincides with that of Degiovanni, Marzocchi & Musesti [15, Theorem 5.3] though the assumptions there are different. For the convenience of the reader we adapt these arguments to our situation.

(a) Let $E^t \subset \mathcal{Q}_{\mathbb{R}^N}$ denote the closed N-interval with coordinates $a_j = -t$, $b_j = t$, $j = 1, \dots, N$, $t > 0$. We fix any point $x \in C$ and set $Q_x^t := x + E^t$. Clearly there is a $t_0 = t_0(x) > 0$ such that $Q_x^t \Subset C$ for all $t \in (0, t_0)$. Using Fubini's theorem we find that the function

$$\alpha(t) := \int_{\partial Q_x^s} \tau \cdot \nu_{Q_x^s} \, d\mathcal{H}^{N-1}$$

belongs to $\mathcal{L}^1(0, t_0)$. Since a.e. $t \in (0, t_0)$ is a Lebesgue point of $\alpha(\cdot)$ and since $Q_x^t \in \mathcal{Q}^{\mathrm{f}}$ for a.e. $t \in (0, t_0)$, we obtain by (4.2) that

$$f(Q_x^t, C) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} \alpha(s) ds = \alpha(t) \quad \text{for a.e. } t \in (0, t_0)\,. \tag{5.6}$$

(b) With $|x|_\infty := \max\{|x_1|, \dots, |x_N|\}$ we define

$$\varrho(x) := \frac{N+1}{2^N} (1 - |x|_\infty)^+ , \quad \varrho_m(x) := m^N \varrho(mx) \quad \text{for } x \in \mathbb{R}^N$$

where $(\cdot)^+$ denotes the positive part. Notice that ϱ is supported on E^1 and that $\int \varrho \, dx = 1$. Let $K_m \subset C$, $m \in \mathbb{N}$, be an increasing sequence of compact subsets of C with $C = \bigcup_{m=1}^\infty \operatorname{int} K_m$ and let $\vartheta_m \in \mathcal{C}_0^\infty(C)$ with $\vartheta_m(x) \in [0,1]$ on C and $\vartheta_m(x) = 1$ on K_m. Then the tensor fields

$$\tau_m(x) := \int_C \varrho_m(x-y) \vartheta_m(y) \tau(y) \, d\mathcal{L}^N(y) \tag{5.7}$$

belong to $C^1(C, \mathbb{R}^{M \times N})$, converge to τ in $\mathcal{L}^1_{\mathrm{loc}}(C)$, and

$$\operatorname{div} \tau_m(x) = \int_C \vartheta_m(y) \tau(y) \cdot D\varrho_m(x-y) \, d\mathcal{L}^N(y)\,.$$

(c) We fix any open $B \subset C$ with compact closure in C and now show that

$$\Big| \int_C \tau \cdot D\varphi \, d\mathcal{L}^N \Big| \le \mu(B) \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(C) \text{ with } \operatorname{spt} \varphi \subset B, \ \max_B |\varphi| \le 1 \tag{5.8}$$

for $\mu := |f|(\cdot, C)$, which then implies that τ has divergence measure.

For any φ as in (5.8) with $\tilde{B} := \operatorname{spt}\varphi$ we have that

$$\begin{aligned}
\Big|\int_C \tau\cdot D\varphi\,d\mathcal{L}^N\Big| &= \lim_{m\to\infty}\Big|\int_C \tau_m\cdot D\varphi\,d\mathcal{L}^N\Big| \\
&\le \liminf_{m\to\infty}\int_C |\operatorname{div}\tau_m|\,|\varphi|\,d\mathcal{L}^N \\
&\le \liminf_{m\to\infty}\int_{\tilde{B}} |\operatorname{div}\tau_m|\,d\mathcal{L}^N\,. \qquad (5.9)
\end{aligned}$$

Setting $t_m := (N+1)m^N/2^N$ we see that

$$\{y\in\mathbb{R}^N \mid \varrho_m(x-y) > t\} = x + E^{\frac{t_m-t}{mt_m}} \quad \text{for } t\in[0,t_m)\,.$$

For sufficiently large m we ensure that $\tilde{B}+E^{\frac{1}{m}} \subset B$ and that $\vartheta_m = 1$ on $\tilde{B}+E^{\frac{1}{m}}$. The change of variable formula (cf. Evans & Gariepy [17, Theorem 2, p. 117]) implies that

$$\begin{aligned}
|\operatorname{div}\tau_m(x)| &= \Big|\int_{x+E^{\frac{1}{m}}} \tau(y)\cdot\frac{D\varrho_m(x-y)}{|D\varrho_m(x-y)|}|D\varrho_m(x-y)|\,d\mathcal{L}^N(y)\Big| \\
&= \Big|\int_0^{t_m}\int_{\partial(x+E^{\frac{t_m-t}{mt_m}})} \tau(y)\cdot\frac{D\varrho_m(x-y)}{|D\varrho_m(x-y)|}\,d\mathcal{H}^{N-1}(y)\,dt\Big| \\
&\overset{(5.6)}{=} \Big|\int_0^{t_m} f(x+E^{\frac{t_m-t}{mt_m}},C)\,dt\Big| \\
&\le \int_0^{t_m}\mu(x+E^{\frac{t_m-t}{mt_m}})\,dt \\
&= t_m\int_0^1 \mu(x+E^{\frac{s}{m}})\,ds \quad \text{for } x\in\tilde{B}\,. \qquad (5.10)
\end{aligned}$$

Fubini's Theorem implies that

$$\begin{aligned}
\int_{\tilde{B}}|\operatorname{div}\tau_m(x)|\,d\mathcal{L}^N(x) &\le t_m\int_{\tilde{B}}\int_0^1 \mu(x+E^{\frac{s}{m}})\,ds\,d\mathcal{L}^N(x) \\
&= t_m\int_0^1\int_{\tilde{B}}\int_B \chi_{x+E^{\frac{s}{m}}}(y)\,d\mu(y)\,d\mathcal{L}^N(x)\,ds \\
&= t_m\int_0^1\int_B\int_{\tilde{B}} \chi_{x+E^{\frac{s}{m}}}(y)\,d\mathcal{L}^N(x)\,d\mu(y)\,ds \\
&= t_m\int_0^1\int_B \frac{2^N s^N}{m^N}\,d\mu(y)\,ds \;=\; \mu(B)\,,
\end{aligned}$$

which, together with (5.9), implies (5.8).

(d) For $Q\in\mathcal{Q}^{\mathrm{f}}$ we set

$$\varphi_\varepsilon(x) := \begin{cases} 1 & \text{on } Q, \\ 0 & \text{on } Q_\sigma \text{ for } \sigma\ge\varepsilon\,, \\ 1-\frac{\sigma}{\varepsilon} & \text{on } Q_\sigma \text{ for } \sigma\in[0,\varepsilon]\,. \end{cases}$$

Then, by (4.2), (5.2),

$$f(Q,C) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{Q_\varepsilon\setminus Q}\tau\cdot\nu_Q\,d\mathcal{L}^N \;=\; -\lim_{\varepsilon\downarrow 0}\int_{Q_\varepsilon\setminus Q}\tau\cdot D\varphi_\varepsilon\,d\mathcal{L}^N$$

$$
\begin{aligned}
&= -\lim_{\varepsilon\downarrow 0}\int_C \tau\cdot D\varphi_\varepsilon\, d\mathcal{L}^N \;=\; \lim_{\varepsilon\downarrow 0}\int_{Q_\varepsilon}\varphi_\varepsilon\, d(\operatorname{div}\tau)\\
&= (\operatorname{div}\tau)(Q)\,.
\end{aligned}
$$

Since $\mathcal{Q}^{\mathrm{f}}$ generates $\mathcal{B}$, we get $\operatorname{div}\tau = f(\cdot, C)$. $\square$

As in Degiovanni et al. [15, Theorem 5.4] we obtain the following formula for partial integration. Recall that $\mathcal{P}$ denotes the sets of finite perimeter in the set C where C is assumed to be open.

Proposition 5.11 *Let $\tau \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R}^{M\times N})$ be a tensor field with divergence measure. Then there exists a nonnegative real-valued function $h \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R})$ such that*

$$
\int_P \tau\cdot D\varphi^T\, d\mathcal{L}^N = \int_{\partial_* P}\varphi\,\tau\cdot\nu_P\, d\mathcal{H}^{N-1} - \int_{P_*}\varphi\, d(\operatorname{div}\tau) \tag{5.12}
$$

for all $P\in\mathcal{P}$ with $\int_{\partial_* P} h\, d\mathcal{H}^{N-1} < \infty$, $|\operatorname{div}\tau|(\partial_* P) = 0$, all locally Lipschitz continuous functions $\varphi : C \mapsto \mathbb{R}^M$, and such that either $\operatorname{cl} P \subset C$ is compact or φ has compact support in C.

Proof. We briefly repeat the main steps of the proof from [15, Theorem 5.2, 5.4] both for the convenience of the reader and for later reference, since we will extend some arguments in our subsequent analysis.

We define a sequence $\tau_n \in \mathcal{C}^\infty(C, \mathbb{R}^{M\times N})$ as in (5.7) but with a nonnegative function $\varrho \in \mathcal{C}^\infty_0(\mathbb{R}^N)$ satisfying $\int \varrho\, d\mathcal{L}^N = 1$. By Brezis [7, Theorem IV.9] there is a nonnegative function $h \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R})$ and a subsequence denoted the same way such that

$$
\lim_{n\to\infty}\tau_n = \tau \quad \text{in } \mathcal{L}^1_{\mathrm{loc}}(C)\,, \quad \lim_{n\to\infty}\tau_n(x) = \tau(x) \quad \text{for all } x\in C \text{ with } \; h(x) < \infty\,, \tag{5.13}
$$

$$
|\tau_n(x)| \le h(x) \quad \text{for all } x\in C\,,\ n\in\mathbb{N}\,. \tag{5.14}
$$

By partial integration and Fubini's theorem we obtain for P and φ as in the proposition and for $n\in\mathbb{N}$ sufficiently large that

$$
\begin{aligned}
\int_P \varphi(x)\operatorname{div}\tau_n(x)\, d\mathcal{L}^N(x) &= \int_P \varphi(x)\Big(\int_C \tau(y)\cdot D\varrho_n(x-y)\, d\mathcal{L}^N(y)\Big)\, d\mathcal{L}^N(x)\\
&= \int_P \varphi(x)\Big(\int_C \varrho_n(x-y)\, d(\operatorname{div}\tau)(y)\Big)\, d\mathcal{L}^N(x)\\
&= \int_C \Big(\int_P \varphi(x)\varrho_n(x-y)\, d\mathcal{L}^N(x)\Big)\, d(\operatorname{div}\tau)(y)\,.
\end{aligned} \tag{5.15}
$$

The inner integral on the right-hand side converges to $\varphi(y)$ on P_* and to 0 on $(C\setminus P)_*$. For a compact set $K\subset C$ with either $\operatorname{cl} P\subset \operatorname{int} K$ or $\operatorname{spt}\varphi \subset \operatorname{int} K$ we have the estimate that

$$
\Big|\int_P \varphi(x)\varrho_n(x-y)\, d\mathcal{L}^N(x)\Big| \le \chi_K(y)\ \max_K |\varphi|\,.
$$

Then Lebesgue's theorem implies that

$$
\lim_{n\to\infty}\int_P \varphi\operatorname{div}\tau_n\, d\mathcal{L}^N = \int_{P_*}\varphi\, d(\operatorname{div}\tau)\,. \tag{5.16}
$$

On the other hand, φ is Lipschitz continuous on K and

$$\int_P \tau_n \cdot D\varphi \, d\mathcal{L}^N = \int_{\partial_* P} \varphi \, \tau_n \cdot \nu_P \, d\mathcal{H}^{N-1} - \int_P \varphi \operatorname{div} \tau_n \, d\mathcal{L}^N . \tag{5.17}$$

By (5.13), (5.14), and the assumption that $\int_{\partial_* P} h \, d\mathcal{H}^{N-1} < \infty$ we get that $\int_{\partial_* P} |\tau| \, d\mathcal{H}^{N-1} < \infty$. We obtain (5.12) by taking the limit in (5.17) where we apply Lebesgue's theorem to the first two integrals and apply (5.16) to the last one. $\square$

Proposition 5.11 enables us to extend the representation formula (4.2) to a much larger class of subbodies with respect to the first argument and to provide a usual representation formula by means of surface integrals on a suitable subclass of subbodies.

First for any $B \in \mathbb{R}^N$ we introduce a Lipschitz continuous function on $\mathbb{R}^N$ by

$$\varphi_B^\varepsilon(x) := \begin{cases} 1 & \text{if } \operatorname{dist}_B x = 0 \,, \\ 0 & \text{if } \operatorname{dist}_B x \geq \varepsilon \,, \\ 1 - \frac{1}{\varepsilon} \operatorname{dist}_B x & \text{if } 0 < \operatorname{dist}_B x < \varepsilon \,, \end{cases} \tag{5.18}$$

and we define the *outer normals* of B relative to its distance function by

$$\nu_B^{\mathrm{d}}(x) := \frac{\partial}{\partial x} \operatorname{dist}_B x \quad \text{for a.e. } x \in \mathbb{R}^N \text{ with } \operatorname{dist}_B x > 0 \,.$$

The normals $\nu_B^{\mathrm{d}}(x)$ are unit vectors for all $x \in \mathbb{R}^N$ with $\operatorname{dist}_B x > 0$ where the gradient exists. By $B_\varepsilon^{\mathrm{d}}$ we denote the usual open ε-neighborhood of B, i.e., all points with $\operatorname{dist}_B x < \varepsilon$. Note that $D\varphi_B^\varepsilon = 0$ for a.e. $x \in \mathbb{R}^N$ satisfying $\varphi_B^\varepsilon(x) = 0$ or $\varphi_B^\varepsilon(x) = 1$ (cf. Gilbarg & Trudinger [21, Lemma 7.7]) and that $\varepsilon D\varphi_B^\varepsilon(x) = -\nu_B^{\mathrm{d}}(x)$ for a.e. $x \in \mathbb{R}^N$ with $0 < \operatorname{dist}_B x < \varepsilon$.

We say that a property is true for almost every $P \subset \mathcal{P}$ if there exist a nonnegative real function $h \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R})$ and a nonnegative measure μ on C such that the property holds for all $P \in \mathcal{P}$ belonging to

$$\mathcal{P}_{h\mu} := \Big\{ P \in \mathcal{P} \,\Big|\, \int_{\partial_* P} h \, d\mathcal{H}^{N-1} < \infty, \ \mu(\partial_* P) = 0 \Big\} . \tag{5.19}$$

Theorem 5.20 *Let $f : \mathcal{B} \times \mathcal{A} \mapsto \mathbb{R}^M$ be a locally bounded contact interaction on the open set $C \subset \mathbb{R}^N$ with $\mathcal{R}^{\mathrm{f}} \subset \mathcal{A}$ and let $\tau_A \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R}^{M \times N})$ denote the interaction tensor corresponding to the partial restriction $f_{(A)}$ of f on $A \in \mathcal{A}$.*

(1) *For any bounded $B \in \mathcal{B}$ that is closed relatively to C and for any $A \in \mathcal{A}$,*

$$f(B, A) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{(B_\varepsilon^{\mathrm{d}} \setminus B)} \tau_A \cdot \nu_B^{\mathrm{d}} \, d\mathcal{L}^N . \tag{5.21}$$

(2) *For any $A \in \mathcal{A}$,*

$$\begin{gathered} f(P, A) = \textstyle\int_{\partial_* P} \tau_A \cdot \nu_P \, d\mathcal{H}^{N-1} \\ \textit{for a.e. bounded normalized } P \in \mathcal{P} . \end{gathered} \tag{5.22}$$

(3) *For each $A \in \mathcal{A}$ there is a full subsystem $\check{\mathcal{Q}} \subset \mathcal{Q}^{\mathrm{f}}$ such that*

$$f(Q, A) = \int_{\partial Q} \tau_A \cdot \nu_Q \, d\mathcal{H}^{N-1} \quad \textit{for all } Q \in \check{\mathcal{Q}} .$$

The right-hand side in the previous three formulas is always called *normal trace* of τ_A on the corresponding set B, P, or Q.

Remark 5.23 (1) Note that for each $A \in \mathcal{A}$ the value $f(B, A)$ is well defined for *all* Borel sets $B \in \mathcal{B}$ though we have a representation as some normal trace merely for "good" sets B. In particular, this means that, in our setting, contact interactions are defined not only on sufficiently regular surfaces.

(2) By (4.7) we can replace the tensor τ_A with $\tau = \tau_C$ in the representation formulas in Theorem 5.20 for all $A \in \mathcal{R}^{\mathrm{f}}$ if we then restrict the integration on the set A. In the next section we will see that this is even true for all $A \in \mathcal{A}$.

Proof. Let $\breve{\tau}_A$ always denote the extension of τ_A to $\mathbb{R}^N$ by zero. By Corollary 4.6 we know that this is the interaction tensor of the zero extension $\breve{f}_{(A)}$ of $f_{(A)}$ on $\mathbb{R}^N$. Clearly $\breve{\tau}_A \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$. Taking Remark 5.5.2 into account we see that the measure $\operatorname{div} \breve{\tau}_A$ is just the extension of the measure $\operatorname{div} \tau_A$ on all Borel sets of $\mathbb{R}^N$ by zero, i.e., $|\operatorname{div} \breve{\tau}_A|(\mathbb{R}^N \setminus C) = 0$. Hence

$$f(B, A) = f_{(A)}(B, A) = \operatorname{div} \tau_A(B) = \operatorname{div} \breve{\tau}_A(B) = \breve{f}_{(A)}(B, A).$$

Therefore it is sufficient to verify the assertions for the interactions $\breve{f}_{(A)}$ on $\mathbb{R}^N$ and the corresponding tensors $\breve{\tau}_A$.

For the first assertion we choose some large $P \in \mathcal{Q}_{\mathbb{R}^N}$ such that $\operatorname{cl} B \subset \operatorname{int} P$ and that (5.12) with $\breve{\tau}_A$ holds for this P. Then we evaluate (5.12) with φ^ε_B. If we observe that φ^ε_B vanishes on ∂P for sufficiently small $\varepsilon > 0$ and that $D\varphi^\varepsilon_B = 0$ for a.e. $x \in \mathbb{R}^N$ satisfying $\varphi^\varepsilon_B(x) = 0$ or $\varphi^\varepsilon_B(x) = 1$ (cf. Gilbarg & Trudinger [21, Lemma 7.7]), then we get (5.21). For the second assertion we choose $\varphi \equiv 1$ in (5.12) with $\breve{\tau}_A$ and observe that $\operatorname{cl} P \subset \mathbb{R}^N$ is compact according to our boundedness assumption. The last assertion is a direct consequence of Degiovanni et al. [15, Theorem 7.2]. □

6 Extension of contact interactions

In our previous investigations we have occasionally considered contact interactions f that are merely defined on $\mathcal{B} \times \mathcal{R}^{\mathrm{f}}$. But in Example 2 of Section 3 we have seen that in the case of concentrations it is interesting to know the interaction $f(B, A)$ for sets A that are cones. This means that the algebra $\mathcal{R}^{\mathrm{f}}$ is too small for a detailed description of concentrations. Therefore let us consider the extent to which f can be extended to an algebra $\mathcal{A}$ larger than $\mathcal{R}^{\mathrm{f}}$ with respect to the second argument. Recall that it is impossible to extend $f(B, \cdot)$ to all Borel sets $\mathcal{B}$ (cf. Example 3 in Section 3). Thus we seek a rich algebra between $\mathcal{R}^{\mathrm{f}}$ and $\mathcal{B}$. It turns out that the algebra $\mathcal{P}$ of sets of finite perimeter or at least a suitable subalgebra $\mathcal{A} \subset \mathcal{P}$ is a reasonable class. The difficulty for such an extension is that we cannot use usual approximation arguments for measures due to the lack of σ-additivity with respect to the second argument. Furthermore, notice that the axioms (C1)–(C3) give only a very weak information about the coupling between the measures $f(\cdot, A)$ for different sets A. Hence the analysis is more sophisticated and we have to exploit the additional information provided by the tensorial structure of contact interactions.

If f is a locally bounded contact interaction on C relative to some algebra $\mathcal{A}$, then we already know by Corollary 4.6 that the partial restriction $f_{(A)}$ is again a locally bounded contact interaction possessing an interaction tensor τ_A for any $A \in \mathcal{A}$. By the representation formula (4.2), the extension of f on $\mathcal{A}$ is uniquely described if we determine the tensors τ_A for all $\mathcal{A}$.

Studying such extensions we in fact want to answer two different questions. First we assume that we already have some contact interaction f on a "large" algebra $\mathcal{A}$. From Theorem 4.1 we obtain the interaction tensor τ that describes f uniquely at least on $\mathcal{B} \times \mathcal{R}^{\mathrm{f}}$, since τ uniquely determines the interaction tensors τ_R for all $R \in \mathcal{R}^{\mathrm{f}}$ according to Corollary 4.6. However, we do not yet know how far the other tensors τ_A, that exist for all $A \in \mathcal{A} \setminus \mathcal{R}^{\mathrm{f}}$, are uniquely determined by τ. In Theorem 6.1 below we will also see that these τ_A are the expected restriction of τ. This, in particular, implies the uniqueness of extensions. Secondly we can assume that a contact interaction f is merely given on the "small" system $\mathcal{B} \times \mathcal{R}^{\mathrm{f}}$. In this case we have to look for a reasonable "large" algebra $\mathcal{A}$ on which f can be extended.

Let us start with the first question where we assume that a contact interaction f on C relative to some algebra $\mathcal{A}$ larger than $\mathcal{R}^{\mathrm{f}}$ is given and we want to determine the tensors τ_A.

Theorem 6.1 *Let f be a locally bounded contact interaction on a Borel set $C \subset \mathbb{R}^N$ relative to some algebra $\mathcal{A}$ of Borel sets containing a full system $\mathcal{Q}^{\mathrm{f}}$ and let τ denote the corresponding interaction tensor. Then the interaction tensor τ_A of the partial restriction $f_{(A)}$ of f on A is given, up to a set of $\mathcal{L}^N$-measure zero, by*

$$\tau_A(x) = \begin{cases} \tau(x) & \textit{for } x \in A\,, \\ 0 & \textit{for } x \in C \setminus A \end{cases} \tag{6.2}$$

for all $A \in \mathcal{A}$.

Thus the interaction tensor τ_A and therefore also the interaction on all of $\mathcal{B} \times \mathcal{A}$ is uniquely determined by the interaction tensor τ. By Corollary 5.4 we obtain that

$$f(B, A) = (\operatorname{div} \tau_A)(B) \quad \text{for all } B \in \mathcal{B}\,, \;\; A \in \mathcal{A} \tag{6.3}$$

whith τ_A given by (6.2). This new fundamental *representation formula* completely describes the interaction f by means of the tensor field τ. It can be considered as a replacement for Cauchy's classical formula (1.1). Note that (6.3) *does not contain any surfaces or normal fields*! Nevertheless we recover the classical formula in (5.22) for sufficiently regular subbodies. Moreover, since the representation formula (4.2) is satisfied for all partial restrictions $f_{(A)}$ with the corresponding tensor τ_A, it readily follows that a locally bounded contact interaction f with corresponding interaction tensor τ satisfies the trace-like formula that

$$\begin{aligned} f(Q, A) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{(Q_\varepsilon \setminus Q) \cap A} \tau \cdot \nu_Q \, d\mathcal{L}^N \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\partial Q_\sigma \cap A} \tau \cdot \nu_Q \, d\mathcal{H}^{N-1} \, d\sigma \end{aligned} \tag{6.4}$$

for all $Q \in \mathcal{Q}^{\mathrm{f}}$, $A \in \mathcal{A}$. The proof of the theorem is deferred to the end of this section.

Let us now study the second question, i.e., we assume that a contact interaction f is given on $\mathcal{B} \times \mathcal{R}^{\mathrm{f}}$ and we seek a suitable algebra $\mathcal{A}$ larger than $\mathcal{R}^{\mathrm{f}}$ on which f can be extended with respect to the second argument. First let us discuss conditions that have to be satisfied by such an algebra $\mathcal{A}$. From Theorem 6.1 we know that the interaction tensors τ_A corresponding to the partial restrictions $f_{(A)}$ must have the form (6.2) for all $A \in \mathcal{A}$. Since all these $f_{(A)}$ are also locally bounded contact interactions, all these τ_A must have divergence measure according to Corollary 5.4. Hence we have to seek for an algebra $\mathcal{A}$ such that all tensors τ_A, $A \in \mathcal{A}$, of the form (6.2) have divergence measure. We already know that this algebra $\mathcal{A}$ cannot contain the collection $\mathcal{B}$ of all Borel sets. But, in analogy to (5.19), we consider

$$\mathcal{P}_h := \{P \in \mathcal{P} | \int_{\partial_* P} h \, d\mathcal{H}^{N-1} < \infty\}$$

for some nonnegative $h \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R})$.

Proposition 6.5 *Let $C \subset \mathbb{R}^N$ be open and let $\tau \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R}^{M \times N})$ be a tensor field with divergence measure. Then there exists a nonnegative $h \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R})$ (the same as in Proposition 5.11) such that*

$$\tau_P(x) = \begin{cases} \tau(x) & \text{for } x \in P\,, \\ 0 & \text{for } x \in C \setminus P \end{cases} \tag{6.6}$$

has divergence measure for all $P \in \mathcal{P}_h$ and the system $\mathcal{P}_h$ is an algebra. For each $P \in \mathcal{P}_h$ there is some $g_P \in \mathcal{L}^\infty(C, |\mathrm{div}\, \tau|)$ with $0 \le g_P(x) \le 1$ such that

$$\mathrm{div}\, \tau_P = g_P \mathrm{div}\, \tau - \tau \cdot \nu_P \, \mathcal{H}^{N-1} \lfloor \partial_* P \tag{6.7}$$

($(g_P \mathrm{div}\, \tau)(B) := \int_B g_P \, d(\mathrm{div}\, \tau)$) in the sense of measures and

$$g_P(x) = \mathrm{dens}_P(x) \quad \text{for all } x \in C \text{ where } \mathrm{dens}_P(x) \text{ exists.} \tag{6.8}$$

The proof of this proposition is carried out at the end of this section. Note that

$$(\mathrm{div}\, \tau_P) \lfloor P_* = (\mathrm{div}\, \tau) \lfloor P_*\,, \quad (\mathrm{div}\, \tau_P) \lfloor (C \setminus P)_* = 0 \tag{6.9}$$

for all $P \in \mathcal{P}_h$ by (6.7).

Proposition 6.5 provides $\mathcal{P}_h$ as a convenient algebra for an extension of a locally bounded contact interaction f with corresponding interaction tensor τ. By Corollary 5.4 and Theorem 6.1 we know that an extension of f must be given by

$$f(B, P) := (\mathrm{div}\, \tau_P)(B) \quad \text{for all } B \in \mathcal{B}\,, \;\; P \in \mathcal{P}_h \tag{6.10}$$

where the τ_P are related to the tensor τ according to (6.6). We still have to check whether this really provides a contact interaction on $\mathcal{B} \times \mathcal{P}_h$:

Theorem 6.11 *Let f be a locally bounded contact interaction on the open set C relative to a full subsystem $\mathcal{R}^{\mathrm{f}}$. Then there exists a nonnegative function $h \in \mathcal{L}^1_{\mathrm{loc}}(C)$ such that f can be uniquely extended to a contact interaction on $\mathcal{B} \times \mathcal{P}_h$ by (6.10).*

Here h is the same as in Proposition 6.5. Moreover, in addition to (6.10), the representation formula (6.4) is certainly valid for the extended contact interaction f. Note finally that $\mathcal{P}_h$ is a quite rich algebra, but it is still open whether contact interactions may be extended to larger algebras $\mathcal{A}$. The proof of this theorem is given below.

Proof of Theorem 6.1. Since we can argue for each component of $\tau = (\tau^{ij})$ separately, it is sufficient to consider the scalar case $M = 1$ and to restrict attention to the component τ^N of $\tau = (\tau^1, \dots, \tau^N)$. We can assume that $C = \mathbb{R}^N$, since otherwise we can use the corresponding zero extension. For notational convenience we set $f_0 := f$ and $\tau_0 := \tau$.

Let us fix any $A \in \mathcal{A}$, let A_* denote its measure-theoretic interior, and recall that $A^c = C \setminus A \in \mathcal{A}$. By $f_1 := f_{(A)}$ and $f_2 := f_{(A^c)}$ we denote the partial restrictions of f on A and A^c, respectively, and let $\tau_1 = \tau_A$ and $\tau_2 = \tau_{A^c}$ be the corresponding interaction tensors. Applying Corollary 4.4 to the f_k, $k = 0, 1, 2$, we find a full set $\check{H}^{\mathrm{f}} \subset \mathbb{R}$ independent of k such that, with $C' := \mathbb{R}^{N-1}$,

$$f_k(Q' \times \{\xi\}, C' \times [\xi, \infty)) = \int_{Q'} \tau_k^N(y', \xi)\, d\mathcal{H}^{N-1}(y') \tag{6.12}$$

for all $\xi \in \check{H}^{\mathrm{f}}$, $Q' \in \mathcal{Q}'$, $k = 0, 1, 2$. Let $L \subset C$ denote the set of all points $x \in C$ such that each $x \in L$ is a Lebesgue point for all $\tau_k^N(\cdot)$ and such that all $\tau_k^N(\cdot)$ are approximately continuous at each $x \in L$ (cf. Evans & Gariepy [17, p. 47]). Obviously, $\mathcal{H}^{N-1}((C' \times \{\xi\}) \setminus L) = 0$ for all $\xi \in \check{H}^{\mathrm{f}}$.

We now fix any $x = (x', \xi) \in A_* \cap L \cap (C' \times \{\xi\})$ with $\xi \in \check{H}^{\mathrm{f}}$. Let $Q'_\delta \subset C'$ denote the closed cube centered at x' with edges of length 2δ and set $Q_\delta := Q'_\delta \times [\xi - \delta, \xi + \delta] \subset C$. Then, by (6.12),

$$\begin{aligned} \int_{Q_\delta} \tau_k^N(y)\, d\mathcal{L}^N &= \int_{-\delta}^{\delta} \int_{Q'_\delta} \tau_k^N(y', \zeta)\, d\mathcal{H}^{N-1}(y')\, d\zeta \\ &= \int_{-\delta}^{\delta} f_k(Q'_\delta \times \{\zeta\}, C' \times [\zeta, \infty))\, d\zeta \qquad \text{for all } \delta > 0\,. \end{aligned}$$

For $\zeta \in \check{H}^{\mathrm{f}}$, additivity implies that

$$\begin{aligned} & f_0(Q'_\delta \times \{\zeta\}, C' \times [\zeta, \infty)) \\ &\quad = f_0(Q'_\delta \times \{\zeta\}, C' \times [\zeta, \infty) \cap A) + f_0(Q'_\delta \times \{\zeta\}, C' \times [\zeta, \infty) \cap A^c) \\ &\quad = f_1(Q'_\delta \times \{\zeta\}, C' \times [\zeta, \infty)) + f_2(Q'_\delta \times \{\zeta\}, C' \times [\zeta, \infty))\,. \end{aligned}$$

Thus,

$$\int_{Q_\delta} \tau_0^N(y)\, d\mathcal{L}^N = \int_{Q_\delta} \tau_1^N(y)\, d\mathcal{L}^N + \int_{Q_\delta} \tau_2^N(y)\, d\mathcal{L}^N \quad \text{for all } \delta > 0\,. \tag{6.13}$$

Since τ_2 vanishes outside of A^c and since A^c has density 0 at x, we get that

$$\operatorname{ap}\lim_{y \to x} \tau_2^N(y) = 0$$

where $\operatorname{ap}\lim$ denotes the approximate limit. Therefore $\tau_2^N(x) = 0$, since $\tau_2^N(\cdot)$ is approximately continuous for all $x \in L$. Observing that x is a Lebesgue point of $\tau_k^N(\cdot)$ for all $k = 0, 1, 2$, we derive from (6.13) that

$$\begin{aligned}
\tau_0^N(x) &= \lim_{\delta\downarrow 0} \frac{1}{\mathcal{L}^N(Q_\delta)} \int_{Q_\delta} \tau_0^N(y)\, d\mathcal{L}^N \\
&= \lim_{\delta\downarrow 0} \frac{1}{\mathcal{L}^N(Q_\delta)} \int_{Q_\delta\cap A} \tau_1^N(y)\, d\mathcal{L}^N + \lim_{\delta\downarrow 0} \frac{1}{\mathcal{L}^N(Q_\delta)} \int_{Q_\delta\cap A^{\mathrm{c}}} \tau_2^N(y)\, d\mathcal{L}^N \\
&= \tau_1^N(x)\,.
\end{aligned} \tag{6.14}$$

But this means that

$$\tau^N(x) = \tau_A^N(x) \;\text{ for all }\; x = (x',\xi) \in A_* \cap L \cap (C' \times \{\xi\}) \;\text{ with }\; \xi \in \check{H}^{\mathrm{f}}\,. \tag{6.15}$$

Since $\mathcal{L}^N(A \setminus A_*) = 0$ and since the sets L and $C' \times \check{H}^{\mathrm{f}}$ have full measure in $\mathbb{R}^N$, we get that $\tau^N(x) = \tau_A^N(x)$ a.e. on A. Since $\tau_A(x) = 0$ for a.e. $x \in A^{\mathrm{c}}$ by Corollary 4.6, we have shown (6.2) and the proof is complete. □

Proof of Proposition 6.5. The proof of the first assertion is a combination of arguments from the proofs of Degiovanni et al. [15, Theorem 5.2, 5.4] adopted to our situation.

We define a sequence $\tau_n \in \mathcal{C}^\infty(C, \mathcal{R}^{M\times N})$ as in the proof of Proposition 5.11 such that (5.13), (5.14) are satisfied with the same nonnegative $h \in \mathcal{L}^1_{\mathrm{loc}}(C)$. Now we choose $P \in \mathcal{P}_h$, a compact set $K \subset C$, and choose $\varphi \in \mathcal{C}_0^\infty(C)$ with $\operatorname{spt}\varphi \subset K$. Furthermore, let $B_\varepsilon(K) \subset C$ denote the ε-neighborhood of K for a suitable fixed $\varepsilon > 0$. Then, for all $n \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}
&\Big| \int_P \varphi(x) \operatorname{div} \tau_n(x)\, d\mathcal{L}^N \Big| \\
&\qquad = \Big| \int_{P\cap K} \varphi(x) \Big(\int_C \tau(y) \cdot D\varrho_n(x-y)\, d\mathcal{L}^N(y) \Big)\, d\mathcal{L}^N(x) \Big| \\
&\qquad = \Big| \int_{P\cap K} \varphi(x) \Big(\int_C \varrho_n(x-y)\, d(\operatorname{div}\tau)(y) \Big)\, \mathcal{L}^N(x) \Big| \\
&\qquad = \Big| \int_C \Big(\int_{P\cap K} \varphi(x) \varrho_n(x-y)\, d\mathcal{L}^N(x) \Big)\, d(\operatorname{div}\tau)(y) \Big| \\
&\qquad \le |\operatorname{div}\tau|(B_\varepsilon(K))\, \max_{x\in K} |\varphi(x)|\,.
\end{aligned} \tag{6.16}$$

By (5.14) we know that τ_n is $\mathcal{H}^{N-1}$-integrable on $\partial_* P$ and, thus,

$$\int_P \tau_n \cdot D\varphi\, d\mathcal{L}^N = \int_{\partial_* P} \varphi\, \tau_n \cdot \nu_P\, d\mathcal{H}^{N-1} - \int_P \varphi \operatorname{div} \tau_n\, d\mathcal{L}^N\,.$$

Using (5.13), (5.14), and Lebesgue's theorem we can pass to the limit in the first two integrals and the limit of the third integral can be estimated by (6.16). Hence,

$$\begin{aligned}
\Big| \int_C \tau_P \cdot D\varphi\, d\mathcal{L}^N \Big| &= \Big| \int_P \tau \cdot D\varphi\, d\mathcal{L}^N \Big| \\
&\le \Big| \int_{\partial_* P} \varphi\, \tau \cdot \nu_P\, d\mathcal{H}^{N-1} \Big| + |\operatorname{div}\tau|(B_\varepsilon(K))\, \max_K |\varphi| \\
&\le \max_K |\varphi| \int_{\partial_* P} h\, d\mathcal{H}^{N-1} + \tilde{c}_K\, \max_K |\varphi| \\
&\le c_K\, \max_K |\varphi|
\end{aligned} \tag{6.17}$$

for some constant $c_K > 0$ depending only on K, i.e., τ_P has divergence measure.

Let us now consider the structure of $\mathcal{P}_h$. We know that $\mathcal{P}$ is an algebra. The definition of the measure-theoretic boundary implies that

$$\partial_* A = \partial_* A_* = \partial_* A^{\mathrm{c}}\,, \quad (A \cap B)_* = A_* \cap B_*$$

for all $\mathcal{L}^N$-measurable sets $A, B \subset \mathbb{R}^N$ (cf. Appendix and [28, p. 3]). By Šilhavý [55, (2.4), (2.9)] we have for normalized sets $A, B \subset \mathbb{R}^N$ (i.e., $A = A_*$, $B = B_*$) that

$$\partial_*(A \cap B) \;\subset\; \partial_* A \cup \partial_* B\,.$$

Hence

$$\begin{aligned} \partial_*(A \cap B) \;&=\; \partial_*(A \cap B)_* = \partial_*(A_* \cap B_*) \subset \partial_* A_* \cup \partial_* B_* = \partial_* A \cup \partial_* B\,, \\ \partial_*(A \cup B) \;&=\; \partial_*(A \cup B)^{\mathrm{c}} = \partial_*(A^{\mathrm{c}} \cap B^{\mathrm{c}}) \\ &=\; \partial_*(A^{\mathrm{c}} \cap B^{\mathrm{c}})_* = \partial_*((A^{\mathrm{c}})_* \cap (B^{\mathrm{c}})_*) \\ &\subset\; \partial_*(A^{\mathrm{c}})_* \cup \partial_*(B^{\mathrm{c}})_* = \partial_*(A^{\mathrm{c}}) \cup \partial_*(B^{\mathrm{c}}) = \partial_* A \cup \partial_* B \end{aligned} \tag{6.18}$$

for all $A, B \in \mathcal{P}$. But this readily implies that P^{c}, $P_1 \cup P_2$, $P_1 \cap P_2 \in \mathcal{P}_h$ as long as $P, P_1, P_2 \in \mathcal{P}_h$ which shows that $\mathcal{P}_h$ is an algebra.

We now choose any $P \in \mathcal{P}_h$ and, with some modifications, we again carry out the proof of Proposition 5.11. First we can proceed until (5.15) while we restrict our attention to the case in which φ has compact support in C. Obviously,

$$\begin{aligned} &\int_P \varphi(x) \varrho_n(x - y)\, d\mathcal{L}^N(x) \\ &\qquad = \varphi(y) \int_P \varrho_n(x - y)\, d\mathcal{L}^N(x) + \int_P (\varphi(x) - \varphi(y))\, \varrho_n(x - y)\, d\mathcal{L}^N(x) \end{aligned} \tag{6.19}$$

for all $y \in C$. Since φ has compact support, φ is uniformly continuous on C. Thus the rightmost integral in (6.19) tends to zero as $n \to \infty$ for all $y \in C$. For the evaluation of the limit in (5.15) we thus have to study the limit of

$$\int_C \varphi(y) g_n(y)\, d(\operatorname{div} \tau)(y)$$

with the nonnegative continuous functions

$$g_n(y) := \int_P \varrho_n(x - y)\, d\mathcal{L}^N(x)\,, \quad y \in C\,.$$

Certainly $\{g_n\}$ is a sequence in $\mathcal{L}^\infty(C, |\operatorname{div} \tau|)$ that is bounded by 1. Hence, at least for a subsequence denoted the same way, $\{g_n\}$ weakly* converges to some $g_P \in \mathcal{L}^\infty(C, |\operatorname{div} \tau|)$, i.e., instead of (5.16),

$$\begin{aligned} \lim_{n \to \infty} \int_P \varphi \operatorname{div} \tau_n\, d\mathcal{L}^N \;&=\; \lim_{n \to \infty} \int_C \varphi(y) g_n(y)\, d(\operatorname{div} \tau)(y) \\ &=\; \int_C \varphi(y) g_P(y)\, d(\operatorname{div} \tau)(y) \end{aligned} \tag{6.20}$$

for all continuous φ with compact support. The limit g_P belongs to the closed convex hull of the g_n and, thus, $0 \le g_P(y) \le 1$ for all $y \in C$. Obviously,

$$g_n(y) \to \mathrm{dens}_P(y) \quad \text{on } C_P \subset C$$

where C_P denotes the set of all $y \in C$ where $\mathrm{dens}_P(y)$ is defined. C_P is a Borel set, since $\mathrm{dens}_P(\cdot)$ is Borel measurable. Therefore we can consider the convergence in (6.20) separatly on C_P and on $C \setminus C_P$ instead of C. Lebesgue's theorem and the uniqueness of the weak limit g_P imply (6.8). If we now take the limit in (5.17), then

$$\int_P \tau \cdot D\varphi \, d\mathcal{L}^N = \int_{\partial_* P} \varphi \, \tau \cdot \nu_P \, d\mathcal{H}^{N-1} - \int_C \varphi g_P \, d(\operatorname{div} \tau) \, .$$

Since

$$\int_P \tau \cdot D\varphi \, d\mathcal{L}^N = \int_C \tau_P \cdot D\varphi \, d\mathcal{L}^N = -\int_C \varphi \, d(\operatorname{div} \tau_P)$$

for all $\varphi \in \mathcal{C}_0^\infty(C)$ according to (5.2), we get (6.7). □

Proof of Theorem 6.11. Let τ be the interaction tensor of f that exists by Theorem 4.1 and that has divergence measure by Theorem 5.3. By Proposition 6.5 there exists a nonnegative function $h \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R})$ such that τ_P according to (6.6) has divergence measure for all $P \in \mathcal{P}_h$. Corollary 5.4 and Theorem 6.1 imply the unique definition for an extension of f, denoted the same way, by

$$f(B, P) := \operatorname{div} \tau_P(B) \quad \text{for all } B \in \mathcal{B}\,, \;\; P \in \mathcal{P}_h \, . \tag{6.21}$$

Certainly, $f(\cdot, P)$ is a measure for all $P \in \mathcal{P}_h$.

Now let $P_1, P_2 \in \mathcal{P}_h$ with $P_1 \cap P_2 = \emptyset$. We know by Proposition 6.5 that τ_{P_1}, τ_{P_1}, and $\tau_{P_1 \cup P_2} = \tau_{P_1} + \tau_{P_2}$ have divergence measure. Hence, by 5.2,

$$\begin{aligned} \int_C \varphi \, d(\operatorname{div} \tau_{P_1} + \operatorname{div} \tau_{P_2}) \;&=\; -\int_C (\tau_{P_1} + \tau_{P_2}) \cdot D\varphi \, d\mathcal{L}^N \\ &=\; \int_C \varphi \, d(\operatorname{div} \tau_{P_1 \cup P_2}) \end{aligned}$$

for all Lipschitz continuous φ with compact support. Therefore,

$$\operatorname{div} \tau_{P_1} + \operatorname{div} \tau_{P_2} = \operatorname{div} \tau_{P_1 \cup P_2} \, .$$

But this implies finite additivity of $f(B, \cdot)$ for any $B \in \mathcal{B}$ by (6.21), i.e., the extended f is an interaction.

We still have to verify that f is a contact interaction. For $Q \in \mathcal{Q}^{\mathrm{f}}$, $P \in \mathcal{P}_h$, $\varepsilon > 0$ with $Q_\varepsilon \in \mathcal{Q}^{\mathrm{f}}$, and φ_Q^σ according to (5.18) we have by (5.2) and (6.6) that

$$\begin{aligned} f(Q, P) \;&=\; \operatorname{div} \tau_P(Q) \\ &= \lim_{\sigma \downarrow 0} \int_C \varphi_Q^\sigma \, d(\operatorname{div} \tau_P) \;=\; -\lim_{\sigma \downarrow 0} \int_C \tau_P \cdot D\varphi_Q^\sigma \, d\mathcal{L}^N \\ &= -\lim_{\sigma \downarrow 0} \int_{(Q_\varepsilon \setminus Q) \cap P} \tau \cdot D\varphi_Q^\sigma \, d\mathcal{L}^N \end{aligned} \tag{6.22}$$

$$
\begin{aligned}
&= -\lim_{\sigma\downarrow 0}\int_C \tau_{(Q_\varepsilon\setminus Q)\cap P}\cdot D\varphi_Q^\sigma\, d\mathcal{L}^N \\
&= \lim_{\sigma\downarrow 0}\int_C \varphi_Q^\sigma\, d(\operatorname{div}\tau_{(Q_\varepsilon\setminus Q)\cap P}) \;=\; f(Q,(Q_\varepsilon\setminus Q)\cap P)
\end{aligned}
$$

which verifies (C1). For any $P\in\mathcal{P}_h$ with $\mathcal{L}^N(P)=0$, we get from (5.2) and (6.6) that

$$
\int_C \varphi\cdot d(\operatorname{div}\tau_P) = -\int_P \tau\cdot D\varphi\, d\mathcal{L}^N = 0
$$

for all Lipschitz continuous functions φ on C with compact support. Hence $\operatorname{div}\tau_P=0$ in the sense of measures, which implies (C2). Now we fix any $P\in\mathcal{P}_h$, $Q\in\mathcal{Q}^{\mathrm{f}}$, and $Z\in\mathcal{B}$ with $\mathcal{L}^N(Z)=0$. Then for a.e. $\varepsilon>0$ we have that $P\setminus Q_\varepsilon\in\mathcal{P}_h$, $|\operatorname{div}\tau|(\partial Q_\varepsilon)=0$, and $\mathcal{H}^{N-1}(\partial Q_\varepsilon\cap Z)=0$. Thus

$$
\begin{aligned}
f(Q_\varepsilon\setminus Z,P\setminus Q_\varepsilon) &\overset{(C1)}{=} f(\partial Q_\varepsilon\setminus Z,P\setminus Q_\varepsilon) \;=\; (\operatorname{div}\tau_{P\setminus Q_\varepsilon})(\partial Q_\varepsilon\setminus Z)\\
&\overset{(6.7)}{=} (g_{P\setminus Q_\varepsilon}\operatorname{div}\tau)(\partial Q_\varepsilon\setminus Z)\\
&\qquad -\Big(\tau\cdot\nu_{P\setminus Q_\varepsilon}\mathcal{H}^{N-1}\lfloor\partial_*(P\setminus Q_\varepsilon)\Big)(\partial Q_\varepsilon\setminus Z)\\
&= -\int_{(\partial_*(P\setminus Q_\varepsilon))\cap(\partial Q_\varepsilon\setminus Z)} \tau\cdot\nu_{P\setminus Q_\varepsilon}\,\mathcal{H}^{N-1}\\
&= -\int_{(\partial_*(P\setminus Q_\varepsilon))\cap\partial Q_\varepsilon} \tau\cdot\nu_{P\setminus Q_\varepsilon}\,\mathcal{H}^{N-1}\\
&= -\Big(\tau\cdot\nu_{P\setminus Q_\varepsilon}\mathcal{H}^{N-1}\lfloor\partial_*(P\setminus Q_\varepsilon)\Big)(\partial Q_\varepsilon)\\
&= (\operatorname{div}\tau_{P\setminus Q_\varepsilon})(\partial Q_\varepsilon) \;=\; f(\partial Q_\varepsilon,P\setminus Q_\varepsilon)\\
&= f(Q_\varepsilon,P) \qquad \text{for a.e. } \varepsilon>0\,.
\end{aligned}
$$

Since $\lim_{\varepsilon\to 0} f(Q_\varepsilon,P)=f(Q,P)$ as a basic property of measures, (C3) follows and completes the proof. □

7 Boundedness condition

In our previous investigations we have seen that locally bounded contact interactions f on C can be described by tensor fields $\tau\in\mathcal{L}^1_{\mathrm{loc}}(C)$ having divergence measure. Now we ask whether each such tensor field τ provides a contact interaction.

Theorem 7.1 *Let $\tau\in\mathcal{L}^1_{\mathrm{loc}}(C,\mathbb{R}^{M\times N})$ be a tensor field with divergence measure on the open set $C\subset\mathbb{R}^N$. Then, for the nonnegative function $h\in\mathcal{L}^1_{\mathrm{loc}}(C,\mathbb{R})$ according to Proposition* 6.5*, there is a contact interaction f on C relative to $\mathcal{P}_h$ given by*

$$
f(B,P):=(\operatorname{div}\tau_P)(B)\quad\text{for all } B\in\mathcal{B}\,,\;\; P\in\mathcal{P}_h\,,
$$

where τ_P is defined as in (6.6)*. Moreover f satisfies the representation formulas*

$$
f(Q,P)=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{Q_\varepsilon\setminus Q}\tau_P\cdot\nu_Q\, d\mathcal{L}^N\quad\text{for all } Q\in\mathcal{Q}\,,\;\; P\in\mathcal{P}_h\,, \tag{7.2}
$$

and

$$f(Q,P) = \int_{\partial Q} \tau_P \cdot \nu_Q \, d\mathcal{L}^N \quad \textit{for a.e. } Q \in \mathcal{Q}\,, \;\; P \in \mathcal{P}_h\,. \tag{7.3}$$

Proof. Proposition 6.5 tells us that all tensor fields τ_P with $P \in \mathcal{P}_h$ have divergence measure and, thus, f is well defined. For the verification that f is a contact interaction we can argue exactly as in the proof of Theorem 6.11. As in (6.22) we obtain that

$$f(Q,P) = -\lim_{\varepsilon \downarrow 0} \int_C \tau_P \cdot D\varphi_Q^\varepsilon \, d\mathcal{L}^N = \lim_{\varepsilon \downarrow 0} \int_{Q_\varepsilon \setminus Q} \tau_P \cdot \nu_Q \, d\mathcal{L}^N$$

for any $Q \in \mathcal{Q}$, $P \in \mathcal{P}_h$, which verifies (7.2). Taking Proposition 5.11 with $\varphi \equiv 1$ and τ_P instead of τ we obtain (7.3). $\square$

This result raises the question how restrictive is the physically motivated condition of local boundedness for contact interactions, i.e., whether it rules out certain tensor fields $\tau \in \mathcal{L}^1_{\mathrm{loc}}(C)$ having divergence measure. This is really the case as can be seen from the following example (which was pointed out me by M. Šilhavý, cf. [57, Example 9.1]) providing such a tensor field τ where the corresponding contact interaction is not locally bounded.

Example 4. Let $C := (-1,1) \times (-1,1) \subset \mathbb{R}^2$ and set

$$\tau(x) := \frac{1}{|x|^2} \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \quad \text{for all } \; x \in C\,.$$

Then $\tau \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R}^2)$ and $\operatorname{div} \tau = 0$ in the sense of distributions. With $R := (-1,1) \times [0,1)$ and $Q_\sigma := (-1,\sigma) \times [0,1)$, formula (7.2) implies for $\sigma > 0$ that

$$f(Q_\sigma, R) = \int_0^1 \frac{x_2}{\sigma^2 + x_2^2} \, dx_2 = \frac{1}{\ln(1+\sigma^2) - \ln \sigma^2} \overset{\sigma \to 0}{\longrightarrow} \infty\,.$$

But this violates the condition of local boundedness for the interaction f.

Now we may ask how restrictive the condition of local boundedness really is and whether it possibly prevents the measures $f(\cdot, A)$ from having a singular part with respect to $\mathcal{L}^N$. For this reason we recall the tensor field τ from Example 2 in Section 3 and let f be the corresponding contact interaction on $C = \mathbb{R}^2$ relative to the algebra $\mathcal{P}_h$ according to Theorem 7.1. Obviously, there is some full system $\mathcal{Q}^{\mathrm{f}} = \mathcal{Q}(H^{\mathrm{f}}) \subset \mathcal{P}_h$ for which we can assume that $0 \notin H^{\mathrm{f}}$. By (6.7) we then have that

$$|f(B,Q)| \le |f|(C,Q) = |\operatorname{div} \tau_Q|(C) \le |\operatorname{div} \tau|(Q) + |\tau \cdot \nu_Q| \mathcal{H}^1 \lfloor \partial_* Q$$

for $B \in \mathcal{B}$, $Q \in \mathcal{Q}^{\mathrm{f}}$. If we fix some $\breve{Q} \in \mathcal{Q}^{\mathrm{f}}$, then the right-hand side is uniformly bounded for all $Q \subset \breve{Q}$. But this means that f is a locally bounded contact interaction while $f(\cdot, C) = \operatorname{div} \tau$ has a concentration at the origin, i.e., the condition of local boundedness still allows concentrations of the measures $f(\cdot, A)$. Notice that both in this example and in Example 4 the values $\tau(x)$ can be arbitrarily large. On the other hand, interactions $f(B,Q)$ for bounded Q can also be arbitrarily large in Example 4 while they are uniformly bounded here.

Let us finally discuss some weaker mathematical boundedness conditions for a contact interaction f. First, for $Q \in \mathcal{Q}$ with coordinates a_i, b_i, $i = 1, \dots, N$, we define $Q^j_{[\xi,\xi]} \in \mathcal{Q}$ as the N-interval with the same coordinates as Q for $i \neq j$ but ξ, ξ instead of a_j, b_j and, analogously, $Q^j_{[\xi,b_j]} \in \mathcal{Q}^{\mathrm{f}}$ with ξ instead of a_j. Then we say that f is *weakly locally bounded* if

$$\int_{a_j}^{b_j} |f|(Q^j_{[\xi,\xi]}, Q^j_{[\xi,b_j]})\, d\xi < \infty \quad \text{for all } Q \in \mathcal{Q}^{\mathrm{f}},\ j = 1, \dots, N\,. \tag{7.4}$$

It turns out that all our arguments still work for weakly locally bounded contact interactions f instead of locally bounded ones. In particular, this means that each such f can be represented by a tensor field $\tau \in \mathcal{L}^1_{\mathrm{loc}}(C)$ and, if we identify τ with its extension on $\mathbb{R}^N$ by zero, we even get that $\tau \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^{M\times N})$. On the other hand, we readily conclude from (7.3) that each tensor field $\tau \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^{M\times N})$ defines a contact interaction f that is weakly locally bounded. Thus we have a one-to-one correspondence between weakly locally bounded contact interactions and tensor fields in $\tau \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^{M\times N})$ with divergence measure. We easily see that now Example 4 is covered. Obviously, a requirement like

$$|f(Q,R)| \le \int_{\partial Q \cap R} h\, d\mathcal{H}^{N-1} \quad \text{for a.e. } Q, R \in \mathcal{Q}^{\mathrm{f}} \tag{7.5}$$

with some suitable nonnegative $h \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$ implies (7.4). Notice that (7.5) is the translation of one of the fundamental assumptions for a Cauchy flux to our setting (cf. Ziemer [66], Šilhavý [55], Degiovanni et al. [15]).

Analogously, we can get a one-to-one correspondence of locally bounded contact interactions with tensor fields $\tau \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R}^{M\times N})$ for any open $C \subset \mathbb{R}^N$ if we require (7.4) only for bounded $Q \in \mathcal{Q}^{\mathrm{f}}$ with $\mathrm{cl}\, Q \in C$. However, in this case we have to be a little careful with arguments where the boundary of C is involved. In particular, N-intervals Q containing parts of ∂C have to be treated with caution. Despite these technicalities this case has the disadvantage that it might be very difficult or even impossible to treat boundary conditions in a reasonable way. Therefore we refrain from exploring this case further.

In summary, we can say that it is a question of one's point of view which boundedness condition is preferred. From the physical point of view the local boundedness seems to be natural for contact interactions while the weaker condition (7.5) has no physical motivation but provides some mathematical equivalence.

8 Appendix

For the convenience of the reader we first summarize some material from measure theory as necessary for our purposes (for a more comprehensive presentation we refer to Ambrosio et al. [1], Evans & Gariepy [17], and Federer [18]). Then we verify two general results for real functions that we need for our analysis.

For any set $A \subset \mathbb{R}^n$ the *density* of A at $x \in \mathbb{R}^N$ is

$$\mathrm{dens}_A(x) := \lim_{r\downarrow 0} \frac{\mathcal{L}^n(B_r(x) \cap A)}{\mathcal{L}^n(B_r(x))}\,,$$

if the limit exists. The *measure-theoretic interior* of A is given by

$$A_* := \left\{ x \in \mathbb{R}^n \,\middle|\, \mathrm{dens}_A(x) = 1 \right\},$$

and the *measure-theoretic boundary* of A by

$$\partial_* A := \mathbb{R}^n \setminus (A_* \cup (A^c)_*)\,.$$

A_* and $\partial_* A$ are always Borel sets, $\mathcal{L}^n(A_* \setminus A) = 0$, $\mathcal{L}^n(\partial_* A) = 0$, and $\mathrm{int}\, A \subset A_* \subset \mathrm{cl}\, A$. If A is $\mathcal{L}^n$-measurable, then $\mathcal{L}^n(A \setminus A_*) = 0$ and if A has a Lipschitz boundary, then $\partial A = \partial_* A$. The vector $\nu \in \mathbb{R}^n$ with $|\nu| = 1$ is said to be an *outer unit normal* of A at $x \in \partial_* A$ if

$$\begin{aligned} 0 &= \lim_{r \downarrow 0} \frac{\mathcal{L}^n(\{y \in B_r(x) \cap A \,|\, (y-x)\cdot \nu > 0\})}{\mathcal{L}^n(B_r(x))} \\ &= \lim_{r \downarrow 0} \frac{\mathcal{L}^n(\{y \in B_r(x) \setminus A \,|\, (y-x)\cdot \nu < 0\})}{\mathcal{L}^n(B_r(x))} \end{aligned}$$

which is unique if it exists. For $x \in \partial_* A$ we define $\nu_A(x)$ to be the outer unit normal of A at x if it exists and $\nu_A(x) = 0$ otherwise. $\nu_A(\cdot)$ is a bounded Borel map.

We say that $A \subset \mathbb{R}^n$ has *finite perimeter* in the open set $C \subset \mathbb{R}^n$ if $\mathcal{H}^{n-1}(C \cap \partial_* A) < \infty$ or, equivalently, if the distributional derivative of the characteristic function χ_A on C is a Radon measure on C. The sets of finite perimeter on C are $\mathcal{L}^n$-measurable, they form an algebra, and $\nu_A(x) \neq 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in C \cap \partial_* A$. For each set A of finite perimeter in C the density $\mathrm{dens}_A(x)$ is well defined for $\mathcal{H}^{N-1}$-a.e. $x \in C$ and $\mathrm{dens}_A(x) \in \{0, \frac{1}{2}, 1\}$ for $\mathcal{H}^{N-1}$-a.e. $x \in C$.

For a function $g : \mathbb{R} \mapsto \mathbb{R}$ we say that $\lambda \in \mathbb{R}$ is the *approximate limit from above* at t, written $\lambda = \mathrm{ap}\lim_{s \downarrow t} g(s)$, if

$$\lim_{r \downarrow 0} \frac{\mathcal{L}^1((t, t+r) \cap \{s \,|\, |g(s) - \lambda| \geq \varepsilon\})}{\mathcal{L}^1((t, t+r))} = 0 \quad \text{for all } \varepsilon > 0\,.$$

In analogy to Evans & Gariepy [17, Theorem 2, p. 46] we obtain that this limit is unique if it exists. Since this limit is not influenced by values of the function $g(s)$ on a set of $\mathcal{L}^1$-measure zero, we can consider this limit also for functions g that are merely defined up to a set of $\mathcal{L}^1$-measure zero.

Proposition 8.1 (1) *Let $H \subset (a,b) \subset \mathbb{R}$ be such that $\mathcal{L}^1(H) = 0$ and let $\varphi : (a,b) \setminus H \mapsto \mathbb{R}$ be a real-valued function satisfying*

$$\mathrm{ap}\lim_{s \downarrow t} \varphi(s) = 0 \quad \textit{for all } t \in (a,b) \setminus H\,. \tag{8.2}$$

Then φ is $\mathcal{L}^1$-measurable and the set $\{t \in (a,b) \,|\, \varphi(t) \neq 0\}$ has $\mathcal{L}^1$-measure zero.

(2) *Let $\varphi : (a,b) \mapsto \mathbb{R}$ be essentially bounded and let $\lambda = \mathrm{ap}\lim_{s \downarrow t} \varphi(s)$ for $t \in (a,b)$. Then*

$$\lambda = \lim_{\delta \downarrow 0} \frac{1}{\delta} \int^*_{(t, t+\delta)} \varphi \, d\mathcal{L}^1$$

where $\int^*$ denotes the upper integral (cf. [17, p. 18]*).*

Note that (8.2) does not force φ to be identically zero. Consider, for example,

$$\varphi(\tfrac{m}{n}) := \tfrac{1}{n} \quad \text{for } t = \tfrac{m}{n} \in \mathbb{Q},\; m,n \text{ coprime}, \qquad \varphi(t) := 0 \quad \text{otherwise.}$$

Proof. (1) For technical simplicity we set $\varphi(s) = 0$ on H and, without any loss of generality, we can assume that

$$0 \le \varphi(s) \le 1 \quad \text{on } (a,b),$$

since otherwise we can replace φ with $\arctan(|\varphi(\cdot)|)$. The difficulty now is that φ might not be $\mathcal{L}^1$-measurable.

In analogy to FEDERER [18, Theorem 2.3.3] we can represent φ as

$$\varphi(s) = \sum_{k=1}^{\infty} \alpha_k \chi_{H_k}(s) \quad \text{for all } s \in (a,b)$$

where $\alpha_k := 1/2^k$ and the χ_{I_k} are the characteristic functions of the recursively defined sets

$$H_k := \{s \in (a,b) |\, \varphi(s) \ge \alpha_k + \sum_{j<k} \alpha_j \chi_{H_j}(s)\}\,.$$

Now we can choose $\mathcal{L}^1$-measurable sets $H_k^* \subset (a,b)$ with $H_k \subset H_k^*$ and $\mathcal{L}^1(H_k) = \mathcal{L}^1(H_k^*)$ which allows us to define the $\mathcal{L}^1$-measurable function

$$\varphi^*(s) := \sum_{k=1}^{\infty} \alpha_k \chi_{H_k^*}(s) \quad \text{for all } s \in (a,b)\,.$$

Obviously,

$$0 \le \varphi(s) \le \varphi^*(s) \le 2 \quad \text{for all } s \in (a,b)\,. \tag{8.3}$$

Note that φ^* is integrable on (a,b) and, by the monotone convergence theorem,

$$\int_{(a,b)} \varphi^* \, d\mathcal{L}^1 = \sum_{k=1}^{\infty} \alpha_k \mathcal{L}^1(H_k^*)\,.$$

Moreover, the upper integral $\int_{\tilde{H}}^* \varphi \, d\mathcal{L}^1$ (cf. EVANS & GARIEPY [17, p.18]) is well defined and finite for each $\mathcal{L}^1$-measurable set $\tilde{H} \subset (a,b)$. In particular,

$$\int_{(a,b)}^* \varphi \, d\mathcal{L}^1 = \sum_{k=1}^{\infty} \alpha_k \mathcal{L}^1(H_k)\,,$$

since φ can be estimated by integrable simple functions according to

$$\sum_{k=1}^{l} \alpha_k \chi_{H_k}(s) \le \varphi(s) \le \sum_{k=1}^{l} \alpha_k \chi_{H_k}(s) + \frac{1}{2^l} \chi_{H_{>l}}(s) \quad \text{with } H_{>l} := \bigcup_{k>l} H_k$$

for all $l \in \mathbb{N}$. Consequently,

$$\int_{(a,b)}^* \varphi \, d\mathcal{L}^1 = \int_{(a,b)} \varphi^* \, d\mathcal{L}^1\,. \tag{8.4}$$

From (8.3) and the definition of the upper integral we get for all $\mathcal{L}^1$-measurable sets $\tilde{H}, \tilde{H}_1, \tilde{H}_2 \subset (a,b)$ with $\tilde{H} = \tilde{H}_1 \cup \tilde{H}_2$, $\tilde{H}_1 \cap \tilde{H}_2 = \emptyset$ that

$$\int^*_{\tilde{H}} \varphi \, d\mathcal{L}^1 \le \int_{\tilde{H}} \varphi^* \, d\mathcal{L}^1 \,, \qquad \int^*_{\tilde{H}} \varphi \, d\mathcal{L}^1 = \int^*_{\tilde{H}_1} \varphi \, d\mathcal{L}^1 + \int^*_{\tilde{H}_2} \varphi \, d\mathcal{L}^1 \,. \tag{8.5}$$

Since $0 \le \varphi(s) \le \varphi^*(s)$ on (a,b) and (8.4), we conclude that even

$$\int^*_{\tilde{H}} \varphi \, d\mathcal{L}^1 = \int_{\tilde{H}} \varphi^* \, d\mathcal{L}^1 \quad \text{for all } \mathcal{L}^1\text{-measurable sets } \tilde{H} \subset (a,b) \,.$$

Hence

$$\varPhi(t) := \int^*_{(a,t)} \varphi \, d\mathcal{L}^1 = \int_{(0,t)} \varphi^* \, d\mathcal{L}^1 \,, \quad t \in (a,b) \,,$$

is absolutely continuous on (a,b).

Now consider $t \in (a,b) \setminus H$ such that $\varPhi'(t)$ exists. Then, by (8.2),

$$\begin{aligned} &\lim_{\delta \downarrow 0} \tfrac{1}{\delta} \mathcal{L}^1(H_{\delta,\varepsilon}) = 0 \quad \text{for all } \varepsilon > 0 \\ &\text{where} \quad H_{\delta,\varepsilon} := \{ s \in (t, t+\delta) \,|\, \varphi(s) \ge \varepsilon \} \,. \end{aligned} \tag{8.6}$$

We choose $\mathcal{L}^1$-measurable sets $H^*_{\delta,\varepsilon}$ with $H_{\delta,\varepsilon} \subset H^*_{\delta,\varepsilon}$ and $\mathcal{L}^1(H_{\delta,\varepsilon}) = \mathcal{L}^1(H^*_{\delta,\varepsilon})$. Thus

$$\begin{aligned} \int^*_{(t,t+\delta)} \varphi \, d\mathcal{L}^1 &= \int^*_{H^*_{\delta,\varepsilon}} \varphi \, d\mathcal{L}^1 + \int^*_{(t,t+\delta)\setminus H^*_{\delta,\varepsilon}} \varphi \, d\mathcal{L}^1 && (8.7) \\ &\le \int^*_{H^*_{\delta,\varepsilon}} d\mathcal{L}^1 + \int^*_{(t,t+\delta)\setminus H^*_{\delta,\varepsilon}} \varepsilon \, d\mathcal{L}^1 && (8.8) \\ &\le \mathcal{L}^1(H_{\delta,\varepsilon}) + \varepsilon\delta \,. && (8.9) \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, we conclude that, by(8.6),

$$\varPhi'(t) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \int^*_{(t,t+\delta)} \varphi \, d\mathcal{L}^1 = 0 \,.$$

Consequently, $\varPhi'(t) = 0$ $\mathcal{L}^1$-a.e. on (a,b), which implies that $\varPhi(t) = 0$ for all $t \in (a,b)$ (observe that $\varPhi(a) = 0$). Hence, $\varphi^*(s) = 0$ and also $\varphi(s) = 0$ $\mathcal{L}^1$-a.e. on (a,b), which proves the assertion.

(2) We set $H_{\delta,\varepsilon} := \{ s \in (t,t+\delta) \,|\, |\varphi(s) - \lambda| \ge \varepsilon \}$ and choose $\mathcal{L}^1$-measurable sets $H^*_{\delta,\varepsilon}$ with $H_{\delta,\varepsilon} \subset H^*_{\delta,\varepsilon}$ and $\mathcal{L}^1(H_{\delta,\varepsilon}) = \mathcal{L}^1(H^*_{\delta,\varepsilon})$. Using the additivity of the upper integral as in (8.5) we get that

$$\begin{aligned} \Big| \frac{1}{\delta} \int^*_{(t,t+\delta)} \varphi \, d\mathcal{L}^1 - \lambda \Big| &= \Big| \frac{1}{\delta} \int^*_{H^*_{\delta,\varepsilon}} (\varphi - \lambda) \, d\mathcal{L}^1 + \frac{1}{\delta} \int^*_{(t,t+\delta)\setminus H^*_{\delta,\varepsilon}} (\varphi - \lambda) \, d\mathcal{L}^1 + \frac{1}{\delta} \int^*_{(t,t+\delta)} \lambda \, d\mathcal{L}^1 - \lambda \Big| \\ &\le \frac{1}{\delta} \int^*_{H^*_{\delta,\varepsilon}} |\varphi - \lambda| \, d\mathcal{L}^1 + \frac{1}{\delta} \int^*_{(t,t+\delta)\setminus H^*_{\delta,\varepsilon}} |\varphi - \lambda| \, d\mathcal{L}^1 \\ &\le \frac{c}{\delta} \mathcal{L}^1(H^*_{\delta,\varepsilon}) + \varepsilon \;=\; \frac{c}{\delta} \mathcal{L}^1(H_{\delta,\varepsilon}) + \varepsilon \end{aligned}$$

for some constant $c > 0$. Since we can choose $\varepsilon > 0$ arbitrarily small, the assertion readily follows from the assumption. $\square$

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