Benjamini-Schramm convergence and pointwise convergence of the spectral measure

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Abstract. In this note we discuss Benjamini-Schramm (or local) convergence for sequences of finite graphs of bounded degree. We prove a continuity result on the expected spectral measure on sofic random rooted graphs. In particular, we prove that the spectral measure converges for every interval of the real line, i.e., the integrated density of states converges pointwise.

As an application we show how this implies the original Lück Approximation theorem for the growth of Betti numbers on covering towers of complexes.

Contents

1. Benjamini-Schramm convergence

1.1. The space of edge labeled rooted graphs. For an integer $D > 0$ let $\mathcal{G}_D$ denote the set of (isomorphism classes of) connected, undirected graphs where every vertex has at most D neighbours, together with a labeling function $E(G) \to \mathbb{Z}$ such that every label has absolute value at most D.

Let $\mathcal{RG}_D$ denote the set of graphs G in $\mathcal{G}_D$ together with a distinguished vertex, called the *root* of G. For $G_1, G_2 \in \mathcal{RG}_D$ let the *rooted distance* of G_1 and G_2 be $1/k$ where k is the maximal integer such that the k-balls around the roots of G_1 and G_2 are isomorphic, as rooted edge-labeled graphs. The rooted distance turns $\mathcal{RG}_D$ to a compact, totally disconnected metric space.

1.2. Random rooted graphs, neigbourhood statistics and local convergence. By a *random rooted graph of degree* D we mean a Borel probability distribution on $\mathcal{RG}_D$. Since $\mathcal{RG}_D$ is a complete metric space, the space of Borel probability measures is compact in the weak topology. We say that a sequence of random rooted graphs G_n defined on $\mathcal{RG}_D$ *converges to* G, if it converges in the weak topology, that is, if for every continous function $f : \mathcal{RG}_D \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \int_{\mathcal{RG}_D} f(x) d\lambda_n(x) = \int_{\mathcal{RG}_D} f(x) d\lambda(x)$$

where λ_n is the distribution of G_n and λ is the distribution of G.

The set of k-neighbourhoods of the root ($k > 0$) give us a closed-open base of $\mathcal{RG}_D$, and in the presence of a clopen base, weak convergence of measures translates to convergence of the measures of the base sets. This gives us the following. For a random rooted graph G, a finite rooted graph α and $k > 0$ let

$$P(G, \alpha) = \mathbf{P}\left(B(o, k) \cong \alpha\right)$$

be the probability that the k-ball around the root of G is isomorphic to α. Then G_n converges to G if and only if for all finite rooted graphs α and $k > 0$, we have

$$\lim_{n\to\infty} P(G_n, \alpha) = P(G, \alpha).$$

That is, weak convergence means convergence in neigbourhood sampling statistics.

Any finite graph $G \in \mathcal{G}_D$ gives rise to a random rooted graph by assigning the root of G uniformly randomly. We denote the distribution of this random rooted graph by λ_G.

Definition 1. *We say that a sequence of finite graphs $G_n \in \mathcal{G}_D$ is* Benjamini-Schramm convergent*, if λ_{G_n} weakly converges, or equivalently, if $P(G_n, \alpha)$ converges for every sample α. The* local limit *of G_n is defined as the weak limit of λ_{G_n}.*

This convergence notion has been introduced by Benjamini, Schramm [3], Aldous and Lyons [2].

1.3. Examples for local convergence. The easiest example is cycles of length tending to infinity: these converge to the infinite rooted line. Also, the $n \times n$ grid converges to the infinite rooted square grid. It takes a bit more work to construct a sequence of graphs that converges to the infinite d-regular tree. As we shall see later, any chain of normal subgroups with trivial interection in a free group provides such sequences. Another way is to let G_n to be a random d-regular graph on n vertices.

1.4. Covering towers. One of the most general examples for local convergence is covering towers. A *covering tower* is a sequence $G_n \in \mathcal{RG}_D$ of finite connected graphs, such that there is a covering map (a surjective map which is a local isomorphism) from G_{n+1} to G_n $(n > 1)$. The root is irrelevant here.

For every $k > 0$, let

$$X_k = \{G \in \mathcal{RG}_D \mid d(o,x) \leq k \text{ for all } x \in V(G)\}$$

that is, the set of possible k-balls in $\mathcal{RG}_D$. Then X_k is finite and we can introduce a 'covering hierarchy' on X_k as follows: we say that $G_1 \in X_k$ *covers* $G_2 \in X_k$ is there is a covering map from G_1 to G_2 that maps the root of G_1 to the root of G_2. This defines a partial ordering $<$ on X_k. Now, for any covering map $G \to H$, the masses

$$\{P(G,\alpha) \mid \alpha \in X_k\} \text{ and } \{P(H,\alpha) \mid \alpha \in X_k\}$$

travel along this ordering, i.e., for any $\alpha \in X_k$, we have

$$\sum_{\beta > \alpha} P(G,\beta) \leq \sum_{\beta > \alpha} P(H,\beta)$$

Since X_k is finite, this implies that for a covering tower, the masses converge, in particular, every covering tower is Benjamini-Schramm convergent.

1.5. Unimodularity. Of course, λ_G of a finite graph G, or a weak limit of such λ_G-s can not be just any distribution on $\mathcal{RG}_D$. The only known condition so far, that these measures automatically satisfy is called *unimodularity.* A vague, but short description: a graph is unimodular, if choosing a uniform random directed edge and then reverting it gives us a uniform random directed edge.

To make this precise, let λ be a probability distribution on $\mathcal{RG}_D$. We introduce a new space $\overrightarrow{\mathcal{RG}_D}$, the space of flagged graphs as follows. A flagged graph is a rooted graph with a directed edge starting from the root, called the flag. There is a natural map from $\overrightarrow{\mathcal{RG}_D}$ to $\mathcal{RG}_D$ by forgetting the directed edge. We can lift the measure λ to a measure $\overrightarrow{\lambda}$ on $\overrightarrow{\mathcal{RG}_D}$ as follows: we take a λ-random G and flag it in all possible ways; the corresponding measure is $\overrightarrow{\lambda}$. Note that $\overrightarrow{\lambda}$ is not a probability measure anymore, except when G is d-regular, because higher degrees get more attention – this can be helped by biasing, if one wants to. There is a distinguished operator * on $\overrightarrow{\mathcal{RG}_D}$ that reverses the direction of the flag and moves the root to the other end of the flag. We say that the measure λ is unimodular, if $\overrightarrow{\lambda}$ is invariant under *.

What are the graphs G such that the Dirac measure on G is unimodular? It is easy to see that such a graph has to be vertex transitive. Somewhat surprisingly though, not every vertex transitive graph is unimodular. A nice counterexample is the so-called grandmother graph, which can be described as follows. Take a 3-regular tree and direct its edges towards a chosen boundary point. Connect every vertex to its unique second neighbour with respect to this direction (its grandmother) and then forget all the directions. This new graph will be 8-regular and vertex transitive, but not unimodular [10]. It is a nice exercise to check this against the precise definition of unimodularity.

Unimodularity of vertex transitive graphs also has other equivalent characterizations. One such is that for any two vertices $x, y \in V(G)$, the orbit of x of under the stabilizer of y in $\mathrm{Aut}(G)$ has the same size as the orbit of y of under the stabilizer of x in $\mathrm{Aut}(G)$. Another is that $\mathrm{Aut}(G)$ is a unimodular locally compact group, that is, its left Haar measure is also right invariant. For details see [10].

One can easily show that for any finite graph G, λ_G is unimodular. Also, every Cayley graph is unimodular. Another rich source of unimodular random graphs is percolation theory. For instance, let G be the connected component of the root under an independent edge percolation of the lattice $\mathbb{Z}^d$. Then G is unimodular. This works for other Cayley graphs as well.

1.6. Soficity. A random rooted graph G is *sofic*, if it can be obtained as the limit of finite graphs, that is, if the distribution of G is in the weak closure of the set

$$\{\lambda_G \mid G \text{ is finite}\}.$$

It is easy to see that unimodularity is preserved by taking a weak limit. Thus, every sofic random rooted graph is unimodular. A famous open problem is whether the converse also holds. This is already open for Cayley graphs.

A group is *sofic* if (one of) its Cayley graphs are sofic respecting additional labels coming from a finite generating set. The notion has been introduced by Gromov and been further clarified by Weiss [14]. It is easy to see that amenable and residually finite groups are sofic. Besides their intristic interest, sofic groups are investigated for two reasons. On one hand, some deep conjectures in group theory, like Gottschalk's Conjecture, Kaplansky's Direct Finiteness Conjecture, the Determinant Conjecture and the Connes Embedding Conjecture hold for all sofic groups. On the other hand, no one knows a group that is not sofic. See [12] and [6] for details.

2. Spectral measure

2.1. The adjacency operator. Let $G \in \mathcal{RG}_D$ and denote by $\ell^2(G)$ the set of square summable real functions on the vertex set $V(G)$. Then one can associate the *adjacency operator* $A : \ell^2(G) \to \ell^2(G)$ as follows. For $f \in \ell^2(G)$ and $x \in V(G)$ let

$$(Af)(x) = \sum_{(x,y)\in E(G)} l((x,y))f(y),$$

where $l\colon E(G) \to \mathbb{Z}$ denotes the labelling function associated with G. Then A is a self-adjoint, bounded operator with $\|A\| \leq D^2$. Note that A is independent of the root of G.

2.2. The eigenvalue distribution of a finite graph and its moments. When G is finite of size n, then A is an n by n symmetric matrix and by the spectral theorem, $\ell^2(G)$ has an orthonormal base that consists of A-eigenvectors. Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues and let $b_1, \ldots, b_n$ be the orthonormal eigenbase with $Ab_i = \lambda_i b_i$. Let the *eigenvalue distribution*

$$\mu_G = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}$$

where δ_x denotes the Dirac measure concentrated on x. One can easily compute the k-th moment of μ_G as

$$\int_{\mathbb{R}} x^k d\mu_G(x) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i^k = \frac{1}{n} \operatorname{tr}(A^k) = \frac{1}{n} \sum_{o \in V(G)} m_{G,o,k}$$

where $m_{G,o,k}$ denotes the number of walks in G of length k starting and ending at o, counting with multiplicities, that is,

$$m_{G,o,k} = \sum_{\substack{(e_1,\ldots,e_k) \\ \text{is a walk } o \to o}} \prod_{i=1}^{k} l(e_i) = A^k_{o,o}.$$

where $A^k_{o,o}$ is the (o,o)-entry of the k-th power of the matrix A. Thus $m_{G,o,k}$ only depends on the k-neighbourhood of o in G.

2.3. The spectral measure of a graph at a vertex. When G is infinite, the above notions do not make sense, in particular, there may not be any eigenvalues or eigenfunctions of A. However, it makes sense to talk about the spectral measure $\mu_{G,o}$ with respect to the root o of G. One can take a pragmatic point of view, define $\mu_{G,o}$ by setting its k-th moment to be $m_{G,o,k}$ and showing that the corresponding moment problem can be solved in a unique way. Alternatively, the spectral theorem for bounded self-adjoint operators gives a projection valued measure

$$\left\{P_X : \ell^2(G) \to \ell^2(G) \mid X \subseteq \mathbb{R} \text{ Borel}\right\}$$

such that for any polynomial p, we have

$$p(A) = \int_{\mathbb{R}} p(x) dP_x$$

where $P_x = P_{(-\infty,x]}$. The projection $P_X : \ell^2(G) \to \ell^2(G)$ can be thought of as the orthogonal projection to the 'span of eigenvectors with eigenvalues in X'. For any fixed vector $f \in \ell^2(G)$ one can define a measure $\mu_{G,f}$ by setting

$$\mu_{G,f}(X) = \langle P_X f, f \rangle \quad (X \subseteq \mathbb{R} \text{ Borel})$$

that is, we assign the length square of the projection of f.

Definition 2. *Let $G \in \mathcal{RG}_D$. The* spectral measure of G at $o \in V(G)$ *is*

$$\mu_{G,o} = \mu_{G,\chi_o}$$

where χ_o is the characteristic function of o.

One can easily show that $\mu_{G,o}$ is a probability measure and applying $p(x) = x^k$ we see that the k-th moment of $\mu_{G,o}$ is equal to $m_{G,o,k}$.

The value $\mu_{G,o}(\{0\})$ is of special interest and we do not need the spectral theorem to describe it. Indeed, let v denote the orthogonal projection of χ_o to the closed subspace $\ker A$. Then we have

$$\text{(MAX)} \qquad \mu_{G,o}(\{0\}) = \langle v, v \rangle = \max\left\{ f(o)^2 \mid f \in \ker A,\ \langle f, f \rangle = 1 \right\}.$$

In general, $\mu_{G,o}$ may depend on the choice of the root o. However, when G is vertex transitive (for instance, when G is a Cayley graph), this measure is independent of o and is called the *spectral measure of* G, denoted by μ_G.

Spectral measure of Cayley graphs has received considerable attention in the literature. In particular, the spectral measures of certain lamplighter groups (with respect to various generating sets) have been explicitely computed. An interesting problem here is due to Atiyah who asked whether the presence of an atom in the spectral measure implies that the group has torsion.

2.4. The expected spectral measure. In the case when G is finite, the definition of spectral measure specializes to

$$\mu_{G,o} = \sum_{i=1}^{n} \langle \chi_o, b_i \rangle^2 \delta_{\lambda_i} = \sum_{i=1}^{n} b_i(o)^2 \delta_{\lambda_i}.$$

Using the orthonormality of b_i we get

$$\begin{aligned} \frac{1}{n} \sum_{o \in V(G)} \mu_{G,o} &= \frac{1}{n} \sum_{o \in V(G)} \sum_{i=1}^{n} b_i(o)^2 \delta_{\lambda_i} = \\ &= \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{o \in V(G)} b_i(o)^2 \right) \delta_{\lambda_i} = \mu_G \end{aligned}$$

which can be also formulated as

$$\mu_G = \mathrm{E}(\mu_{G,o})$$

where o is a uniform random vertex of G. Now this definition makes sense for an arbitrary random rooted graph.

DEFINITION 3. *Let G be a random rooted graph. We define the* expected spectral measure of G *as*

$$\mu_G = \mathrm{E}(\mu_{G,o})$$

where o is the root of the random graph G.

We have seen above that when G is finite, μ_G equals the eigenvalue distribution of G, so there is no ambiguity in the definition of μ_G.

3. Lück's Approximation Theorem

3.1. The expected spectral measure is local. Let R be an invariant of finite graphs. Abstractly, this means a map from the set of finite graphs to a topological space – most often the real line. We call R *local*, if $R(G_n)$ converges whenever G_n is a locally convergent sequence of finite graphs.

A trivial example for a local invariant is the average degree. A much less trivial example is the matching ratio, that is, the size of a maximal matching in G normalized by the size of G. This has been first proved in [**11**], see also [**5**] where the limit of the matching ratios is identified with a matching invariant of the limiting random rooted graph.

A quick example for a non-local invariant is the independence ratio, that is, the maximal size of an independent subset, normalized by the size of the graph. The counterexample is the standard one in this field: both the random d-regular graph on n points and the random d-regular *bipartite* graph on n points converge to the d-regular tree, but the independence ratio of the random d-regular graph is bounded away from $1/2$.

Another interesting invariant, that is connected to statistical physics via the so-called Potts model, is the *coloring entropy*

$$t_G(q) = \frac{\log\left(\#\text{ of legal colorings of } G \text{ with } q \text{ colors}\right)}{|G|}$$

where a legal coloring is a vertex coloring where no two adjacent vertices have the same color. Borgs, Chayes, Kahn and Lovász [**4**] showed that $t_G(q)$ is a local invariant for all q greater than twice the maximal degree. Recently Abert and Hubai [**1**] showed that the real moments of the uniform distribution on the roots of the chromatic polynomial are also local, which in particular implies the above result on coloring entropy.

Here we show that the eigenvalue distribution (or, more generally, the expected spectral measure for sofic random rooted graphs) is a local invariant, both in the weak topology on probability measures and the topology obtained by evaluating the measure of a point.

THEOREM 4 (Lück Approximation for Graphs). *Let G_n be a sequence of sofic random rooted graphs converging to G. Then μ_{G_n} weakly converges to μ_G and*

$$\lim_{n\to\infty} \mu_{G_n}(\{x\}) = \mu_G(\{x\})$$

for every $x \in \mathbb{R}$.

Weak convergence and convergence at $x = 0$ was proved by Lück in the case when G_n is a covering tower of finite Cayley graphs, where the corresponding chain of normal subgroups has trivial intersection [**9**]. Farber [**7**] later extended this for subgroup chains where the Schreier graphs converge to the Cayley graph of the fundamental group. The second author [**13**] further extended this for all $x \in \mathbb{R}$ and an arbitrary sofic approximation of the fundamental group. We mainly follow the arguments in [**13**].

Theorem 4 leads to the following interesting corollary.

COROLLARY 5. *Let G be a sofic random rooted graph. Then every atom of μ_G is an algebraic integer.*

This in particular is true for the spectral measure of any sofic Cayley graph, which was first proved in [**13**]. This corollary may prove to be useful in finding a unimodular random rooted graph (or even a Cayley graph) that is not sofic.

Now we start to prove Theorem 4 with a series of lemmas. The first one is an easy exercise; we skip the proof.

LEMMA 6. *Let X be a compact space and let μ_n be a sequence of probability measures on X that weakly converges to μ. Then for any closed set $Y \subseteq X$, we have*

$$\limsup_{n\to\infty}\ \mu_n(Y) \le \mu(Y).$$

For $x \in R$ and $\varepsilon > 0$ let

$$I_{x,\varepsilon} = (x-\varepsilon, x+\varepsilon)\backslash\{x\}$$

The next lemmas give uniform bounds on the spectral measure of $I_{x,\varepsilon}$ for sofic random rooted graphs.

LEMMA 7. *Let α be an algebraic integer. Then*

$$\mu_G(\{\alpha\}) \le \frac{1}{\deg(\alpha)}$$

For any finite graph $G \in \mathcal{G}_D$.

PROOF. By the irreducibility of the minimal polynomial of α, $\mu_G(\{\alpha\}) = \mu_G(\{\alpha'\})$ for any Galois conjugate of α. $\square$

LEMMA 8. *For every $n > 0$ there are only finitely many algebraic integers α of degree at most n such that all Galois conjugates of α lie in $[-D^2, D^2]$.*

PROOF. We get an upper bound on the degree and the absolute values of the coefficients of the characteristic polynomial in terms of n and D. $\square$

The following lemma is the core of Theorem 4. A more explicit version using Diophantine approximation has been proved in [**13**] but to conclude Theorem 4 this version suffices.

LEMMA 9. *Let $D > 0$ be an integer. Then for every $x \in [-D^2, D^2]$ there exists a sequence ε_k of positive real numbers converging to 0, such that*

$$\mu_G(I_{x,\varepsilon_k}) \leq \left(\frac{\log(2D^2)}{\log(1/2\varepsilon_k)} + \frac{1}{k}\right)^{1/2}$$

for all sofic random rooted graphs G and $k > 0$.

PROOF. Let X_k be the set of algebraic integers α of degree k such that all Galois conjugates of α are in $[-D^2, D^2]$. By Lemma 8, X_k is finite. Let $\varepsilon_k < 1/k$ be a positive number such that $I_{x,\varepsilon_k} \cap X_k = \varnothing$. Let G be a sofic random rooted graph. We claim that the inequality of the lemma holds.

Let us first assume that G is finite. For abbreviation, denote $\varepsilon = \varepsilon_k$, $I = I_{x,\varepsilon_k}$, $\mu(I) = \mu_G(I_{x,\varepsilon_k})$ and $\mu(y) = \mu_G(\{y\})$ for $y \in \mathbb{R}$. Let A be the adjacency matrix of G and let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of A. Then $|\lambda_i| \leq D^2$. Let

$$m = \frac{1}{n^2}\left|\{(i,j) \mid 1 \leq i,j \leq n,\, \lambda_i = \lambda_j \neq x\}\right| = \sum_{\lambda \in I} \mu^2(\lambda).$$

Then by the definition of ε and Lemma 7, we have $\mu(\lambda) < 1/k$ for all $\lambda \in I$, which yields

$$m < \frac{1}{k}\sum_{\lambda \in I}\mu(\lambda) \leq \frac{1}{k} \tag{A}$$

and the number of pairs

$$\begin{aligned} |\{(i,j) \mid \lambda_i, \lambda_j \in (x-\varepsilon, x+\varepsilon), \lambda_i \neq \lambda_j\}| &= (n(\mu(I) + \mu(x)))^2 - n^2(m + \mu^2(x)) \\ &= n^2(\mu^2(I) + 2\mu(x)\mu(I) - m). \end{aligned}$$

The product

$$\prod_{\substack{1 \leq i,j \leq n \\ \lambda_i \neq \lambda_j}} (\lambda_i - \lambda_j)$$

is invariant under Galois-conjugation and hence it is a non-zero integer. So we have

$$1 \leq \left| \prod_{\substack{1 \leq i < j \leq n \\ \lambda_i \neq \lambda_j}} (\lambda_i - \lambda_j) \right| \leq (2\varepsilon)^{n^2(\mu^2(I)+2\mu(x)\mu(I)-m)} (2D^2)^{n^2}$$

Using (A) this yields

$$\mu^2(I) + 2\mu(x)\mu(I) \leq \frac{\log(2D^2)}{\log(1/2\varepsilon)} + m < \frac{\log(2D^2)}{\log(1/2\varepsilon)} + \frac{1}{k}$$

which gives

$$\mu(I) \leq \left(\mu^2(x) + \frac{\log(2D^2)}{\log(1/2\varepsilon)} + \frac{1}{k}\right)^{1/2} - \mu(x) \leq \left(\frac{\log(2D^2)}{\log(1/2\varepsilon)} + \frac{1}{k}\right)^{1/2}$$

as claimed.

We proved the Lemma for finite graphs. Now the same follows for any sofic random rooted graph G, using Lemma 6 on the complement $[-D^2, D^2] \setminus I_{x,\varepsilon_k}$. □

We are ready to prove Theorem 4.

Proof of Theorem 4. For any unimodular random graph H, the k-th moment of μ_H equals

$$\int_{\mathbb{R}} x^k d\mu_H(x) = \mathrm{E}\left(\int_{\mathbb{R}} x^k d\mu_{H,o}(x)\right) = \mathrm{E}\left(m_{H,o,k}\right)$$

where o is the root of H. But $m_{H,o,k}$ only depends on the k-neighbourhood of o in H, so convergence in sampling probability implies that for all $k \geq 0$, the k-th moment of μ_{G_n} converges to the k-th moment of μ_G. Hence, μ_{G_n} weakly converges to μ_G.

We need to prove pointwise convergence. In short, weak convergence of μ_{G_n} implies pointwise convergence, except when there is a positive mass traveling into 0, which is impossible by Lemma 9. More precisely, by Lemma 6 we have

$$\limsup_{n\to\infty} \mu_{G_n}(\{x\}) \leq \mu_G(\{x\})$$

and

$$\liminf_{n\to\infty} \mu_{G_n}((x-\varepsilon, x+\varepsilon)) \geq \mu_G((x-\varepsilon, x+\varepsilon))$$

for all $\varepsilon > 0$. By Lemma 9, there exists $\varepsilon_k > 0$ with $\varepsilon_k \to 0$ $(k \to \infty)$ such that for all n we have

$$\mu_{G_n}(I_{x,\varepsilon_k}) \leq \left(\frac{\log(2D^2)}{\log(1/2\varepsilon_k)} + \frac{1}{k}\right)^{1/2}.$$

This gives us

$$\begin{aligned}\limsup_{n\to\infty} \mu_{G_n}(\{x\}) &\le \mu_G(\{x\}) \le \mu_G((x-\varepsilon_k, x+\varepsilon_k)) \\ &\le \liminf_{n\to\infty} \mu_{G_n}((x-\varepsilon_k, x+\varepsilon_k)) \\ &\le \liminf_{n\to\infty} \mu_{G_n}(\{x\}) + \left(\frac{\log(2D^2)}{\log(1/2\varepsilon_k)} + \frac{1}{k}\right)^{1/2}\end{aligned}$$

for all $k > 0$. Letting k tend to infinity, this implies

$$\lim_{n\to\infty} \mu_{G_n}(\{x\}) = \mu_G(\{x\})$$

as claimed. □

Remark. It is natural to ask whether the Lück approximation result already holds in the space of (deterministic) rooted graphs. Namely, whether for a sequence $G_n \in \mathcal{RG}_D$ converging to $G \in \mathcal{RG}_D$, we have μ_{G_n,o_n} converging to $\mu_{G,o}$, both in weak topology and pointwise. Here o_n denotes the root of G_n and o the root of G. This in particular would imply Theorem 4.

Looking at moments, it is easy to see that weak convergence holds again. However, convergence at the point 0 fails in general, as the following example shows. Take a long cycle of odd length rooted at o. Label edges of even distance from o by 1 and -2 otherwise, except at the edge farthest away from o. Use equality (MAX) above. We leave the reader to the details.

3.2. The point $z = 0$.

The point $z = 0$ is of special interest for most of the applications. In this case, we get a stronger result, because we can use another measuring function, simpler than the product of $\lambda_i - \lambda_j$.

Lemma 10. *Let G be a sofic unimodular random graph. Then for every $\varepsilon > 0$ we have*

$$\mu(I_{0,\varepsilon}) \le \frac{\log(D)}{\log(1/\varepsilon)}.$$

Proof. It is enough to prove the Lemma for μ equal to distribution of roots of an integer monic polynomial with all its roots in the disc of radius D. Let $\lambda_1, \ldots, \lambda_n$ be the roots of f. The product

$$\prod_{\lambda_i \ne 0} \lambda_i$$

is the first nonzero coefficient of f and hence is a non-zero integer. This gives

$$1 \le \prod_{\lambda_i \ne 0} |\lambda_i| \le \varepsilon^{n\mu(I_{0,\varepsilon})} D^n$$

which yields the lemma. □

This yields an explicite estimate on the measure of $\{0\}$ in terms of the moments of μ_G, assuming that the measure is supported on the real line.

Lemma 11. *Let $D > 1$ and let μ_G be the spectral measure of a unimodular random graph G. Then we have*

$$\left|\mu_G(\{0\}) - \int\limits_{|x|\leq R} \left(1 - \frac{x^2}{D^2}\right)^n d\mu_G(x)\right| \leq \frac{\log(D)+2}{\log(n)}.$$

Proof. Let

$$f(x) = \left(1 - \frac{x^2}{D^2}\right)^n.$$

Then $f(0) = 1$, $0 \leq f(x) < f(\varepsilon)$ for all $\varepsilon < |x| \leq D$, and $|f(x)| \leq 1$ for all $|x| \leq \varepsilon$. This implies

$$\left|\mu_G(\{0\}) - \int\limits_{|x|\leq D} f(x) d\mu_G(x)\right| \leq \mu_G(I_{0,\varepsilon}) + f(\varepsilon)$$

which, by Lemma 10, implies

$$\left|\mu_G(\{0\}) - \int\limits_{|x|\leq D} f(x) d\mu_G(x)\right| \leq \frac{\log(D)}{\log(1/\varepsilon)} + \left(1 - \frac{\varepsilon^2}{D^2}\right)^n$$

Setting $\varepsilon = n^{-1}$ yields

$$\frac{\log(D)}{\log(1/\varepsilon)} + \left(1 - \frac{\varepsilon^2}{D^2}\right)^{2n} \leq \frac{\log(D)}{\log(n)} + \exp(-n^2/D^2)$$

However,

$$\exp(-n^2/D^2) \leq \frac{2}{\log(n)}.$$

and the lemma is proved. □

In particular, this gives that if the first $2n$ even moments of two spectral measures of radius D are the same (or very close), then the measures of $\{0\}$ differ by at most $O(1/\log n)$. Another consequence is Lück's approximation theorem, whose original proof in [8] has been stimulating for the results in [13] and the present paper.

Theorem 12. *Let X be a finite connected simplicial complex of dimension d with fundamental group $\Gamma := \pi_1(X)$. Let $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \dots$ be a chain of normal subgroups of finite index in Γ such that $\cap_n \Gamma_n = 1$ and let $X_n = \widetilde{X}/\Gamma_n$ where $\widetilde{X}$ is the universal cover of X. Then*

$$\lim_{n\to\infty} \frac{b_k(X_n)}{|\Gamma : \Gamma_n|} = \beta_k^{(2)}(X)$$

where $\beta_k^{(2)}(X)$ is the k-th L^2 Betti number of X.

PROOF. For each $n \in \mathbb{N}$, consider the simplicial chain complex of X_n

$$0 \to C_d(X_n) \xrightarrow{\delta_d} C_{d-1}(X_n) \xrightarrow{\delta_{d-1}} \cdots \to C_0(X_n) \to 0.$$

Each group $C_k(X_n)$ comes equipped with a prefered basis formed by the k-simplices. For fixed $k \in \mathbb{N}$, we can associate to each integer n a random rooted graph $G(k)_n$, whose vertices are the k-simplices of X_n. We put a $\mathbb{Z}$-labelled edge between vertices according to the matrix cofficients of the k-th Laplace operator $\Delta_k = \delta_k^*\delta_k + \delta_{k-1}\delta_{k-1}^* \colon C_k(X_n) \to C_k(X_n)$. It is clear that $G(k)_n$ converges to a random rooted graph $G(k)_\infty$. Moreover, by the Hodge decomposition theorem

$$\mu_{G(k)_n}(\{0\}) = \frac{b_k(X_n)}{|\Gamma : \Gamma_n|},$$

whereas $\mu_{G(k)_\infty}(\{0\}) = \beta_k^{(2)}(X)$. Now, Theorem 4 for $x = 0$ implies the claim. $\square$

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