Interactions in continuum physics

Friedemann Schuricht

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1. Introduction

In continuum physics as continuum mechanics and thermodynamics the behavior of a body is determined by all the interactions taking place between its parts, which are also called subbodies. Thus we need an effective way to describe interactions based on a suitable concept of subbodies. If we consider the foundations of continuum physics as found in the literature we observe that they are insufficient to describe satisfactorily singularities arising in modern applications. In particular the usually used system of subbodies, which somehow disregards boundary points, does not allow an efficient and comprehensive description of concentration effects. Moreover it seems that the nature of interactions as short-range and long-range interactions is not yet completely understood. The new foundation for contact interactions given in Schuricht [10] tries to resolve some of these difficulties.

In this survey we claim to make accessible the new basic ideas from [10] to a broad audience. For that we not only summarize the theory developed in [10] but we also illuminate its relation to the classical treatment of contact interactions in a way going beyond the discussion in [10]. In addition we extend the central ideas to interactions which are mostly called body interactions in the literature but which we suggest to call distant interactions, since they describe long-range phenomena. Moreover we demonstrate by means of a classical example that, in contrast to the usual opinion, concentrations are not only caused by singular external actions but we have to be aware of them even in very "harmless" looking situations.

As an essential difference to the classical approach we assume that subbodies correspond to subsets in the set-theoretical sense. Moreover we consider all interactions as set functions on pairs of subbodies. On this basis we start with additivity assumptions as the most elementary properties of interactions and we investigate what remains to require in order to characterize different kinds of interactions by comparing the new arguments with the classical ones. In particular we illuminate the significance of σ-additivity for interactions. This aspect seems to be somehow hidden in previous treatments and it turns out that σ-additivity in the second argument is a characterizing difference between distant and contact interactions. But our analysis also shows that the choice of a system of subbodies is guided by the analysis and may differ for different arguments. Thus the selection of a class of subbodies is not a matter of choice as it appears in some former treatments.

The presentation given here provides a new view to the subject and might help for a further understanding of the fundamental phenomena relevant in continuum physics. For a more comprehensive treatment and for more references we refer to [10].

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Notation. By $\mathcal{L}^n$ we denote the n-dimensional Lebesgue measure and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure on $\mathbb{R}^n$. Readers that are not familiar with these notions can just consider $\mathcal{L}^n$ as volume measure and $\mathcal{H}^{n-1}$ as area measure on $(n-1)$-dimensional surfaces. Correspondingly, $\int_A g\, d\mathcal{L}^n$ is a volume integral and $\int_{\partial A} g\, d\mathcal{H}^{n-1}$ is a surface integral. We write $\mathcal{L}^1(C)$ for the space of (Lebesgue-) integrable functions on C and $\mathcal{L}^1_{\mathrm{loc}}(C)$ contains all functions that are integrable on each compact subset of C. Here, in contrast to the usual practice, we do not identify functions that differ on a set of measure zero. $\mathcal{C}_0^\infty(C)$ stands for the usual space of smooth test functions having compact support and $\operatorname{spt} g$ denotes the support of function g.

For a set $A \subset \mathbb{R}^n$ we denote the measure theoretic interior by A_* and the measure theoretic boundary by $\partial_* A$. These notions agree with the "usual" (topological) interior $\operatorname{int} A$ and boundary ∂A for "nice" sets A as, e.g., sets with piecewise smooth boundary. A is said to be normalized if $A = A_*$. For a function $g : \mathbb{R} \mapsto \mathbb{R}$ we call $\lambda \in \mathbb{R}$ the *approximate limit from above* at t, written $\lambda = \operatorname{ap}\lim_{s\downarrow t} g(s)$, if

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^1\big((t,t+r) \cap \{s \,|\, |g(s)-\lambda| \ge \varepsilon\}\big)}{\mathcal{L}^1\big((t,t+r)\big)} = 0 \quad \text{for all } \varepsilon > 0\,.$$

This approximate limit can be considered, roughly speaking, as a limit from above that disregards values of g on a set of measure zero.

2. Bodies and subbodies

Let us assume that the material points of a body correspond to the points of a set $C \subset \mathbb{R}^n$. In particular we may choose the material description, where the points of C correspond to the positions of the material points in a distinguished reference configuration, or the spatial description, where the points of C correspond to the positions of the material points at a given time. Note that our subsequent analysis can be carried out for any appropriate choice.

For the investigation of the behavior of a body we have to analyze the behavior of its parts, which we also call *subbodies*. Thus we have to ask what we should choose as a reasonable system of subbodies. Of course, the simplest and most natural way to define subbodies of a body is to take the system of all subsets or a suitable subsystem of it. Since it appears to be both conceptually reasonable and analytically useful, we assume that the system $\mathcal{A}$ of subbodies is an *algebra* on C, i.e.,

$$\emptyset, C \in \mathcal{A}, \quad A_1 \setminus A_2 \in \mathcal{A}, \quad \bigcup_{i=1}^{k} A_i \in \mathcal{A}, \quad \bigcap_{i=1}^{k} A_i \in \mathcal{A}$$

whenever $A_1, \ldots, A_k \in \mathcal{A}$. If, in addition,

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}, \quad \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$$

for any sequence $A_1, A_2, ... \in \mathcal{A}$, then the algebra $\mathcal{A}$ is called *σ-algebra*. The smallest σ-algebra containing a given algebra $\mathcal{A}$ is said to be the σ-algebra generated by $\mathcal{A}$. Analogously we say that the algebra $\mathcal{A}$ is generated by a system of subsets of C if it is the smallest algebra containing this system.

The systems of subbodies considered in the literature are usually not assumed to be an algebra but merely a Boolean algebra, since, roughly speaking, boundary points of the subbodies are disregarded and, thus, the union and the intersection have to be defined in a more general way. In our subsequent analysis we want to demonstrate that there is no need for that. Even more, we obtain a richer and more powerful theory by using algebras. In particular, certain conceptual difficulties in the traditional approach disappear naturally that way. Consider, e.g., a concentrated force exerted to some point of the body A. If we now cut A through that point into two pieces A_1 and A_2, then, within the classical approach, one could not decide to which of the two parts the force is exerted.

All systems of subbodies considered in the literature before have in common that they in fact generate the σ-algebra $\mathcal{B}$ of Borel subsets of C (which is the σ-algebra generated by the sets that are open or closed relative to C). Thus it seems to be reasonable for our further treatment to assume that C is a Borel set in $\mathbb{R}^n$ and that we choose $\mathcal{B}$ as system of subbodies of C. But it turns out that the σ-algebra $\mathcal{B}$ is too large for certain arguments. For a comprehensive understanding of certain questions we have to consider not only small and rich algebras generating $\mathcal{B}$ but also suitable generators of certain algebras. For that reason we introduce a number of subsystems of $\mathcal{B}$.

First let $\mathcal{Q}'$ denote the collection of all closed n-dimensional *intervals* Q' having the form

$$Q' = \{(x_1, \dots, x_n) \in \mathbb{R}^n |\, a_i \le x_i \le b_i,\ a_i, b_i \in \mathbb{R},\ i = 1, \dots, n\}$$

where the a_i, b_i are called the *coordinates* of Q'. The *ε-neighborhood* of Q' is the interval $Q'_\varepsilon \in \mathcal{Q}'$ having coordinates $a_i - \varepsilon$, $b_i + \varepsilon$. By $\nu_{Q'} : \mathbb{R}^n \mapsto \mathbb{R}^n$ we assign a *normal field* to $Q' \in \mathcal{Q}'$ such that $\nu_{Q'}(x)$ is the outer unit normal of Q'_ε if $x \in \partial Q'_\varepsilon$ for some $\varepsilon \in \mathbb{R}$. Obviously $\nu_{Q'}$ is well defined up to a set of $\mathcal{L}^n$-measure zero. The subsystem $\mathcal{Q}'(I) \subset \mathcal{Q}'$ contains all intervals having coordinates a_i, b_i, $i = 1, \dots, n$, confined to the subset $I \subset \mathbb{R}$.

Now we define the collection of *boxes* on C by

$$\mathcal{Q} := \{Q' \cap C |\, Q' \in \mathcal{Q}'\}\,.$$

For $Q \in \mathcal{Q}$ with $Q = Q' \cap C$ and $Q' \in \mathcal{Q}'$, the box $Q_\varepsilon := Q'_\varepsilon \cap C$ is the *ε-neighborhood* of Q on C, $\nu_Q := \nu_{Q'}$ is the *normal field* of Q, and $\operatorname{int} Q := \operatorname{int} Q' \cap C$. We set $\mathcal{Q}(I) := \{Q' \cap C |\, Q' \in \mathcal{Q}'(I)\}$ and call $\mathcal{Q}^f \subset \mathcal{Q}$ a *full* subsystem of $\mathcal{Q}$ if $\mathcal{Q}^f = \mathcal{Q}(I)$ for some $I \subset \mathbb{R}$ with $\mathcal{L}^1(\mathbb{R} \setminus I) = 0$. The algebras generated by $\mathcal{Q}$ or by some $\mathcal{Q}^f$ are denoted by $\mathcal{R}$ or $\mathcal{R}^f$, respectively. For open C we also consider the system $\mathcal{P}$ of all sets of *finite perimeter* in C, i.e.,

$$\mathcal{P} := \{P \cap C |\, \mathcal{H}^{n-1}(C \cap \partial_* P) < \infty\}\,,$$

which is an algebra on C. By ν_P we denote the normal field that is defined on $\partial_* P$. Notice that $\mathcal{Q} \subset \mathcal{R} \subset \mathcal{P} \subset \mathcal{B}$. Occasionally we write $\mathcal{Q}_{\tilde{C}}$, $\mathcal{B}_{\tilde{C}}$, etc. to indicate that the system is a collection of corresponding subsets of $\tilde{C}$ instead of C. For an algebra $\mathcal{A}$ on C and any $\tilde{A} \in \mathcal{A}$ we define the *restriction* of $\mathcal{A}$ on $\tilde{A}$ by $\mathcal{A}_{|\tilde{A}} := \{A \cap \tilde{A} | \, A \in \mathcal{A}\}$, which is a subalgebra of $\mathcal{A}$.

For our further treatment we always assume that $\mathcal{A}$ is some not yet specified algebra on C containing a full system $\mathcal{Q}^f$. Note that the last requirement is in fact met by all systems of subbodies studied in the literature.

3. Interactions

The behavior of a body is determined by all the interactions that its parts and its neighborhood exert on each other. Though in classical continuum physics such interactions are usually formulated in terms of density functions, we have to realize that merely resultants $f(A, A')$ exerted from part A' on part A can be measured. Thus it is very natural to formulate the foundations of continuum physics in terms of set functions $(A, A') \mapsto f(A, A')$ instead of densities.

In order to introduce some basic properties let us first consider a set function $g : \mathcal{A} \mapsto \mathbb{R}^m$ with a single argument. According to our experience set functions in continuum physics should be *additive* with respect to disjoint decompositions of its argument, i.e.,

$$g\Big(\bigcup_{i=1}^{k} A_i\Big) = \sum_{i=1}^{k} g(A_i)$$

for all pairwise disjoint $A_1, \ldots, A_k \in \mathcal{A}$. For an additive g we readily see that

$$g(\emptyset) = 0\,.$$

From the analytical point of view it is desirable to have available the powerful tools of measure and integration theory. But for that we need g to be even *σ-additive*, i.e.,

$$g\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} g(A_i)$$

for any sequence $A_1, A_2, ... \in \mathcal{A}$ of pairwise disjoint sets with $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ (here we implicitly assume absolute convergence on the right hand side, since the left hand side is independent of a rearrangement of the A_i). A σ-additive set function g that is defined on a σ-algebra (which is always the class of Borel sets $\mathcal{B}$ in our treatment) is called a (vector-valued) *measure*. It is a *Radon measure* if it is finite on compact sets.

Since resultants of actions in bounded physical systems should be finite, it is reasonable to consider set functions g that are *locally bounded*, i.e., for any $\tilde{Q} \in \mathcal{Q}$ there is a constant $c_{\tilde{Q}} > 0$ such that

$$|g(Q)| \le c_{\tilde{Q}} \quad \text{for any } Q \in \mathcal{Q}^f \text{ with } Q \subset C \cap \tilde{Q}\,. \tag{3.1}$$

Notice that any locally bounded σ-additive set function $g : \mathcal{A} \mapsto \mathbb{R}^m$ can be uniquely extended to a *measure* on $\mathcal{B}$ (cf. Proposition 6.1 in the Appendix). Usually we identify such set functions with its extension.

We call a set function $f : \mathcal{A} \times \mathcal{A} \mapsto \mathbb{R}^m$ an *interaction* if it is *biadditive*, i.e., additive with respect to both of its arguments. We say that f is *locally bounded* if for any $\tilde{Q} \in \mathcal{Q}$ there is some $c_{\tilde{Q}} > 0$ such that

$$|f(Q, Q')| \le c_{\tilde{Q}} \quad \text{for all } Q, Q' \in \mathcal{Q}^f \text{ with } Q, Q' \subset C \cap \tilde{Q}\,.$$

From the physics point of view we can distinguish long-range or *distant* interactions f_d and short-range or *contact* interactions f_c. Contact interactions as traction or heat flux are thought to act across the common boundary for subbodies that touch each other. Distant interactions such as gravity or electromagnetic forces act also on subbodies that are far away from the “source”.

In the literature distant interactions are usually called body or volume interactions, since they are traditionally assumed to become small if the volume of one of its arguments becomes small. Contact interactions are sometimes identified with surface interactions depending on the common boundary of the touching arguments, and they are assumed to become small if the area of the common boundary becomes small. Though contact interactions as described in the previous paragraph are much more general objects, here we consider them in the sense of surface interactions.

There is a variety of conditions in the literature defining distant or contact interactions. Additivity, as supposed above for any interaction, is certainly the most natural and most elementary requirement for such a mathematical object due to its agreement with our experience. This requirement is usually supplemented by different kinds of estimates. It is certainly a question of taste how far these estimates are considered to be physically elementary. Moreover we have to realize, as a particular aspect, that distant interactions are defined completely independently while contact interactions are basically assumed to be balanced by a distant interaction (also if the condition to be "balanced" is meanwhile quite weak). This coupling leads to the (explicit or implicit) assumption that there should be only the two mentioned kinds of interactions. While, on the one hand, phenomena like surface tension do not seem to be covered in this way, we want to show in our subsequent treatment that, on the other hand, there is no need for such a coupling in defining different kinds of interactions.

In our further treatment we discuss distant and contact interactions from a special point of view where we both summarize classical arguments and provide a new approach to the subject. We in particular illuminate the significance of σ-additivity which expresses some kind of continuity for set functions and which might be considered as one of the most elementary conditions beyond (finite) additivity. Since the classical systems of subbodies do not seem to be appropriate for the investigation of σ-additivity, we discuss the classical arguments by using the subbodies as introduced in the previous section. That way we will see that distant and contact interactions can be defined independently from each other and that both are σ-additive in the first argument. However, *σ-additivity in the second argument turns out to be the characterizing difference of distant (long-range) and contact (short-range) interactions.*

4. Distant interactions

Let us start with the discussion of distant interactions. Some typical condition used in the literature for defining a distant (or body) interaction f is, e.g., that C is assumed to be bounded and that for any $A' \in \mathcal{A}$ there is some $c_{A'} > 0$ with

$$|f(A, A')| \leq c_{A'} \, \mathcal{L}^n(A) \quad \text{for all } \; A, A' \in \mathcal{A}$$

such that the $c_{A'}$ are bounded and tend to zero with $\mathcal{L}^n(A')$. Here we clearly recognize the classical idea of volume dependence. This condition implies the existence of an integrable function $b : C \times C \mapsto \mathbb{R}^m$ such that

$$|f(A,A')| \leq \int_{A\times A'} b(x,y)\, d\mathcal{L}^n(x,y) \quad \text{for all } \ A,A' \in \mathcal{A}$$

(cf. Gurtin, Williams & Ziemer [6]). It turns out that concentration effects, that may occur for distant interactions, are not covered in this way. Therefore we generalize the previous condition by assuming that there is some Radon measure μ on $C \times C$ such that

$$|f(A,A')| \leq \mu(A,A') \quad \text{for all } \ A,A' \in \mathcal{A} \tag{4.1}$$

(cf. Marzocchi & Musesti [8]).

It seems that all definitions of distant interactions found in the literature have in common that they imply a condition like (4.1). Since C is usually assumed to be bounded, it is certainly not restrictive for applications to assume that the Radon measure μ be finite. Thus, by using Corollary 6.4 in the Appendix, we discover that any distant interaction f satisfying (4.1) for a finite Radon measure μ can be extended uniquely on $\mathcal{B} \times \mathcal{B}$ such that the extension is *bi-σ-additive*, i.e., σ-additive in both of its arguments. Note that this in particular implies that f itself has to be locally bounded and bi-σ-additive already on $\mathcal{A} \times \mathcal{A}$.

Let us now consider any interaction f on $\mathcal{A}\times\mathcal{A}$ which be even bi-σ-additive and locally bounded. By Proposition 6.3 in the Appendix there is a measure $f^\times$ on the Borel sets of $C\times C$ such that $f(A,A') = f^\times(A\times A')$ for all $A, A' \in \mathcal{A}$. Defining μ as the total variation $|f^\times|$ of the measure $f^\times$ we readily obtain an estimate as (4.1).

Summarizing we can say that a variety of conditions used in the literature to define a distant interaction f is equivalent to the condition that f is a bi-σ-additive interaction. While our discussion has shown that such a distant interaction f is uniquely determined by its specification on $\mathcal{A} \times \mathcal{A}$, it can be uniquely extended on $\mathcal{B}\times\mathcal{B}$ such that for any $B \in \mathcal{B}$ the mappings $f(B,\cdot)$ and $f(\cdot,B)$ are measures on $\mathcal{B}$. Note that, in this way, we can naturally account for concentrations and that there is no need for Boolean algebras as in previous treatments.

5. Contact interactions

Let us now consider contact interactions. Here we first illuminate the classical approach from the point of view of additivity. Then we present an example showing that the traditional approach does not have enough structure to describe concentrations sufficiently well. In this way we motivate the main ideas of the new approach given in Schuricht [10] that we summarize in the last part of this section.

5.1. Classical approach

We formulate the main ideas of the classical approach by taking subbodies in the sense described in Section 2 (and not in the traditional sense where boundary points are disregarded), since that allows to analyze σ-additivity and it illuminates the connection to the new theory presented later on. In order not to obscure our arguments by technicalities we assume that C is open and we restrict our discussion to subbodies $A \in \mathcal{R}$ (recall that $\mathcal{R}$ is the algebra generated by the boxes $\mathcal{Q}$). Note that these simplifications do not influence the generality of our conclusions. To be able to easily identify the hypotheses demanded we number them separately by using "C" for (explicit or implicit) conditions in the classical approach and by using "S" for supplementing conditions.

Traditionally contact interactions f are interactions that are merely considered for arguments (A, A') with

$$\mathcal{L}^n(A \cap A') = 0\,. \tag{C.0}$$

The most elementary condition characterizing a contact (or short-range) interaction is that the action $f(A, A')$ exerted from A' on A should vanish if the bodies have no contact, i.e.,

$$f(A, A') = 0 \quad \text{if} \quad \partial A \cap \partial A' = \emptyset\,. \tag{C.1}$$

Thus, by additivity, the action $f(A, A')$ exerted by A' on A depends merely on that material of A' which lies within an arbitrarily small neighborhood of A and the material of A' outside of a small neighborhood of A can be disregarded without changing the interaction.

Contact interactions are thought to describe phenomena as, e.g., traction and heat flux and, according to our experience, only some "substantial" material can exert a nonvanishing action. Therefore subbodies are basically assumed to correspond to open sets in most treatments, i.e., one implicitly imposes the elementary condition that

$$f(A,A')=0 \quad \text{if} \quad \mathcal{L}^n(A')=0\,. \tag{C.2}$$

It is a simple but important observation that already the basic conditions (C.1) and (C.2) prevent the set function $f(A,\cdot)$ from being σ-additive. This can be seen by choosing any $A, A' \in \mathcal{Q}$ respecting (C.0) and a sequence $A'_j \in \mathcal{R}$ of pairwise disjoint elements such that

$$A' \setminus A = \bigcup_{j=1}^{\infty} A'_j\,, \quad \partial A \cap \partial A'_j = \emptyset \;\text{ for all }\; j \in \mathbb{N}\,.$$

Then σ-additivity of $f(A,\cdot)$ would imply that

$$f(A,A') \overset{(C.2)}{=} f(A,A'\setminus A) = \sum_{j=1}^{\infty} f(A,A'_j) \overset{(C.1)}{=} 0\,.$$

But this is only possible for the trivial contact interaction $f \equiv 0$.

Though it is not explicitly stated in all treatments, the traditional idea of a contact interaction $f(A,A')$ is that it should depend merely on the common boundary $\partial A \cap \partial A'$ of A and A' and, even more, it should not change with A, A' as long as $\partial A \cap \partial A'$ does not change, i.e.,

$$f(A_1,A'_1) = f(A_2,A'_2) \quad \text{if} \quad \partial A_1 \cap \partial A'_1 = \partial A_2 \cap \partial A'_2\,.$$

This tacit assumption is slightly stronger than (C.1) and suggests, as done in most treatments, to identify a contact interaction f with an additive set function $g : \mathcal{S} \mapsto \mathbb{R}^m$ defined on a suitable class $\mathcal{S}$ of (oriented) surfaces such that

$$f(A,A') = g(\partial A \cap \partial A') \quad \text{for all }\; A, A'\,.$$

It is this identification which motivates to require that a contact interaction f should become small if the area of $\partial A \cap \partial A'$ becomes small, i.e.,

$$|f(A,A')| \le c\,\mathcal{H}^{n-1}(\partial A \cap \partial A') \quad \text{for all }\; A, A'$$

with some constant $c > 0$. In more recent treatments this classical condition is replaced with the weaker requirement that there is some integrable function $h : C \mapsto [0, \infty)$ and a full subsystem $\mathcal{Q}^f \subset \mathcal{Q}$ such that

$$\text{(C.3)} \qquad |f(A,A')| \le \int_{\partial A \cap \partial A'} h \, d\mathcal{H}^{n-1} \quad \text{for all } A, A' \in \mathcal{Q}^f$$

or, in terms of the function g, that

$$\text{(C.3}^*\text{)} \qquad |g(S)| \le \int_S h \, d\mathcal{H}^{n-1} \text{ for all } S \in \mathcal{S}^f := \{S \in \mathcal{S} | \, S \subset \partial A \, , A \in \mathcal{Q}^f\}$$

(cf. Šilhavý [11], [12]). An additive set function $g : \mathcal{S} \mapsto \mathbb{R}^m$ satisfying estimate (C.3*) is called *Cauchy flux*. It can be shown that contact interactions satisfying (C.0)-(C.3) and Cauchy fluxes correspond to each other (cf. Marzocchi & Musesti [8]).

Remark 5.1 - Note that (C.3) is not a completely elementary generalization of the original area estimate. It would be a more natural extension to assume that for any $S \in \mathcal{S}$ there is some ($\mathcal{H}^{n-1}$-) integrable function $h^S : S \mapsto \mathbb{R}^m$ such that (C.3) holds with h^S instead of h as long as $\partial A \cap \partial A' \subset S$. However, this weaker condition is analytically not sufficient to derive the desired results. Thus the significance of (C.3) seems not to be completely clear (cf. Šilhavý [11] for a discussion of that point).

Note that we always considered A' as the subbody that *exerts* an action and A as the part that *resists* this action. The usual "reduction" of a contact interaction f to its Cauchy flux g and the usual restriction to subbodies corresponding to open sets express that the classical approach tacitly employs the *symmetry* that the role of A and A' in $f(A, A')$ is interchangeable. But notice that *there is no need to specify the nature of the reaction by A.* If, e.g., A' consists of some elastic material and exerts the traction $f(A, A')$ on A, then it does not matter whether the reaction is caused by a rigid body, the tip of a needle, some magnetic film on ∂A, or also by some elastic material. Consequently there is no necessity to demand a condition like (C.2) with respect to the first argument A. Hence let us illuminate the classical theory from the point of view where we neglect this traditional symmetry, i.e., we allow that $f(A, A') \neq 0$ for $\mathcal{L}^n(A) = 0$. In particular we want to analyze whether the theory becomes consistent with σ-additivity of f in the first argument in this way.

Let us fix A' where, for technical simplicity, we assume that $A' \in \mathcal{Q}$. By (C.1) we have that $f(A, A') = 0$ for all $A \subset C \setminus A'$, $A \in \mathcal{Q}$, since $C \setminus A'$ is open and A is closed. As a necessary condition to get σ-additivity of $f(\cdot, A')$ we thus have to assume that

$$\text{(S.1)} \qquad f(A, A') = 0 \quad \text{for all } A \subset C \setminus A' \;\; A \in \mathcal{R}\,.$$

Hence only the points of $\partial A'$ can give a nonvanishing contribution to $f(\cdot, A')$. Thus we rediscover the classical idea that the mapping $A \mapsto f(A, A')$ corresponds to a mapping $S \mapsto g(S)$ for surfaces $S \subset \partial A'$. By Proposition 6.2 we now observe that (C.3) or, equivalently, (C.3*) already implies that g has to be σ-additive on any algebra of surfaces $S \subset \partial A'$. Consequently, condition (S.1) ensures that

$$\text{(5.1)} \qquad A \mapsto f(A, A') \quad \text{is } \sigma\text{-additive for all } A \in \mathcal{R} \text{ respecting (C.0).}$$

Thus $f(\cdot, A')$ can be even extended to a measure on $\mathcal{B}$ by Proposition (6.1). Note carefully that these implications do not contradict our arguments excluding σ-additivity of f with respect to the second argument.

The previously made hypotheses are not yet sufficient to derive Cauchy's famous representation formula that there is some tensor field $\tau : C \mapsto \mathbb{R}^{n \times m}$ such that

$$\text{(5.2)} \qquad f(A, A') = \int_{\partial A \cap \partial A'} \tau \cdot \nu \, d\mathcal{H}^{n-1} \quad \text{for all } A, A' \in \mathcal{Q}^f$$

where ν is the outer unit normal field of A'. Therefore only *balanced* contact interactions f are usually considered, i.e., f is assumed to satisfy a balance law

$$f(A, C \setminus A) + f_d(A, C) = 0 \quad \text{for all } A \in \mathcal{R}$$

with some distant interaction f_d. Note that this condition, in some sense, provides a relation between the measures $f(\cdot, A')$ for different A'. Since the specification of $f_d(A, B)$ for $B \neq C$ is not necessary for the derivation of Cauchy's formula, more recent treatments require the equivalent condition that there is some Borel measure η such that

$$\text{(C.4)} \qquad |f(A, C \setminus A)| \leq \eta(A) \quad \text{for all } A \in \mathcal{R}\,.$$

Let us study also this condition in the light of σ-additivity. For that reason we drop (C.0) and, thus, we have to clarify what $f(A, A')$ means in the case $A' \subset A$. We intend that, for (closed) boxes $A, A' \in \mathcal{Q}$, $f(A, A')$ provides the contact interaction exerted by the material of A' on A *from outside*. Hence we assume that

$$f(A, A') = 0 \quad \text{for all } \; A' \subset A\,, \;\; A, A' \in \mathcal{Q}\,, \tag{S.2}$$

where, of course, f is assumed to be biadditive on all of $\mathcal{R} \times \mathcal{R}$. After this extension we can replace (C.4) with

$$|f(A, C)| \le \eta(A) \quad \text{for all } \; A \in \mathcal{R}\,.$$

Using Proposition 6.2 we conclude that $f(\cdot, C)$ is σ-additive on $\mathcal{Q}$. Let us now investigate σ-additivity of $f(\cdot, A')$ in that extended case where, for technical simplicity, we assume that $A' \in \mathcal{Q}$. For any fixed $A \in \mathcal{R}$ we choose a pairwise disjoint decomposition $A = \bigcup_{j=1}^{\infty} A_j$ with $A_j \in \mathcal{R}$ and we set $A_j^1 := A_j \setminus \operatorname{int} A'$, $A_j^2 := A_j \cap \operatorname{int} A'$. By additivity,

$$f(A, A') = f(A \setminus \operatorname{int} A', A') + f(A \cap \operatorname{int} A', A')\,.$$

By property (5.1),

$$f(A \setminus \operatorname{int} A', A') = \sum_{j=1}^{\infty} f(A_j^1, A')\,.$$

From (S.1) we derive that $f(\tilde{A}, C \setminus A') = 0$ for all $\tilde{A} \subset \operatorname{int} A'$. Since $f(\cdot, C)$ is σ-additive, we thus get

$$f(A \cap \operatorname{int} A', A') = f(A \cap \operatorname{int} A', C) = \sum_{j=1}^{\infty} f(A_j^2, C) = \sum_{j=1}^{\infty} f(A_j^2, A')\,.$$

Consequently,

$$f(A, A') = \sum_{j=1}^{\infty} f(A_j^1, A') + \sum_{j=1}^{\infty} f(A_j^2, A') = \sum_{j=1}^{\infty} f(A_j, A')\,.$$

Thus, in the case where (C.0) is disregarded we can extend (5.1) to

$$A \mapsto f(A, A') \quad \text{is } \sigma\text{-additive for all } A \in \mathcal{R}. \tag{5.3}$$

Let us summarize our discussion. The traditional approach takes it as granted that contact interactions f are only considered for pairs of (basically) disjoint subbodies (A, A') and that the role of A and A' is somehow interchangeable. However a more careful distinction between acting and reacting body and, correspondingly, a more careful treatment of subbodies makes visible a fundamental *asymmetry* in the arguments of contact interactions which was hidden in classical treatments. Moreover we see that contact interactions f are in fact σ-additive with respect to the first argument while σ-additivity in the second argument is not possible. Thus it turns out that *σ-additivity in the second argument is a fundamental difference between distant and contact interactions*. While here we have derived the σ-additivity of $f(\cdot, A')$ from the classical hypotheses (C.1) - (C.4) supplemented by (S.1), (S.2), we will show below what remains to require if we demand σ-additivity of $f(\cdot, A')$ from the beginning. But before let us illuminate some aspects related to concentrations.

5.2. Concentrations

If we press the tip of a needle against some deformable body, then it is convenient to idealize the exerted force as concentrated at one point. In the foundations of continuum physics such concentrations had been disregarded for a long time. They are somehow taken into account in the treatment of Degiovanni et al. [2], but the surfaces where they act have to be neglected. Here we demonstrate, by means of a classical example, that the usual idea where the contact interaction $f(A, A')$ merely depends on the common boundary $\partial A \cap \partial A'$ is too restrictive for a complete treatment of concentrations. Moreover the example demonstrates, independently from the permanent discussion how far concentrations really occur in nature, that we might be confronted with concentrations very naturally even in “harmless” looking situations where the presence of concentrations becomes visible only by a more careful analysis. For a more comprehensive discussion of related questions we refer to Podio-Guidugli [9].

Let us consider a planar version of the problems studied by Boussinesq [1] and Flamant [5]. More precisely, we consider an elastic body occupying the half plane $C = \{(x_1, x_2) \in \mathbb{R}^2 | \, x_1 \geq 0\}$ and we assume that an external unit point load directed as $(1, 0)$ is exerted at the origin. The corresponding stress distribution is given by the locally integrable tensor field $\tau : \mathbb{R}^2 \mapsto \mathbb{R}^4$ with

$$\tau(x) := \frac{2x_1}{\pi|x|^4} \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix} \quad \text{for } x_1 \geq 0\,. \tag{5.4}$$

Let us extend τ by zero on all of $\mathbb{R}^2$. Then the divergence of this τ in the weak sense equals the measure

$$\operatorname{div}\tau = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta_{(0,0)}$$

where $\delta_{(0,0)}$ is the scalar Dirac measure concentrated at $x = (0,0)$.

We now want to analyze the balance of forces for closed circular sectors $V = V(\theta_1, \theta_2)$ of radius $\varrho > 0$ with vertex at the origin as shown in Figure 1 where $0 \leq \theta_1 < \theta_2 \leq \pi$.

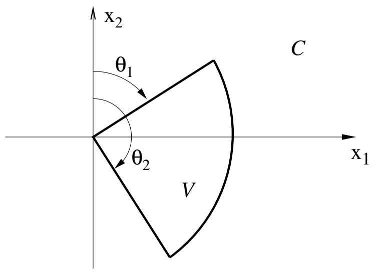

Figure 1: Circular sectors $V = V(\theta_1, \theta_2)$.

It turns out that the material of $C \setminus V$ exerts a resultant elastic contact force

$$f(V, C \setminus V) = -\frac{1}{\pi} \begin{pmatrix} (\theta_2 - \theta_1) - \frac{1}{2}\sin 2(\theta_2 - \theta_1) \\ \sin^2\theta_2 - \sin^2\theta_1 \end{pmatrix}$$

on V depending on θ_1, θ_2 but not on ϱ. Obviously, $f(V(0,\pi), C \setminus V(0,\pi)) = (-1, 0)$ for all $\varrho > 0$. Since this traction is balanced by the external force $(1,0)$ exerted on $V(0,\pi)$ at the origin, the material of $V(0,\pi)$ has to exert the resultant elastic contact force $(-1,0)$ on the single point $(0,0)$, i.e.,

$$f(\{(0,0)\}, V(0,\pi)) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

is a concentrated force. By analogous arguments we obtain that

$$f(\{(0,0)\}, V(0,\pi/2)) = \begin{pmatrix} -1/2 \\ -1/\pi \end{pmatrix}$$

which balances only some part of the external force and, suprisingly, it even has a component parallel to the boundary ∂C of the half plane. In general it turns out that $f(\{(0,0)\}, V(\theta_1,\theta_2))$ depends on the opening angle of the sector V. Consequently a theory where the contact interaction $f(A,A')$ merely depends on the common boundary of A and A' is too restrictive to describe such concentrations in full detail. Therefore we drop this traditional restriction in the more general approach for contact interactions given later on.

One might think that these difficulties with concentrations can only occur on the boundary of the body in the case of concentrated external forces. But let us consider the tensor field $\tilde{\tau}$ that equals τ for $x_1 \geq 0$ and that is the reflection of τ for $x_1 \leq 0$, i.e.,

$$\tilde{\tau}(x) := (\operatorname{sgn} x_1)\, \frac{2x_1}{\pi|x|^4} \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix} \quad \text{for all } \ (x_1,x_2) \in \mathbb{R}^2\,.$$

The weak divergence of this tensor field $\tilde{\tau}$ is identical zero everywhere on $\mathbb{R}^2$. Hence we can regard $\tilde{\tau}$ as the stress tensor of an elastic body occupying $\tilde{C} = \mathbb{R}^2$ where no external force is applied. Let us now assume that the material outside the unit ball around the origin is removed and that the stresses of the material on the boundary are balanced by a suitable external boundary traction. That way we obtain an equilibrium for the elastic unit ball with smooth external boundary traction and without applying external forces inside the body. If we now analyze the neighborhood of the origin for this "harmless looking" problem, then we see that all phenomena discussed above are still there. In particular we have concentrated contact forces exerted at the origin by circular sections. Consequently this example indicates that we always should be aware of concentrations and, thus, *we have to account for concentrations in the foundations of continuum physics in any case.*

5.3. Alternative approach

In this section we summarize the foundations for contact interactions as given in Schuricht [10] that substantially differ from previous treatments. The primary concern of this new approach was to develop a theory that satisfactorily describes concentrations as they are discussed in the previous part. But this approach also demonstrates what of the traditional hypotheses for contact interactions remains relevant if additivity and σ-additivity (as far as possible) are postulated from the beginning. In this way the new theory certainly provides a new insight into the nature of contact interactions.

Let $\mathcal{A}$ be an algebra on the Borel set $C \subset \mathbb{R}^n$ such that $\mathcal{A}$ contains a full system $\mathcal{Q}^f$ of boxes and let $f : \mathcal{B}\times\mathcal{A} \mapsto \mathbb{R}^m$ be (the extension of) an interaction that is σ-additive in the first argument. We call f a *contact interaction* on C relative to $\mathcal{A}$ if for any $Q \in \mathcal{Q}^f$, $A \in \mathcal{A}$:

(H1) $f(Q, A) = 0$ as long as $Q_\varepsilon \cap A = \emptyset$ for some $\varepsilon > 0$,

(H2) $f(Q, A) = 0$ as long as $\mathcal{L}^n(A \setminus Q) = 0$,

(H3) $f(Q, A) = \operatorname{ap}\lim_{\varepsilon\downarrow 0} f(Q_\varepsilon \setminus Z, A \setminus Q_\varepsilon)$ for all $Z \in \mathcal{B}$ with $\mathcal{L}^n(Z) = 0$.

Note that (H1) and (H2) slightly differ from those given in Schuricht [10]. But, as a simple consequence of additivity, (H1) and (H2) are equivalent to the original conditions

(H1') $f(Q, A) = f(Q, (Q_\varepsilon \setminus Q) \cap A)$ for all $\varepsilon > 0$ with $Q_\varepsilon \in \mathcal{Q}^f$,

(H2') $f(Q, A) = 0$ as long as $\mathcal{L}^n(A) = 0$.

The fundamental feature of a contact interaction to be a short-range phenomenon is expressed by the locality condition (H1). Taking into account also (H2) we see that the action exerted on Q by A depends only on that material of A which is outside of Q and which belongs to an arbitrarily small neighborhood of Q. Note that this already includes the condition that

$$f(Q, A) = 0 \quad \text{if } A \subset Q,\ \ A \in \mathcal{A},\ \ Q \in \mathcal{Q}^f \tag{5.5}$$

(cf. also (S.2)). In addition to these elementary postulates merely certain singular cases are excluded that seem to be untypical for "usual" contact interactions. (H2) says in particular that only a "thick" body A can exert a nonvanishing action. The more technical condition (H3) expresses, roughly speaking, that an eventually concentrated action $f(Q \cap Z, A) \neq 0$ cannot "propagate" along the "thin" set Z, but it rather has to diffuse within the "thick" body A. Note that (H3) is always satisfied in the case $Z = \emptyset$, since $f(\cdot, A)$ is a measure and since $f(Q_\varepsilon, A \setminus Q_\varepsilon) = f(Q_\varepsilon, A)$ according to (H2).

Let us demonstrate that it is important to use "closed" boxes Q in the hypotheses. Since $f(\cdot, A)$ is a measure, we first observe that, by (H1),

$$f(\operatorname{int} Q, A) = 0 \quad \text{for all } \ A \subset (C \setminus \operatorname{int} Q)\,, \ \ Q \in \mathcal{Q}^f\,.$$

Using also (H2) we see that

$$f(Q, A) = f(\partial Q, A \setminus Q) \quad \text{for all } \ A \in \mathcal{A}\,, \ \ Q \in \mathcal{Q}^f\,.$$

Consequently, by additivity,

$$\begin{aligned} f(\operatorname{int} Q, A \cap Q) &= f(\operatorname{int} Q, A) - f(\operatorname{int} Q, A \setminus Q) = f(\operatorname{int} Q, A) \\ &= f(Q, A) - f(\partial Q, A) \\ &= f(Q, A) - f(\partial Q, A \setminus Q) - f(\partial Q, A \cap Q) \\ &= -f(\partial Q, A \cap Q)\,. \end{aligned}$$

Since the right hand side does not vanish in general, also the left hand side does not, in contrast to (5.5).

As in the classical case the previous hypotheses ensure that a contact interaction can be represented by means of a tensor. Here it is sufficient to consider the contact interaction on a "small" algebra $\mathcal{A} = \mathcal{R}^f$ that is generated by a full system $\mathcal{Q}^f \subset \mathcal{Q}$.

Theorem 5.2. (Interaction tensor). *Let $f : \mathcal{B} \times \mathcal{R}^f \mapsto \mathbb{R}^m$ be a locally bounded contact interaction on C. Then there exists an* interaction tensor *$\tau \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R}^{m \times n})$ such that, for any $Q \in \mathcal{Q}^f$, $R \in \mathcal{R}^f$,*

$$\begin{aligned} f(Q, R) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{(Q_\varepsilon \setminus Q) \cap R} \tau \cdot \nu_Q \, d\mathcal{L}^n \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\partial Q_\sigma \cap R} \tau \cdot \nu_Q \, d\mathcal{H}^{n-1} \, d\sigma\,. \end{aligned} \tag{5.6}$$

The tensor τ is uniquely determined up to a set of $\mathcal{L}^n$-measure zero.

We are now interested in a more general representation formula for f than (5.6). Moreover we want to look for a preferably large algebra $\mathcal{A}$ such that $f(B,\cdot)$ can be extended on $\mathcal{A}$ for any $B \in \mathcal{B}$. As preparation for these investigation we consider a locally bounded contact interaction $f : \mathcal{B} \times \mathcal{A} \mapsto \mathbb{R}^m$ relative to an algebra $\mathcal{A}$ containing $\mathcal{R}^f$ and we assign special interactions to f. For any $A \in \mathcal{A}$ we define the *partial restriction* $f_{(A)} : \mathcal{B} \times \mathcal{A} \mapsto \mathbb{R}^m$ of f by

$$f_{(A)}(\tilde{B},\tilde{A}) := f(\tilde{B},\tilde{A} \cap A) \quad \text{for all } \tilde{B} \in \mathcal{B},\ \tilde{A} \in \mathcal{A}.$$

Moreover, for any Borel set $\breve{C} \subset \mathbb{R}^n$ with $C \subset \breve{C}$ and for some algebra $\breve{\mathcal{A}}$ on $\breve{C}$ with $\breve{\mathcal{A}}_{|C} \subset \mathcal{A}$ and $\mathcal{Q}^f_{\breve{C}} \subset \breve{\mathcal{A}}$ we define the *zero extension* $\breve{f} : \mathcal{B}_{\breve{C}} \times \breve{\mathcal{A}} \mapsto \mathbb{R}^m$ of f by

$$\breve{f}(\breve{B},\breve{A}) := f(\breve{B} \cap C, \breve{A} \cap C) \quad \text{for all } \breve{B} \in \mathcal{B}_{\breve{C}},\ \breve{A} \in \breve{\mathcal{A}}$$

(notice that $\mathcal{Q}^f = \mathcal{Q}(I) \subset \mathcal{R}^f$ and $\mathcal{Q}^f_{\breve{C}} = \mathcal{Q}_{\breve{C}}(I)$ with the same set $I \subset \mathbb{R}$ of coordinates). It can be shown that both the partial restrictions $f_{(A)}$ and the zero extensions $\breve{f}$ are also locally bounded contact interactions. We can think of the partial restriction $f_{(A)}$ as the interaction on C where the material outside of A is inactive while the zero extension $\breve{f}$ results from f by adding inactive material. Obviously, by the application of the previous theorem, we obtain a tensor $\tau_A \in \mathcal{L}^1_{\text{loc}}(C,\mathbb{R}^{m\times n})$ for each partial restriction $f_{(A)}$ and a tensor $\breve{\tau} \in \mathcal{L}^1_{\text{loc}}(\breve{C},\mathbb{R}^{m\times n})$ for each zero extension $\breve{f}$. The next proposition tells us how the τ_A and $\breve{\tau}$ are related to the tensor τ of f.

Proposition 5.3. *Let f be a locally bounded contact interaction on C relative to the algebra $\mathcal{A}$ and let τ be the corresponding interaction tensor.*

(1) *For any $A \in \mathcal{A}$ the interaction tensor τ_A of the partial restriction $f_{(A)}$ of f is given by*

$$\tau_A(x) = \begin{cases} \tau(x) & \text{for } x \in A\,, \\ 0 & \text{for } x \in C \setminus A\,. \end{cases} \tag{5.7}$$

(2) *If $\breve{f}$ is the zero extension of f on $\breve{C}$, then the interaction tensor $\breve{\tau}$ is given by*

$$\breve{\tau}(x) = \begin{cases} \tau(x) & \text{for } x \in C\,, \\ 0 & \text{for } x \in \breve{C} \setminus C\,. \end{cases}$$

For our further treatment we confine our attention to contact interactions on open sets $C \subset \mathbb{R}^n$. Notice that this is not restrictive, since otherwise we can use a suitable zero extension. We say that the tensor field $\tau \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R}^{m\times n})$ has *divergence measure* if for each compact set $K \subset C$ there exists a constant $c_K \geq 0$ such that

$$\left| \int_C \tau \cdot D\varphi \, d\mathcal{L}^n \right| \leq c_K \max_K |\varphi|$$

for all $\varphi \in \mathcal{C}_0^\infty(C, \mathbb{R})$ with $\operatorname{spt} \varphi \subset K$. This means that the distributional divergence $\operatorname{div} \tau$ can be interpreted as vector-valued Radon measure on C such that

$$-\int_C \tau \cdot D\varphi \, d\mathcal{L}^n = \int_C \varphi \, d(\operatorname{div} \tau)$$

for all Lipschitz continuous functions $\varphi : C \mapsto \mathbb{R}^m$ having compact support. Notice that the measure $\operatorname{div} \tau$ is independent of a change of τ on a set of $\mathcal{L}^n$-measure zero.

Theorem 5.4. (Representation formula). *Let f be a locally bounded contact interaction on the open set C relative to the algebra $\mathcal{A}$. Then the interaction tensor τ of f and the interaction tensors τ_A of the partial restrictions $f_{(A)}$ for any $A \in \mathcal{A}$ have divergence measure. Moreover,*

$$f(B, A) = \operatorname{div} \tau_A(B) \tag{5.8}$$

for all $B \in \mathcal{B}$, $A \in \mathcal{A}$.

Relation (5.8) can be considered as a replacement for Cauchy's famous representation formula (5.2). In contrast to the classical case, (5.8) is independent of a normal field on the boundary of B and, thus, it provides the interaction for any Borel set B. However, we recover Cauchy's identity for "nice" sets B.

Corollary 5.5. *Let f, τ, τ_A be as in the previous theorem. Then, for any $A \in \mathcal{A}$, there is some nonnegative $h \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R})$ and some nonnegative real measure μ on C such that*

$$f(P, A) = \int_{\partial_* P} \tau_A \cdot \nu_P \, d\mathcal{H}^{n-1}$$

for all $P \in \mathcal{P}$ with $\int_{\partial_* P} h \, d\mathcal{H}^{n-1} < \infty$ and $\mu(\partial_* P) = 0$.

At the beginning we have seen that we get a unique interaction tensor τ already for locally bounded contact interactions f that are merely defined on some quite "small" algebra $\mathcal{R}^f$ with respect to the second argument. However, for a comprehensive understanding of contact interactions it is certainly useful to look for extensions of f on a preferably large algebra $\mathcal{A}$.

Let us first assume that we can reach $\mathcal{A} = \mathcal{B}$. Then we choose boxes $Q, Q' \in \mathcal{Q}^f$ and construct $\tilde{Q}'$ from Q' by removing the points of ∂Q_ε for all $\varepsilon \in (\frac{1}{4k+2}, \frac{1}{4k})$, $k \in \mathbb{N}$. Obviously $\tilde{Q}' \in \mathcal{B}$ and $f(Q_\varepsilon, \tilde{Q}') = 0$ for all $\varepsilon_k^1 := \frac{1}{4k+1}$ by (H1). Since $f(\cdot, Q)$, $f(\cdot, \tilde{Q}')$ have to be measures, we obtain with $\varepsilon_k^3 := \frac{1}{4k+3}$ that

$$\begin{aligned} f(Q, Q') &= \lim_{k\to\infty} f(Q_{\varepsilon_k^3}, Q') = \lim_{k\to\infty} f(Q_{\varepsilon_k^3}, \tilde{Q}') \\ &= f(Q, \tilde{Q}') = \lim_{k\to\infty} f(Q_{\varepsilon_k^1}, \tilde{Q}') = 0\,. \end{aligned}$$

Hence f has to be the trivial interaction $f \equiv 0$. Therefore $f(B, \cdot)$ cannot be extended consistently on all of $\mathcal{B}$ for a nontrivial f. But now Theorem 5.4 provides the necessary condition that we have to look for a "large" algebra $\mathcal{A}$ such that the tensor fields τ_A according to (5.7) have divergence measure for all $A \in \mathcal{A}$.

Proposition 5.6. (Extension). *Let f be a locally bounded contact interaction on the open set C relative to an algebra $\mathcal{R}^f$. Then there is a nonnegative function $h \in \mathcal{L}^1_{\mathrm{loc}}(C, \mathbb{R})$ such that, with the algebra*

$$\mathcal{P}_h := \{P \in \mathcal{P} \mid \int_{\partial_* P} h \, d\mathcal{H}^{N-1} < \infty\}\,,$$

f can be uniquely extended to a contact interaction on $\mathcal{B} \times \mathcal{P}_h$ by (5.8).

The algebra $\mathcal{P}_h$ is certainly rich enough for applications, but it is still open whether contact interactions f may be extended to larger algebras $\mathcal{A}$.

Summarizing we can say that we obtain a new theory that has a richer structure than the traditional approach and that precisely accounts for boundary points of subbodies. That way the new theory is able to describe concentrations in more detail than this was possible before. Finally we observe that σ-additivity in the second argument turns out to be a characterizing difference between distant and contact interactions, that have been introduced completely independently from each other in our approach.

6. Appendix

In this appendix we provide some facts from measure theory. Here we assume that $\mathcal{A}$ is an algebra on a Borel set $C \subset \mathbb{R}^n$ and that $g : \mathcal{A} \mapsto \mathbb{R}^m$ is a set function.

Proposition 6.1. *If g is locally bounded and σ-additive on $\mathcal{A}$, then it can be extended uniquely to a σ-additive set function on $\mathcal{B}$.*

PROOF – Let us sketch the essential arguments. We assume that $m = 1$, since otherwise we can argue for each component of g separately. The assigned functions $g^{\pm} : \mathcal{A} \mapsto [0, \infty)$, given by

$$g^+(A) := \sup_{A' \in \mathcal{A},\, A' \subset A} g(A'), \quad g^-(A) := \sup_{A' \in \mathcal{A},\, A' \subset A} -g(A'), \tag{6.1}$$

are additive on $\mathcal{A}$ and satisfy $g(A) = g^+(A) - g^-(A)$ for all $A \in \mathcal{A}$ by Dunford & Schwartz [4, p. 98] (note that $g^{\pm}(A) \geq \pm g(\emptyset) = 0$). Let now $A = \bigcup_{i=1}^{\infty} A_i$ be a pairwise disjoint decomposition within $\mathcal{A}$. Then, for any $\varepsilon > 0$, there are subsets $A^\varepsilon \subset A$, $A_i^\varepsilon \subset A_i$ in $\mathcal{A}$ such that

$$g^+(A) \leq g(A^\varepsilon) + \varepsilon = \sum_{i=1}^{\infty} g(A^\varepsilon \cap A_i) + \varepsilon \leq \sum_{i=1}^{\infty} g^+(A_i) + \varepsilon,$$

$$\sum_{i=1}^{\infty} g^+(A_i) \leq \sum_{i=1}^{\infty} \Big(g(A_i^\varepsilon) + \frac{\varepsilon}{2^i} \Big) = g(\bigcup_{i=1}^{\infty} A_i^\varepsilon) + \varepsilon \leq g^+(A) + \varepsilon.$$

The arbitrariness of ε implies σ-additivity of g^+ and, analogously, of g^- on $\mathcal{A}$.

Since g is locally bounded, $g^{\pm}$ are locally bounded too and, thus, σ-finite. Therefore $g^{\pm}$ can both be extended uniquely to a measure on $\mathcal{B}$ which we denote again by $g^{\pm}$ (cf. Halmos [7, p. 54]). Then $g^+ - g^-$ is the desired extension of the function g. $\square$

Proposition 6.2. *Let $g : \mathcal{A} \mapsto \mathbb{R}^m$ be additive and let*

$$|g(A)| \leq \mu(A) \quad \textit{for all } A \in \mathcal{A}$$

where μ is some finite Radon measure on C. Then g is even σ-additive on $\mathcal{A}$.

Proof – Let $A'_1, A'_2, \dots \in \mathcal{A}$ be a sequence of pairwise disjoint sets. Additivity implies

$$g(\bigcup_{j=1}^{\infty} A'_j) = \sum_{j=1}^{k-1} g(A'_j) + g(\bigcup_{j=k}^{\infty} A'_j)\,.$$

Since μ is finite, $\mu(\bigcup_{j=k}^{\infty} A'_j) \to 0$ for $k \to \infty$. Thus $g(\bigcup_{j=k}^{\infty} A'_j) \to 0$ and σ-additivity follows. □

We say that the set function $f : \mathcal{A} \times \mathcal{A} \mapsto \mathbb{R}^m$ is *biadditive* if the set functions $f(A, \cdot)$ and $f(\cdot, A')$ are additive on $\mathcal{A}$ for each $A, A' \in \mathcal{A}$. If both set functions are even σ-additive on $\mathcal{A}$, then we call f *bi-σ-additive*. By $\mathcal{A}^\times$ we denote the algebra on $C \times C$ generated by $\mathcal{A} \times \mathcal{A}$ (note that $\mathcal{A}^\times$ consists of all finite unions of product sets $A \times A'$ with $A, A' \in \mathcal{A}$, cf. Doob [3, II.5]).

Proposition 6.3. *Let $f : \mathcal{A} \times \mathcal{A} \mapsto \mathbb{R}^m$ be biadditive. Then:*

(1) *There is a unique additive set function $f^\times : \mathcal{A}^\times \mapsto \mathbb{R}^m$ such that*

$$f^\times(A \times A') = f(A, A') \quad \textit{for all } A, A' \in \mathcal{A}\,.$$

(2) *If f is even bi-σ-additive, then $f^\times$ from* (1) *is σ-additive on $\mathcal{A}^\times$.*

(3) *If there is some finite Radon measure $\mu^\times$ on $C \times C$ with*

$$|f(A, A')| \leq \mu^\times(A \times A') \quad \textit{for all } A, A' \in \mathcal{A}\,,$$

then $f^\times$ from (1) *is σ-additive on $\mathcal{A}^\times$.*

(4) *If f is locally bounded and $f^\times$ from* (1) *is σ-additive on $\mathcal{A}^\times$, then $f^\times$ can be extended uniquely to a measure on the Borel sets of $C \times C$.*

Proof – Let us start with (1). We first define $f^\times(A \times A') := f(A, A')$ for all $A, A' \in \mathcal{A}$. For fixed $A, A' \in \mathcal{A}$ we now choose any pairwise disjoint decomposition $A \times A' = \bigcup_{j=1}^{k} A_j \times A'_j$ with $A_j, A'_j \in \mathcal{A}$. Then there are pairwise disjoint decompositions $A = \bigcup_{i=1}^{l} \tilde{A}_i$ and $A' = \bigcup_{i'=1}^{l'} \tilde{A}'_{i'}$ with $\tilde{A}_i, \tilde{A}'_{i'} \in \mathcal{A}$ such that $A_j = \bigcup_{i \in \iota_j} \tilde{A}_i$ and $A'_j = \bigcup_{i' \in \iota'_j} \tilde{A}'_{i'}$ for suitable index sets $\iota_j \subset \{1, \ldots, l\}$ and $\iota'_j \subset \{1, \ldots, l'\}$ (cf. Doob [3, II.3]). Hence

$$\begin{aligned} f^\times(A \times A') \;&=\; f(A, A') = \sum_{i=1}^{l} f(\tilde{A}_i, A') = \sum_{i=1}^{l} \sum_{i'=1}^{l'} f(\tilde{A}_i, \tilde{A}'_{i'}) \\ &= \sum_{j=1}^{k} \sum_{i \in \iota_j} \sum_{i' \in \iota'_j} f(\tilde{A}_i, \tilde{A}'_{i'}) = \sum_{j=1}^{k} f(A_j, A'_j) = \sum_{j=1}^{k} f^\times(A_j \times A'_j)\,. \end{aligned}$$

But this verifies that $f^\times$ is additive on the set of all product sets $A \times A'$ with $A, A' \in \mathcal{A}$. Therefore $f^\times$ can be extended uniquely to an additive set function on the algebra $\mathcal{A}^\times$ (cf. Doob [3, III.2]).

Let us now show (2). Since each element in $\mathcal{A}^\times$ is the finite union of product sets $A \times A'$ with $A, A' \in \mathcal{A}$, it is sufficient to show σ-additivity of $f^\times$ for pairwise disjoint decompositions $A \times A' = \bigcup_{j=1}^{\infty} A_j \times A'_j$ with $A, A_j, A', A'_j \in \mathcal{A}$. But this follows as in the previous part of the proof with the only difference that all occurring decompositions might be countable.

We now show (3). Since any $A^\times \in \mathcal{A}^\times$ has the form $A^\times = \bigcup_{j=1}^{k} A_j \times A'_j$ with $A_j, A'_j \in \mathcal{A}$, we have for $f^\times$ from (1) that

$$|f^\times(A^\times)| \le \sum_{j=1}^{k} |f(A_j, A'_j)| \le \sum_{j=1}^{k} \mu^\times(A_j \times A'_j) = \mu^\times(A^\times)\,.$$

The assertion now follows from Proposition 6.2.

Statement (4) is a consequence of Proposition 6.1. □

The next corollary is a direct consequence of the previous proposition.

Corollary 6.4. *Let $f : \mathcal{A} \times \mathcal{A} \mapsto \mathbb{R}^m$ be biadditive and locally bounded and let f satisfy the assumptions from (2) or (3) of Proposition 6.3. Then f can be extended uniquely on $\mathcal{B} \times \mathcal{B}$ such that $f(\cdot, B')$ and $f(B, \cdot)$ are σ-additive.*

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Mathematisches Institut
Universität zu Köln
Weyertal 86-90
50931 Köln, Germany
e-mail: fs@math.uni-koeln.de