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Abstract

The paper studies differential equations of the form $u'(x) = f(x, u(x), \lambda(x))$, $u(x_0) = u_0$, where the right hand side is merely measurable in x. In particular sufficient conditions for the continuous and the differentiable dependence of solution u on the data and on the parameter λ are stated.

1 Introduction

In many applications we are confronted with ordinary differential equations of the type

$$u'(x) = g(x, u(x), \lambda)\,, \quad u(x_0) = u_0\,,$$

where the right hand side depends on a parameter λ. Here the dependence on the data and on the parameter λ is a fundamental task which is usually studied in the classical sense, i.e., the function $g(\cdot,\cdot,\cdot)$ on the right hand side is assumed to be at least continuous in all of its arguments (cf. Coddington & Levinson [1], Hestenes [2], Walter [7]). It turns out that this is too restrictive for several applications. In particular certain problems for elastic rods lead to equations of the type

$$u'(x) = f(x, u(x), \lambda(x))\,, \quad u(x_0) = u_0\,,$$

where $\lambda : I \subset \mathbb{R} \mapsto \mathbb{R}^m$ is a parameter function that is merely measurable. Thus the right hand side is merely measurable in x and the differential equation has to be considered in the sense of Carathéodory. While general existence results for that type of problem are available in the literature (cf. Walter [7]), results about the dependence of the solution on the parameter seem to be quite rare. The case of parameters $\lambda \in \mathbb{R}^n$, which is contained in Kurzweil [3], is insufficient for the intended applications. For $\lambda \in L^\infty(I)$ the problem is covered by the general results in Schuricht & v.d. Mosel [5] which are applied to the investigation of self-contact for nonlinearly elastic rods in Schuricht & v.d. Mosel [6]. However, for further applications, it is necessary to consider also the case $\lambda \in \mathcal{L}^p(I)$ with $1 < p < \infty$ (cf. Schuricht [4]). In the present paper we study a general situation which is sufficient for the desired applications. In particular we study both the continuous dependence and the differentiable dependence of the solution u on the data and on the parameter. While the continuous dependence is considered in Section 2 and includes the case $p = 1$, the differentiable dependence is investigated in Section 3. Let us still mention that a special case of the present results is already announced and applied in Schuricht [4] where, however, Theorem 5.8 is only valid for $p > 1$ and the δ in the theorems might be smaller than that in the assumptions.

Notation. The closure of a set A is denoted by $\operatorname{cl} A$ and $|a|$ stands for the norm of a vector $a \in \mathbb{R}^n$. We write $C(I)$ for the space of continuous functions on I. The Lebesgue space of p-integrable functions is denoted by $L^p(I)$ and $L^{p'}(I)$ with $\frac{1}{p} + \frac{1}{p'}$ is its dual. $g_{|I}$ denotes the restriction of the function g on I.

2 Continuous dependence on the data and the parameter

We consider an initial value problem for ordinary differential equations of the form

$$u'(x) = f(x, u(x), \lambda(x)) \quad \text{for a.e. } x \in J, \quad u(x_0) = u_0\,, \tag{2.1}$$

which depends on a parameter λ and where the right-hand side is merely measurable in the argument $x \in \mathbb{R}$. In this section we show existence and uniqueness of a solution u and continuous dependence of the solution on the initial data and the parameter.

Let $f : J\times K\times\mathbb{R}^m \mapsto \mathbb{R}^n$ be a mapping where $K \subset \mathbb{R}^n$ is open and $J \subset \mathbb{R}$ is an open bounded interval. Moreover, let $x_0 \in J$, $u_0 \in K$, and $\lambda \in L^p(J,\mathbb{R}^m)$, $1 \le p \le \infty$. We assume that f is a Carathéodory function, i.e., $(u, \lambda) \mapsto f(x, u, \lambda)$ is continuous on $K \times \mathbb{R}^m$ for a.e. $x \in J$ and $x \mapsto f(x, u, \lambda)$ is measurable on J for any $(u, \lambda) \in K \times \mathbb{R}^m$. Consequently, $x \mapsto f(x, u(x), \lambda(x))$ is measurable on J for any $\lambda \in L^p(J)$ and any $u \in C(J)$ with $u(J) \subset K$. Thus the right-hand side of (2.1) is merely measurable in x and we have to consider solutions in the sense of Carathéodory, i.e., we are looking for solutions u that are absolutely continuous. In particular we are looking for local solutions on suitable subintervals $I \subset J$.

For our further analysis we fix $\tilde{x} \in J$, $\tilde{u} \in K$, $\tilde{\lambda} \in L^p(J,\mathbb{R}^m)$ and we choose $\delta_0 > 0$ such that

$$\tilde{J} := B_{\delta_0}(\tilde{x}) \subset J\,, \quad \tilde{K} := \operatorname{cl} B_{\delta_0}(\tilde{u}) \subset K\,. \tag{2.2}$$

We assume that there exists a constant $c_0 > 0$ such that for a.e. $x \in \tilde{J}$

(f1) $\quad |f(x, u, \lambda)| \le c_0(1 + |\lambda|^p) \quad$ for all $u \in \tilde{K}$, $\lambda \in \mathbb{R}^m$,

(f2) $\quad |f(x, u, \lambda) - f(x, v, \lambda)| \le c_0(1 + |\lambda|^p)\,|u - v| \quad$ for all $u, v \in \tilde{K}$, $\lambda \in \mathbb{R}^m$,

(f3) $\quad |f(x, u, \lambda) - f(x, u, \mu)| \le c_0(|\lambda - \mu| + |\lambda - \mu|^p) \quad$ for all $u \in \tilde{K}$, $\lambda, \mu \in \mathbb{R}^m$.

In the case $p = \infty$ we assume (f1)-(f3) with some exponent $q \in [1,\infty)$ instead of p.

We claim to verify solutions of (2.1) on some open interval

$$I^* := B_\varrho(\tilde{x}) \subset B_{\delta_0}(\tilde{x})$$

with some suitable $\rho > 0$ and for initial values and parameters satisfying

$$x_0 \in I^*\,, \quad u_0 \in K^* := B_{\delta_0/2}(\tilde{u})\,, \quad \lambda \in \Lambda^* := B_{\delta_0^*}(\tilde{\lambda}_{|I^*}) \subset L^p(I^*,\mathbb{R}^m)$$

with some $\delta_0^* \in (0, \delta_0]$. Obviously $f(\cdot, u(\cdot), \lambda(\cdot)) \in L^1(I^*)$ for $\lambda \in L^p(I^*)$ and $u \in C(I^*)$ with $u(I^*) \subset \tilde{K}$ by (f1). Therefore $u \in C(I^*)$ with $u(I^*) \subset \tilde{K}$ solves (2.1) on I^* if and only if it solves

$$u(x) = u_0 + \int_{x_0}^{x} f(s, u(s), \lambda(s))\,\mathrm{d}s \quad \text{for all } x \in I^*\,. \tag{2.3}$$

Let $u(x; x_0, u_0, \lambda)$ denote the solution of (2.1) for initial values (x_0, u_0) and parameter λ.

Theorem 2.4 *Let $1 \le p \le \infty$ and let f satisfy* (f1)-(f3). *Then there exists some $\rho > 0$ with $I^* = B_\rho(\tilde{x}) \subset \tilde{J}$ and some $\delta_0^* \in (0, \delta_0]$ such that for each $(x_0, u_0, \lambda) \in I^* \times K^* \times \Lambda^*$ there exists a unique solution $u(\cdot; x_0, u_0, \lambda)$ of* (2.1) *on I^* and $u \in C(I^* \times I^* \times K^* \times \Lambda^*, \mathbb{R}^n)$.*

As preparation for the proof we start with some preliminary considerations. Notice that the case $p = \infty$, which we do not treat explicitly, is covered by replacing p with q in the exponents. First we choose $\delta_0^* > 0$ such that

$$\delta_0^* \le \min\left\{\delta_0, \left(\frac{\delta_0}{2^{p+1}c_0}\right)^{1/p}, \left(\frac{1}{2^{p+1}c_0}\right)^{1/p}\right\}.$$

Then we choose $\varrho > 0$ so small that

$$2\varrho + \int_{I^*} |\tilde{\lambda}(s)|^p \, \mathrm{d}s \le \min \left\{ \frac{\delta_0}{4}, \frac{\delta_0}{4c_0}, \frac{1}{4c_0} \right\} .$$

Consequently,

$$\begin{aligned} \int_{I^*} (1 + |\lambda|^p) \, \mathrm{d}s \;&\le\; 2\varrho + \int_{I^*} |\, (\lambda - \tilde{\lambda}) + \tilde{\lambda} \,|^p \, \mathrm{d}s \\ &\le\; 2\varrho + 2^{p-1} \int_{I^*} \Big(|\lambda - \tilde{\lambda}|^p + |\tilde{\lambda}|^p \Big) \, \mathrm{d}s \\ &\le\; 2\varrho + 2^{p-1} (\delta_0^*)^p + \int_{I^*} |\tilde{\lambda}|^p \, \mathrm{d}s \\ &\le\; \min \left\{ \frac{\delta_0}{2c_0}, \frac{1}{2c_0} \right\} \qquad \text{for all } \; \lambda \in \Lambda^* . \end{aligned} \tag{2.5}$$

Now we introduce the Banach space

$$C^* := \{ u \in C(I^* \times I^* \times K^* \times \Lambda^*, \mathbb{R}^n) | \, \|u\|_\infty < \infty \}$$

with the norm

$$\|u\|_\infty := \sup_{(x,x_0,u_0,\lambda) \,\in\, I^* \times I^* \times K^* \times \Lambda^*} |u(x; x_0, u_0, \lambda)| \, .$$

In order to exploit conditions (f1)-(f3), we are in particular interested in $u \in C^*$ belonging to the closed subset

$$C^*_{\tilde{K}} := \{ u \in C^* | \, u(x; x_0, u_0, \lambda) \in \tilde{K} \; \text{ for all } \; (x, x_0, u_0, \lambda) \in I^* \times I^* \times K^* \times \Lambda^* \} \, . \tag{2.6}$$

We define an operator T on $C^*_{\tilde{K}}$ such that $Tu : I^* \times I^* \times K^* \times \Lambda^* \mapsto \mathbb{R}^n$ and

$$(Tu)(x; x_0, u_0, \lambda) := u_0 + \int_{x_0}^{x} f(s, u(s; x_0, u_0, \lambda), \lambda(s)) \, \mathrm{d}s \tag{2.7}$$

for all $(x, x_0, u_0, \lambda) \in I^* \times I^* \times K^* \times \Lambda^*$. The next lemma verifies properties of T that allow the application of Banach's fixed point theorem below.

Lemma 2.8 *Let T be given as in* (2.7). *Then:*
(1) $Tu \in C^*$ *for all* $u \in C^*_{\tilde{K}}$.
(2) *We have that*

$$\|Tu - Tv\|_\infty \le \frac{1}{2} \, \|u - v\|_\infty \quad \textit{for all } \; u, v \in C^*_{\tilde{K}} \, . \tag{2.9}$$

Proof. Let us start with (1). The mapping $Tu : I^* \times I^* \times K^* \times \Lambda^* \mapsto \mathbb{R}^n$ is well-defined for all $u \in C^*_{\tilde{K}}$ by (f1). In order to show continuity of $Tu(\cdot)$ we fix $u \in C^*_{\tilde{K}}$. Using the abbreviations $(x, \nu) = (x, x_0, u_0, \lambda)$ and $(\bar{x}, \bar{\nu}) = (\bar{x}, \bar{x}_0, \bar{u}_0, \bar{\lambda})$ for elements in $I^* \times I^* \times K^* \times \Lambda^*$ we obtain by (f1)-(f3) that

$$
\begin{aligned}
&|Tu(x;x_0,u_0,\lambda)-Tu(\bar{x};\bar{x}_0,\bar{u}_0,\bar{\lambda})|\\
&\qquad\le |u_0-\bar{u}_0|+\Big|\int_{x_0}^{x} f(s,u(s;\nu),\lambda(s))\,\mathrm{d}s-\int_{\bar{x}_0}^{\bar{x}} f(s,u(s;\bar{\nu}),\bar{\lambda}(s))\,\mathrm{d}s\Big|\\
&\qquad\le |u_0-\bar{u}_0|+\Big|\int_{x_0}^{x} f(s,u(s;\nu),\lambda(s))\,\mathrm{d}s-\int_{\bar{x}_0}^{x_0} f(s,u(s;\bar{\nu}),\bar{\lambda}(s))\,\mathrm{d}s\\
&\qquad\quad -\int_{x_0}^{x} f(s,u(s;\bar{\nu}),\bar{\lambda}(s))\,\mathrm{d}s-\int_{x}^{\bar{x}} f(s,u(s;\bar{\nu}),\bar{\lambda}(s))\,\mathrm{d}s\Big|\\
&\qquad\le |u_0-\bar{u}_0|+\Big|\int_{x_0}^{x}\Big(f(s,u(s;\nu),\lambda(s))-f(s,u(s;\bar{\nu}),\bar{\lambda}(s))\Big)\,\mathrm{d}s\Big|\\
&\qquad\quad +\Big|\int_{\bar{x}_0}^{x_0} f(s,u(s;\bar{\nu}),\bar{\lambda}(s))\,\mathrm{d}s\Big|+\Big|\int_{x}^{\bar{x}} f(s,u(s;\bar{\nu}),\bar{\lambda}(s))\,\mathrm{d}s\Big|\\
&\qquad\le |u_0-\bar{u}_0|+\Big|\int_{x_0}^{x}|f(s,u(s;\nu),\lambda(s))-f(s,u(s;\nu),\bar{\lambda}(s))|\,\mathrm{d}s\Big|\\
&\qquad\quad +\Big|\int_{x_0}^{x}|f(s,u(s;\nu),\bar{\lambda}(s))-f(s,u(s;\bar{\nu}),\bar{\lambda}(s))|\,\mathrm{d}s\Big|\\
&\qquad\quad +\Big|\int_{\bar{x}_0}^{x_0}|f(s,u(s;\bar{\nu}),\bar{\lambda}(s))|\,\mathrm{d}s\Big|+\Big|\int_{x}^{\bar{x}}|f(s,u(s;\bar{\nu}),\bar{\lambda}(s))|\,\mathrm{d}s\Big|\\
&\qquad\le |u_0-\bar{u}_0|+c_0\Big|\int_{x_0}^{x}\Big(|\lambda(s)-\bar{\lambda}(s)|+|\lambda(s)-\bar{\lambda}(s)|^p\Big)\,\mathrm{d}s\Big|\\
&\qquad\quad +c_0\Big|\int_{x_0}^{x}(1+|\bar{\lambda}(s)|^p)\,|u(s;\nu)-u(s;\bar{\nu})|\,\mathrm{d}s\Big|\\
&\qquad\quad +c_0\Big|\int_{\bar{x}_0}^{x_0}(1+|\bar{\lambda}(s)|^p)\,\mathrm{d}s\Big|+c_0\Big|\int_{x}^{\bar{x}}(1+|\bar{\lambda}(s)|^p)\,\mathrm{d}s\Big|\,.
\end{aligned}
\tag{2.10}
$$

By the continuity of $u(x;\cdot)$ the right hand side tends to zero if $(x,x_0,u_0,\lambda)\to(\bar{x},\bar{x}_0,\bar{u}_0,\bar{\lambda})$ in $I^*\times I^*\times K^*\times\Lambda^*$. But this implies that $Tu\in C^*$ and verifies assertion (1).

In order to show (2) we choose $u,v\in C^*_{\tilde{K}}$. By (f2) and (2.5) we can estimate for every $(x,x_0,u_0,\lambda)=(x,\nu)\in I^*\times I^*\times K^*\times\Lambda^*$ that

$$
\begin{aligned}
&|Tu(x;x_0,u_0,\lambda)-Tv(x,x_0,u_0,\lambda)|\\
&\qquad\le \Big|\int_{x_0}^{x}|f(s,u(s;\nu),\lambda(s))-f(s,v(s;\nu),\lambda(s))|\,\mathrm{d}s\Big|\\
&\qquad\le c_0\Big|\int_{x_0}^{x}(1+|\lambda(s)|^p)\,|u(s;\nu)-v(s;\nu)|\,\mathrm{d}s\Big|\\
&\qquad\le c_0\,\|u-v\|_\infty\int_{I^*}(1+|\lambda(s)|^p)\,\mathrm{d}s\\
&\qquad\le \frac{1}{2}\,\|u-v\|_\infty\,.
\end{aligned}
$$

Taking the supremum over all $(x,x_0,u_0,\lambda)\in I^*\times I^*\times K^*\times\Lambda^*$ on the left-hand side, we obtain the contraction property (2.9). □

Proof of Theorem 2.4. By Lemma 2.8 we know that $T : C^*_{\tilde K} \mapsto C^*$. By (f1) and (2.5) we obtain that

$$\begin{aligned} |Tu(x,x_0,u_0,\lambda) - u_0| &= \Big| \int_{x_0}^{x} f(s,u(s;x_0,u_0,\lambda),\lambda(s))\,\mathrm{d}s \Big| \\ &\le c_0 \int_{I^*} (1+|\lambda(s)|^p)\,\mathrm{d}s \\ &\le \frac{\delta_0}{2} \end{aligned}$$

for all $(x,x_0,u_0,\lambda) \in I^* \times I^* \times K^* \times \Lambda^*$. Hence $T : C^*_{\tilde K} \mapsto C^*_{\tilde K}$. The operator T is contractible on $C^*_{\tilde K}$ by Lemma 2.8. Since $C^*_{\tilde K}$ is closed in C^*, the operator T has a unique fixed point $u \in C^*_{\tilde K}$ by Banach's fixed point Theorem, i.e.,

$$u(x;x_0,u_0,\lambda) = u_0 + \int_{x_0}^{x} f(s,u(s;x_0,u_0,\lambda),\lambda(s))\,\mathrm{d}s$$

for all $(x,x_0,u_0,\lambda) \in I^* \times I^* \times K^* \times \Lambda^*$. Obviously $u(\cdot;x_0,u_0,\lambda)$ uniquely solves (2.1) according to (2.3). Moreover $u \in C^* = C(I^* \times I^* \times K^* \times \Lambda^*, \mathbb{R}^n)$ by construction. Notice that the case $p = \infty$ is covered by the previous arguments. □

3 Differentiable dependence on the parameter

In this section we consider situations in which the solution u of

$$u'(x) = f(x,u(x),\lambda(x)) \quad \text{for a.e. } x \in J, \quad u(x_0) = u_0\,, \tag{3.1}$$

is continuously differentiable with respect to the initial data and the parameter. Since the right hand side is again merely assumed to be measurable in x, we are looking again for solutions that are absolutely continuous in x. Thus we cannot expect a continuous derivative of u with respect to x and x_0. Hence we study differentiability of u only with respect to u_0 and λ.

Let $f : J \times K \times \mathbb{R}^m \mapsto \mathbb{R}^n$ be a Carathéodory function as in the second paragraph of the previous section. Moreover, let $(u,\lambda) \mapsto f(x,u,\lambda)$ be continuously differentiable on $K \times \mathbb{R}^m$ for a.e. $x \in J$. As in the previous section we fix $\tilde x \in J$, $\tilde u \in K$, $\tilde\lambda \in L^p(J,\mathbb{R}^m)$ and we define $\tilde J$, $\tilde K$ as in (2.2) where, however, we here restrict our attention to $1 < p \le \infty$. In addition to (f1)-(f3), we assume that for a.e. $x \in \tilde J$ and all $u, v \in \tilde K$, $\lambda \in \mathbb{R}^m$

(f4) $|f_u(x,u,\lambda)| \le c_0(1+|\lambda|^p)\,, \quad |f_\lambda(x,u,\lambda)| \le c_0\,,$

(f5) $|f_u(x,u,\lambda) - f_u(x,v,\lambda)| \le c_0(1+|\lambda|^p)\,|u-v|\,, \quad |f_\lambda(x,u,\lambda) - f_\lambda(x,v,\lambda)| \le c_0|u-v|\,.$

In the case $p = \infty$ we assume (f4), (f5) with some exponent $q \in [1,\infty)$ instead of p.

Again we consider solutions of (3.1) on $I^* := B_\rho(\tilde x) \subset \tilde J$, where $\rho > 0$ is not yet specified, and we take into account

$$x_0 \in I^*\,, \quad u_0 \in K^* := B_{\delta_0/2}(\tilde u)\,, \quad \lambda \in \Lambda^* := B_{\delta_0^*}(\tilde\lambda_{|I^*}) \subset L^p(I^*,\mathbb{R}^m)$$

for some $\delta_0^* \in (0, \delta_0]$. For $\lambda \in L^p(I^*)$ and $u \in C(I^*)$ with $u(I^*) \subset \tilde{K}$ we now not only have $f(\cdot, u(\cdot), \lambda(\cdot)) \in L^1(I^*)$ but also $f_u(\cdot, u(\cdot), \lambda(\cdot)) \in L^1(I^*, \mathbb{R}^{n\times n})$ and $f_\lambda(\cdot, u(\cdot), \lambda(\cdot)) \in L^\infty(I^*, \mathbb{R}^{n\times m})$ by (f4). As before $u(.; x_0, u_0, \lambda)$ denotes the solution of (3.1) for the data (x_0, u_0, λ) and $D_{(u_0,\lambda)}u(\cdot)$ denotes the derivative of u with respect to (u_0, λ).

Theorem 3.2 *Let $1 < p \leq \infty$, let f satisfy* (f1)-(f5), *and let $u = u(\cdot; x_0, u_0, \lambda)$ be the unique solution of* (3.1) *according to Theorem* 2.4. *Then, with some possibly smaller ϱ and δ_0^* than in Theorem* 2.4, *the solution $u : I^* \times I^* \times K^* \times \Lambda^* \to \mathbb{R}^n$ is differentiable with respect to (u_0, λ) on $K^* \times \Lambda^*$ for all $x, x_0 \in I^*$. The derivative $D_{(u_0,\lambda)}u(\cdot; \cdot, \cdot, \cdot)$ is continuous on $I^* \times I^* \times K^* \times \Lambda^*$ and*

$$D_{(u_0,\lambda)}u(x; x_0, u_0, \lambda)\, w \;=\; w_1 + \int_{x_0}^{x} \Big(f_u(s, u(s; x_0, u_0, \lambda), \lambda(s))\, D_{(u_0,\lambda)}u(s; x_0, u_0, \lambda)\, w$$
$$+ f_\lambda(s, u(s; x_0, u_0, \lambda), \lambda(s))\, w_2(s) \Big)\, \mathrm{d}s \tag{3.3}$$

for all $w = (w_1, w_2) \in \mathbb{R}^n \times L^p(I^*, \mathbb{R}^m)$. Moreover, for all $(x_0, u_0, \lambda) \in I^* \times K^* \times \Lambda^*$ and $w = (w_1, w_2) \in \mathbb{R}^n \times L^p(I^*, \mathbb{R}^m)$,

$$\frac{d}{dx} D_{(u_0,\lambda)}u(x; x_0, u_0, \lambda)\, w = D_{(u_0,\lambda)}\Big(\frac{d}{dx} u(x; x_0, u_0, \lambda)\Big)\, w \tag{3.4}$$

for a.e. $x \in I^*$.

The only point in the proof of the theorem where the dependence on x_0 has to be considered explicitly is the continuity of $D_{(u_0,\lambda)}u(\cdot)$. But for that we have to argue the same way as with respect to x. Therefore we suppress the dependence of u on x_0 in the subsequent considerations and we merely consider $u = u(x; u_0, \lambda)$. We also use the abbreviation $\nu = (u_0, \lambda)$ for elements in $K^* \times \Lambda^*$ and we write $D_\nu u(\cdot)$ for $D_{(u_0,\lambda)}u(\cdot)$.

As preparation for the proof we again start with some preliminary considerations. The case $p = \infty$ is covered by replacing p with q in the exponents. We choose $\delta_0^* > 0$ such that

$$\delta_0^* \leq \min \left\{ \delta_0, \left(\frac{\delta_0}{2^{p+4} c_0} \right)^{1/p}, \left(\frac{1}{2^{p+4} c_0} \right)^{1/p} \right\}.$$

Then we choose $\varrho > 0$ so small that

$$2\varrho + (2\varrho)^{1/p'} + \int_{I^*} |\tilde{\lambda}(s)|^p \, \mathrm{d}s \leq \min \left\{ \frac{\delta_0}{4}, \frac{\delta_0}{32 c_0}, \frac{1}{32 c_0} \right\}$$

where we have to use $p' = 1$ in the case $p = \infty$. In particular we assume that δ_0^* and ϱ are not larger than taken in the previous section. Instead of (2.5) we then have

$$\int_{I^*} (1 + |\lambda|^p)\, \mathrm{d}s \leq \min \left\{ \frac{\delta_0}{16 c_0}, \frac{1}{16 c_0} \right\} \qquad \text{for all } \lambda \in \Lambda^*. \tag{3.5}$$

Analogously to the previous section we define the Banach space

$$C^* := \{ u \in C(I^* \times K^* \times \Lambda^*, \mathbb{R}^n) |\, \|u\|_\infty < \infty \},$$

$$\|u\|_\infty := \sup_{(x,\nu)\in I^*\times K^*\times\Lambda^*} |u(x;\nu)|\,.$$

In addition we introduce the Banach space

$$C^L := \{U \in C(I^* \times K^* \times \Lambda^*,\, L(\mathbb{R}^n \times L^p(I^*,\mathbb{R}^m),\mathbb{R}^n)\,)|\ \|U\|_\infty < \infty\}\,,$$

$$\|U\|_\infty := \sup_{(x,\nu)\in I^*\times K^*\times\Lambda^*} \|U(x;\nu)\|_{L(\mathbb{R}^n\times L^p(I^*,\mathbb{R}^m),\mathbb{R}^n)}\,.$$

Recall that

$$\|U(x;\nu)\|_{L(\mathbb{R}^n\times L^p(I^*),\mathbb{R}^n)} := \sup_{\|w\|_{\mathbb{R}^n\times L^p(I^*)}\le 1} |U(x;\nu)\,w|\,,$$

$$\|w\|_{\mathbb{R}^n\times L^p(I^*)} := |w_1| + \|w_2\|_{L^p(I^*)} \quad \text{for } w = (w_1,w_2) \in \mathbb{R}^n \times L^p(I^*,\mathbb{R}^m)\,.$$

Moreover we consider the space

$$C^{1*} := \{u \in C^*|\ D_\nu u \in C^L\}\,, \quad \|u\|_{1,\infty} := \|u\|_\infty + \|D_\nu u\|_\infty\,.$$

Lemma 3.6 *The space C^{1*} is a Banach space.*

Proof. Obviously C^{1*} is a linear normed space and it remains to verify completeness. Let $\{u_n\} \subset C^{1*}$ be a Cauchy sequence. Then there are $u \in C^*$ and $U \in C^L$ such that $u_n \to u$ in C^* and $D_\nu u_n \to U$ in C^L. For any $(x,\nu) \in I^* \times K^* \times \Lambda^*$, $w \in \mathbb{R}^n \times L^p(I^*,\mathbb{R}^m)$, and $t \in \mathbb{R}$ with $\nu + tw \in K^* \times \Lambda^*$ we have that

$$\begin{aligned} u_n(x;\nu+tw) - u_n(x;\nu) &= \int_0^t D_\nu u_n(x;\nu+sw)w\,\mathrm{d}s \\ &= \int_0^t \Big(D_\nu u_n(x;\nu+sw)w - U(x;\nu+sw)w\Big)\,\mathrm{d}s + \int_0^t U(x;\nu+sw)w\,\mathrm{d}s\,. \end{aligned}$$

In the limit we get

$$u(x;\nu+tw) - u(x;\nu) = \int_0^t U(x;\nu+sw)w\,\mathrm{d}s\,.$$

Since the right hand side is differentiable in t and $U(x;\nu) \in L(\mathbb{R}^n \times L^p(I^*,\mathbb{R}^m),\mathbb{R}^n)$, we conclude that $u(x;\cdot)$ is Gâteaux differentiable at ν with $D_\nu u(x;\nu)w = U(x;\nu)w$ for all $w \in \mathbb{R}^n \times L^p(I^*,\mathbb{R}^m)$, i.e., $D_\nu u(x;\nu) = U(x;\nu)$ for all $(x,\nu) \in I^* \times K^* \times \Lambda^*$. Since $U(\cdot;\cdot)$ is continuous, $D_\nu u(x;\nu)$ is even a Fréchet derivative and $u \in C^{1*}$. Thus $D_\nu u_n \to D_\nu u$ in C^L and, hence, $u_n \to u$ in C^{1*}. □

In order to exploit conditions (f1)-(f5) we are particularly interested in u belonging to

$$C^{1*}_{\tilde K} := \{u \in C^{1*}|\ u(x;u_0,\lambda) \in \tilde K \text{ for all } (x,u_0,\lambda) \in I^* \times K^* \times \Lambda^*\,,\ \|D_\nu u\|_\infty \le 2\}\,.$$

Obviously, $C^{1*}_{\tilde K}$ is a closed subset of C^{1*}. From the previous section we know that any solution u of (3.1) is a fixed point of the operator T given by

$$(Tu)(x;\nu) := u_0 + \int_{x_0}^x f(s,u(s;\nu),\lambda(s))\,\mathrm{d}s \tag{3.7}$$

for all $(x,\nu) = (x,u_0,\lambda) \in I^* \times K^* \times \Lambda^*$. The next lemma provides differentiability properties of the mapping $\nu \mapsto Tu(x;\nu)$.

Lemma 3.8 *For T according to (3.7) we have that $Tu \in C^{1*}$ for all $u \in C^{1*}_{\tilde{K}}$ and*

$$D_\nu Tu(x;\nu)w = w_1 + \int_{x_0}^{x} f_u(s,u(s;\nu),\lambda(s))D_\nu u(s;\nu)w + f_\lambda(s,u(s;\nu),\lambda(s))w_2(s)\,\mathrm{d}s$$

for all $(x,\nu) = (x,u_0,\lambda) \in I^* \times K^* \times \Lambda^*$ and $w = (w_1,w_2) \in \mathbb{R}^n \times L^p(I^*,\mathbb{R}^m)$.

Proof. By Lemma 2.8 we know that $Tu \in C^*$ for any $u \in C^{1*}_{\tilde{K}}$. Thus we have to study the differentiability of $\nu \mapsto Tu(x;\nu)$. For that we fix $u \in C^{1*}_{\tilde{K}}$, $\nu = (u_0,\lambda) \in K^* \times \Lambda^*$, and $w = (w_1,w_2) \in \mathbb{R}^n \times L^p(I^*,\mathbb{R}^m)$ and consider the function

$$\alpha(\sigma) := Tu(x;\nu+\sigma w) = u_0 + \sigma w_1 + \int_{x_0}^{x} \Big(f(s,u(s;\nu+\sigma w),\lambda(s)+\sigma w_2(s)) \Big)\,\mathrm{d}s \tag{3.9}$$

for small $\sigma \in \mathbb{R}$ with $\nu + \sigma w \in K^* \times \Lambda^*$. Since $f(x,\cdot,\cdot)$ is continuously differentiable for a.e. x and since $u \in C^{1*}$, the integrand on the right-hand side is differentiable with respect to σ for a.e. $s \in I^*$ and, by (f4), we can estimate

$$\begin{aligned}
&\Big|\frac{d}{d\sigma}\, f(s,u(s;\nu+\sigma w),\lambda(s)+\sigma w_2(s))\Big| \\
&\quad = |f_u(s,u(s;\nu+\sigma w),\lambda(s)+\sigma w_2(s))D_\nu u(s;\nu+\sigma w)w \\
&\qquad\qquad + f_\lambda(s,u(s;\nu+\sigma w),\lambda(s)+\sigma w_2(s))w_2(s)| \\
&\quad \le |f_u(s,u(s;\nu+\sigma w),\lambda(s)+\sigma w_2(s))D_\nu u(s;\nu+\sigma w)w| \\
&\qquad\qquad + |f_\lambda(s,u(s;\nu+\sigma w),\lambda(s)+\sigma w_2(s))w_2(s)| \\
&\quad \le c_0\left((1+|\lambda(s)+\sigma w_2(s)|^p)\ \|D_\nu u\|_\infty\, \|w\|_{\mathbb{R}^n\times L^p} + |w_2(s)|\,\right) \qquad (3.10)\\
&\quad \le c_0\Big(\left(1+2^{p-1}(|\lambda(s)|^p + |w_2(s)|^p)\right)\ \|D_\nu u\|_\infty\, \|w\|_{\mathbb{R}^n\times L^p} + |w_2(s)|\Big)
\end{aligned}$$

for a.e. $s \in I^*$ as long as $|\sigma| \le 1$. Thus the derivative on the left-hand side is bounded by an integrable function. Hence we can differentiate under the integral in (3.9) (cf. [8, p. 1018]) and obtain a linear operator $A : \mathbb{R}^n \times L^p(I^*,\mathbb{R}^m) \to \mathbb{R}^n$ with

$$Aw := \alpha'(0) = w_1 + \int_{x_0}^{x} \Big(f_u(s,u(s;\nu),\lambda(s))D_\nu u(s;\nu)w + f_\lambda(s,u(s;\nu),\lambda(s))w_2(s) \Big)\,\mathrm{d}s$$

for all $(x,\nu) = (x,u_0,\lambda) \in I^* \times K^* \times \Lambda^*$ and $w = (w_1,w_2) \in \mathbb{R}^n \times L^p(I^*,\mathbb{R}^m)$. Using (3.10) with $\sigma = 0$ we obtain that

$$\begin{aligned}
|Aw| &\le |w_1| + c_0 \int_{I^*} \Big((1+|\lambda(s)|^p)\ \|D_\nu u\|_\infty\, \|w\|_{\mathbb{R}^n\times L^p} + |w_2(s)| \Big)\,\mathrm{d}s \\
&\le \|w\|_{\mathbb{R}^n\times L^p} + c_0 \|D_\nu u\|_\infty \|w\|_{\mathbb{R}^n\times L^p} \int_{I^*} (1+|\lambda(s)|^p)\,\mathrm{d}s + c_0 \int_{I^*} |w_2(s)|\,\mathrm{d}s \\
&\le \|w\|_{\mathbb{R}^n\times L^p} \left(1 + c_0\ \|D_\nu u\|_\infty\, (2\varrho + \|\lambda\|^p_{L^p}) + \tilde{c}_0\right)
\end{aligned}$$

for some $\tilde{c}_0 > 0$. Hence $A \in L(\mathbb{R}^n \times L^p(I^*,\mathbb{R}^m),\mathbb{R}^n)$. Thus $D_\nu Tu(x;\nu) = A$ is the Gâteaux derivative of $Tu(x;.\,)$.

It remains to show that $(x,\nu) \mapsto D_\nu Tu(x;\nu)$ is continuous on $I^* \times K^* \times \Lambda^*$. For that we choose a sequence $(x_n,\nu_n) = (x_n, u_{0,n}, \lambda_n) \subset I^* \times K^* \times \Lambda^*$ converging to $(x,\nu) = (x, u_0, \lambda) \in I^* \times K^* \times \Lambda^*$. Using (f4) and Hölder's inequality we can estimate

$$
\begin{aligned}
&|D_\nu Tu(x;\nu)w - D_\nu Tu(x_n;\nu_n)w| \\
&\quad\le |D_\nu Tu(x;\nu)w - D_\nu Tu(x_n;\nu)w| + |D_\nu Tu(x_n;\nu)w - D_\nu Tu(x_n;\nu_n)w| \\
&\quad\le \Big| \int_{x_0}^{x} \Big(f_u(s,u(s;\nu),\lambda(s)) D_\nu u(s;\nu) w + f_\lambda(s,u(s;\nu),\lambda(s)) w_2(s) \Big)\, \mathrm{d}s \\
&\qquad\qquad - \int_{x_0}^{x_n} \Big(f_u(s,u(s;\nu),\lambda(s)) D_\nu u(s;\nu) w + f_\lambda(s,u(s;\nu)\lambda(s)) w_2(s) \Big)\, \mathrm{d}s \Big| \\
&\qquad + \Big| \int_{x_0}^{x_n} \Big(f_u(s,u(s;\nu),\lambda(s)) D_\nu u(s;\nu) w + f_\lambda(s,u(s;\nu),\lambda(s)) w_2(s) \Big)\, \mathrm{d}s \\
&\qquad\qquad - \int_{x_0}^{x_n} \Big(f_u(s,u(s;\nu_n),\lambda_n(s)) D_\nu u(s;\nu_n) w + f_\lambda(s,u(s;\nu_n),\lambda_n(s)) w_2(s) \Big)\, \mathrm{d}s \Big| \\
&\quad\le \Big| \int_{x_n}^{x} |f_u(s,u(s;\nu),\lambda(s))\, D_\nu u(s;\nu)\, w + f_\lambda(s,u(s;\nu),\lambda(s))\, w_2(s)|\, \mathrm{d}s \Big| \\
&\qquad + \Big| \int_{x_0}^{x_n} \Big(|f_u(s,u(s;\nu),\lambda(s))\, D_\nu u(s;\nu)\, w - f_u(s,u(s;\nu_n),\lambda_n(s))\, D_\nu u(s;\nu_n)\, w| \\
&\qquad\qquad\qquad + |f_\lambda(s,u(s;\nu),\lambda(s))\, w_2(s) - f_\lambda(s,u(s;\nu_n),\lambda_n(s))\, w_2(s)| \Big)\, \mathrm{d}s \Big| \\
&\quad\le \Big| \int_{x_n}^{x} |f_u(s,u(s;\nu),\lambda(s))\, D_\nu u(s;\nu)\, w| + |f_\lambda(s,u(s;\nu),\lambda(s))\, w_2(s)|\, \mathrm{d}s \Big| \\
&\qquad + \int_{I^*} |(f_u(s,u(s;\nu),\lambda(s)) - f_u(s,u(s;\nu_n),\lambda_n(s)))\, D_\nu u(s;\nu)\, w|\, \mathrm{d}s \\
&\qquad + \int_{I^*} |f_u(s,u(s;\nu_n),\lambda_n(s))\, (D_\nu u(s;\nu) - D_\nu u(s;\nu_n)) w|\, \mathrm{d}s \\
&\qquad + \int_{I^*} |(f_\lambda(s,u(s;\nu),\lambda(s)) - f_\lambda(s,u(s;\nu_n),\lambda_n(s)))\, w_2(s)|\, \mathrm{d}s \\
&\quad\le \Big| c_0 \int_{x_n}^{x} (1 + |\lambda(s)|^p) |D_\nu u(s;\nu_n) w| + |w_2(s)|\, \mathrm{d}s \Big| \\
&\qquad + \int_{I^*} |f_u(s,u(s;\nu),\lambda(s)) - f_u(s,u(s;\nu_n),\lambda_n(s))|_{L(\mathbb{R}^n,\mathbb{R}^n)}\ |D_\nu u(s;\nu)\, w|\, \mathrm{d}s \\
&\qquad + c_0 \int_{I^*} (1 + |\lambda_n(s)|^p)\ |(D_\nu u(s;\nu) - D_\nu u(s;\nu_n)) w|\, \mathrm{d}s \\
&\qquad + \int_{I^*} |f_\lambda(s,u(s;\nu),\lambda(s)) - f_\lambda(s,u(s;\nu_n),\lambda_n(s))|_{L(\mathbb{R}^m,\mathbb{R}^n)}\ |w_2(s)|\, \mathrm{d}s \\
&\quad\le c_0 \|D_\nu u\|_\infty \|w\|_{\mathbb{R}^n \times L^p} \Big| \int_{x_n}^{x} (1 + |\lambda(s)|^p)\, \mathrm{d}s \Big| + c_0 \|w_2\|_{L^p} |x - x_n|^{1/p'} \\
&\qquad + \|D_\nu u\|_\infty \|w\|_{\mathbb{R}^n \times L^p} \int_{I^*} |f_u(s,u(s;\nu),\lambda(s)) - f_u(s,u(s;\nu_n),\lambda_n(s))|_{L(\mathbb{R}^n,\mathbb{R}^n)}\, \mathrm{d}s \\
&\qquad + c_0 \|w\|_{\mathbb{R}^n \times L^p} \int_{I^*} (1 + |\lambda_n(s)|^p)\ \|D_\nu u(s;\nu) - D_\nu u(s;\nu_n)\|_{L(\mathbb{R}^n \times L^p,\mathbb{R}^n)}\, \mathrm{d}s \\
&\qquad + \|w_2\|_{L^p} \left(\int_{I^*} |f_\lambda(s,u(s;\nu),\lambda(s)) - f_\lambda(s,u(s;\nu_n),\lambda_n(s))|^{p'}_{L(\mathbb{R}^m,\mathbb{R}^n)}\ \mathrm{d}s \right)^{\frac{1}{p'}} .
\end{aligned}
$$

Taking the supremum over all $w \in \mathbb{R}^n \times L^p(I^*, \mathbb{R}^m)$ with $\|w\|_{\mathbb{R}^n \times L^p} \leq 1$ we obtain

$$\begin{aligned}
&\|D_\nu Tu(x;\nu) - D_\nu Tu(x_n;\nu_n)\|_{L(\mathbb{R}^n \times L^p(I^*,\mathbb{R}^m),\mathbb{R}^n)} \\
&\quad \leq c_0 \|D_\nu u\|_\infty \Big| \int_{x_n}^{x} (1 + |\lambda(s)|^p)\,\mathrm{d}s \Big| + c_0 |x - x_n|^{1/p'} \qquad (3.11)\\
&\quad + \|D_\nu u\|_\infty \int_{I^*} |f_u(s,u(s;\nu),\lambda(s)) - f_u(s,u(s;\nu_n),\lambda_n(s))|_{L(\mathbb{R}^n,\mathbb{R}^n)}\,\mathrm{d}s \qquad (3.12)\\
&\quad + c_0 \int_{I^*} (1 + |\lambda_n(s)|^p)\, \|D_\nu u(s;\nu) - D_\nu u(s;\nu_n)\|_{L(\mathbb{R}^n \times L^p,\mathbb{R}^n)}\,\mathrm{d}s \qquad (3.13)\\
&\quad + \Big(\int_{I^*} |f_\lambda(s,u(s;\nu),\lambda(s)) - f_\lambda(s,u(s;\nu_n),\lambda_n(s))|^{p'}_{L(\mathbb{R}^m,\mathbb{R}^n)}\,\mathrm{d}s \Big)^{\frac{1}{p'}} . \qquad (3.14)
\end{aligned}$$

Let us now show that the right-hand side tends to zero for $(x_n, \nu_n) \to (x, \nu)$. This is immediately clear for the two terms in (3.11). Since $\lambda_n \to \lambda$ in $L^p(I^*)$, for a subsequence (denoted the same way) we have $\lambda_n(s) \to \lambda(s)$ for a.e. $s \in I^*$. If we use that $u \in C^{1*}$, then we get that the integrands in (3.12)-(3.14) converge to zero for a.e. $s \in I^*$. By (f4)

$$|f_u(s,u(s;\nu),\lambda(s)) - f_u(s,u(s;\nu_n),\lambda_n(s))|_{L(\mathbb{R}^n,\mathbb{R}^n)} \leq c_0(2 + |\lambda(s)|^p + |\lambda_n(s)|^p)$$

and

$$\begin{aligned}
&|f_\lambda(s,u(s;\nu),\lambda(s)) - f_\lambda(s,u(s;\nu_n),\lambda_n(s))|^{p'}_{L(\mathbb{R}^m,\mathbb{R}^n)} \\
&\quad \leq 2^{p'-1} \Big(|f_\lambda(s,u(s;\nu),\lambda(s))|^{p'}_{L(\mathbb{R}^m,\mathbb{R}^n)} + |f_\lambda(s,u(s;\nu_n),\lambda_n(s))|^{p'}_{L(\mathbb{R}^m,\mathbb{R}^n)} \Big) \ \leq \ 2^{p'} c_0^{p'}
\end{aligned}$$

for a.e. $s \in I^*$. Moreover

$$(1 + |\lambda_n(s)|^p)\, \|D_\nu u(s;\nu) - D_\nu u(s;\nu_n)\|_{L(\mathbb{R}^n \times L^p,\mathbb{R}^n)} \leq 2\|D_\nu u\|_\infty (1 + |\lambda_n(s)|^p)$$

for a.e. $s \in I^*$. Since $\lambda_n \to \lambda$ in $L^p(I^*)$, the generalized dominated convergence theorem yields the desired convergence in (3.12)-(3.14) at least for a subsequence. Notice that our previous arguments also show that any subsequence of (x_n, ν_n) has a subsequence $(x_{n'}, \nu_{n'})$ such that $D_\nu Tu(x_{n'};\nu_{n'}) \to D_\nu Tu(x;\nu)$ as $n' \to \infty$. This subsequence principle implies the convergence for the complete sequence (x_n, ν_n) which verifies the continuity of $D_\nu Tu(\cdot;\cdot)$, i.e., $u \in C^{1*}$ and the proof is complete. □

Lemma 3.15 *For T according to* (3.7) *we have that*

$$Tu \in C^{1*}_{\tilde{K}} \quad \textit{and} \quad \|Tu - Tv\|_{1,\infty} \leq \frac{3}{4} \|u - v\|_{1,\infty} \quad \textit{for all } u, v \in C^{1*}_{\tilde{K}} .$$

Proof. Let us fix $u, v \in C^{1*}_{\tilde{K}}$. We know that $Tu \in C^{1*}$ by Lemma 3.8 and $Tu \in C^{*}_{\tilde{K}}$ by the proof of Theorem 2.4. In order to show that $\|D_\nu Tu\|_\infty \leq 2$ we use (f4), the representation from Lemma 3.8, and Hölder's inequality and estimate for $(x, \nu) \in I^* \times K^* \times \Lambda^*$ and $w \in \mathbb{R}^n \times L^p(I^*, \mathbb{R}^m)$

$$\begin{aligned}
&|D_\nu Tu(x;\nu)w| \\
&\quad \leq |w_1| + \int_{x_0}^{x} \Big(|f_u(s,u(s;\nu),\lambda(s)) D_\nu u(s;\nu) w| + |f_\lambda(s,u(s;\nu),\lambda(s)) w_2(s)| \Big)\,\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\le \|w\|_{\mathbb{R}^n\times L^p} + c_0 \int_{x_0}^{x} \Big((1+|\lambda(s)|^p)\, |D_\nu u(s;\nu)w| + |w_2(s)| \Big)\, \mathrm{d}s \\
&\le \|w\|_{\mathbb{R}^n\times L^p} + c_0 \|w\|_{\mathbb{R}^n\times L^p} \int_{x_0}^{x} (1+|\lambda(s)|^p)\, \|D_\nu u(s;\nu)\|_{L(\mathbb{R}^n\times L^p,\mathbb{R}^n)}\, \mathrm{d}s \\
&\quad + c_0 \|w_2\|_{L^p} (2\varrho)^{1/p'} .
\end{aligned}$$

The supremum over all w with $\|w\|_{\mathbb{R}^n\times L^p} \le 1$, relation (3.5), the properties of ϱ, and $\|D_\nu u\|_\infty \le 2$ give

$$\|D_\nu T u(x;\nu)\|_{L(\mathbb{R}^n\times L^p,\mathbb{R}^n)} \le 1 + c_0 \Big(\|D_\nu u\|_\infty \frac{1}{16 c_0} + \frac{1}{32 c_0} \Big) \le 2 .$$

The supremum over all $(x,\nu) \in I^* \times K^* \times \Lambda^*$ implies $\|D_\nu T u\|_\infty \le 2$ and, thus, $Tu \in C^{1*}_{\tilde{K}}$.

For the contraction property we carry out a similar estimate as in the previous proof. Using (f4) and (f5) we obtain for $(x,\nu) \in I^* \times K^* \times \Lambda^*$ and $w \in \mathbb{R}^n \times L^p(I^*, \mathbb{R}^m)$

$$\begin{aligned}
&|D_\nu T u(x;\nu) w - D_\nu T v(x;\nu) w| \\
&\quad \le \Big| \int_{x_0}^{x} \Big(f_u(s,u(s;\nu),\lambda(s)) D_\nu u(s;\nu) w + f_\lambda(s,u(s;\nu),\lambda(s)) w_2(s) \\
&\qquad\qquad - f_u(s,v(s;\nu),\lambda(s)) D_\nu v(s;\nu) w - f_\lambda(s,v(s;\nu)\lambda(s)) w_2(s) \Big)\, \mathrm{d}s \Big| \\
&\quad \le \Big| \int_{x_0}^{x} \Big(f_u(s,u(s;\nu),\lambda(s)) - f_u(s,v(s;\nu),\lambda(s)) \Big)\, D_\nu u(s;\nu)\, w\, \mathrm{d}s \Big| \\
&\qquad + \Big| \int_{x_0}^{x} f_u(s,v(s;\nu),\lambda(s)) \Big(D_\nu u(s;\nu) - D_\nu v(s;\nu) \Big)\, w\, \mathrm{d}s \Big| \\
&\qquad + \Big| \int_{x_0}^{x} \Big(f_\lambda(s,u(s;\nu),\lambda(s)) - f_\lambda(s,v(s;\nu),\lambda(s)) \Big)\, w_2(s)\, \mathrm{d}s \Big| \\
&\quad \le c_0 \int_{x_0}^{x} (1+|\lambda(s)|^p)\, |u(s;\nu) - v(s;\nu)|\, |D_\nu u(s;\nu)\, w|\, \mathrm{d}s \\
&\qquad + c_0 \int_{x_0}^{x} (1+|\lambda(s)|^p) \Big| (\, D_\nu u(s;\nu) - D_\nu v(s;\nu)\,)\, w \Big|\, \mathrm{d}s \\
&\qquad + c_0 \int_{x_0}^{x} |u(s;\nu) - v(s;\nu)|\, |w_2(s)|\, \mathrm{d}s \\
&\quad \le c_0 \|u-v\|_\infty \int_{x_0}^{x} (1+|\lambda(s)|^p)\, \|D_\nu u(s;\nu)\|_{L(\mathbb{R}^n\times L^p,\mathbb{R}^n)}\, \|w\|_{\mathbb{R}^n\times L^p}\, \mathrm{d}s \\
&\qquad + c_0 \int_{x_0}^{x} (1+|\lambda(s)|^p)\, \|\, D_\nu u(s;\nu) - D_\nu v(s;\nu)\|_{L(\mathbb{R}^n\times L^p,\mathbb{R}^n)}\, \|w\|_{\mathbb{R}^n\times L^p}\, \mathrm{d}s \\
&\qquad + c_0 \|u-v\|_\infty \|w\|_{\mathbb{R}^n\times L^p} |x - x_0|^{1/p'} .
\end{aligned}$$

Taking the supremum over all $w \in \mathbb{R}^n \times L^p(I^*, \mathbb{R}^m)$ with $\|w\|_{\mathbb{R}^n\times L^p} \le 1$ we obtain

$$\begin{aligned}
&\|D_\nu T u(x;\nu) - D_\nu T v(x;\nu)\|_{L(\mathbb{R}^n\times L^p,\mathbb{R}^n)} \\
&\qquad \le c_0 \Big((\|D_\nu u\|_\infty + 1) \int_{x_0}^{x} (1+|\lambda(s)|^p)\, \mathrm{d}s + (2\varrho)^{1/p'} \Big) \|u-v\|_{1,\infty} .
\end{aligned} \tag{3.16}$$

Taking the supremum over all $(x;\nu) \in I^* \times K^* \times \Lambda^*$ and using (3.5), the properties of ρ, and the definition of $C^{1*}_{\tilde{K}}$, we get

$$\|D_\nu Tu - D_\nu Tv\|_\infty \le c_0 \left(\frac{3}{16c_0} + \frac{1}{32c_0} \right) \|u-v\|_{1,\infty} \le \frac{1}{4} \|u-v\|_{1,\infty} .$$

By Lemma 2.8 we have that

$$\|Tu - Tv\|_\infty \le \frac{1}{2} \|u-v\|_{1,\infty}$$

which implies the assertion. □

Proof of Theorem 3.2. Using Lemma 3.8, Lemma 3.15, and Banach's fixed point theorem, we obtain a unique fixed point $u \in C^{1*}_{\tilde{K}}$ of the operator T defined in (3.7). Thus $u = u(\cdot; u_0, \lambda)$ is a unique solution of (3.1) on I^* for all $(u_0, \lambda) \in K^* \times \Lambda^*$ (recall (2.3)) and, consequently, u has to agree with the solution verified in Theorem 2.4. By $u \in C^{1*}$ the mapping $(u_0, \lambda) \mapsto u(x; u_0, \lambda)$ is continuously differentiable and $(x, u_0, \lambda) \mapsto D_\nu u(x; u_0, \lambda)$ is continuous. Since u is a fixed point of T, we can differentiate the identity $Tu(x;\nu) = u(x;\nu)$ with respect to ν and obtain that

$$D_\nu Tu(x;\nu)w = D_\nu u(x;\nu)w$$

for all $(x, \nu) \in I^* \times K^* \times \Lambda^*$, $w \in \mathbb{R}^n \times L^p(I^*, \mathbb{R}^m)$. Now we can derive (3.3) directly from Lemma 3.8.

For fixed $\nu \in K^* \times \Lambda^*$, $w \in \mathbb{R}^n \times L^p(I^*, \mathbb{R}^m)$ the right hand side in (3.3) is absolutely continuous in x and, thus,

$$\frac{d}{dx} D_\nu u(x;\nu)w = f_u(x, u(x;\nu), \lambda(x)) D_\nu u(x;\nu)w + f_\lambda(x, u(x;\nu), \lambda(x)) w_2(x) \tag{3.17}$$

for a.e. $x \in I^*$. Since $f(x, \cdot, \cdot)$ and $u(x; \cdot)$ are continuously differentiable, we readily obtain that

$$D_\nu f(x, u(x;\nu), \lambda(x))w = f_u(x, u(x;\nu), \lambda(x)) D_\nu u(x;\nu)w + f_\lambda(x, u(x;\nu), \lambda(x)) w_2(x) \tag{3.18}$$

for a.e. $x \in I^*$. From the differential equation (3.1) we now see that $D_\nu(\frac{d}{dx} u(x;\nu))w$ exists and equals the right hand side in (3.18). Recalling (3.17) we get (3.4) which completes the proof. □

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