Andreas Berthold Thom

Connective $\mathbf{E}$-theory and bivariant homology for C^*-algebras

2003

Reine Mathematik

Connective $\mathbf{E}$-theory and bivariant homology for C^*-algebras

Inaugural-Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften im Fachbereich Mathematik
der Mathematisch-Naturwissenschaftlichen Fakultät
der Westfälischen Wilhelms-Universität Münster

vorgelegt von
Andreas Berthold Thom
aus Dresden
– 2003 –

Dekan:	Prof. Dr. F. Natterer
Erster Gutachter:	Prof. Dr. J. Cuntz
Zweiter Gutachter:	Prof. Dr. S. Echterhoff
Tag der mündlichen Prüfungen:	24. Juli 2003
Tag der Promotion:	30. Juli 2003

Zusammenfassung

Die vorliegende Arbeit konzentriert sich auf die Untersuchung bivarianter und triangulierter Homologietheorien auf der Kategorie separabler C^*-Algebren. Insbesondere enthält die Arbeit eine Definition von bivarianter konnektiver $\mathbf{E}$-theorie und bivarianter Homologie. Beide Theorien erlauben ein besseres Verständnis der Homotopietheorie nicht-kommutativer Zellkomplexe. Die algebraischen Eigenschaften der beiden Theorien werden mit Hilfe von Spektralsequenzen untersucht.
In verschiedenen Berechnungen werden Matrixbündel und nicht-kommutative Algebren, welche man auf natürliche Weise zu kompakten, lokal Hausdorffschen Räumen assoziieren kann, untersucht. Desweiteren werden Hindernisse zur Existenz eines rationalen Chern-Charakters von der bivarianten Homologie zur bivarianten $\mathbf{K}$-Theorie identifiziert.
Der letzte Teil der Arbeit beschäftigt sich mit einer Verbindung zur Theorie der Modulspektren über dem konnektiven $\mathbf{K}$-Theorie Spektrum.

Contents

Ich widme diese Arbeit meinen lieben Eltern,
welche mir Kraft, Zuversicht und Freude gegeben haben.

1 Introduction

The aim of this thesis is to analyze the variety of bivariant homology theories on the category of separable C^*-algebras. This is motivated by the importance of the study of E-theory and KK-theory which are both bivariant homology theories. This thesis follows the general scheme of studying topological or operator theoretical problems by algebraic means. A bivariant (or triangulated) homology theory (for a definition see section 2.3) is thought of as a linear approximation of a category.

KK-theory is a result of the work of G.G. Kasparov [35] in the early 80s. The character of KK-theory as a bivariant theory was central for its success throughout operator theory. Later, in the work of J. Cuntz [16] KK-theory was identified as the universal bivariant homology theory satisfying an excision property for all semi-split extensions, and stability with respect to the algebra of compact operators. (These notions will be explained in section 2.3). This was a real surprise, since the definition given by G.G. Kasparov was not at all abstract but very concrete and geometrically motivated. It is well known that not all extensions of C^*-algebras are semi-split. Much less obvious is the conclusion of G. Skandalis in [62] which says that KK-theory indeed does not satisfy excision for certain non-semi-split extensions. It is therefore natural to ask whether there exists a bivariant homology theory that satisfies excision for all extensions.

E-theory was developed by N. Higson and A. Connes in [13] as a concrete realization of the universal bivariant homology theory satisfying excision for all extensions and stability. The existence of such a theory was known before by an abstract construction using categories of fractions, see N. Higson's work in [30]. The important news was that E-theory can be described using the concrete picture of asymptotic morphisms. Furthermore, a lot of asymptotic morphisms arise in geometrically significant contexts like deformation quantization and so on. KK-theory and E-theory have applications towards a proof of the Baum-Connes conjecture, e.g. in [28].

Our starting point in the discussion of bivariant homology theories on the category of separable C^*-algebras is stable homotopy theory which was defined independently by M. Dădărlat in [20] and A. Connes in [13, 14]. There had been earlier definitions of non-commutative analogues of stable homotopy by J. Rosenberg [51] and different other authors (see [51] for complete references) but they did not have sufficient exactness properties. Only the usage of asymptotic morphisms ensures the excision property for all extensions which is what one requires in the operator algebraic setting. The introduction of asymptotic morphisms by N. Higson and A. Connes in [13] was the remaining crucial ingredient to make the theory work.

The foundation of the circle of ideas in stable homotopy of course goes back to E. Spanier and J.H.C. Whitehead (see e.g. [72]). Stable homotopy for C^*-algebras is not so well known to operator algebraists as a bivariant homology theory, since only few computations for non-commutative algebras have been done and the geometric significance is much less obvious than in the classical case. It remains open to connect the deep work in (classical) stable homotopy theory that has been done since its foundation to the results in (non-commutative) stable homotopy. An important question which remains open is the one about the stable homotopy groups of matrix algebras. Those questions and possible applications are related to the fine structure in bu-theory which will be analyzed in a forthcoming article.

Before going into details, our viewpoint will be an abstract one. We want to study properties of bivariant homology theories. Most of the known bivariant homology theories turn out to be what we call triangulated homology theories. Those triangulated homology theories carry a lot more structure and a whole well-developed machinery from the theory of triangulated categories can be applied (see e.g. the book by A. Neeman, [43]). Later, we want to study some bivariant homology theories in more detail and discover that they are natural generalizations of well known bivariant homology theories on the category of finite CW-complexes to the non-commutative setting. We stick to the requirement that the bivariant theories ought to satisfy excision with respect to all extensions, although a similar approach leads to theories satisfying excision for semi-split extensions (see [34]).

The category of bivariant homology theories (morphisms being natural transformations of bivariant theories) on the category of finite CW-complexes is equivalent to a homotopy category of ring spectra. The identification goes as follows.

$$(E_n, \sigma_n : E_n \to \Omega E_{n+1}) \leftrightarrow \{(X, Y) \mapsto [X, E_* \wedge Y]\}$$

The importance of the category of ring spectra would be another good starting point for the motivation of the study of bivariant homology theories. We are going to explain the relationship between bivariant homology theories on CW-complexes and ring spectra in section 2.3. Instead of studying the whole category of bivariant homology theories on the category of C^*-algebras we are going to study certain bivariant homology theories satisfying interesting universal properties.

The thesis is organized as follows. Starting with preliminaries to the definition of stable homotopy, first, we generalize some theorems which are well known in the classical world of finite CW-complexes. This will occupy most of section 2. In particular, we will show how the usage of the notion of co-fibrations is circumvented in the setting of operator algebras. This section also contains a definition of the terms 'bivariant and triangulated homology theory' (see 2.3).

In section 3 we define stable homotopy theory for separable C^*-algebras and show that it is the universal triangulated homology theory on the category of separable C^*-algebras. In this section we heavily use the extension category of the category of separable C^*-algebras which is defined in appendix C.3. Indeed, we show that the extension category is equivalent to the stable homotopy category which was defined using asymptotic morphisms. The usage of the extension category simplifies the proof of the universal properties of stable homotopy theory.

In section 4 we construct certain triangulated and bivariant homology theories which are generalizations of well known bivariant homology theories on CW-complexes. In particular, this section contains the definition of connective $\mathbf{E}$-theory (which is denoted by $\mathbf{bu}$) and bivariant homology for C^*-algebras. Connective $\mathbf{E}$-theory and bivariant homology give extensions of connective $\mathbf{K}$-homology and $\mathbf{K}$-theory resp. singular homology and co-homology from the category of finite CW-complexes to the category of (possibly non-commutative) separable C^*-algebras. Our description provides a new perspective on those theories from a non-commutative point of view. Furthermore, it shows how natural theories, such as $\mathbf{K}$-theory and connective $\mathbf{K}$-theory, occur. The idea to regard cohomology and homology as the defect of multiplication with the Bott map in connective $\mathbf{K}$-theory can be found in the work of Larry Smith [63] in the late 70s.

There has been previous work towards a definition of connective $\mathbf{K}$-theory for C^*-algebras by J. Rosenberg [51]. The definition in [51] lacked the appropriate exactness property. To our knowledge there has not been a definition of a bivariant theory before. There has also been no study of the defect of the Bott map in this context. There had been attempts to define cellular cohomology for C^*-algebras by R. Exel and T.A. Loring [26] but they obviously did not have the desired exactness properties.

In section 4 we are also going to define the notion of a (strict) non-commutative cell complex. (Strict) non-commutative cell complexes will constitute a suitable substitute for CW-complexes in the non-commutative setting. Strict non-commutative cell complexes (we will omit the term 'non-commutative' in most of the situations) are recursively defined type 1 algebras with a bound on the dimension of their irreducible representations. We conjecture that the unitalizations of strict cell complexes are recursively sub-homogenous algebras in the sense of C. Phillips [48] but we have not tried to prove this.

The triangulated categories corresponding to $\mathbf{E}$-theory and connective $\mathbf{E}$-theory are only partially understood by homological algebra. This is due to the fact that there are possible obstructions to the convergence of a universal coefficient (UC) spectral sequence. The obstruction is identified with the possibility of the existence of algebras with vanishing $\mathbf{K}$-groups or connective $\mathbf{K}$-groups, which are non-trivial in the corresponding bivariant homology theories (i.e. $\mathbf{K}_*(A) = 0$ but $\mathbf{E}_*(A, A) \neq 0$ or $\mathbf{bu}_*(A) = 0$ but $\mathbf{bu}_*(A, A) \neq 0$). Nonetheless, the behavior on the sub-category of cell complexes is rather nice.

We study the thick sub-category of algebras with finitely generated connective $\mathbf{K}$-groups in more detail. The localization of this sub-category with respect to the possible obstruction sub-category turns out to be a Bousfield localization. The quotient (in the sense of triangulated categories) gives a triangulated homology theory which always satisfies convergence of the UC spectral sequence and hence is much better understood algebraically. (There is one problem with this construction. Since we cannot exclude the possibility that $\mathbf{bu}_*(A) = 0$ but $\mathbf{bu}_*(A \otimes A) \neq 0$, the localization is not necessarily monoidal. But it is at least equivalent to a triangulated category with compatible monoidal structure. Indeed, it is equivalent (as a triangulated category) to the full triangulated sub-category of cell complexes which is monoidal.)

Among other computations we compute the $\mathbf{bu}$-type of matrix bundles over compact pointed spaces. Furthermore, we give a procedure of associating a separable C^*-algebra up to $\mathbf{bu}$-equivalence to any locally Hausdorff pointed compact topological space. This allows to compute certain invariants of those spaces which give obstructions to the existence of a stable homotopy equivalence to a global Hausdorff space. The existence of a global Hausdorff structure up to homotopy is an interesting property. It seems that the question of its existence has not been considered before. There are no further applications so far.

We show that negative connective $\mathbf{E}$-theory coincides with negative algebraic $\mathbf{K}$-theory on a category of strict non-commutative cell-complexes. This is based on results by J. Rosenberg [52]. In [53] he conjectures that homotopy invariance of negative algebraic $\mathbf{K}$-theory holds for C^*-algebras which would relate the two theories even closer. This remains open. We mention those results, since they provide another view on connective $\mathbf{E}$-theory.

In the algebraic study of bivariant homology we establish a Bockstein-Chern spectral sequence with E^2-term given by the bivariant homology groups and converging towards (non-connective) $\mathbf{E}$-theory. This spectral sequence encodes all the information about the torsion of the Bott element. If we rationalize, the spectral sequence is in some sense what remains of the rational Chern character after passage to the non-commutative setting. We compute the ring of cohomology operations of bivariant homology and use it to detect algebras which are not equivalent to commutative ones in connective $\mathbf{E}$-theory. We prove the existence of a bivariant rational Chern character which is an isomorphism in the case that the algebras in question are $\mathbf{bu}$-equivalent to commutative algebras. A converse is proved in the rational case. Furthermore, we construct an Adams spectral sequence converging to connective $\mathbf{E}$-theory.

Finally, in section 5, we are going to identify the triangulated categories found so far with homotopy categories of module spectra (see e.g. the work of M. Hovey, B. Shipley and J. Smith in [33]) over the non-connective and connective $\mathbf{K}$-theory spectrum (the latter is denoted by $\underline{bu}$). Further algebraic study of C^*-algebras, considered as objects in $\mathbf{bu}$, can be carried out in the homotopy category of $\underline{bu}$-module spectra as well (see e.g. the work of J. Wolbert in [73]). This category has the advantage that it is the homotopy category of a simplicial model category. The spectral sequences which we have considered can be constructed in the category of $\underline{bu}$-spectra as well. The author hopes that the usage of C^*-algebras allows to give a more concrete picture of $\underline{bu}$-module spectra. So far there are no applications of this alternative description.

The construction of a suitable model structure on the category of pro-objects of separable C*-algebras is content of further study. Although such a model structure might be useful for abstract purposes we do not expect it to be very useful for concrete computations. A first step towards a useful model structure would be a strict category of asymptotic morphisms which is rather difficult to obtain.

This thesis partially answers questions which were asked by B. Blackadar [8], M. Dădărlat and J. McClure [24] and J. Wolbert [73].

I want to thank my supervisor Professor Joachim Cuntz for his guidance through the topic and for a lot of interesting comments. Furthermore, I want to thank Christian Ausoni, Paul Balmer, Arthur Bartels, Tilman Bauer, Michael Joachim, Mark Johnson, Eberhard Kirchberg, Michel Matthey, Ralf Meyer, Roman Sauer, Stefan Schwede, Thomas Timmermann, Julius Verrel, Christian Voigt and Wilhelm Winter for fruitful and interesting discussions which I have enjoyed very much. I also want to thank the SFB 478 for its hospitality during almost three years in Münster. In particular, my work was supported by the

- EU-Network Quantum Spaces - Non-commutative Geometry (Contract No.HPRN-CT-2002-00280) and the
- Deutsche Forschungsgemeinschaft (SFB 478).

2 Preliminaries

This section gives the preliminaries to the definition of stable homotopy theory for C^*-algebras and the formulation of its properties. The definition of stable homotopy will be given in the next section. For introductory information about C^*-algebras see appendix C.1. There can be found a definition of all symbols and short notations that are used throughout the text. Almost all C^*-algebras that appear in these notes are separable (see appendix C.1 for a definition).

In the first part we want to introduce the theory of asymptotic morphisms. We discuss the notion of homotopy of asymptotic morphisms. The concept of an asymptotic morphism is central for a concrete understanding of a suitable homotopy theory of C^*-algebras. We will clarify what 'suitable' means in this context. The original definition of stable homotopy theory for C^*-algebras was based on asymptotic morphisms and given by A. Connes and N. Higson in [13] and M. Dădărlat in [20].

The second part clarifies the relation between extensions of C^*-algebras and asymptotic morphisms. This requires a notion of homotopy of extensions. The ideas in this part are taken from [34].

In a third part we give a definition of the concepts of bivariant and triangulated homology theories. The definition of triangulated homology theory seems to be new in the context of operator algebras. We do not claim any originality, since the idea of approximating categories with a notion of homotopy using triangulated categories is indeed old (e.g. in the context of topological spaces, chain complexes etc.) and has a long history. Triangulated categories were used implicitly in the construction of E-theory by N. Higson in [30] as we will explain at the end of section 3.

2.1 What is an asymptotic morphism ?

An asymptotic morphism between C^*-algebras is a generalization of a $*$-homomorphism. The idea of asymptotic morphisms goes back to the work of A. Connes and N. Higson [13] and was further developed in the articles [20] by M. Dădărlat and [34] by K. Thomson, and T.G. Houghton-Larsen. A self-contained treatment of the theory of asymptotic morphisms can be found in [28]. The concept of asymptotic morphisms is far more flexible than the notion of $*$-homomorphism and allows to construct a homotopy category of separable C^*-algebras with the desired exactness properties.

Let us start with the definition of the asymptotic envelope of a C^*-algebra.

Definition 2.1.1 *For any C^*-algebra A define $\mathfrak{a}(A)$ to be the quotient in the following exact sequence.*

$$0 \longrightarrow A[0,1) \longrightarrow A_b[0,1) \longrightarrow \mathfrak{a}(A) \longrightarrow 0$$

We call the algebra $a(A)$ *the* <u>*asymptotic envelope*</u> *of* A. *There is a natural inclusion* $\alpha_A : A \to a(A)$ *which sends the element* $a \in A$ *to the image of the constant function* $\{[0,1) \ni t \mapsto a\} \in A_b[0,1)$.

Note that there is an injection $A \otimes C_b[0,1) \hookrightarrow A_b[0,1)$ which is not surjective unless A is finite dimensional. In particular, $A_b[0,1)$ (and therefore also $a(A)$) cannot be easily described as the algebra of A-valued functions on a compact space. Note further that the asymptotic envelope of an algebra (except in the trivial case of the zero algebra) is not separable. These will be about the only non-separable C^*-algebras that appear in these notes.

Lemma 2.1.2 *The assignment* $a : A \mapsto a(A)$ *of a* C^**-algebra to its asymptotic envelope extends to a functor* $a : \mathcal{C} \to \mathcal{C}$ *from the category of* C^**-algebras to itself.*

Proof: Let $\phi : A \to B$ be a $*$-homomorphism between C^*-algebras. It induces natural $*$-homomorphisms $A[0,1) \to B[0,1)$ and $A_b[0,1) \to B_b[0,1)$. The uniqueness of the induced $*$-homomorphism between the quotients proves the functoriality. This finishes the argument.

Lemma 2.1.3 *The inclusion* $A \hookrightarrow a(A)$ *of a* C^**-algebra into its asymptotic envelope extends to a natural transformation* $\alpha : \mathrm{id}_{\mathcal{C}} \to a$.

The proof is obvious.

Using the asymptotic envelope we are able to define the notion of an asymptotic morphism.

Definition 2.1.4 *An* <u>*asymptotic morphism*</u> *between* C^**-algebras* A *and* B *is a* $*$*-homomorphism* $A \to a(B)$. *(We will either speak about a* $*$*-homomorphism* $A \to a(B)$ *or an asymptotic morphism* $A \to B$.*)*

Note that any ordinary $*$-homomorphism $A \to B$ gives rise to an asymptotic morphism by composition with the natural map $B \stackrel{(B)}{\to} a(B)$.

Having a definition, we want to review another picture of asymptotic morphisms which is quite useful for concrete constructions and computations. By the theorem of Bartle-Graves (see appendix C.1.14), there is a bounded continuous (not necessarily linear, $*$-preserving or multiplicative) split of the surjection $B_b[0,1) \to a(B)$. Composing any asymptotic morphism with co-domain B with this split we get a continuous map $A \to B_b[0,1)$. Considering the evaluations at the points in $[0,1)$, one can interpret the asymptotic morphism as a family of maps satisfying the relation of a $*$-homomorphisms if the parameter t tends to 1. The precise statement is subsumed in the following theorem.

Theorem 2.1.5 (A. Connes, N. Higson in [13]) *Let* $\phi_t : A \to B$ *(for* $t \in [0,1)$*) be a uniformly bounded continuous family of continuous maps satisfying*

$$\lim_{t\to 1} \|\phi_t(a+b) - \phi_t(a) - \phi_t(b)\| = 0$$
$$\lim_{t\to 1} \|\phi_t(ab) - \phi_t(a)\phi_t(b)\| = 0$$
$$\lim_{t\to 1} \|\phi_t(a^*) - \phi_t(a)^*\| = 0$$

for all $a, b \in A$*. It defines an asymptotic morphism by regarding the family as a map* $A \to B_b[0,1)$ *and passing to the quotient* $\mathfrak{a}(B)$*. Conversely any asymptotic morphism from* A *to* B *is represented by such a family, and the family is unique up to addition of a map* $A \to B[0,1)$*.*

To understand the nature of asymptotic morphisms geometrically one should look at what it means to have an asymptotic morphism between commutative algebras. The following theorem was proved by M. Dădărlat and is taken from [20].

Theorem 2.1.6 (M. Dădărlat in [20]) *Let* X *and* Y *be compact topological spaces and* f *be an asymptotic morphism from* $C(X)$ *to* $C(Y)$*. Denote by* $M(X)$ *the space of probability measures on* X *equipped with the weak-*$*$*-topology.*

An asymptotic morphism f *gives rise to a continuous family of continuous maps* $\mu_t : Y \to M(X)$ *such that* $\mu_t(y)$ *converges to* X *(embedded as the space of point measures) for all* $y \in Y$ *as the parameter* t *tends to one.*

Conversely any such family defines an asymptotic morphism from $C(X)$ *to* $C(Y)$ *and two such families give rise to the same class of asymptotic morphisms if and only if their difference converges pointwise to zero in the weak-* $*$*-topology of* $C(X)'$*.*

The last theorem tells that an asymptotic morphism is quite close to an ordinary $*$-homomorphism. Indeed, there is a description of asymptotic morphisms using strong shape theory (see the work of M. Dădărlat in [21]). We are not considering strong shape theory here but we want to emphasize that it is very useful, since it allows to find obstructions to the existence of asymptotic morphisms which is difficult in the other pictures. Theorem 2.2.15 allows a further clarification of the geometric meaning of an asymptotic morphism in the case of finite CW-complexes or more generally ANRs.

Next we want to define the notion of an asymptotic homotopy between asymptotic morphisms. The basic question here is which interval functor one should take. There are essentially two possibilities. Either one takes $\mathfrak{a}(A)[0,1]$ or $\mathfrak{a}(A[0,1])$. It turns out that the second possibility is suited much better for our purposes.

Definition 2.1.7 *Two morphisms*

$$\phi, \psi : A \to \mathfrak{a}(B)$$

are called asymptotically homotopic*, if there is a morphism* $H : A \to \mathfrak{a}(B[0,1])$ *such that* $\mathfrak{a}(ev_0) \circ H = \phi$ *and* $\mathfrak{a}(ev_1) \circ H = \psi$*.*

In order to ensure that the relation of homotopy is an equivalence relation we have to remark for the transitivity that, by lemma 2.5 in [28], there is a natural isomorphism $\mathfrak{a}(A[0,1]) \oplus_{\mathfrak{a}(A)} \mathfrak{a}(A[0,1]) \cong \mathfrak{a}(A[0,1])$. Reflexivity and symmetry are obvious.

We denote the set of asymptotic homotopy classes of asymptotic morphisms from A to B by $[[A, B]]$. Given an asymptotic morphism $\alpha : A \to B$, we denote its asymptotic homotopy class by $[[\alpha]] \in [[A, B]]$.

Note that any ray in $\hom(A, B)$ gives rise to a morphism $A \to \mathfrak{a}(B)$. Furthermore, given an asymptotic morphism, any reparametrisation of the half-open interval will produce a new one. Only our choice of the notion of homotopy ensures that one can show that these trivially constructed asymptotic morphisms are all asymptotically homotopic.

Composition of asymptotic morphisms is in general a subtle issue. It is obvious how one can compose an asymptotic morphism with an ordinary $*$-morphism. It is much less obvious how to compose two asymptotic morphisms. Some remarks about composition of asymptotic morphisms are in order. To define the composition we introduce a third picture of asymptotic morphisms along the lines of [28] (see definition 2.7 in [28]). First of all, instead of morphisms $A \to \mathfrak{a}(B)$ up to homotopies $A \to \mathfrak{a}(B[0,1])$ we consider morphisms $A \to \mathfrak{a}^n(B)$ up to homotopies $A \to \mathfrak{a}^n(B[0,1])$ for any $n \in \mathbb{N}$. We denote the homotopy relation of morphisms $A \to \mathfrak{a}^n(B)$ by $\sim_n$. All those morphisms assemble in the co-limit

$$\mathrm{colim}_n \hom(A, \mathfrak{a}^n(B))/\sim_n$$

where the co-limit is taken in the category of sets and with respect to the maps

$$\hom(A, \mathfrak{a}^n(B[0,1])) \xrightarrow{(\mathfrak{a}^n(B[0,1]))\circ ?} \hom(A, \mathfrak{a}^{n+1}(B[0,1])).$$

The following result relates this apparently more general situation to what we already considered. A proof can be found in [28].

Lemma 2.1.8 *Let* A *and* B *be separable* C^**-algebras. The canonical map*

$$\hom(A, \mathfrak{a}(B))/\sim_1 \longrightarrow \mathrm{colim}_n \hom(A, \mathfrak{a}^n(B))/\sim_n$$

is an isomorphism of sets.

The only purpose of considering this co-limit is that it allows to define a composition product. Given $f : A \to \mathfrak{a}^n(B)$ and $g : B \to \mathfrak{a}^m(C)$, we can define a composition $\mathfrak{a}^n(g) \circ f : A \to \mathfrak{a}^{n+m}(C)$. In order to get an associative composition product up to homotopy one still has to prove some things. The statement with a full proof can be found as proposition 2.12 in [28]. We only state the final conclusion.

Theorem 2.1.9 *Let* A,B *and* C *be separable* C^**-algebras. There is an associative composition product* $[[A,B]] \times [[B,C]] \to [[A,C]]$*. It allows to define a category* $\mathfrak{as}\mathcal{C}$ *with objects separable* C^**-algebras and morphisms* $\mathfrak{as}\mathcal{C}(A,B) = [[A,B]]$*. The category* $\mathfrak{as}\mathcal{C}$ *is called the* asymptotic homotopy category*.*

The canonical map $\hom(A,B) \to [[A,B]]$ *can be extended to a functor* $s\mathcal{C} \to \mathfrak{as}\mathcal{C}$ *from the category of separable* C^**-algebras into the asymptotic homotopy category.*

In [28] it is shown that the symmetric monoidal structure which comes from the maximal tensor product (see appendix C.4) extends to the category $\mathfrak{as}\mathcal{C}$ such that the functor $s\mathcal{C} \to \mathfrak{as}\mathcal{C}$ becomes a symmetric monoidal functor. For a proof and related results see chapter 4 in [28]. We state theorem 4.6 in [28].

Theorem 2.1.10 *The category* $\mathfrak{as}\mathcal{C}$ *is symmetric monoidal with respect to the maximal tensor product.*

Up to now, it has not become clear why we did not take the ordinary notions of morphism and homotopy. The next section will clarify the advantages of the approach using asymptotic morphism and asymptotic homotopy.

It would be very pleasant to construct the asymptotic homotopy category as the 'homotopy category' of a strict category with nice properties. In order to make this precise one has to explain what one means by the term 'homotopy category'. The right framework to do this is the context of 'model categories' (see e.g. [31]). The construction of a suitable set-up will be content of a forthcoming article. This involves ideas coming from strong shape theory which were developed in the context of C^*-algebras by M. Dădărlat in [21].

2.2 Extensions vs. asymptotic morphisms

Important examples of asymptotic morphisms arise via extensions. In fact, there is a very close connection between asymptotic morphisms and extensions of C^*-algebras which will be explained in this part. For the main statements in this part it is important to assume that all C^*-algebras in question are separable. Let us start with the definition of the concept of extension (compare also appendix C.3).

Definition 2.2.1 *Let* A, B *and* C *be* C^**-algebras. An* extension *of* C *by* A *is a diagram of the form*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

where $A \xrightarrow{f} B$ *is a kernel of* $B \xrightarrow{g} C$ *and* $B \xrightarrow{g} C$ *is a co-kernel of* $A \xrightarrow{f} B$*. (This is equivalent to saying that* f *is injective,* $f(A)$ *is a closed 2-sided ideal in* B*, the composition* $g \circ f$ *is equal to zero and the induced map* $B/f(A) \xrightarrow{g} C$ *is an isomorphism of* C^**-algebras.)*

There is a manifoldness and ambiguity in the class of extensions. Indeed, the class of extensions between two algebras does not even form a set. In order to rectify this ambiguity we want to introduce a notion of isomorphism and homotopy of extensions (the treatment is taken from [34]). The equivalence classes with respect to those relations do form sets.

Definition 2.2.2 *Let*

$$0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$$

and

$$0 \longrightarrow A \longrightarrow C' \longrightarrow B \longrightarrow 0$$

be extensions of B *by* A.

- *The extensions above are called isomorphic, if there exists a commutative diagram as follows.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & C' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

  (Note that the morphism $f : C \to C'$ *clearly has to be an isomorphism. Thence the relation is an equivalence relation.)*

- *The extensions above are called homotopic, if there is an extension*

$$0 \longrightarrow A[0,1] \longrightarrow C'' \longrightarrow B \longrightarrow 0$$

  such that there exists a commutative diagram of the following form.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & B & \longrightarrow & 0 \\ & & {\scriptstyle ev_0}\uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & A[0,1] & \longrightarrow & C'' & \longrightarrow & B & \longrightarrow & 0 \\ & & {\scriptstyle ev_1}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & C' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Remark 2.2.3 *The notion of homotopy we are using in this context is called 'weak homotopy' in [34] but we want to omit the word 'weak', since we are not going to use any stronger notions.*

Lemma 2.2.4 *The relation of homotopy of extensions is an equivalence relation.*

Proof: Given two composable homotopies, we obtain a diagram with five rows. We can take the pull-back of each column of the middle three rows. One easily checks that the resulting middle row is again exact and, since $A \oplus_A A \cong A$ and $B[0,1] \oplus_B B[0,1] \cong B[0,1]$, this proves the transitivity of the relation. Symmetry follows by the existence of the following extension.

$$0 \longrightarrow A[0,1] \longrightarrow C[0,1] \oplus_{B[0,1]} B \longrightarrow B \longrightarrow 0$$

(The pull-back in the middle is taken with respect to the canonical map $C[0,1] \to B[0,1]$ and the inclusion of the constant functions $B \to B[0,1]$.) The constructed extension maps to

$$0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$$

either by evaluating at 0 or 1. Reflexivity is obvious. (An alternative proof can be found in [34].)

Definition 2.2.5 *Let A and B be separable C^*-algebras. Denote by $\mathrm{ext}(A,B)$ the set of homotopy classes of extensions of B by A. (Note that this is indeed a set, since the category of separable C^*-algebras is skeletally small and isomorphic extensions are, in particular, homotopic.)*

Definition 2.2.6

- *Denote by $\Sigma : s\mathcal{C} \to s\mathcal{C}$ the functor which is given by suspension (i.e. $\Sigma(A) = C(S^1, 1; A)$, see definition C.1.5).*
- *Denote by $t : \Sigma \to \Sigma$ the natural transformation which is given by the twist of the suspension (i.e. the map which is induced by complex conjugation on $S^1 \subset \mathbb{C}$).*
- *Let*

  $$0 \longrightarrow A \xrightarrow{g} B \xrightarrow{h} C \longrightarrow 0$$

  be an extension. We call the extension

  $$0 \longrightarrow \Sigma(A) \xrightarrow{t\ \ (f)} \Sigma(B) \xrightarrow{(h)} \Sigma(C) \longrightarrow 0$$

  the twisted suspended extension.
- *Denote by $\Sigma : \mathrm{ext}(A,B) \to \mathrm{ext}(\Sigma(A), \Sigma(B))$ the map which assigns to an extension the twisted suspended extension. (One easily checks that suspending is compatible with homotopy.)*

Our next aim is to relate $\mathrm{ext}(A,B)$ to $[[B,A]]$. We follow the lines of K. Thomsen and T.G. Houghton-Larsen in [34] and introduce, first of all, a procedure of assigning an asymptotic morphism to an extension and, secondly, a procedure which is inverse up to suspension.

Following N. Higson and A. Connes in [13] we can naturally associate to any homotopy class of extensions represented by, say

$$0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$$

a homotopy class of asymptotic morphisms $[[\alpha]] : \Sigma(B) \to A$ such that the homotopy class depends only on the homotopy class of the extension and induces the connecting morphism in the long exact sequence that arises in homology theories such as **K**-theory. We want to recall this construction, since it is central for the theory.

Lemma 2.2.7 (Connes-Higson-Construction) *Let*

$$0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$$

be an extension of separable C^**-algebras. We can assign to it a* $*$*-homomorphism* $\gamma : \Sigma(B) \to \mathfrak{a}(A)$*, such that the asymptotic homotopy class of* γ *does not depend on the choice of the representative in the homotopy class of extensions given by the extension.*

The statement of the lemma is a little vague, since we cannot formulate the interesting properties of this assignment yet. They will become clear later. The statement of the lemma ensures only that it is well-defined for our purposes.

Proof: Choose a continuous approximate unit of A which is quasi-central in C. More precisely we choose a ray $u : [0,1) \to A$ such that

$$\lim_{t \to 1} \|x - u_t x\| = 0$$

for all $x \in A$ and

$$\lim_{t \to 1} \|c u_t - u_t c\| = 0$$

for all $c \in C$. (Such a ray can be obtained by linear interpolation of a quasi-central approximate unit. The existence of quasi-central approximate units is classical and proved in [47].) We define γ by

$$\gamma_t(f \otimes b) = f(u_t)s(b)$$

for $f \in \Sigma$ and $b \in B$ and where $s : B \to C$ is any continuous bounded split of the surjection from C to B. (Such splits exist by the theorem of Bartle-Graves (see C.1.14).) By the properties of the approximate unit, the continuous family (γ_t) induces a $*$-homomorphism $\Sigma \otimes_{\mathrm{alg}} B \to \mathfrak{a}(C)$ which can be extended to a $*$-homomorphism from the full tensor product (see e.g. theorem C.4.3). Furthermore, its homotopy class is independent of the choice of the split s and independent of the choice of the approximate unit. Indeed, the set of choices was convex in both cases.

A homotopy of extensions defines a homotopy of asymptotic morphisms in an obvious way. This proves the homotopy invariance and hence finishes the proof. Full details can be found in [28] or [20].

The existence of quasi-central approximate units was crucial in the last proof. The proof of its existence relies on the separability of the algebras in question. This is the only but crucial and indispensable step where we really need the separability condition.

Definition 2.2.8 *We denote the assignment of a homotopy class of asymptotic morphisms to any extension which was established in the last lemma by*

$$\lambda_{A,B} : ext(A, B) \to [[\Sigma(B), A]].$$

Now, we come to a sort of inverse procedure. It is not quite an inverse, but it is an inverse up to suspension as will become clear in the next theorem. We associate to any asymptotic morphism $\beta : A \to B$ an extension

$$0 \longrightarrow \Sigma(B) \longrightarrow E \longrightarrow A \longrightarrow 0$$

such that the homotopy class of the extension only depends on the homotopy class of the asymptotic morphisms. This result appears as theorem 5.12 in [28]. We give a short argument, since the construction is central for the construction of stable homotopy theory.

Lemma 2.2.9 *Let A and B be separable C^*-algebras and let $\alpha : B \to \mathfrak{a}(A)$ be an asymptotic morphism from B to A. There is a short exact sequence*

$$0 \longrightarrow \Sigma(A) \longrightarrow E \longrightarrow B \longrightarrow 0$$

with

$$E = \{(b, f) \in B \oplus A_b([0, 1), 0) : \alpha(b) = q(f) \in \mathfrak{a}(A)\} = B \oplus_{\mathfrak{a}(A)} A_b([0, 1), 0)$$

where $q : A_b([0, 1), 0) \to \mathfrak{a}(A)$ denotes the canonical quotient map. The algebra E is separable. Furthermore, the homotopy class of the extension depends only on the homotopy class of the asymptotic morphism.

Proof: The exactness of the sequence follows from the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma(A) & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Sigma(A) & \longrightarrow & A_b([0,1),0) & \xrightarrow{q} & \mathfrak{a}(A) & \longrightarrow & 0 \end{array}$$

Note that the right hand square is a pull-back by definition of E. This implies the exactness of the upper sequence. The algebra E is separable, since B and $\Sigma(A)$ are separable. To see this choose inverse images $\{b_n, n \in \mathbb{N}\}$ of a countable dense subset of B. Let $\{a_n, n \in \mathbb{N}\}$ be a countable dense subset of $\Sigma(A)$. The set $\{a_n + b_m, (n, m) \in \mathbb{N} \times \mathbb{N}\}$ is dense in E by a standard argument using the definition of the quotient norm.

Let $h : B \to a(A[0,1])$ be a homotopy such that $h_0 = \alpha$. Since our construction of an extension is functorial, we get a commutative diagram as follows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma(A) & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow {\scriptstyle ev_0} & & \uparrow & & \| & & \\
0 & \longrightarrow & \Sigma(A[0,1]) & \longrightarrow & E_h & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle ev_1} & & \downarrow & & \| & & \\
0 & \longrightarrow & \Sigma(A) & \longrightarrow & E_{h_1} & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

This establishes the existence of the required homotopy and hence finishes the proof.

Definition 2.2.10 *We denote the assignment of a homotopy class of extensions to an asymptotic morphism which was established in the last lemma by*

$$\mu_{A,B} : [[B, A]] \to \mathrm{ext}(\Sigma(A), B).$$

Composing these two constructions we end up essentially with a suspended asymptotic morphism or a twisted suspended extension as will be made precise in the next theorem. A proof of this can be found in [34].

Theorem 2.2.11 (K. Thomsen, L. Houghton-Larsen in [34]) *Let* A *and* B *be separable* C^**-algebras. Let*

$$\lambda_{A,B} : \mathrm{ext}(A, B) \to [[\Sigma(B), A]]$$

and

$$\mu_{A,B} : [[B, A]] \to \mathrm{ext}(\Sigma(A), B)$$

be the maps as defined before.

The following two diagrams are commutative.

$$\begin{array}{ccc}
\mathrm{ext}(A,B) & \xrightarrow{\ A,B\ } & [[\Sigma(B), A]] \\
\downarrow & \swarrow {\scriptstyle A,\ (B)} & \\
\mathrm{ext}(\Sigma(A), \Sigma(B)) & &
\end{array}
\qquad
\begin{array}{ccc}
 & & [[B, A]] \\
 & \swarrow {\scriptstyle A,B} & \downarrow \\
\mathrm{ext}(\Sigma(A), B) & \xrightarrow{\ (A),B\ } & [[\Sigma(B), \Sigma(A)]]
\end{array}$$

This theorem is absolutely crucial to most of the constructions in the first sections of this thesis. It allows to identify asymptotic morphisms between separable algebras and 1-step extensions. It helped us to recognize how important extensions are and how they can be used to construct universal bivariant homology theories. Note that the notion of extension is much more general than the notion of asymptotic morphism. To make the theory of asymptotic morphisms work one uses very special properties of separable C^*-algebras (i.e. the existence of quasi-central approximate units etc.). The insight gained from this theorem has led to the definition of the notion of extension categories (see appendix C.3). The possibility of constructing an extension category is not particular to the category of separable C^*-algebras. First of all, it immediately applies to all sorts of topological algebras but also to completely different situations. See the work of J. Cuntz in [18], where he uses similar constructions and arguments which motivated our definition of extension category very much. We hope that it has fruitful applications not only in the context of operator algebras.

We continue with some technical lemmas which are important if one wants to understand how the assignment $\lambda_{?,?}$ behaves in standard situations. They were proved by M. Dădărlat in [20].

Lemma 2.2.12 • *Let*

$$0 \longrightarrow \Sigma(A) \longrightarrow c(A) \xrightarrow{ev_1} A \longrightarrow 0$$

be the cone extension of A *(see definition C.1.6). Its image under* $\lambda_{\ (A),A}$: $\mathrm{ext}(\Sigma(A), A) \to [[\Sigma(A), \Sigma(A)]]$ *is the class of* $\mathrm{id}_{\ (A)}$.

- *Consider extensions* γ *of* C *by* A *and* γ' *of* C' *by* A. *Suppose there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle f} & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

with f *injective. Then* $\lambda_{A,C'}(\gamma') = \lambda_{A,C}(\gamma) \circ \Sigma(f)$ *as elements in* $[[\Sigma(C'), A]]$

- *Consider extensions* γ *of* C *by* A *and* γ' *of* C *by* A'. *Suppose there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle f}\uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

with f *injective. Then* $\lambda_{A,C}(\gamma) = f \circ \lambda_{A',C}(\gamma)$ *as elements in* $[[\Sigma(C), A]]$.

Proof: The proofs are not difficult and can be found in [20].

Having provided these technical properties we are going to prove an important lemma which is frequently used in further constructions. Let us start with a definition.

Definition 2.2.13 *Let* $f : A \to B$ *be a homomorphism. The cone of* f *which will be denoted by* $c(f)$ *is given by the pull-back* $c(B) \oplus_B A$ *(see definition C.1.6) along the obvious maps. Note that there is always a canonical map* $\ker(f) \to c(f)$ *given by* $a \mapsto (0, a)$.

The next lemma will make apparent why we use asymptotic morphisms. The construction of long exact sequences in bivariant homology theories falls into two parts. First of all, one has to construct the mapping cone sequence. Its exactness is obvious in most of the situations. Secondly, one has to identify the cone of a surjection with its kernel. In **KK**-theory this was achieved for surjections with completely positive split by J. Cuntz and G. Skandalis in [19]. The next lemma shows that the required statement is true in the asymptotic homotopy category after suspending.

Lemma 2.2.14 *Let* $f : A \to B$ *be a surjective homomorphism. The canonical map*

$$\pi : \ker(f) \to c(f)$$

is an asymptotic homotopy equivalence after suspending once.

Proof: We have to construct an inverse of $\Sigma(\pi)$. Let ψ be the asymptotic morphism that corresponds to the following extension.

$$0 \longrightarrow \Sigma(\ker(f)) \longrightarrow c(A) \longrightarrow c(f) \longrightarrow 0$$

It is an asymptotic left inverse up to homotopy of $\Sigma(\pi)$ by the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma(\ker(f)) & \longrightarrow & c(A) & \longrightarrow & c(f) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Sigma(\ker(f)) & \longrightarrow & c(\ker(f)) & \longrightarrow & \ker(f) & \longrightarrow & 0
\end{array}$$

and lemma 2.2.12 which shows that $\psi \circ \Sigma(\pi) = \mathrm{id}_{\ker(f)}$.

To show that ψ is also a right inverse consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma(c(f)) & \longrightarrow & c(c(f)) & \longrightarrow & c(f) & \longrightarrow & 0 \\
 & & (\ )\uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \Sigma(\ker(f)) & \longrightarrow & c(A) & \longrightarrow & c(f) & \longrightarrow & 0
\end{array}$$

where $c(A) \to c(c(f))$ takes a_t to $(f(a_{s+t-st}), a_t)$. Again, the claim follows from lemma 2.2.12 which shows that $\Sigma(\pi) \circ \psi = \mathrm{id}_{c(f)}$. This finishes the proof.

The last lemma is the only purpose of introducing asymptotic morphisms, since it allows to construct a suitable homotopy category without having the notion of cofibration.

In the category of compactly generated spaces there is a distinguished class of maps which are called cofibrations. Cofibrations (depending on the model structure one chooses) are essentially retractions of cellular extensions and they are always closed embeddings and hence correspond to certain surjections of the associated algebras of functions. Since most interesting objects in algebraic topology can be made out of cells by gluing them together, this is a sufficiently rich class of maps. The properties of cofibrations suffice to prove that the exact analogue of lemma 2.2.14 holds in the category of compactly generated spaces. The analogue statement is that the cone of a cofibration identifies up to homotopy with the quotient. We are getting this result in the asymptotic homotopy category only after suspending once but this will be sufficient for our purposes, since we are considering only stable phenomena.

Since there is no obvious analogue of the notion of a cofibration for C^*-algebras and the approach of cellular extensions seems to be inappropriate for several reasons, one has to find a suitable way around this lack of structure. The method proposed by A. Connes and N. Higson enlarges the set of morphisms so that there exists the required homotopy equivalence for any surjection after suspending once. This is about the best thing that could possibly happen.

The next theorem which was proved by M. Dădărlat shows that although we have enlarged the class of morphisms we did not change the homotopy category of finite CW-complexes.

Lemma 2.2.15 *Let (X, x) and (Y, y) be compact Hausdorff pointed topological spaces and let, furthermore, X be an ANR, locally contractible at x. The canonical map*

$$[Y, X]_+ \to [[C(X, x), C(Y, y)]]$$

is an isomorphism.

Proof: Let $\phi : C(X, x) \to \mathfrak{a}(C(Y, y))$ be a $*$-homomorphism. By lemma 2.1.6, there exists a map family of maps $\psi_t : Y \to M(X)$ which approaches X (here, $M(X)$ denotes the space of probability measures on X equipped with the weak-$*$-topology). Since X is an ANR, we can find a retraction of a neighborhood of X inside $M(X)$ (the embedding being the one given by the point measures). The image of Y will lie inside this neighborhood for sufficiently large values of the parameter t by compactness of Y. Composing with the retraction gives a continuous map $Y \to X$ which is not necessarily pointed but the image of y is close to x. Since X is locally contractible at x, we can retract a neighborhood of y to y in order to get a pointed map from Y to X. This proves surjectivity. Injectivity follows by a relative argument. This finishes the (outline of the) proof.

A more general statement which we need later is the following.

Lemma 2.2.16 *Let A be a C^*-algebra and (X, x) be a compact Hausdorff pointed topological space. If $\hom(A, M_n)$ is an ANR, then the canonical map*

$$[A, M_n(C(X,x))] \to [[A, M_n(C(X,x))]]$$

is an isomorphism.

Proof: The proof of this statement goes along the lines of the proof of lemma 2.2.15. It can be found in detail in the work of M. Dădărlat [20].

The last lemmas are also of some importance, since they allow the computation of some sets or groups of homotopy classes of asymptotic morphisms by classical means. In general, our scheme of the construction of an interesting bivariant homology theory falls into two parts. First of all one has to construct it and to show that is satisfies certain abstract properties, maybe even a universal property. Secondly, one has to compute at least the coefficients to give a geometric interpretation of the theory. The computation of the coefficients of stable homotopy and connective **E**-theory for C^*-algebras relies on the last lemmas, since they allow to reduce the questions about homotopy classes of asymptotic morphisms to questions about homotopy classes of ordinary $*$-homomorphisms.

The next part will answer some questions that might have occurred after the last comments. It will give definitions of the main object of study in this thesis.

We finish this part by providing another result about extensions. According to lemma 2.4 in [28] the functor $a : \mathcal{C} \to \mathcal{C}$ is exact. This result is recalled in the following proposition which we do not want to prove here.

Proposition 2.2.17 *The functor* $a : \mathcal{C} \to \mathcal{C}$ *is exact. This means that for every extension*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

the diagram

$$0 \longrightarrow a(A) \xrightarrow{a(f)} a(B) \xrightarrow{a(g)} a(C) \longrightarrow 0$$

is also an extension. For a fixed extension γ *we denote the application of the functor* a *to it by* $a(\gamma)$. *(Note that this assignment is in general not compatible with the notion of homotopy of extensions.)*

2.3 What is a bivariant theory ?

In this part we want to introduce a partially new aspect of bivariant homology theories on the category of separable C^*-algebras such as **E**-theory, **KK**-theory and so on. In the following definitions we want to abstract the properties from those theories which make the algebraic machinery work. There have been other definitions, and we do not claim any originality of the concepts.

We begin this part with a definition of what we understand as a bivariant (resp. triangulated) homology theory. For the definition and properties of triangulated categories see appendix A.1.

Definition 2.3.1 *A* <u>*bivariant homology theory*</u> *on* $s\mathcal{C}$ *is a functor* $H : s\mathcal{C} \to (R, \Sigma)$ *into an additive category* R *together with an automorphism* $\Sigma : R \to R$ *which satisfies the following conditions.*

- *The functor* H *is homotopy-invariant (i.e.* f *and* g *homotopic implies* $H(f) = H(g)$*).*
- *For any extension*

  $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

  and any $D \in ob(R)$ *the following induced sequences of Abelian groups are exact.*

  $$hom_R(D, H(A)) \to hom_R(D, H(B)) \to hom_R(D, H(C))$$

  $$hom_R(H(C), D) \to hom_R(H(B), D) \to hom_R(H(A), D)$$

- *There is a natural isomorphism of functors* $H \circ \Sigma \cong \Sigma \circ H$*. A choice of this isomorphism is part of the data of a bivariant homology theory. (We want to omit the explicit mentioning of this isomorphism in many contexts although it might not be the identity.)*

Remark 2.3.2 *The second property is usually referred to as 'excision' or 'half-exactness' (see e.g. [7]).*

Definition 2.3.3 *A* <u>*triangulated homology theory*</u> *on* $s\mathcal{C}$ *is a functor* $H : s\mathcal{C} \to (T, \Sigma^{-1})$ *into a triangulated category* (T, Σ^{-1}) *such that the following conditions are satisfied.*

- *The functor* H *is homotopy invariant.*
- *For any extension*

  $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

  there is a natural choice of a map $\alpha : \Sigma(H(C)) \to H(A)$ *such that the triangle*

  $$\Sigma(H(C)) \longrightarrow H(A) \xrightarrow{H(f)} H(B) \xrightarrow{H(g)} H(C)$$

  is distinguished. (Naturality of the choice means naturality with respect to mappings of extensions.)

One might wonder why the triangulated structure is defined with respect to the inverse of Σ. This is of course pure convention. We have chosen this apparently annoying variant, since it prevents us from having more trouble later. Note, for example, that the category of finite CW-complexes sits contravariantly inside $s\mathcal{C}$. Hence one expects a contravariant functor from the stable homotopy category of finite CW-complexes into a triangulated homology theory of C^*-algebras. Contravariant functors between triangulated categories indeed make sense, since the opposite of a triangulated category is indeed triangulated but with the inverse automorphism.

Proposition 2.3.4 *Any triangulated homology theory is, in particular, a bivariant homology theory.*

Proof: Note that the exactness properties of a bivariant homology theory are automatically satisfied if the conditions in the preceding definition of a triangulated homology theory hold, since the hom-groups of a triangulated category T are homological resp. co-homological (see appendix 2 for definitions) with respect to distinguished triangles.

It remains to provide a natural transformation. Given a C^*-algebra A, there is a natural extension

$$0 \longrightarrow \Sigma(A) \longrightarrow c(A) \xrightarrow{ev_1^A} A \longrightarrow 0$$

(see definition C.3.9 in the appendix) which gives rise to a natural map $\Sigma(H(A)) \to H(\Sigma(A))$ which is an isomorphism, since $c(A)$ is contractible and H is a homotopy invariant functor. This finishes the proof.

A bivariant (resp. triangulated) homology theory can also be defined with respect to a distinguished class of extensions in an obvious way. Here, the class of semi-split extensions is the most important one to mention. It would lead to a stable homotopy category which is closer related to KK-theory. In fact, also the parallel theory of asymptotic morphisms has an analogue. It could be replaced by the theory of 'completely positive' asymptotic morphisms as defined in [34]. We do not follow this line of arguments here.

It is also obvious that the concept of a bivariant (resp. triangulated) homology theory can be used in similar situations, since it is not using special properties of C^*-algebras at all.

Remark 2.3.5 *Most of the time the objects of a bivariant (or triangulated) homology theory will be separable C^*-algebras or pairs of a separable C^*-algebra and an integer. In both cases we do not distinguish between the algebra and its image in the bivariant (or triangulated) homology theory for simplicity.*

We will show that most of the known bivariant homology theories on $s\mathcal{C}$ are in fact naturally triangulated. Moreover any bivariant homology theory on $s\mathcal{C}$ is a triangulated module (see appendix 2 for a definition) over a triangulated homology theory on $s\mathcal{C}$.

Definition 2.3.6 *Let* T *be a triangulated homology theory. We call the bivariant homology theory which is given by forgetting the triangulated structure the* *associated* *bivariant homology theory.*

Definition 2.3.7 *Let* (R, Σ) *be a bivariant homology theory. We use the notation*

$$R_n(A, B) = \hom_R(A, \Sigma^n(B)).$$

Note that this definition enriches the category R *over* $\mathbb{Z}$*-graded Abelian groups (i.e. we can think of the category* R *as a category with morphisms* $\mathbb{Z}$*-graded Abelian groups* $R_*(A, B)$ *rather than just Abelian groups* $\hom_R(A, B) = R_0(A, B)$*).*

Now, we are going to remind the reader about the consequences of the definition of a bivariant homology theory. Theorems in this spirit can be found in articles by C. Schochet [57, 58] and by J. Cuntz and G. Skandalis [19].

Theorem 2.3.8 *Let* (R, Σ) *be a bivariant homology theory. Furthermore, let* A *be a separable* C^**-algebra and*

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

be an extension of separable C^**-algebras. The following sequences are exact.*

$$\cdots \longrightarrow R_{n+1}(A, D) \longrightarrow R_n(A, B) \longrightarrow R_n(A, C) \longrightarrow R_n(A, D) \longrightarrow \cdots$$

$$\cdots \longrightarrow R_{n+1}(B, A) \longrightarrow R_n(D, A) \longrightarrow R_n(C, A) \longrightarrow R_n(B, A) \longrightarrow \cdots$$

Proof: The proof is based on a classical observation about the homotopy type of iterated mapping cones. Observe that the mapping cone of a surjection is isomorphic to the kernel of the surjection in the category R. The proof of this statement is analogous to the proof of lemma 2.2.14. This finishes the proof.

In order to construct long exact sequences one has to identify the cone of a surjection with its kernel. This is not difficult, if one requires the exactness properties (see the proof of lemma 2.2.14 in the preceding section or proposition 21.4.1 and 21.4.1 in [7]). However, the situation is usually a different one. Usually, one has to prove the identification above in order to get the exactness property of the bivariant homology theory. See also the remarks before and after lemma 2.2.14.

In the construction of the long exact sequence it is important to fix a cone and hence a mapping cone construction. We fix the one which comes from the cone $C([0, 1], 0)$. Taking the other choice gives almost the same sequence but with different signs. Our choice ensure that the cone extensions associated to $C([0, 1], 0)$ gives rise to the identity as boundary map as one easily checks.

The existence of long exact sequences is of course of great importance in the computation of certain groups.

Definition 2.3.9 *If one of the entries in the bivariant or triangulated homology theory is just $\mathbb{C}$ we are using the following convenient notations.*

$$R_*(A) = R_*(\mathbb{C}, A)$$

$$R^*(A) = R_*(A, \mathbb{C})$$

The groups are called R*-homology resp.* R*-cohomology groups.*

We are also going to consider homology or co-homology theories which do not necessarily come from a bivariant homology theory. Denote the category of $\mathbb{Z}$-graded Abelian groups by Ab_*. We denote the shift functor which assigns to any graded Abelian group $(A_n)_{n\in\mathbb{N}}$ the shifted group $(A_{n+1})_{n\in\mathbb{N}}$ by $\Sigma : Ab_* \to Ab_*$.

Definition 2.3.10 *A homology theory on the category of separable* C^**-algebras is a homotopy invariant functor* $H : s\mathcal{C} \to Ab_*$ *into the category of* $\mathbb{Z}$*-graded Abelian groups such that for any extension*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

- *there is a natural (with respect to mappings of extensions) morphism*

  $$\Sigma(H(C)) \to H(A)$$

  of graded Abelian groups and

- *the sequence of graded Abelian groups*

  $$\Sigma(H(C)) \to H(A) \to H(B) \to H(C)$$

  is exact at $H(A)$ *and* $H(B)$.

There are other definitions but they are all equivalent and it is maybe possible to weaken the condition even further. The example that one should have in mind is **K**-theory or maybe **K**-theory with coefficients in some finitely generated Abelian group. Again, the obvious long exact sequences of Abelian groups can be constructed. The definition of co-homology theory is dual.

Definition 2.3.11 *A bivariant or triangulated homology theory or a homology theory is said to satisfy*

- *matrix stability, if the canonical rank* 1 *inclusion into the upper left corner*

  $$id_A \otimes \iota_2 : A \to M_2A$$

  induces an isomorphism.

- *stability, if the canonical rank* 1 *inclusion into the upper left corner*

$$\mathrm{id}_A \otimes \iota_\infty : A \to A \otimes K$$

  induces an isomorphism.

The next proposition states a standard fact about the existence of Mayer-Vietoris sequences. A neat proof of the following proposition in a general context was given by R. Meyer in [40].

Proposition 2.3.12 *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{k} & D \end{array}$$

be a pull-back square in the category of separable C^**-algebras and let* $H : s\mathcal{C} \to Ab_*$ *be a homology theory. If the* $*$*-homomorphism* $h : B \to D$ *is surjective then there exists a boundary map* $\delta : \Sigma H(D) \to H(A)$ *such that the following Mayer-Vietoris sequence is exact.*

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(B) \oplus H_n(C) \longrightarrow H_n(D) \xrightarrow{n-1} H_{n-1}(A) \longrightarrow \cdots$$

The morphisms in the sequence are (up to sign) the ones which are induced by the pull-back diagram. Furthermore, the Mayer-Vietoris sequence is natural with respect to mappings of pull-back diagrams.

Most of the examples of bivariant or triangulated homology theories carry an additional structure which comes from the maximal tensor product of C^*-algebras. The abstract setting is fixed in the next definition.

Definition 2.3.13 *We speak of monoidal bivariant homology theories* $H : s\mathcal{C} \to (R, \Sigma)$ *(or monoidal triangulated homology) theories if the respective categories are symmetric monoidal (or triangulated monoidal (see definition A.1.5 in the appendix)) and the functors preserve the symmetric monoidal structure.*

This means that there are natural isomorphisms in the category R.

$$H(A) \otimes H(B) \to H(A \otimes B)$$

which are compatible with the symmetry and associativity isomorphisms of the respective symmetric monoidal structures.

There are several notions of monoidal triangulated categories (see e.g. P. May in [38]). We only require a naive notion of compatibility which is explained in definition A.1.5 in the appendix. Our main examples turn out to satisfy the stronger requirements stated in [38].

It often happens that bivariant homology theories appear naturally as modules over triangulated homology theories (see definition A.1.10). (Later, we will see that this is always the case.) In the next definition and proposition we give an explicit construction of such bivariant homology theories arising from monoid objects. We abbreviate $A \otimes B$ by AB.

Definition 2.3.14 *Let* T *be a symmetric monoidal category with unit* U*. An object* A *together with morphisms* $m : A \otimes A \to A$ *and* $\iota : U \to A$ *is called a* *monoid object* *if*

- $m \circ (m \otimes id_A) = m \circ (id_A \otimes m) : A^3 \to A$
- $m \circ (\iota \otimes id_A) = m \circ (id_A \otimes \iota) = id_A : A \to A$

The morphisms are called *multiplication* *and* *unit* *respectively.*

Proposition 2.3.15 *Let* (T, Σ) *be a triangulated monoidal category and* A *be a monoid object in* T*. There is an associated triangulated module* (T^A, Σ) *with* $T^A(U, U) \cong T(U, A)$ *and a functor* $(T, \Sigma) \to (T^A, \Sigma)$.

Proof: Define T^A to be a category with the same objects as T. Define

$$hom_{T^A}(B, C) =_{def} hom_T(B, CA)$$

and let the composition product be given as follows.

$$hom_{T^A}(B, C) \times hom_{T^A}(C, D) =_{def} hom_T(B, CA) \times hom_T(C, DA) \stackrel{id \times ? \otimes A}{\longrightarrow}$$

$$hom_T(B, CA) \times hom_T(CA, DA^2) \to hom_T(B, DA^2) \stackrel{(id_D \otimes m)_*}{\longrightarrow} hom_T(B, DA)$$

$$=_{def} hom_{T^A}(B, D)$$

The identities in T^A are given by tensoring the identities in T with the unit of the multiplication. The associativity of the composition follows from the associativity of the multiplication of the monoid object. The unit property of the identities follows from properties of the unit of the multiplication. The functor $T \to T^A$ is given by the tensor product with the unit. The properties are now obvious. This finishes the proof.

It is now clear that every ring spectrum - seen as an object in the stable homotopy category of spectra (see appendix B for definitions) - gives rise to a bivariant homology theory. On the other hand it follows from the representability theorem of E.H.

Brown and its variants (see e.g. [37]) that any bivariant homology theory on finite CW-complexes can be obtained from a ring spectrum. We do not want to give precise statements and just refer to the classical sources (e.g. [65, 37]). The representability theorem of E.H. Brown says that all co-homology theories on the category of spectra are in some sense inner (i.e. represented by a spectrum). This is far from being true for the category of finite CW-complexes. Theories such as $\mathbf{K}$-theory or even singular cohomology cannot be represented by a finite complex.

The situation for C^*-algebras is in some sense a little better. It will turn out that theories such as connective $\mathbf{K}$-theory, $\mathbf{K}$-theory or singular cohomology can be easily described using homotopy classes of (asymptotic) morphisms and certain representing C^*-algebras like M_n or the algebra of compact operators on a separable Hilbert space. We refer to section 4.5 for precise statements of the results.

3 Stable homotopy theory

In this section we want to define stable homotopy theory for separable C^*-algebras as a bivariant homology theory. Furthermore, we want to show that it is the universal bivariant homology theory and in fact triangulated with respect to a natural class of distinguished triangles (see definitions 2.3.1, 2.3.3 and appendix A.1).

First, we recall the original approach and the approach of K. Thomsen and G.T. Houghten-Larsen (see [34]) which is more conceptional and explains the universal properties. Thirdly we give a new picture using the concept of extension categories (see appendix C.3). This picture simplifies the proof of the universal properties even further and is used in the main theorem.

Having introduced stable homotopy theory we want to show that it is the universal triangulated homology theory on the category of separable C^*-algebras. These results seem to be partially folklore or considered to be abstract nonsense. We want to give complete proofs, since these results have appeared to our knowledge nowhere in the literature and are used in the subsequent sections.

3.1 Definition

The theory of triangulated categories was developed in order to apply to the case of CW-complexes and stable maps of those. It is therefore natural to look at the opposite of the category of separable C^*-algebras rather than at the category itself. We do not want to take this point of view, since there is also some tradition of bivariant theories of algebras.

Definition 3.1.1 *Let A and B be separable C^*-algebras. We define the stable homotopy classes of morphisms from A to B to be the Abelian group*

$$\{A, B\} = \mathrm{colim}_n[[\Sigma^n A, \Sigma^n B]].$$

The co-limit is taken with respect to the suspension of asymptotic morphisms as follows

$$\hom(\Sigma^n(A), \mathfrak{a}(\Sigma^n B)) \overset{\mathrm{id}\otimes ?}{\to} \hom(\Sigma^{1+n}(A), \Sigma(\mathfrak{a}(\Sigma^n B))) \to \hom(\Sigma^{1+n}(A), \mathfrak{a}(\Sigma^{1+n} B)).$$

Remark 3.1.2 *The set $\{A, B\}$ of homotopy classes of asymptotic morphisms carries the natural structure of an Abelian group and composition distributes over addition.*

Proof: This follows essentially from the existence of the pinching map $\Sigma \oplus \Sigma \to \Sigma$ and the observation that the functor $s\mathcal{C} \to \mathfrak{a}s\mathcal{C}$ preserves finite products (see lemma 2.5 in [28]).

This definition was first given independently by A. Connes and M. Dădărlat in analogy to the classical case. The crucial difference to the classical case was that they were considering homotopy classes of asymptotic morphisms in order to gain the desired exactness properties. By lemma 2.2.15, we recover the classical definition of the stable homotopy groups for finite CW-complexes which was first given by E.H. Spanier and J.H.C. Whitehead in the early 50s. The aim of these notes is to generalize some aspects of classical stable homotopy theory to the realm of non-commutative geometry.

In the context of C^*-algebras we refer to the description of stable homotopy given in definition 3.1.1 as the standard picture.

Proposition 3.1.3 *There is an additive category $\underline{S}$ with objects separable C^*-algebras and morphisms stable homotopy classes of asymptotic morphisms. There is a functor $s : s\mathcal{C} \to \underline{S}$ which sends a $*$-homomorphism to its stable homotopy class.*

Proof: For the additivity of $\underline{S}$ it remains to show that there exist products. Note that this is obvious, since the product $\oplus$ commutes with suspension. Again, we use that the product in $as\mathcal{C}$ coincides with the one in $s\mathcal{C}$ (lemma 2.5 in [28]).

Definition 3.1.4 *Define S to be the category with objects pairs (A, n) - where A is a separable C^*-algebra and n is an integer - and morphisms $S((A,n),(B,m)) = \{\Sigma^{p+n}A, \Sigma^{p+m}B\}$ for p sufficiently large.*

Note that

$$S((A,n),(B,m)) =_{\text{def}} \operatorname{colim}_p[[\Sigma^{p+n}A, \Sigma^{p+m}B]]$$

canonically with the obvious composition product.

The next proposition and remark clarify the relation between S and $\underline{S}$.

Proposition 3.1.5

- *There is an additive automorphism $\Sigma : S \to S$ which maps $(A,n) \mapsto (A, n+1)$.*
- *The functor $\Phi : (A,n) \mapsto (\Sigma(A), n-1)$ is naturally equivalent to id_S. (We will use this equivalence implicitly but not mention it explicitly in most contexts.)*

Proof: It is obvious that $\Sigma : S \to S$ defines an additive automorphism. Its inverse Σ^{-1} maps (A,n) to $(A, n-1)$.

In order to prove the second claim we show that there are natural transformations $\alpha : \mathrm{id}_S \to \Phi$ and $\beta : \Phi \to \mathrm{id}_S$ which are inverse to each other. To define α and β consider

$$\alpha : S((A,n),(B,m)) =_{\text{def}} \operatorname{colim}_p[[\Sigma^{p+n}A, \Sigma^{p+m}B]] \cong$$

$$\operatorname{colim}_p[[\Sigma^{p+n-1}\Sigma A, \Sigma^{p+m-1}\Sigma B]] =_{\text{def}} S((\Sigma A, n-1),(\Sigma B, m-1)) : \beta.$$

Since the isomorphism is canonical, α and β define natural transformations and are inverse to each other by definition. This finishes the proof.

Remark 3.1.6 *The preceding proposition shows that the category* S *differs from the category* $\underline{S}$ *essentially only with respect to the invertibility of the suspension (as a functor). The functor* $\Sigma : S \to S$ *is invertible,* $\iota : \underline{S} \to S$ *is a full and faithful inclusion and we have a natural equivalence of functors* $\Sigma \circ \iota \cong \iota \circ \Sigma$*. (It is obvious that* S *is universal with respect to these properties.)*

Proposition 3.1.7 *The categories* $\underline{S}$ *(resp.* S*) are symmetric monoidal with the maximal tensor product (resp. a canonical extension of the maximal tensor product to S according to the assignment* $(A, n) \otimes (B, m) = (A \otimes B, m + n)$*.)*

Proof: This is just an easy check of definitions.

In the book by H.R. Margolis [37] the notion of pre-triangulated category is defined. A pre-triangulated category differs from a triangulated category only with respect to the invertibility of the endomorphism Σ. (Note that Margolis' notion of 'pre-triangulated category' differs from the one used by Neeman in [43].) Most of the results which can be proved in triangulated categories are also true in pre-triangulated categories. In some sense it is more natural to work with pre-triangulated categories but we do not want to take this point of view.

3.2 Three different pictures of stable homotopy

In this section we want to give three different pictures of stable homotopy theory and compare them. These are

- the standard picture using homotopy classes of asymptotic morphisms and stabilizing with respect to the suspension
- the Thomsen-picture using homotopy classes of 1-step extensions and stabilizing with respect to the suspension and
- the Yoneda-picture using homotopy classes of extensions and stabilizing with respect to the cone extensions (for a definition see appendix C.3).

The standard picture is of course the one which was first introduced by A. Connes and M. Dădărlat. In the last section (theorem 2.2.11) we have seen that stable homotopy for separable C^*-algebras can be described using 1-step extensions of algebras, i.e. the second picture. But note that there is no obvious composition product on 1-step extensions up to suspension and only the identification of the two first approaches allows to define a composition product on those. Therefore the second picture seems to be rather artificial. Yet, there is a composition product on extensions which is the Yoneda product. But composing two 1-step extensions using the Yoneda product gives a 2-step extension and not a 1-step extension. In appendix C.3 the definition of extension categories is given and its elementary properties are developed. The composition product in extension categories is based on the Yoneda product.

This extension category will again be equivalent to stable homotopy for separable C^*-algebras. It will be of great use in proving the universal properties of stable homotopy theory.

As already mentioned the construction of an extension category is not particular to the category of separable C^*-algebras. There is a general procedure which produces a triangulated category out of data of very general type. Actually, to do the construction requires only the existence of a suitable interval functor and a class of extensions which is closed under certain constructions. We do not develop this theory here and want to prove the existence of the triangulated structure only in the special case of the category of separable C^*-algebras. The general construction will be content of a forthcoming article.

An immediate implication of theorem 2.2.11 is the following result which we implicitly stated in the preceding remarks.

Theorem 3.2.1 *Let A and B be separable C^*-algebras. The set of stable homotopy classes of maps $\{A, B\}$ can be described using 1-step extensions. More precisely*

$$\{A, B\} \cong \mathrm{colim}_n \mathrm{ext}(\Sigma^{n+1}(B), \Sigma^n(A))$$

where the co-limit is taken with respect to the connecting maps $\mathrm{ext}(\Sigma^{n+1}(B), \Sigma^n(A)) \to \mathrm{ext}(\Sigma^{n+2}(B), \Sigma^{n+1}(A))$ given by taking the twisted suspended extension.

Proof: This is obvious in the light of theorem 2.2.11.

Now, we turn to an identification of the extension category of separable C^*-algebras with stable homotopy theory. This will also rely upon theorem 2.2.11. Let $\underline{S}$ be the stable homotopy category (without artificial desuspension) as defined in section 3.1. Denote by S' the extension category of the category of separable C^*-algebras as defined in appendix C.3. We want to construct functors $P : \underline{S}^{op} \to S'$ and $Q : S' \to \underline{S}^{op}$ which are inverse equivalences.

First of all, using lemma 2.2.14, we want to prove a strengthening of lemma 2.2.12.

Lemma 3.2.2 *Consider a commutative diagram of extensions*

$$\begin{array}{cccccccccc}
a' : & & 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
\downarrow & & & & \downarrow f & & \downarrow g & & \downarrow h & & \\
a : & & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

The diagram of asymptotic morphisms

$$\begin{array}{ccc} \Sigma(C') & \xrightarrow{\ A',C'(a')\ } & A' \\ \downarrow{\scriptstyle (h)} & & \downarrow{\scriptstyle f} \\ \Sigma(C) & \xrightarrow{\ A,C(a)\ } & A \end{array}$$

commutes up to asymptotic homotopy after suspending once.

Proof: Consider the following diagram of asymptotic morphisms.

$$\begin{array}{ccccc} \Sigma^2(C') & \longrightarrow & \Sigma c(\alpha) & \longrightarrow & \Sigma A' \\ \downarrow{\scriptstyle ^2(h)} & & \downarrow & & \downarrow{\scriptstyle (f)} \\ \Sigma^2(C) & \longrightarrow & \Sigma c(\beta) & \longrightarrow & \Sigma A \end{array}$$

The horizontal arrows in the right-hand square are the connecting morphisms coming from the extensions

$$0 \longrightarrow \Sigma(A') \longrightarrow c(B') \longrightarrow c(\alpha) \longrightarrow 0$$

and

$$0 \longrightarrow \Sigma(A) \longrightarrow c(B) \longrightarrow c(\beta) \longrightarrow 0.$$

By lemma 2.2.14, they are the asymptotic homotopy inverses to the suspensions of the canonical inclusions $A' \to c(\alpha)$ and $A \to c(\beta)$. The left-hand square in the diagram above commutes, since the diagram of extensions was commutative. The right-hand square commutes, since the diagram

$$\begin{array}{ccc} A' & \longrightarrow & c(\alpha) \\ \downarrow{\scriptstyle f} & & \downarrow \\ A & \longrightarrow & c(\beta) \end{array}$$

clearly commutes (by the commutativity of the diagram of extensions) and the horizontal arrows become asymptotic homotopy equivalences after suspending. Finally, we have to convince ourselves that the composition of the vertical arrows is exactly what we want.

Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma(A) & \longrightarrow & \Sigma(B) & \xrightarrow{\ (\ )\ } & \Sigma(C) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Sigma(A) & \longrightarrow & c(B) & \longrightarrow & c(\beta) & \longrightarrow & 0. \end{array}$$

There is an analogous diagram concerning α. By lemma 2.2.12, this proves our claims.

This already implies that stable homotopy is a bivariant homology theory but we do not want to give the argument and instead, we are going to show directly that it is triangulated. The construction of the functor $Q : S' \to \underline{S}^{op}$ is a special case of the universal property of the extension category. We will formulate the result in a greater generality, since we will use related constructions several times. Note that the preceding results concerning boundary morphisms can be obtained in any bivariant homology theory. We will formulate the crucial results in the following theorem. In the statement of the following theorem we do not distinguish between objects in $s\mathcal{C}$ and their images in a bivariant homology theory (R, Σ).

Theorem 3.2.3 *Let* $H : s\mathcal{C} \to (R, \Sigma)$ *be a bivariant homology theory. Then the following conditions hold:*

- *Let*
  $$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$
  be an extension of separable C^**-algebras. There is a natural (natural with respect to mappings of extensions) morphism* $\Sigma(C) \to A$ *in the category* R *(i.e. an element in* $\hom_R(\Sigma(C), A)$*) which induces the boundary map in the long exact sequences.*

- *The cone extension of the algebra* A *gives rise to the identity element in* $\hom_R(\Sigma(A), \Sigma(A))$ *as a boundary map.*

- *The boundary map which is induced by the twisted suspension of a given extension is the suspension of the boundary map of the extension.*

Proof: The first part is a combination of the naturality of the long exact sequence and the Yoneda lemma. A proof of results which immediately imply our claims can be found in [19, 57, 58].

The second part is somewhat circular, since the construction of the long exact sequence is just made such that the condition holds. (Again, this is about the question whether one works with $C([0,1],0)$ or $C([0,1],1)$ as the cone. Note that the procedure of forming the mapping cone is crucial in the proof of the existence of a long exact sequence. Both choices of the cone will do, but one has to work consistently with one choice in order to prevent problems with signs. We choose to work with $C([0,1],0)$.)

The third part is an easy and classical observation about the homotopy type of certain mapping cones and the well known fact that the flip on the 2-sphere induces $-[\mathrm{id}_{S^2}]$ in $\pi_2(S^2)$.

The conditions in theorem 3.2.3 do not imply the exactness properties of a bivariant homology theory. Although they are weaker they are sufficient to make the following theorem work. Recall that there is a functor $s_{ext} : s\mathcal{C} \to S'^{op}$ according to lemma C.3.17. It maps a $*$-homomorphism to its cone extension.

Theorem 3.2.4 *Let* $H : s\mathcal{C} \to (R, \Sigma)$ *be a functor from the category of separable* C^**-algebras which satisfies the conditions in theorem 3.2.3. There exists a functor* $H_S : S'^{op} \to (R, \Sigma)$ *such that*

$$H_s \circ s_{ext} = H.$$

Proof: Let a be a representative of a class in $S'(A, B)$. It is given by a n-step extension in $Ext^n(\Sigma^n A, B)$. We decompose it into n 1-step extensions $a_1, \dots, a_n$ with $a_i \in Ext^1(A_{i-1}, A_i)$, $A_0 = \Sigma^n A$ and $A_n = B$. Let $f_i \in hom_R(\Sigma(A_i), A_{i-1})$ be the corresponding boundary morphism (see theorem 3.2.3) for $i = 1, \dots, n$. Consider their composition

$$f_1 \circ \Sigma(f_2) \circ \Sigma^2(f_3) \circ \dots \circ \Sigma^{n-1}(f_n) \in hom_R(\Sigma^n(B), \Sigma^n(A)) = hom_R(B, A).$$

First of all, the constructed bivariant homology class is independent of the decomposition we have chosen. Indeed, this follows by an easy application of the naturality of the boundary map. By the homotopy invariance, it is also independent of the representative in the homotopy class of extensions.

The second result in theorem 3.2.3 tells us that cone extensions induce the identity as a boundary morphism. Note further that the boundary of the twisted suspended extension is the suspension of the boundary map. This together implies that the assignment is well-defined on $S'(A, B)$. Thence we get a map $S'(A, B) \to hom_R(B, A)$.

We still have to show functoriality but this is obvious, since a particular decomposition of a Yoneda product is given by concatenating the decompositions of the factors. This finishes the proof of the existence of the functor H_s.

Corollary 3.2.5 *There is a functor* $Q : S' \to \underline{S}^{op}$ *that is compatible with the maps* $\lambda_{?,?}$ *considered in theorem 2.2.11.*

Proof: This is implied by lemma 2.2.12 and lemma 3.2.2 which show that stable homotopy theory satisfies the properties of theorem 3.2.3. Hence we can apply theorem 3.2.4.

Next we want to construct a functor $P : \underline{S}^{op} \to S'$ which will serve as an inverse for Q. Note that $\underline{S}(B, A)$ can be described by the co-limit of

$$hom(\Sigma^m B, a^n(\Sigma^m A))/\sim_n$$

where $\sim_n$ identifies morphisms which are homotopic via elements in

$$hom(\Sigma^m B, a^n(\Sigma^m A[0, 1])).$$

The construction proceeds in several steps. First of all, we construct a functor that assigns to an asymptotic morphism an extension of possibly non-separable algebras.

For this purpose we denote the extension category of non-separable algebras by S'_{ns}. (A priori it is not clear that S'_{ns} is a category (i.e. that the class of homotopy classes of extensions always form sets etc.). This is shown by an argument similar to lemma 3.2.7. We omit a proof, since we may as well work with the essentially small category of algebras having a dense subset of cardinality $2^{\mathbb{N}}$.)

The second step is a reduction to extensions of separable algebras (i.e. we are going to show that $S' \to S'_{ns}$ is the inclusion of a full sub-category). Let us start by constructing an assignment $\underline{S}(B, A) \to S'_{ns}(A, B)$. We use the abbreviation A_b for $A_b([0,1), 0)$.

Consider the 1-step extension

$$0 \longrightarrow \Sigma(A) \longrightarrow A_b \longrightarrow a(A) \longrightarrow 0$$

for any C^*-algebra A and denote it by $\alpha_A \in \mathrm{Ext}^1(\Sigma(A), a(A))$. Consider the n-step extension which is given by the Yoneda product

$$\alpha_{\ {}^{n-1}(A)} \cdot a(\alpha_{\ {}^{n-2}(A)}) \cdot a^2(\alpha_{\ {}^{n-3}(A)}) \cdot \cdots \cdot a^{n-1}(\alpha_A) \in \mathrm{Ext}^n(\Sigma^n(A), a^n(A))$$

and denote it by α^n_A (see proposition 2.2.17 for an explanation of the notation). Note that $\alpha^n_A = \alpha^{n-1}_{\ (A)} \cdot a^{n-1}(\alpha_A)$.

An element in $\underline{S}(B, A) = \{B, A\}$ is represented by a homomorphism $f : \Sigma^m(B) \to a^n(\Sigma^m(A))$ for some n and m. First of all, we can assign to it the following pull-back extension

$$\begin{array}{ccccccccc}
0 \longrightarrow & \Sigma^{n+m}(A) & \longrightarrow \cdots \longrightarrow & a^{n-1}(\Sigma^m(A)_b) \oplus_{a^n(\ ^m(A))} \Sigma^m(B) & \longrightarrow & \Sigma^m(B) & \longrightarrow 0 \\
 & \| & & \downarrow & & \downarrow f & \\
0 \longrightarrow & \Sigma^{n+m}(A) & \longrightarrow \cdots \longrightarrow & a^{n-1}(\Sigma^m(A)_b) & \longrightarrow & a^n(\Sigma^m(A)) & \longrightarrow 0,
\end{array}$$

which we denote by $\alpha^n_{\ ^m(B)} \circ f \in \mathrm{Ext}^n(\Sigma^{n+m}(A), \Sigma^m(B))$, and map it to $\mathrm{Ext}^{n+m}(\Sigma^{n+m}(A), B)$ using the canonical element in $\mathrm{Ext}^m(\Sigma^m(B), B)$. (We are using a notation for the pull-back of an extension which is introduced in appendix C.3, definition C.3.7.) We have to show that this assignment is well-defined as a map $\underline{S}(B, A) \to S'_{ns}(A, B)$. We do this in successive steps.

1. It is well-defined on $\mathrm{hom}(\Sigma^m(B), a^n(\Sigma^m(A)))/\sim_n$.

   Proof: Indeed, an element in $\mathrm{hom}(\Sigma^m(B), a^n(\Sigma^m(A)[0,1]))$ is assigned to a homotopy of extensions which obviously restricts to the respective extensions which are assigned to the evaluations.

2. It is well-defined on $\mathrm{colim}_n \mathrm{hom}(\Sigma^m(B), a^n(\Sigma^m(A)))/\sim_n$.

   Proof: Let $f : \Sigma^m(B) \to a^n(\Sigma^m(A))$ be a $*$-homomorphism. We have to show that

$$\alpha(a^n(\Sigma^m(A)) \circ f : \Sigma^m(B) \to a^{n+1}(\Sigma^m(A))$$

gives rise to the same class in $S'_{ns}(A, B)$. Consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^{n+m+1}(A) & \longrightarrow & c(\Sigma^{n+m}(A)) & \longrightarrow & \Sigma^{n+m}(A) & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow {\scriptstyle (\ \ {}^{n+m}(A))} & & \\
0 & \longrightarrow & \Sigma^{n+m+1}(A) & \longrightarrow & \Sigma^{n+m}(A)_b & \longrightarrow & a(\Sigma^{n+m}(A)) & \longrightarrow & 0.
\end{array}$$

We can extend this to a diagram of $(n+1)$-step extensions using the naturality of α as follows.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{n+m+1}(A) & \longrightarrow & c(\Sigma^{n+m}(A)) & \longrightarrow & \cdots \\
 & & \Vert & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{n+m+1}(A) & \longrightarrow & (\Sigma^{n+m}(A))_b & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & a^{n-1}(\Sigma^m(A)_b) & \longrightarrow & a^n(\Sigma^m(A)) & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle (a^{n-1}(\ \ {}^m(A)_b))} & & \downarrow {\scriptstyle (a^n(\ \ (A)))} & & \\
\cdots & \longrightarrow & a^n(\Sigma^m(A)_b) & \longrightarrow & a^{n+1}(\Sigma^m(A)) & \longrightarrow & 0
\end{array}$$

The diagram shows that there is a congruence between $C_{\ \ {}^{n+m}(A)} \cdot (\alpha^n{}_{m(A)} \circ f)$ and $\alpha^{n+1}_{\ \ m(A)} \circ (\alpha(a^n(\Sigma^m(A))) \circ f)$ in $\mathrm{Ext}^{n+1}(\Sigma^{n+m+1}(A), \Sigma^m(B))$. This is what we wanted to show.

3. It is well-defined on $\underline{S}(A, B)$.

   Proof: We have to show that the assignment is compatible with suspension. Let again $f : \Sigma^m(B) \to a^n(\Sigma^m(A))$ be a homomorphism. Its suspension in the sense of asymptotic morphisms is given by the composition $\Sigma^{1+m}(B) \to \Sigma(a^n(\Sigma^m(A))) \to a^n(\Sigma^{1+m}(A))$. Consider the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^{m+1}(B) & \longrightarrow & c(\Sigma^m(B)) & \longrightarrow & \Sigma^m(B) & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle (f)} & & \downarrow {\scriptstyle c(f)} & & \downarrow {\scriptstyle f} & & \\
0 & \longrightarrow & a^n(\Sigma^{m+1}(A)) & \longrightarrow & a^n(c(\Sigma^m(A))) & \longrightarrow & a^n(\Sigma^m(A)) & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow {\scriptstyle a^n(\ (\ \ {}^m(A)))} & & \\
0 & \longrightarrow & a^n(\Sigma^{m+1}(A)) & \longrightarrow & a^n((\Sigma^m(A))_b) & \longrightarrow & a^{n+1}(\Sigma^m(A)) & \longrightarrow & 0
\end{array}$$

   The diagram shows that $(\alpha^n{}_{m+1(A)} \circ \Sigma(f)) \cdot C_{\ \ {}^m(B)}$ is congruent to $\alpha^{n+1}_{\ \ m+1(A)} \circ (a^n(\alpha(\Sigma^m(A))) \circ f)$. This is not quite what we wanted, since a priori $a^n(\alpha(\Sigma^m(A))) \circ f$ and $\alpha(a^n(\Sigma^m(A))) \circ f$ might differ. But by proposition 2.8 in [28] the compositions $\alpha(a^n(\Sigma^m(A))) \circ f$ and $a^n(\alpha(\Sigma^m(A)) \circ f$ are $\sim_{n+1}$-equivalent.

Therefore we can conclude by step 1 in the proof that $\Sigma(f)$ and $\alpha(a^n(\Sigma^m(B))) \circ f$ are sent to the same element in $S'_{ns}(A, B)$. Thence we are done by step 2. This finishes the proof of the existence of an assignment $\underline{S}(B, A) \to S'_{ns}(A, B)$ which is compatible with $\mu_{?,?}$.

Lemma 3.2.6 *The assignment* $\underline{S}(B, A) \to S'_{ns}(A, B)$ *defines a functor* $P : \underline{S}^{op} \to S'_{ns}$.

Proof: First of all, without loss of generality, we restrict to the case where we do not stabilize with respect to suspensions. Secondly, we simplify the problem quite a bit by noting that by lemma 2.1.8 any homotopy class of asymptotic morphisms in $[[B, A]]$ is represented by a morphism $f : B \to a(A)$, since A and B are separable algebras. Given another class in $[[C, B]]$ represented by $g : C \to a(B)$, their composite is given by the class $a(f) \circ g : C \to a^2(A)$.

These morphisms are sent to extensions

$$0 \longrightarrow \Sigma(A) \longrightarrow A_b \oplus_{a(A)} B \longrightarrow B \longrightarrow 0$$

$$0 \longrightarrow \Sigma(B) \longrightarrow B_b \oplus_{a(B)} C \longrightarrow C \longrightarrow 0$$

Consider now the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^2(A) & \longrightarrow & \Sigma(A_b \oplus_{a(A)} B) & \longrightarrow & B_b \oplus_{a(B)} C & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Sigma^2(A) & \longrightarrow & (\Sigma(A))_b & \longrightarrow & a(A_b) \oplus_{a^2(A)} C & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \downarrow {\scriptstyle a(f)\circ g} & & \\
0 & \longrightarrow & \Sigma^2(A) & \longrightarrow & (\Sigma(A))_b & \longrightarrow & a(A_b) & \longrightarrow & a^2(A) & \longrightarrow & 0
\end{array}$$

The extension in the middle is the image of the composite $a(f) \circ g$. Indeed, by definition it is the pull back of α^2_C along $a(f) \circ g$. The upper extension is the composite of the images of f and g in the extension category. The diagram shows that there is a congruence between the two. This proves the functoriality of the assignment P.

It remains to show that everything already happens with extensions of separable algebras instead of extensions of arbitrarily large algebras. The following lemma will allows us to draw this conclusion.

Lemma 3.2.7 *Let*

$$a : \qquad 0 \longrightarrow A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow 0$$

be a n-step extension of C^*-algebras such that A_0 and A_n are separable. There exists a n-step extension

$$b: \qquad 0 \longrightarrow A_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_{n-1} \longrightarrow A_n \longrightarrow 0$$

with B_i separable for $i \in \{1, \ldots, n-1\}$ and a morphism of extensions

$$\begin{array}{ccccccccccccc}
b: & & 0 & \longrightarrow & A_0 & \longrightarrow & B_1 & \longrightarrow \cdots \longrightarrow & B_{n-1} & \longrightarrow & A_n & \longrightarrow & 0 \\
\downarrow & & & & \| & & \downarrow & & \downarrow & & \| & & \\
a: & & 0 & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow \cdots \longrightarrow & A_{n-1} & \longrightarrow & A_n & \longrightarrow & 0.
\end{array}$$

Moreover given two n-step extension b and b' consisting of separable algebras and morphisms of extensions $b \to a$ and $b' \xrightarrow{'} a$ there exists a n-step extension b'' consisting of separable algebras and a morphism of extensions $b'' \xrightarrow{''} a$ such that both α and α' factorize through α''.

Proof: Given an extension

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and a sub-algebra B' of B, the following diagram

$$0 \longrightarrow \psi(A) \cap B' \longrightarrow B' \longrightarrow \phi(B') \longrightarrow 0$$

is also an extension.

We apply this principle from the right to the left. First choose a separable sub-algebra B_{n-1} of A_{n-1} which maps onto A_n. This defines a separable sub-algebra of $\ker(A_{n-1} \to A_n) = \operatorname{coker}(A_{n-3} \to A_{n-2})$. Now choose again a separable sub-algebra B_{n-2} of A_{n-2} that maps onto the sub-algebra and so on. At the last step we may choose a separable sub-algebra of A_1 which contains A_0, since A_0 is separable itself. This finishes the first part of the proof.

Given two morphisms as stated in the lemma, we may choose the sub-algebras of A_i always such that they also contain the separable images of the morphisms, if we again do it from the right to the left by induction. This proves also the second part of the lemma and finishes the proof.

It is now obvious that we can always reduce to the situation of extensions consisting of separable algebras. The following theorem puts everything together.

Proposition 3.2.8 *The functor from the extension category of separable C^*-algebras to the extension category of C^*-algebras is the inclusion of a full sub-category.*

Proof: This is what the last lemma shows.

Theorem 3.2.9 *The functors* $Q : S' \to \underline{S}^{op}$ *and* $P : \underline{S}^{op} \to S'$ *are inverse equivalences.*

Proof: This is obvious from the fact that the functors are well-defined and compatible with the maps considered in theorem 2.2.11.

Now, we succeeded in identifying two possible definitions of a stable homotopy theory for separable C^*-algebras. The extension picture will simplify proofs regarding universal properties. The picture of asymptotic morphisms will be useful for concrete computations (see lemma 2.2.15).

3.3 Stable homotopy is triangulated

In this section we are going to show that stable homotopy for separable C^*-algebras is a triangulated homology theory. First of all, we show that it is a triangulated category. The properties of a triangulated homology theory are then obvious from the properties of the triangulation. The classical counterpart was proved long ago by D. Puppe in [49].

We start with an important lemma.

Lemma 3.3.1 *Up to desuspension every morphism in* S *is in the essential image of* $s : s\mathcal{C} \to S$*. More precisely, let* $f : (A, n) \to (B, m)$ *be a morphism in* S*. There exist a separable* C^**-algebra* B'*, an integer* $q \in \mathbb{N}$ *and a morphism of* C^**-algebras* $f' : \Sigma^{n+q+1}(A) \to B'$ *such that there exists the following commutative square in which the left vertical arrow is the natural equivalence and the right vertical arrow is an isomorphism in* S*.*

$$\begin{array}{ccc} (A, n) & \xrightarrow{\quad f \quad} & (B, m) \\ \downarrow & & \downarrow \\ (\Sigma^{n+q+1}(A), -q-1) & \xrightarrow{\ ^{-q-1}(s(f'))\ } & (B', -q-1) \end{array}$$

Proof: A morphism $f : (A, n) \to (B, m)$ in S is by definition represented by an asymptotic morphism $f'' : \Sigma^{q+n}(A) \to \Sigma^{q+m}(B)$ for some $q \in \mathbb{N}$. Without loss of generality and to simplify the notation we assume that $q + n = q + m = 0$. According to lemma 2.2.9 we have an extension as follows.

$$0 \longrightarrow \Sigma(B) \longrightarrow E_{f''} \longrightarrow A \longrightarrow 0$$

The cone extension of π looks as follows.

$$0 \longrightarrow \Sigma(A) \longrightarrow c(\pi) \longrightarrow E_{f''} \longrightarrow 0$$

The algebra $c(\pi)$ is isomorphic to $A(0, 1] \oplus_A E_{f''}$. Denote the canonical map $\Sigma(B) \to c(\pi)$ by ι. We want to show that the following diagram commutes up to homotopy as diagram of asymptotic morphisms after suspending once.

$$\begin{array}{ccc} \Sigma(A) & \longrightarrow & c(\pi) \\ \| & & \uparrow \\ \Sigma(A) & \xrightarrow{(f'')} & \Sigma(B) \end{array}$$

Consider also the algebra $c(\gamma)$ which comes with the following extension.

$$0 \longrightarrow \Sigma(c(\pi)) \longrightarrow c(\gamma) \longrightarrow \Sigma(A) \longrightarrow 0$$

There is a commutative diagram of extensions as follows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma(c(\pi)) & \longrightarrow & c(\gamma) & \longrightarrow & \Sigma(A) & \longrightarrow & 0 \\ & & {\scriptstyle (\,)}\uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \Sigma^2(B) & \longrightarrow & \Sigma(E_{f''}) & \longrightarrow & \Sigma(A) & \longrightarrow & 0 \end{array}$$

The existence of the diagram implies the claim by lemma 3.2.2 and theorem 2.2.11. This finishes the proof, since we can now just take $B' = c(\pi)$, $f' = \gamma$ and note that ι becomes an isomorphism in the category S.

Remark 3.3.2 *The last lemma shows not only that we can replace morphisms in S (up to desuspension and isomorphism) by images of $s : s\mathcal{C} \to S$ but that the same is true for finite diagrams with the property that there is at most one ingoing morphism at each object. The proof goes by an easy induction argument.*

Next we want to establish a triangulation on the category S. Note that to give a triangulation on S amounts to specify an additive automorphism and a class of distinguished triangles satisfying certain axioms (see appendix A.1). We take Σ^{-1} as the automorphism. This choice is compatible with the fact that the classical stable homotopy category is supposed to be contravariantly included into the stable homotopy category of C^*-algebras. (Note that the opposite of a triangulated category is naturally triangulated if one takes the inverse automorphism.) We omit the mentioning of the functor s in the next definition.

Definition 3.3.3 *We call a triangle*

$$\Sigma(C) \xrightarrow{a} A \xrightarrow{b} B \xrightarrow{c} C$$

in S distinguished if for some $n \in \mathbb{N}$ and some $*$-homomorphism $f : A' \to B'$ the triangle

$$\Sigma(\Sigma^n C) \xrightarrow{(-\;)^n a} \Sigma^n A \xrightarrow{{}^n b} \Sigma^n B \xrightarrow{{}^n c} \Sigma^n C$$

is isomorphic to a triangle

$$\Sigma(B') \longrightarrow c(f) \longrightarrow A' \xrightarrow{f} B'$$

with the natural maps involved.

In the definition of triangulated categories the triangles are of a different form. Note that we are working with the inverse of Σ as automorphism so that everything fits.

Remark 3.3.4 *Note that it follows from the definition that if one of the triangles*

$$\Sigma(C) \xrightarrow{a} A \xrightarrow{b} B \xrightarrow{c} C$$

and

$$\Sigma(\Sigma(C)) \xrightarrow{-\Sigma(a)} \Sigma(A) \xrightarrow{\Sigma(b)} \Sigma(B) \xrightarrow{\Sigma(c)} \Sigma(C)$$

is distinguished, then so is the other.

Theorem 3.3.5 *The category S together with the functor Σ^{-1} and the class of distinguished triangles which was specified in the last definition form a triangulated category.*

Proof: We have to check the conditions of definition A.1.2. We already noted that Σ is an additive automorphism. Therefore only conditions $1-6$ have to be checked.

1. Condition 1 requires that the class of distinguished triangles is closed under isomorphism. This is part of the definition of the class of distinguished triangles.

2. Condition 2 is fulfilled, since, given an object $(A, n) \in ob(S)$, we can consider the morphism $0 \to A$. A suitable (de)suspension (see remark 3.3.4) of the corresponding distinguished triangle

$$\Sigma(A) \xrightarrow{id} \Sigma(A) \longrightarrow 0 \longrightarrow A$$

is the one we wanted.

3. Condition 3 is obvious after having proved lemma 3.3.1 which shows that up to isomorphism and desuspension every morphism is in the image of $s : s\mathcal{C} \to S$. Since distinguished triangles are closed under isomorphism and desuspension, this proves the claim (modulo the next condition).

4. To show condition 4 we have to prove two things. Up to desuspension and isomorphism we can restrict our considerations to standard triangles. Consider distinguished triangles

$$\Sigma(B) \longrightarrow c(f) \xrightarrow{ev_1} A \xrightarrow{f} B$$

and

$$\Sigma(A) \longrightarrow c(ev_1) \longrightarrow c(f) \xrightarrow{ev_1} A$$

We have a homotopy commutative diagram of $*$-morphisms as follows.

$$\begin{array}{ccccc} \Sigma(A) & \xrightarrow{t\ \ (f)} & \Sigma(B) & \longrightarrow & c(f) \\ \| & & \downarrow & & \| \\ \Sigma(A) & \longrightarrow & c(ev_1) & \longrightarrow & c(f) \end{array}$$

Indeed, the right square commutes by construction. The left square commutes up to homotopy as one easily checks.

Note that we have shown in lemma 2.2.14 that $\Sigma(B)$ is asymptotically homotopy equivalent to $c(ev_1)$ by $\pi : \Sigma(B) \to c(ev_1)$ after suspending once. Putting all together we get a triangle as follows.

$$\Sigma(A) \xrightarrow{-\ \ (f)} \Sigma(B) \longrightarrow c(f) \xrightarrow{ev_1} A$$

The other direction of condition 4 follows by applying our result three times and using remark 3.3.4.

5. The idea of the proof of condition 5 is classical. Even an operator algebraic reformulation can be found in the literature as proposition 2.9 in [57]. Since it is an important step in the proof of the existence of a triangulation, we want to give the proof in some detail.

   Proof: Up to isomorphism the triangles are desuspensions of cone sequences. Hence up to desuspension we may assume that all morphisms are represented by asymptotic morphisms and the situation is the following. Consider the following diagram, where all horizontal morphisms are actually morphisms of C^*-algebras and only the vertical ones are possibly asymptotic.

$$\begin{array}{ccccccc} \Sigma(B) & \longrightarrow & c(f) & \longrightarrow & A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle (g)} & & \vdots & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ \Sigma(B') & \longrightarrow & c(f') & \longrightarrow & A' & \xrightarrow{f'} & B' \end{array}$$

   Given that the right square is a homotopy commutative diagram of asymptotic morphisms, we have to show the existence of the dotted arrow γ making the whole diagram homotopy commutative.

   Let $h_t : A \to B'[0,1]$ be an (asymptotic) homotopy such that $h_{t,0} = g_t \circ f$ and $h_{t,1} = f' \circ h_t$ for all $t \in [0,1)$. Note that $c(f) = B(0,1] \oplus_B A$ and define $\gamma_t : c(f) \to c(f')$ by setting $\gamma_t(b_r, a) = (h_{t,2r}(a), h_t(a))$ for $0 \leqslant r \leqslant \frac{1}{2}$ and $\gamma_t(b_r, a) = g_t(b_{2r-1})$ for $\frac{1}{2} \leqslant r \leqslant 1$.

   One easily checks that this is well-defined and satisfies all the properties needed. This finishes the proof.

6. The last condition, which is commonly called Verdier's Axiom, follows similarly to condition 4 from the classical proof of proposition 2.10 in [57]. Since our formulation is somewhat different from the classical octahedron which is considered in proposition 2.10 of [57], we want to give a complete proof.

   Proof: Consider a diagram as follows.

$$
\begin{array}{ccccccc}
\Sigma^2(A) & \longrightarrow & \Sigma(c(f)) & \longrightarrow & \Sigma(B) & \xrightarrow{(f)} & \Sigma(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & c(g) & \longrightarrow & c(g) & \longrightarrow & 0 \\
\downarrow & & \vdots & & \downarrow & & \downarrow \\
\Sigma(A) & \longrightarrow & c(f \circ g) & \xrightarrow{C} & C & \xrightarrow{f \circ g} & A \\
\downarrow{\scriptstyle id} & & \vdots & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle id} \\
\Sigma(A) & \longrightarrow & c(f) & \xrightarrow{B} & B & \xrightarrow{f} & A
\end{array}
$$

   First we want to argue that this is already the general situation. First of all, we may assume that f and g are represented by morphisms between C^*-algebras. Furthermore, up to desuspension and isomorphisms, we can assume that our triangles are cone triangles.

   Note that the lower right square is commutative. Hence there is a canonical map $\gamma : c(f \circ g) \to c(f)$ which is given by $c(f \circ g) \cong A(0,1] \oplus_A C \ni (a_r, c) \mapsto (a_r, g(c)) \in A(0,1] \oplus_A B \cong c(f)$. The map δ is defined as before in the proof of condition 5. It remains to show that $c(g)$ is naturally homotopy equivalent to $c(\gamma)$. This is a classical argument which we do not want to repeat.

   We finish the proof by showing that the square

$$
\begin{array}{ccc}
c(f \circ g) & \xrightarrow{ev_1} & C \\
\downarrow & & \downarrow{\scriptstyle g} \\
c(f) & \xrightarrow{ev_1} & B
\end{array}
$$

   is homotopy Cartesian. That is, we have to show that there is a triangle of the form

$$
\Sigma(B) \longrightarrow c(f \circ g) \xrightarrow{\left(\ \ ev_1\right)} c(f) \oplus C \xrightarrow{\left(\begin{smallmatrix} -ev_1 \\ g \end{smallmatrix}\right)} B
$$

   for some η.

   Consider the map of C^*-algebras $\eta = \iota \circ \Sigma(f) : \Sigma(B) \to c(f \circ g)$. We get a cone extension as follows.

$$
0 \longrightarrow \Sigma(c(f \circ g)) \longrightarrow c(\eta) \longrightarrow \Sigma(B) \longrightarrow 0
$$

A short computation shows that $c(\eta)$ is isomorphic to $c(\mu : \Sigma(C) \to c(f))$. Furthermore, there is a split $\Sigma(C) \to c(\mu)$. It is given by mapping the element $(t \mapsto c_t) \in \Sigma(C)$ to $(t, s) \mapsto ((f \circ g)(c_{s+t-st}), g(c_t), c_s))$. This implies that $c(\eta)$ becomes isomorphic to $\Sigma(c(f \circ g)) \oplus \Sigma(C)$ in S as we wanted it to be.

We still have to show that the maps induced by this isomorphism are the required ones. The split s is chosen such that $\Sigma(C) \xrightarrow{s} c(\eta) \to \Sigma(B)$ equals $t\Sigma(g)$. The other composition $\Sigma(c(f)) \to c(\eta) \to \Sigma(B)$ is just the suspension of the evaluation ev_1. Consider the composition $\Sigma(c(f \circ g)) \to c(\eta) \to \Sigma(C)$. It is the suspension of the canonical evaluation ev_1 onto C. The map $\Sigma c(f \circ g) \to \Sigma c(f) \to c(\eta)$ is the inclusion in the exact sequence given above. Putting all this information together we get a distinguished triangle of the following form.

$$\Sigma^2(B) \xrightarrow{(\ )} \Sigma(c(f \circ g)) \xrightarrow{(\ \ ev_1)} \Sigma(c(f)) \oplus \Sigma(C) \xrightarrow{\left(\begin{smallmatrix} (ev_1) \\ - \ \ g \end{smallmatrix}\right)} \Sigma(B)$$

After desuspending this distinguished triangle once we get the following distinguished triangle.

$$\Sigma(B) \longrightarrow c(f \circ g) \xrightarrow{(\ \ ev_1)} c(f) \oplus C \xrightarrow{\left(\begin{smallmatrix} -ev_1 \\ g \end{smallmatrix}\right)} B$$

This shows precisely what we wanted to prove. The square above is indeed homotopy Cartesian.

This finishes the proof that (S, Σ^{-1}) forms a triangulated category.

Theorem 3.3.6 *The stable homotopy category S and the canonical functor $s : s\mathcal{C} \to S$ constitute the universal triangulated homology theory.*

Proof: First of all, we have to convince ourselves that $s : s\mathcal{C} \to S$ is indeed a triangulated homology theory. Let

$$0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$

be an extension. We get a triangle

$$\Sigma(C) \longrightarrow c(f) \longrightarrow B \xrightarrow{f} C$$

in S. By lemma 2.2.14, $c(f)$ is naturally isomorphic to $\ker(f) \cong A$ in S such that we get a triangle as follows.

$$\Sigma(C) \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C$$

It is an easy check that it is natural with respect to morphisms of extensions. It is obvious that all triangles (up to isomorphism and (de)suspension) are of this form. This shows that $s : s\mathcal{C} \to S$ is triangulated homology theory.

Let $R : s\mathcal{C} \to (P, \Sigma)$ be another triangulated homology theory. The associated bivariant homology theory satisfies the conditions of theorem 3.2.3 and therefore theorem 3.2.4 can be applied. We get a functor $L' : \underline{S} \to P$ which descends to a functor $L : S \to P$, since the suspension is invertible in P. Since the triangles in S are just the ones which come from extensions in $s\mathcal{C}$, they are mapped to triangles in P. Furthermore, the functor $L : S \to P$ is clearly compatible with the respective suspension functors and hence it is a triangulated functor. This finishes the proof.

Theorem 3.3.7 *Stable homotopy theory of separable C^*-algebras is a monoidal triangulated category and the functor $s : s\mathcal{C} \to S$ is a functor of monoidal categories.*

Proof: This is now an easy consequence of the exactness of the maximal tensor product as discussed in theorem C.4.4. The existence of the monoidal structure was remarked before and the compatibility with the class of distinguished triangles follows, since the tensor product with a fixed algebra preserves extensions and hence distinguished triangles.

Theorem 3.3.8 *Any bivariant homology theory is a triangulated module (see definition A.1.10) over the stable homotopy category.*

Proof: This is again an application of theorem 3.2.4. The existence of the comparison functor was stated there. The exactness properties of definition A.1.10 are obvious, since again triangles in stable homotopy theory correspond to extensions and extensions lead to long exact sequences in a bivariant homology theory.

Having constructed the universal theories, we go on to construct certain theories in the next section which have certain universal properties. A second point will be to find non-trivial applications for the universal properties.

We also want to state and prove a theorem concerning homology theories. In fact, it is implied by theorem 3.2.4 as we will see. There will be one important application of the following theorem in section 4.4.

Theorem 3.3.9 *Let $R : s\mathcal{C} \to Ab_*$ be a homology theory on the category of separable C^*-algebras. There exists a product map*

$$R(A) \otimes S(A, B) \to R(B)$$

which gives the functor R the structure of a S'-module. (This just means that the obvious associativity condition is satisfied.)

Proof: Define $R^{\mathbb{Z}}$ to be the category with objects pairs (A, n), consisting of a separable C^*-algebras A and an integer n, and morphisms between objects (A, n) and (B, m) given by $\hom_{Ab_*}(\Sigma^n R(A), \Sigma^m R(B))$. This is in general not a bivariant homology theory.

But still, it satisfies the conditions of theorem 3.2.3. Hence we get a functor $R'_s : \underline{S} \to R^{\mathbb{Z}}$. It is clear that this functor descents to a functor

$$R_s : S \to R^{\mathbb{Z}}.$$

The existence of the module structure is now obvious. This finishes the proof of the theorem.

Corollary 3.3.10 *Let $R : sC \to Ab_*$ be a homology theory on the category of separable C^*-algebras. Assume that there is an isomorphism $R_0(\mathbb{C}) = \mathbb{Z}$. There exists a natural transformation $S_*(A) \to R_*(A)$.*

Proof: This is obvious by theorem 3.3.9.

The existence of universal bivariant homology theories can be proved by different methods. An abstract existence proof could also follow the lines of the work of Nigel Higson in [30]. Although he is not using the concept of a triangulated category his procedure in proving the existence of E-theory is the following.

- Establish the structure of a triangulated category on the ordinary (without asymptotic morphisms) stable homotopy category for separable C^*-algebras. The triangles in this triangulated category are just the co-fiber sequences. This can be done by completely classical arguments.
- Construct a monad (for a definition see [30] or a classical reference like [59]) corresponding to the idempotent process of stabilization with the compact operators. The category of algebras over the monad will again be triangulated. This only uses properties of rank one projections in the algebra of compact operators.
- Localize at the class of arrows $\ker(f) \to c(f)$ for any surjection f between C^*-algebras. (In [30] a different class is considered but these are the essential morphisms that have to be inverted.) This is done by Verdier localization (see appendix 2).

In [30] the localization process is done by hand and hence the construction gets complicated and seems to be a little artificial. Nevertheless in the framework of triangulated categories it is the canonical thing to do. For a further simplification the second step could be subsumed in the third one if one adds the inclusions $A \to A \otimes K$ to the arrows that have to be inverted. Having the right language at hand Higson's result gets a rather easy consequence of the existence of localizations in triangulated categories.

Note that if one omits the second step in the proof of N. Higson in [30] one ends up with stable homotopy for C^*-algebras.

One might ask why we have not given Higson's proof if it is so much simpler. The point is that it is a major achievement (essentially due to A. Connes and N. Higson

[13]) to have a concrete presentation of the universal homology theory by something which is more tractable than a category of fractions. The description using asymptotic morphisms has already proved to be useful in a lot of contexts, in particular, for concrete computations and examples. For the results in the next section it will be essential that we have a concrete description. The abstract existence would not help very much.

Note that the description using the extension category has the advantage that it can be used for a definition for non-separable algebras as well. (In fact, the existence of a suitable stable homotopy category for all C^*-algebras can not (at least to our knowledge) be achieved by abstract arguments, since set-theoretic problems prevent us from knowing whether the category of fractions exists.) Secondly, the approach using the extension category can be used in other contexts as well and therefore seems to be of interest from an abstract point of view.

4 Connective E-theory and bivariant homology

This section contains material about concrete triangulated and bivariant homology theories.

In the first part we recall properties of E-theory (see [28]). It contains no new results, but our presentation has to be considered as providing a new look at E-theory using the language of triangulated categories.

The second part is about a connective version of E-theory which is called bu. The main feature is that it is no longer 2-periodic and therefore intrinsically much more complicated. The connective theory lacks one important property which is stability. Stability is very useful for computations in (non-connective) E-theory. We restrict our study of connective E-theory to a category of non-commutative cell complexes on which it is computable to some extend. We establish the existence of useful spectral sequences which in some sense replace the Künneth theorem and the universal coefficient theorem in KK-theory. This part also contains a discussion of duality.

The third part contains examples and computations. In particular, we are going to determine the bu-type of matrix bundles and certain non-commutative algebras which are naturally associated to pairs consisting of a compact locally Hausdorff space and a Hausdorff open cover of the space.

In a fourth part we establish an interesting relation between connective E-theory and negative algebraic K-theory which goes back to the work of Jonathan Rosenberg in [53].

The fifth part contains the definition of a bivariant homology theory which generalizes singular homology from the category of finite CW-complexes to the category of separable C^*-algebras. We compute the algebra of cohomology operations and discuss the classical Chern character in this more general context. Finally, we establish the existence of an Adams spectral sequence and provide a Künneth theorem and an universal coefficient theorem for bivariant homology.

Considering bivariant and triangulated homology theories, all our examples will have the property that the categories will have the same objects as $s\mathcal{C}$ (or pairs consisting of a C^*-algebra and an integer) and the functor is just the identity on objects (or the inclusion into the pairs of the form $(A, 0)$). We will therefore omit the mentioning of the functor.

4.1 Revisiting E-theory

Theorem 4.1.1 *The localization (see appendix A.2.2) of stable homotopy theory as a triangulated category with respect to the class of morphisms* $\{\iota_\infty \otimes \mathrm{id}_A : A \to K \otimes A\}$ *is a monoidal triangulated homology theory* $\mathbf{E}$ *whose associated bivariant homology theory is isomorphic to* E*-theory as defined by A. Connes and N. Higson in [13].*

Proof: This follows either from the universal properties that are satisfied by (classical) E-theory or by a direct argument as follows. The morphisms in the localization are by theorem A.2.6 given as

$$\mathbf{E}(A, B) = \mathbf{colim}_{p,n}[[\Sigma^n A, K^p \Sigma^n B]]$$

By classical results in Connes-Higson E-theory (Actually one only needs suitable isomorphisms $K^2 \cong K$ and $K\Sigma \cong K\Sigma^3$ in the homotopy category of asymptotic morphism $as\mathcal{C}$.), one does not need to take the co-limits. The map

$$[[\Sigma A, K\Sigma(B)]] \to \mathbf{colim}_{p,n}[[\Sigma^n A, K^p \Sigma^n B]]$$

is always an isomorphism. This just shows that the localization indeed identifies with classical E-theory.

Theorem 4.1.2 (G.G. Kasparov, C. Schochet) *Let (X, x) and (Y, y) be finite pointed CW-complexes. Then we have the following isomorphism.*

$$\mathbf{E}_n(C(Y, y), C(X, x)) \cong \mathbf{colim}_m[\Sigma^{m+n} X, \underline{BU}_m \wedge Y]_+$$

In the last statement $\underline{BU}_*$ denotes the non-connective topological **K***-theory spectrum.*

This theorem is stated only for sake of completeness, since similar statements will appear later. It was proved by G.G. Kasparov in [35] for **KK**-theory instead of E-theory. The result follows, since we know that for nuclear algebras **KK**-theory and E-theory coincide (see [30]). This finishes the proof.

The coefficient ring of **E** is given by $\mathbf{E}(\mathbb{C}, \mathbb{C}) = \mathbb{Z}[u, u^{-1}]$ where the degree of u is equal to two. In particular, E-theory is 2-periodic. A neat proof of the preceding statement was given by J. Cuntz using the Toeplitz extension and quasi-homomorphisms, see [17].

The element u is the class of the Bott map $\sigma \in \mathrm{hom}(\Sigma, M_2\Sigma^3)$ and induces the periodicity of **E**-theory. The Bott map in **E**-theory can be concretely given by a map $\sigma : \Sigma \to M_2\Sigma^3$ as we will see in detail later. The inverse of the Bott map involves necessarily the compact operators. It is, for example, given by

- connecting morphism of the Toeplitz extension

$$0 \longrightarrow K \longrightarrow \tau_0 \longrightarrow \Sigma \longrightarrow 0$$

or

- a choice of asymptotic morphism $\Sigma^2 \to K$ corresponding to a non-trivial pair of asymptotically commuting unitaries (see [68]).

There is an obvious asymmetry in the presentation of the Bott element and its inverse. One occurs naturally as a map between the 3-sphere and the 1-sphere just by introducing matrices. In certain contexts relatives of this map occur even as maps between spaces (e.g. the Adams map [2]). The inverse of the Bott map appears only after bringing the compact operators into play. In sub-section 4.2 we will use this asymmetry to find finer invariants of C^*-algebras than ordinary **K**-theory which obviously treats the compact operators as being the same as matrices.

The obvious questions that arise with this result are the following.

- 'What triangulated homology theories do we get if we localize with respect to other natural classes of morphisms ?'
- 'What are other natural classes of morphisms ?' and
- 'Are other geometrically motivated or geometrically relevant bivariant homology theories obtainable through localization ?'

There is an important corollary to the characterization of **E**-theory.

Corollary 4.1.3 *Let* $H : s\mathcal{C} \to (R, \Sigma)$ *be a bivariant homology theory. If for any separable* C^**-algebra* A *the map* $H(\mathrm{id}_A \otimes \iota_\infty) \in \hom_R(A, A \otimes K)$ *is an isomorphism, then the bivariant homology theory is 2-periodic (i.e. there exists a natural equivalence* $H \circ \Sigma^2 \cong H$*).*

4.2 Connective E-theory

This sub-section introduces a new triangulated homology theory. It is constructed by a similar procedure as **E**-theory and indeed carries a lot of similar properties. An important computation shows that it generalizes connective **K**-theory. In the following we want to analyze its abstract properties and give a couple of computations for interesting algebras.

Theorem 4.2.1 *The localization (see appendix A.2.2) of stable homotopy theory as a triangulated category with respect to the class of morphisms* $\{\mathrm{id}_A \otimes \iota_2 : A \to M_2(A)\}$ *is a triangulated monoidal homology theory* **bu** *whose associated bivariant homology theory generalizes connective* **K***-theory.*

More precisely the following holds for any finite pointed CW*-complexes* (X, x) *and* (Y, y)*.*

$$\mathbf{bu}_n(C(Y, y), C(X, x)) \cong \mathrm{colim}_m [\Sigma^{m+n} X, \underline{\mathrm{bu}}_m \wedge Y]_+$$

In the last statement $\underline{\mathrm{bu}}$ *denotes the* *connective* **K***-theory spectrum.*

Remark 4.2.2 *Since the category of separable* C^**-algebras is essentially small, we are not going to have any sort of set theoretic problems with the existence of the quotient. Erik Pedersen would kindly remark that the Adams-Bousfield story does not apply.*

Theorem 4.2.3 *The coefficient ring of* $\mathbf{bu}$ *is given by* $\mathbf{bu}(\mathbb{C}, \mathbb{C}) = \mathbb{Z}[u]$ *where the degree of* u *is equal to two. The element* u *is again the Bott map but it is* *<u>not</u>* *invertible.*

Proof of theorem 4.2.1 and 4.2.3: By the concrete form of the localization we have a natural isomorphism (see appendix A.2.2)

$$\mathbf{bu}_p(C(Y,y), C(X,x)) \cong \mathbf{colim}_{m,n}[[\Sigma^m C(Y,y), \Sigma^{m+p} M_n C(X,x)]].$$

On the other hand by M. Dădărlat's characterization of connective K-theory according to theorem 3.5 in [24] one has

$$\mathbf{colim}_m[\Sigma^{m+p} X, \underline{bu}_m \wedge Y]_+ \cong \mathbf{colim}_m[\Sigma^m C(Y,y), \Sigma^{m+p} C(X,x) \otimes K].$$

By lemma C.1.13 the following map is a natural isomorphism

$$\mathbf{colim}_{m,n}[\Sigma^{m+p} C(Y,y), \Sigma^m M_n C(X,x)] \rightarrow$$

$$\mathbf{colim}_m[\Sigma^{m+p} C(Y,y), \Sigma^m C(X,x) \otimes K]$$

whenever (Y,y) is locally contractible at y which is the case for CW-complexes.

This implies that we get a natural transformation of bivariant homology theories (here we are using the term 'bivariant homology theory' informally for the category of finite CW-complexes and 'extensions' given by co-fiber sequences.)

$$\mathbf{colim}_m[\Sigma^{m+p} X, \underline{bu}_m \wedge Y]_+ \rightarrow \mathbf{bu}_p(C(Y,y), C(X,x))$$

which is an isomorphism on finite CW-complexes if we can argue that it is an isomorphism on coefficients. Indeed, the isomorphism on coefficients follows from the fact that

$$[\Sigma^n, M_p \Sigma^m] \rightarrow [[\Sigma^n, M_p \Sigma^m]]$$

is an isomorphism which in turn follows from lemma 2.2.16 and the observation that $\mathrm{hom}(\Sigma^n, M_p)$ is an ANR (indeed, it is even a real algebraic variety). This finishes the proof.

The main part of the proof relies on theorem 3.5 in M. Dădărlat's and J. McClure's foundational work [24] about connective K-theory. One should also mention that the original idea of relating connective K-homology of a pointed space (X,x) to the homotopy groups of the mapping space $\mathrm{hom}(C(X,x), M_n)$ is due to G. Segal [61]. The computation above was also done by J. Rosenberg in [51].

Lemma 4.2.4 *We have a canonical functor* $\iota : \mathbf{bu} \to \mathbf{E}$ *which compares the two theories. Whenever* B *is stable the induced map* $\mathbf{bu}(A, B) \to \mathbf{E}(A, B)$ *is an isomorphism.*

Proof: There is an isomorphism $\phi : K \otimes M_2 \to K \otimes K$ such that the following diagram commutes.

$$
\begin{array}{ccccc}
\mathbb{C} & \xrightarrow{\ \infty\ } & K & = & K \\
\downarrow {\scriptstyle 2} & & \downarrow {\scriptstyle id_K \otimes\ 2} & & \downarrow {\scriptstyle id_K \otimes\ \infty} \\
M_2 & \xrightarrow{i_\infty \otimes id_{M_2}} & K \otimes M_2 & \longrightarrow & K \otimes K
\end{array}
$$

Note that in this diagram all horizontal and the rightmost vertical arrow go to isomorphisms in $\mathbf{E}$-theory. Thence ι_2 is taken to an isomorphism. By the universal property of $\mathbf{bu}$, there is a unique transformation $\mathbf{bu} \to \mathbf{E}$.

This proves the lemma. The existence of the transformation is also obvious from the concrete pictures that we have of the corresponding theories.

Our next definition introduces the notion of a (strong) non-commutative cell complex which will replace finite CW-complexes for many purposes in the non-commutative setting. Strong cell complexes are certain type 1 C^*-algebras which are rather easy to understand from an operator algebraic point of view. Still, their geometry is interesting and captures a lot of phenomena which are quite different from the ones in the commutative setting.

Definition 4.2.5 *We define the category of non-commutative* <u>*cell complexes*</u> *to be the full sub-category of separable* C^**-algebras which contains* M_n *for all* $n \in \mathbb{N}$ *and is closed under*

- *extensions (i.e. 2-out-of-3 for any extension of separable* C^**-algebras),*
- *suspension and*
- *homotopy equivalence.*

The category of <u>*strict cell complexes*</u> *is defined in an analogous way but with the requirements that is has to be closed under suspension and extensions only.*

Lemma 4.2.6 *Let* A *be a cell complex. The* $\mathbf{bu}_*(A)$ *is finitely generated as a* $\mathbb{Z}[u]$*-module.*

Proof: This is true for M_n, $n \geqslant 1$. The result now follows by nice properties of the ring $\mathbb{Z}[u]$. Indeed, any sub-module and any quotient of a finitely generated $\mathbb{Z}[u]$ module is finitely generated. Furthermore, any extension of finitely generated ones is finitely generated. This proves the claim, since the definition of cell complexes is recursive.

4.2.1 Some spectral sequences

In this section we want to prove the existence of a Künneth spectral sequence and a universal coefficient (UC) spectral sequence for connective $\mathbf{E}$-theory. The methods provided in this section will also be useful later.

Our arguments use geometric projective resolutions which were introduced in algebraic topology by F. Adams (see [1]), and brought to the operator algebraic context in articles by J. Rosenberg and C. Schochet in [54].

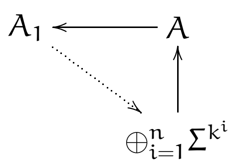
First of all, let us consider algebras A such that $\mathbf{bu}_*(A)$ is finitely generated as a $\mathbb{Z}[u]$-module. This is the case for all finite cell complexes by the last lemma in the preceding section. Denote the full sub-category of $\mathbf{bu}$ which is given by all algebras with finitely generated $\mathbf{bu}$-homology groups by $\mathbf{bu}^f$. In what follows we restrict our study to $\mathbf{bu}^f$. Most of the constructions have analogues in more general situations but we want to stick to the finitely generated case for simplicity. We denote the full sub-category which is given by cell complexes by $\mathbf{bu}^c$. There are obvious inclusions

$$\mathbf{bu}^c \subset \mathbf{bu}^f \subset \mathbf{bu}.$$

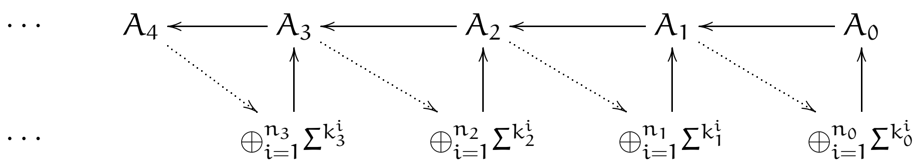
Let A be an object in $\mathbf{bu}^f$. Since $\mathbf{bu}_*(A)$ is finitely generated, there exist integers $k^1, \cdots, k^n \in \mathbb{Z}$ and a map $\oplus_{i=1}^n \Sigma^{k^i} \to A$ which induces a surjection from the free $\mathbf{bu}_*$-module $\mathbf{bu}_*(\oplus_{i=1}^n \Sigma^{k_i})$ onto $\mathbf{bu}_*(A)$. Completing this map to a triangle yields

$$\Sigma(A_1) \longrightarrow \oplus_{i=1}^n \Sigma^{k^i} \longrightarrow A \longrightarrow A_1$$

for which we use the convenient short notation



where the dotted arrow is of degree minus one. Clearly $\mathbf{bu}_*(A_1)$ is also finitely generated and we can continue this process. We obtain a diagram with $A = A_0$ as follows.



Note that we can choose the resolution such that n_3 is zero, since $\mathbb{Z}[u]$ is of projective dimension 2. We only want to consider geometric projective resolutions with this property. It implies that $A_i \to A_{i+1}$ is a $\mathbf{bu}$-equivalence whenever $i \geqslant 3$.

Proposition 4.2.7 *Let A be a separable C^*-algebra. Denote a choice of A_i which appeared in a geometric projective resolution of A for $i \geqslant 3$ by A^{loc}. There is a map $\psi^A : A \to A^{loc}$. The pair (A^{loc}, ψ^A) is unique up to* $\mathbf{bu}$*-equivalence.*

Proof: This is an adaption of a standard proof in homological algebra. Note that the definition of a triangulated category was chosen just to make the important arguments in homological algebra work.

Constructions of more general geometric projective resolutions and more general sequences of this type can be found in an article by Daniel Christensen [12].

We call this a projective geometric resolution of the algebra A, since it induces a sequence

$$\cdots \longrightarrow \mathbf{bu}_{*+2}(\oplus_{i=1}^{n_2}\Sigma^{k_2^i}) \longrightarrow \mathbf{bu}_{*+1}(\oplus_{i=1}^{n_1}\Sigma^{k_1^i}) \longrightarrow \mathbf{bu}_{*}(\oplus_{i=1}^{n_0}\Sigma^{k_0^i}) \longrightarrow \mathbf{bu}_{*}(A)$$

which is a projective (indeed, free, since all projectives over $\mathbb{Z}[u] = \mathbf{bu}_*$ are free) resolution of $\mathbf{bu}_*(A)$.

Remark 4.2.8 *Similarly we can obtain a dual projective geometric resolution of* A *which provides us with a projective resolution of* $\mathbf{bu}^*(A)$ *as a* $\mathbf{bu}_*$*-module after applying the contravariant functor* $\mathbf{bu}^*$ *to it. Therefore duals of the following theorems can be obtained.*

We can apply several functors to this projective geometric resolution in order to obtain exact couples that give rise to homology and cohomology spectral sequences. Before doing so, we want to derive another result which will be of some importance.

Theorem 4.2.9 *Let* A *be a separable* C^**-algebra with* $\mathbf{bu}_*(A)$ *finitely generated as a* $\mathbb{Z}[u]$*-module. There is a cell complex* A^c *and a map* $A^c \overset{A}{\to} A$ *such that the induced map* $\mathbf{bu}_*(A^c) \to \mathbf{bu}_*(A)$ *induces an isomorphism. The pair* (A^c, ϕ^A) *is unique up to a unique* $\mathbf{bu}$*-equivalence.*

Dually there is a cell complex A_c *and a map* $A \overset{A}{\to} A_c$ *such that the induced map* $\mathbf{bu}^*(A_c) \to \mathbf{bu}^*(A)$ *is an isomorphism. Again, the pair* (A_c, ϕ_A) *is unique up to* $\mathbf{bu}$*-equivalence.*

Proof: We only prove the first part of the statement, since the second part is dual. Choose a geometric projective resolution (with $A_0 = A$). Consider the cone of the map

$$\oplus_{i=1}^{n_2}\Sigma^{k_2^i} \longrightarrow \oplus_{i=1}^{n_1}\Sigma^{k_1^i}$$

and denote it by B_1. Since the composition $\oplus_{i=1}^{n_2}\Sigma^{k_2^i} \to \oplus_{i=1}^{n_1}\Sigma^{k_1^i} \to A_1$ is zero, there is an induced map $B_1 \to \oplus_{i=1}^{n_0}\Sigma^{k_0^i}$. Its cone B_0 maps to A_0, since, again, the composition $B_1 \to \oplus_{i=1}^{n_0}\Sigma^{k_0^i} \to A$ is zero. At the level of $\mathbb{Z}[u]$-modules it is easily seen that $B_0 \to A$ is an isomorphism. On the other hand $B_0 = A^c$ is a cell complex. The uniqueness of the pair is obvious, since any transformation

$$\mathbf{bu}(?, A^c) \to \mathbf{bu}(?, A)$$

of functors which is an isomorphism for $\mathbb{C}$ is an isomorphism for all cell complexes by excision. Hence A^c plus the transformation to A is unique by the Yoneda lemma. This finishes the proof.

Proposition 4.2.10 *Let* A *be a separable* C^**-algebra with* $\mathbf{bu}_*(A)$ *finitely generated. There is a distinguished triangle*

$$A^c \xrightarrow{A} A \xrightarrow{A} A^{loc} \longrightarrow \Sigma(A^c)$$

with A^c *and* A^{loc} *as above.*

Proof: This is just how we constructed A^c. For a complete proof one needs the octahedron axiom of triangulated categories. We do not want to give complete details.

We will come back to the consequences of theorem 4.2.9 and proposition 4.2.10 in the next subsection.

Theorem 4.2.11 (Künneth spectral sequence) *Let* A, B *and* C *be separable* C^**-algebras. Moreover let* $\mathbf{bu}_*(A)$ *be finitely generated as a* $\mathbf{bu}$*-module. There is a homology spectral sequence with*

$$E^2_{p,q} = \mathrm{Tor}^{p,q}_{\mathbb{Z}[u]}(\mathbf{bu}_*(B, C), \mathbf{bu}_*(A))$$

converging strongly to $\mathbf{bu}_*(B, A^c \otimes C)$.

Proof: Apply the functor $\mathbf{bu}_*(B, ? \otimes C)$ to a geometric projective resolution of A. We define an exact couple (see section 5.9 in [69] for definitions and properties) as follows.

$$\begin{array}{ccc} \oplus_{p,q}\mathbf{bu}_{p+q}(B, A_p \otimes C) & \xrightarrow{\quad i \quad} & \oplus_{p,q}\mathbf{bu}_{p+q}(B, A_p \otimes C) \\ & {}_{k}\nwarrow \qquad \swarrow_{j} & \\ & \oplus_{p,q}\mathbf{bu}_{p+q}(B, \oplus_{i=1}^{n_p}\Sigma^{k_p^i} \otimes C) & \end{array}$$

In this diagram i is of bi-degree $(1,-1)$, j is of bi-degree $(-1,0)$ and k is of bi-degree $(0,0)$. We get a homology spectral sequence (see e.g. [69] pp. 153) with

$$E^2_{p,q} = \mathrm{Tor}^{p,q}_{\mathbb{Z}[u]}(\mathbf{bu}_*(B, C), \mathbf{bu}_*(A))$$

using that

$$\mathbf{bu}_*(B, \oplus_{i=1}^{n_p}\Sigma^{k_p^i} \otimes C) = \mathbf{bu}_*(B, C) \otimes_{\mathbb{Z}[u]} \mathbf{bu}_*(\oplus_{i=1}^{n_p}\Sigma^{k_p^i}).$$

(In this notation q denotes the degree of the shift and p the dimension of the Tor-functor. For example, $\mathrm{Tor}^{0,q}$ is just $\otimes^q$, i.e. the part of the tensor product which is of degree q.)

In an ideal world this spectral sequence would converge to colim $_i\mathbf{bu}_*(B, A_i \otimes C) = \mathbf{bu}_*(B, A \otimes C)$. But this is obstructed by the possibility that $\lim_i \mathbf{bu}(B, A_i \otimes C) = \mathbf{bu}_*(B, A^{loc} \otimes C)$ does not vanish even though $\mathbf{bu}_*(A^{loc})$ of course vanishes. (Note that clearly $\lim^1_i \mathbf{bu}(B, A_i \otimes C) = 0$)

Nevertheless if we replace A by A^c the spectral sequence does not change but it now clearly $\lim_i \mathbf{bu}(B, A_i \otimes C) = \mathbf{bu}_*(B, (A^c)^{loc} \otimes C)$, since $(A^c)^{loc} = 0$. Note that we obviously have that $\lim^1_i \mathbf{bu}(B, A_i \otimes C) = 0$ and hence the spectral sequence converges conditionally to $\mathbf{bu}_*(B, A^c \otimes C)$ (see [9]). The spectral sequence also converges strongly, since the maps $\mathbf{bu}_*(B, A_{i+1} \otimes C) \to \mathbf{bu}_*(B, A_{i+1} \otimes C)$ become isomorphisms if $i \geqslant 3$ for a suitable geometric resolution, since $\mathbb{Z}[u]$ is of projective dimension 2. This proves the theorem.

Note that there might be other reasons for $\lim_n \mathbf{bu}_*(B, A_n \otimes C)$ to vanish. Of course, it always vanishes if B and C are cell complexes. In this case we have that $\mathbf{bu}_*(B, A^c \otimes C) \cong \mathbf{bu}_*(B, A \otimes C)$.

Theorem 4.2.12 (UC spectral sequence) *Let A and B be separable C^*-algebras. Moreover let $\mathbf{bu}_*(A)$ be finitely generated as a $\mathbb{Z}[u]$-module. There is a cohomology spectral sequence with*

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{\mathbb{Z}[u]}(\mathbf{bu}_*(A), \mathbf{bu}_*(B))$$

converging strongly to $\mathbf{bu}_*(A^c, B)$.

Proof: The proof proceeds similarly to the proof of theorem 4.2.11. We apply the functor $\mathbf{bu}_*(?, B)$ to a geometric projective resolution of A. The proof here uses that

$$\mathbf{bu}(\oplus_{i=1}^{n_l} \Sigma^{k_l^i}, B) = \mathrm{hom}_{\mathbb{Z}[u]}(\mathbf{bu}_*(\oplus_{i=1}^{n_l} \Sigma^{k_l^i}), \mathbf{bu}_*(B)).$$

(A reference for the co-homological spectral sequence constructed from an exact couple is theorem 2.8 in the book by J. McCleary [39].) The arguments which are used to obtain strong convergence are completely analogous to the arguments in the proof of theorem 4.2.11.

Remark 4.2.13 *Note that the ring $\mathbb{Z}[u]$ is of projective dimension 2. This implies that all $\mathrm{Tor}^{p,q}_{\mathbb{Z}[u]}$ and $\mathrm{Ext}^{p,q}_{\mathbb{Z}[u]}$ groups vanish whenever $p \geqslant 3$. Therefore the E_2-term of the spectral sequence carries only one non-vanishing differential and the E_3-term is already equal to the E_∞-term.*

4.2.2 Duality

This part contains a short discussion of duality in the context of **bu**. We show that certain objects are self-dual with respect to the natural notion of duality in **bu**. A short discussion of traces and the natural notions of Euler characteristic and Lefshetz number is also contained in this part.

For a definition of duality in the context of triangulated categories see appendix A.3. We start with the following observation which uses lemma A.3.3.

Proposition 4.2.14 *Let* A *be a separable* C^**-algebra such that* $\mathbf{bu}_*(A)$ *is finitely generated. The algebra is dualizable in* $\mathbf{bu}$ *if it is* $\mathbf{bu}$*-equivalent to a cell complex.*

Proof: Since the generating objects in the category of cell-complexes are dualizable and dualizability is closed under suspensions and extensions (see lemma A.3.3), we have nothing to prove.

Before we start, we have to recall some easy facts from homological algebra. We use these considerations to define a concrete trace on self-maps of objects in $\mathbf{bu}$. The trace takes values in $\mathbb{Z}[u]$. Later, we compare the trace with a trace that comes from the intrinsic notion of duality in $\mathbf{bu}^c$ and show that the traces coincide.

Let M be a finitely generated free module over a graded commutative ring R. Let $f : M \to M$ be a R-module homomorphism. One can define a trace by picking generators of M as free module, writing the homomorphism in matrix form and taking the sum of the diagonal terms weighted by $(-1)^k$ where k is the degree of the generator of the free module. The same procedure works for projective modules (i.e. summands of free ones). If M is not free (but still finitely generated) the definition of a sufficiently interesting trace is more difficult or even impossible. If the ring satisfies the following condition

'Any sub-module of a finitely generated free module is finitely generated.'

we can proceed as follows. The condition ensures that there is a resolution consisting of finitely generated free modules for any finitely generated module. The self-map of the module implies the existence of a self map of the resolution (which is unique up to chain homotopy). Taking the alternating sum of the traces of this self-map we get (either an infinite sum of possibly non-zero terms or in case of finite dimension) an element in in the ring R which does neither depend on the choice of the resolution nor the choice of the lifting of the self-map. This is an easy exercise in homological algebra.

Note that the ring $\mathbb{Z}[u]$ clearly satisfies the condition above. Furthermore, since any projective module over $\mathbb{Z}[u]$ is already free and $\mathbb{Z}[u]$ is of finite projective dimension, we have to consider only finite sums by a 'syzygy' argument. Note further that this is an extension of the concept of trace on finitely generated free modules, since they constitute their own free resolutions.

Definition 4.2.15 *Let* M *be a finitely generated* $\mathbb{Z}[u]$*-module and let* $f : M \to M$ *be* $\mathbb{Z}[u]$*-module map. We denote by* $\tau_c(f) \in \mathbb{Z}[u]$ *the* <u>*concrete trace*</u> *which we can assign to* f *by the procedure discusses above.*

Let $f \in \mathbf{bu}_*(A, A)$ *be a self-map (in the graded sense) of* A. *We define the concrete trace of* f *to by* $\tau_c(\mathbf{bu}_*(f))$.

Note that there is another trace which is defined using the notion of duality in the triangulated category. For a definition see appendix A.3. The next theorem identifies the two notions of trace.

Theorem 4.2.16 *The trace which is given by duality and the concrete trace coincide for cell complexes.*

Proof: This follows, since the traces coincide on generators and satisfy additivity (see appendix A.3 for a definition). The duality trace satisfies additivity, since we can identify this trace with a trace on the homotopy category of $\underline{\text{bu}}$-module spectra which in turn comes from a monoidal model category (see section 5). This result is proved in an article by P. May [38]. The concrete trace satisfies additivity by an easy computation.

One might ask whether one can proof the additivity of the duality trace only using operator theory. We do not try to prove this, since the concrete trace is the more interesting and natural one anyway. In order to organize all the required homotopies it seems to be useful to introduce the structure of a model category on a category of operator algebras, but this has not been achieved yet.

Definition 4.2.17

- *The Euler characteristic* $\chi(A)$ *of a dualizable algebra* A *is defined to be* $\tau(\mathbf{bu}_*(\mathrm{id}_A)) \in \mathbb{Z}$.
- *Let* $f : A \to A$ *be a self-map of a dualizable algebras* A. *Its Lefshetz number is defined to be* $\tau(f) \in \mathbb{Z}$.

Corollary 4.2.18 *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an extension of C^**-algebras. The following relation between the Euler characteristics*

$$\chi(B) = \chi(A) + \chi(C)$$

holds.

Proof: This follows from the additivity of the trace.

A similar additivity result can be obtained for Lefshetz numbers. Since we have no applications, we do not want to state it. The Euler characteristic is the most basic invariant which one can assign to cell complex. We are going to compute the Euler characteristic of a cell complex in theorem 4.5.22.

Now, we come a different aspect of duality. It is the natural notion of self-duality in a triangulated category. Most of the dualities (i.e. Poincaré duality, Atiyah duality, Spanier-Whitehead duality etc.) that appear in algebraic topology can be traced to a self-duality in a certain triangulated category. For a comprehensive guided tour through the history of duality in algebraic topology see [6]. In our context we denote the duality functor by $D : \mathbf{bu}^c \to (\mathbf{bu}^c)^{op}$.

Definition 4.2.19 *A dualizable object* A *is said to satisfy Poincaré duality if there is an integer* $n \in \mathbb{N}$ *and an equivalence between* A *and* $\Sigma^n(D(A))$.

Examples of objects satisfying Poincaré duality in **bu** are of course given by manifolds with stably trivial normal bundle, e.g. Lie groups etc. The next proposition shows that there are also natural non-commutative examples which satisfy Poincaré duality in **bu**.

Denote the cone of the Bott map by Z. In section 2.3 we study the properties of Z more extensively.

Proposition 4.2.20 *The cone of the Bott map satisfies Poincaré duality. Indeed, there is the following isomorphism.*

$$Z \to \Sigma^5(D(Z))$$

Proof: It is a general fact that the dual of a distinguished triangle in $\mathbf{bu}^c$ is distinguished (in $\mathbf{bu}^c$). The cone of the Bott map appears in the triangle

$$\Sigma^2 \xrightarrow{u} \Sigma^4 \longrightarrow Z \longrightarrow \Sigma$$

The dual triangle is just

$$\Sigma^{-2} \xleftarrow{D(u)} \Sigma^{-4} \longleftarrow D(Z) \longleftarrow \Sigma^{-1}$$

If we shift the triangle to the left and suspend (and switch the direction of arrows) we obtain the following the following one (this uses condition 4 in the definition of triangulated categories).

$$\Sigma^2 \xrightarrow{{}^5(D(u))} \Sigma^4 \longrightarrow \Sigma^5(D(Z)) \longrightarrow \Sigma$$

It is clear that $\Sigma^5(D(u))$ is either equal to u or to $-u$. In either case we obtain a morphism $Z \to \Sigma^5(D(Z))$ by condition 5 in the definition of triangulated categories. It has to be an isomorphism by the Yoneda-lemma. This finishes the proof.

A more detailed discussion of properties of the cone of the Bott map is contained in section 4.5.

4.2.3 Localization

In the proof of theorem 4.2.12 we saw that the convergence of the UC spectral sequence is obstructed by the possibility that $\mathbf{bu}_*(A, B)$ is not zero even though $\mathbf{bu}_*(B)$ is zero. Moreover we saw that the obstruction vanishes if A is a cell complex. In this section we want to study the local objects of the category of cell complexes and the localization with respect to this sub-category. It will turn out to be a Bousfield localization. The corresponding quotient category behaves rather nicely with respect to convergence of the UC spectral sequences as will become clear.

Most of the notations and definitions which we are using in this section can be found the book by A. Neeman [43], chapter 8.

Recall that we denoted the full triangulated sub-category which is given by cell complexes (and their desuspensions) by $\mathbf{bu}^c$.

Definition 4.2.21 *An algebra A in* $\mathbf{bu}$ *is called* $\mathbf{bu}^c$*-local for every (de)suspension C of a cell-complex all morphisms* $\mathbf{bu}(C, A) \ni f : C \to A$ *vanish.*

Proposition 4.2.22 *An algebra A is* $\mathbf{bu}^c$*-local if and only if* $\mathbf{bu}_*(\mathbb{C}, A) = 0$.

Proof: One direction is obvious, since all Σ^n are cell complexes. The other direction follows by induction, since the definition of cell complexes is recursive. This finishes the proof.

Definition 4.2.23 *The full sub-category of* $\mathbf{bu}^c$*-local objects in* $\mathbf{bu}^f$ *is a thick sub-category of* $\mathbf{bu}^f$*. We denote the category of* $\mathbf{bu}^c$*-local objects in* $\mathbf{bu}^f$ *by* $\mathbf{bu}^l$*.*

By theorem 9.1.13 in [43] and proposition 4.2.10 the Verdier localization functor $\mathbf{bu}^f \to \mathbf{bu}^f/\mathbf{bu}^c$ has a right adjoint. Those localizations are commonly called Bousfield localizations (see chapter 8 in [43]). Furthermore, by remark 9.1.15 in [43] we have that the localization functor $\mathbf{bu}^f \to \mathbf{bu}^f/\mathbf{bu}^l$ has a left adjoint. Indeed, the composition

$$\mathbf{bu}^l \subset \mathbf{bu}^f \to \mathbf{bu}^f/\mathbf{bu}^c$$

is an equivalence of triangulated categories by theorem 9.1.16 in [43]. Dually also the composition

$$\mathbf{bu}^c \subset \mathbf{bu}^f \to \mathbf{bu}^f/\mathbf{bu}^l$$

is an equivalence of triangulated categories.

The category $\mathbf{bu}^f/\mathbf{bu}^l$ has some nice properties. First of all, the objects are just separable C^*-algebras and it therefore constitutes a triangulated homology theory on the category of separable C^*-algebras. Since any map that induces an isomorphism in bu-homology has to be a bu-equivalence (This is just, since the cone of the map has vanishing bu-homology and is therefore isomorphic to zero in $\mathbf{bu}^f/\mathbf{bu}^l$.), the UC spectral sequence converges always to the correct bivariant group (compare theorem 4.2.12).

The following lemma is a general fact for Bousfield localizations.

Lemma 4.2.24 *The natural map* $\mathbf{bu}^f(A, B) \to (\mathbf{bu}^f/\mathbf{bu}^l)(A, B)$ *is an isomorphism for all* A *if and only if* B *is isomorphic in* $\mathbf{bu}^f$ *to (a desuspension of) a cell complex.*

This is just an application of lemma 9.15 and corollary 9.1.14 in [43].

This, in particular, implies that the UC spectral sequence for $\mathbf{bu}^f/\mathbf{bu}^l$ does not give any new information. It is just transported in a framework which is a little bit more convenient. We do not repeat the statements about the existence of certain spectral sequences which we have made precise in section 4.2.1.

The category $\mathbf{bu}^c$ is of course a triangulated monoidal category. Although $\mathbf{bu}^f/\mathbf{bu}^l$ is equivalent to $\mathbf{bu}^c$ we were not able to show that $\mathbf{bu}^f/\mathbf{bu}^l$ is monoidal as well. This is due to the fact that $\mathbf{bu}^l$ is not a priori a monoidal sub-category, since a priori $\mathbf{bu}_*(A) = 0$ does not imply that $\mathbf{bu}_*(A \otimes A) = 0$. Another way of looking at this phenomenon is to note that the cell replacement which can be made functorial by the remarks above (this is a non-trivial fact) is not necessarily a monoidal. Again, the possibility that $\mathbf{bu}_*(A) = 0$ although $\mathbf{bu}_*(A \otimes A) \neq 0$ obstructs the existence of a monoidal replacement functor.

4.3 Examples and computations

4.3.1 Matrix bundles

Note that the group of algebra automorphisms of M_n is $PSU(n)$ (see lemma C.1.12) which acts by conjugation. Therefore every principal $PSU(n)$-bundle over X gives rise to a bundle over X with fiber M_n. Since $PSU(n)$ consists of algebra automorphisms, the continuous sections of this bundle carry a natural algebra structure. Indeed, one checks easily that the space of continuous sections forms a C^*-algebra. A C^*-algebra of this kind is called n-matrix bundle (or n-homogenous) algebra.

Although these bundles are locally trivial over X they may be globally non-trivial. Indeed, the isomorphism classes (over X) of such n-matrix bundles are in correspondence with $[X, BPSU(n)]$.

Definition 4.3.1 *Let* X *be a compact Hausdorff topological space and* $\phi : X \to BPSU(n)$ *be a continuous map. Denote by* $Prin(\phi)$ *the principal* $PSU(n)$*-bundle which is classified by* ϕ*. Denote by* $E(\phi)$ *the algebra of continuous sections of the bundle* $Prin(\phi) \times_{PSU(n)} M_n \to X$.

Now, we are computing the isomorphism type of the section algebras in the triangulated homology theory **bu**. It turns out that the section algebras are all naturally isomorphic to the algebra of continuous functions on the base space. This implies that it is not possible to detect different bundles using connective **E**-theory.

Lemma 4.3.2 *Let* $n < m$ *be integers and* $\iota_{n,m-n} : PSU(n) \times PSU(m-n) \to PSU(m)$ *be canonical inclusion. Let* $\phi_1 : X \to BPSU(n)$ *and* $\phi_2 : X \to BPSU(m-n)$ *be continuous maps.*

There are induced maps $\psi_1 : E(\phi_1) \to E(B(\iota_{n,m-n} \circ (\phi_1 \times \phi_2)))$ *and* $\psi_2 : E(\phi_2) \to E(B(\iota_{n,m-n}) \circ (\phi_1 \times \phi_2))$ *between the algebras of sections, which induce equivalences in the category* **bu**.

Proof: A bundle is given by patching information. Consider a finite cover $\{X_i, 1 \leqslant i \leqslant r\}$ consisting of closed subsets of X such that the restrictions of $E(\phi_1)$ and $E(\phi_2)$ to the X_i are trivial. The patching information consists of isomorphisms over X

$$\alpha_{1,i,j} : M_n(C(X_i \cap X_j)) \to M_n(C(X_i \cap X_j))$$

for $E(\phi_1)$ and

$$\alpha_{2,i,j} : M_{m-n}(C(X_i \cap X_j)) \to M_{m-n}(C(X_i \cap X_j))$$

for $E(\phi_2)$. The bundle $E(B(\iota_{n,m-n}) \circ (\phi_1 \times \phi_2))$ clearly corresponds to the patching information

$$\mathrm{diag}(\alpha_{1,i,j}, \alpha_{2,i,j}) : M_m(C(X_i \cap X_j)) \to M_m(C(X_i \cap X_j)).$$

The argument goes roughly as follows. From this concrete description we see that the local maps $\iota : M_n(C(X_i)) \to M_m(C(X_i))$ can be glued together to a global map $\psi_1 : E(\phi_1) \to E(B(\iota_{n,m-n}) \circ (\phi_1 \times \phi_2))$. At the same time using Mayer-Vietoris sequences (see proposition 2.3.12) and the 5-lemma one can lift the local **bu**-equivalence to a global **bu**-equivalence.

In fact, to prove the last assertion we make an induction argument. We prove the existence of a natural map which induces a **bu**-equivalence for restrictions of the bundle to any set of the form $A_1 \cup \cdots \cup A_k$ for any positive k, where A_i is of the form $X_{i_1} \cap \cdots \cap X_{i_{l(i)}}$. The induction goes over k. The case $k = 1$ is obvious. Now assume the truth of the assertion for $l < k$ and consider a set of the form $A_1 \cup \cdots \cup A_k$. Denote the restriction of the bundle $E(\psi)$ to a closed subset $Y \subset X$ by $E(\psi)(Y)$. We have a pull-back square

$$\begin{array}{ccc} E(\phi_1)(A_1 \cup \cdots \cup A_k) & \longrightarrow & E(\phi_1)(A_1 \cup \cdots \cup A_{k-1}) \\ \downarrow & & \downarrow \\ E(\phi_1)(A_k) & \longrightarrow & E(\phi_1)((A_1 \cup \cdots \cup A_{k-1}) \cap A_k) \end{array}$$

Note that we can apply the assumption to the sets $A_1 \cup \cdots \cup A_{k-1}$, A_k and $(A_1 \cup \cdots \cup A_{k-1}) \cap A_k$. By the universal property of the pull-back, this implies that there exists a natural map

$$E(\phi_1)(A_1 \cup \cdots \cup A_k) \to E(B(\iota_{n,m-n}) \circ (\phi_1 \times \phi_2))(A_1 \cup \cdots \cup A_k).$$

Considering the Mayer-Vietoris sequences coming with the pull-back squares we conclude that it has to be a bu-equivalence by the naturality of the Mayer-Vietoris sequence, the 5-lemma and the Yoneda-lemma. This finishes the proof, since $X = X_1 \cup \cdots \cup X_r$.

Theorem 4.3.3 *Any two matrix bundles over a compact Hausdorff space* X *are* bu-*equivalent. In particular, all such bundles are equivalent to* $C(X)$.

Proof: This follows from the last lemma.

4.3.2 Compact locally Hausdorff topological spaces

In this section we want to define a functor which essentially assigns to any compact, pointed, locally Hausdorff topological space a C^*-algebra up to bu-equivalence. The prim spectrum with the hull-kernel topology of the constructed C^*-algebras will be homeomorphic to the compact locally Hausdorff space we started with.

To give more precise statements we have to make some definitions. This construction appears in A. Connes work for Hausdorff manifolds [14]. The attempt to construct C^*-algebras with a prescribed non-Hausdorff spectrum is old and early results were obtained by J. Dixmier [25]. The topological spaces which are considered in this section are not much more difficult to understand than Hausdorff spaces themselves, and the fact that there exist C^*-algebras with such as prime spectrum, is well known.

Definition 4.3.4 *Let* $\mathcal{T}$ *be the category with objects triples* $(X, x, \{U_i, i \in \Delta\})$ *consisting of*

- *a compact locally Hausdorff topological space* X,
- *a point* $x \in X$ *and*
- *a finite set* Δ *and a cover* $\{U_i, i \in \Delta\}$ *of* $X - \{x\}$ *by open Hausdorff subsets.*

The morphisms in this category from $(X, x, \{U_i, i \in \Delta\})$ *to* $(X', x', \{U'_i, i \in \Delta'\})$ *are given by pairs* (f, γ) *where*

- $f : X \to X'$ *is a continuous and base-point preserving mapping,*
- $\gamma : \Delta' \to \Delta$ *is an injective mapping of sets which satisfies*
- $f^{-1}(U'_i) \subset U_{\ (i)}$ *for all* $i \in \Delta'$.

The next is to define a functor $A : \mathcal{T}^{\mathrm{op}} \to s\mathcal{C}$. Given the data $(X, x, \{U_i, i \in \Delta\})$, we construct a certain C^*-algebra as a sub-algebra of $B(\oplus_{i\in}\ L^2(U_i^+))$. We consider those $\Delta \times \Delta$-matrices where the (i, j)-entry is given by the multiplication operator induced by an element in $C_0(U_i \cap U_j)$. In order to understand the multiplication we have to consider the multiplication of an (i, j)-entry with a (l, k)-entry. The product is zero if $j \neq l$ and otherwise given by the ordinary pointwise multiplication which defines a map

$$C_0(U_i \cap U_j) \times C_0(U_j \cap U_k) \to C_0(U_i \cap U_j \cap U_k) \hookrightarrow C_0(U_i \cap U_k).$$

The involution is given by conjugating the (i, j)-entry and considering it as a (j, i)-entry.

In order to define a functor we have to say a word about the morphisms between objects in $\mathcal{T}$. Given a morphism $(f, \gamma) : (X, x, \{U_i, i \in \Delta\}) \to (X', x', \{U'_i, i \in \Delta'\})$, we define a mapping

$$A(X', x', \{U'_i, i \in \Delta'\}) \to A(X, x, \{U_i, i \in \Delta\})$$

by sending the entry at $(i, j) \in \Delta' \times \Delta'$ which is given by some $g \in C_0(U'_i \cap U'_j)$ to a $(\gamma(i), \gamma(j))$-entry given by its image in $C_0(U_{\ (i)} \cap U_{\ (j)})$. To see that this algebra contains a natural image of g, note that there is an open inclusion $f^{-1}(U'_i \cap U'_j) = f^{-1}(U'_i) \cap f^{-1}(U'_j) \subset U_{\ (i)} \cap U_{\ (j)}$ and consider the pointed maps

$$(U_i \cap U_j)^+ \to (f^{-1}(U'_i \cap U'_j))^+ \xrightarrow{f} (U'_i \cap U'_j)^+$$

which induce the required map $C_0(U'_i \cap U'_j) \to C_0(U_{\ (i)} \cap U_{\ (j)})$.

This map clearly respects the involution and multiplication of (i, j)-entries with (j, k)-entries. The injectivity of γ ensures that it also respects multiplication of (i, j)-entries with (l, k)-entries in case that $j \neq l$. The functoriality of this assignment is obvious.

We denote the intersection of open sets $U_{i_1}, \ldots, U_{i_n}$ by $U_{i_1,\ldots,i_n}$ and use the convenient notation $M\ \ [C_0(U_{i,j})]_{i,j}$ for the image of $(X, x, \{U_i, i \in \Delta\})$ in $s\mathcal{C}$ under the functor A.

Our goal is to show that the bu-type of the algebra $M\ \ [C_0(U_{i,j})]_{i,j}$ is independent of the choice of the cover. Up to now this is not even obvious if X itself is Hausdorff. Let us consider this case first. We consider the algebras $C_0(U_{i_1,\ldots,i_k})$ naturally as sub-algebras of $C(X, x)$.

Theorem 4.3.5 *Let* $(X, x, \{U_i, i \in \Delta\})$ *be a pointed compact Hausdorff space with a cover. Fix an enumeration of* Δ. *The natural map* $M\ \ [C_0(U_{i,j})]_{i,j} \to M_{|\ |}C(X, x)$ *is a* bu*-equivalence. In particular, the* bu*-type of* $M\ \ [C_0(U_{i,j})]_{i,j}$ *is independent of the cover.*

Before proving theorem 4.3.5, we need some lemmas.

Lemma 4.3.6 *Let* $(X, x, \{U_i, i \in \Delta\})$ *be a triple as above. Let* $U_{(1,i)} = U_{(2,i)} = U_i$ *for* $i \in \Delta$. *The image of the triple* $(X, x, \{U_{(k,i)}, (k,i) \in \{1,2\} \times \Delta\})$ *under* A *is naturally isomorphic to* $M_2 M\ [C_0(U_{i,j})]_{i,j}$. *The natural map* $(X, x, \{U_{(k,i)}, (k,i) \in \{1,2\} \times \Delta\}) \to (X, x, \{U_i, i \in \Delta\})$ *which is the identity on* X *and maps* $\Delta \ni i \mapsto (1,i)$ *corresponds to the inclusion into the upper left corner of the* 2×2*-matrices under this isomorphism.*

Proof: The proof is almost obvious. One maps the $((k,i),(m,l))$-entry to the $e_{k,m} \otimes (i,l)$-entry in $M_2 M\ [C_0(U_{i,j})]_{i,j}$. The identification of the map is easily checked.

Lemma 4.3.7 *Let* $(X, x, \{U_i, i \in \Delta\})$ *be a triple as above. Let* $\Delta \cup \{k\}$ *be a disjoint union and let* U_k *be an open Hausdorff subset which is contained in* U_m *for some* $m \in \Delta$. *The natural map* $(X, x, \{U_i, i \in \Delta \cup \{k\}\}) \to (X, x, \{U_i, i \in \Delta\})$ *which is given by the identity on* X *and the inclusion* $\Delta \hookrightarrow \Delta \cup \{k\}$ *induces a* **bu**-*equivalence under the functor* A.

Proof: There is a map $(X, x, \{U_{(l,i)}, (l,i) \in \{1,2\} \times \Delta\}) \to (X, x, \{U_i, i \in \Delta \cup \{k\}\})$ which is again given by the identity on X but this time it maps $\Delta \ni i \mapsto (1,i)$ and $k \mapsto (2,m)$. This is clearly injective and satisfies the required conditions, since $U_k \subset U_m$ by assumption.

By lemma 4.3.6, the composition

$$M\ [C_0(U_{i,j})]_{i,j} \to M_{\ \cup\{k\}}[C_0(U_{i,j})]_{i,j} \to M_{\{1,2\}\times\ }[C_0(U_{i,j})]_{i,j}$$

is just the inclusion of $M\ [C_0(U_{i,j})]_{i,j}$ into the upper left corner of $M_2 M\ [C_0(U_{i,j})]_{i,j}$ and therefore a **bu**-equivalence. One can check that the other composition is homotopic to the inclusion of $M_{\ \cup\{k\}}[C_0(U_{i,j})]_{i,j}$ into $M_2 M_{\ \cup\{k\}}[C_0(U_{i,j})]_{i,j}$ and hence induces a **bu**-equivalence as well.

Indeed,

$$M_{\ \cup\{k\}}[C_0(U_{i,j})]_{i,j} \to M_{\{1,2\}\times\ }[C_0(U_{i,j})]_{i,j} \to M_{\{1,2\}\times(\ \cup\{k\})}[C_0(U_{i,j})]_{i,j}$$

$$\xrightarrow{\equiv} M_2 M_{\ \cup\{k\}}[C_0(U_{i,j})]_{i,j}$$

differs from the natural inclusion into the upper left corner just by the fact that k is sent to $(2,m)$ instead of $(1,k)$. Note that the map factorizes through the algebra $M_{\{1,2\}\times(\ \cup\{k\})}[C_0(U_{i,j})]_{i,j}$ where $U_{(2,m)} = U_k$. Since in this algebra $U_{(1,k)} = U_{(2,m)}$, this implies that we can change our map by a homotopy given by a rotation into the inclusion into the upper left corner. This finishes the proof. We can now proceed with the proof of theorem 4.3.5

Proof of theorem 4.3.5: We prove the result by induction on the cardinality of Δ. The case $|\Delta| = 1$ is of course obvious. Suppose that the theorem is true if the cardinality of Δ is less that n.

Consider a triple $(X, x, \{U_k, k \in \Delta\})$ such that Δ has precisely n elements. Take an element U_k of the cover. Fix an enumeration of Δ. Consider the following diagram of extensions.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M\ \ [C_0(U_{k,i,j})]_{i,j} & \longrightarrow & M\ \ [C_0(U_{i,j})]_{i,j} & \longrightarrow & M\ \ _{-\{k\}}[C_0(U_{i,j} - U_k)]_{i,j} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_n C_0(U_k) & \longrightarrow & M_n C(X, x) & \longrightarrow & M_n(C(X - U_k, x) & \longrightarrow & 0
\end{array}$$

The left vertical map is a bu-equivalence by iterated application of lemma 4.3.7. Indeed, all sets $U_{i,k}$ are contained in U_k. By lemma 4.3.7, this implies that the inclusion $C_0(U_k) \to M\ \ [C_0(U_{k,i,j})]_{i,j}$ is a bu-equivalence. The composition $C_0(U_k) \to M\ \ [C_0(U_{k,i,j})]_{i,j} \to M_n(C_0(U_k))$ is just a rank 1 inclusion and hence also a bu-equivalence. This implies our assertion about the left vertical map.

The right vertical map is a composition

$$M\ \ _{-\{k\}}[C_0(U_{i,j} - U_k)]_{i,j} \to M_{n-1}(C(X - U_k, x)) \to M_n(C(X - U_k, x)).$$

The first map is a bu-equivalence by induction, since the cover has cardinality strictly less than n. The second map is obviously a bu-equivalence. Hence the proof is finished, since the middle vertical map has to be a bu-equivalence as well.

Theorem 4.3.5 can also be obtained by a different method. Let $(X, x, \{U_i, i \in \Delta\})$ a triple as above. Assume that x is isolated. Choose a partition of unity, say ϕ_i for $i \in \Delta$, subordinate to the cover $\{U_i, i \in \Delta\}$. By sending $f \mapsto [f \cdot \phi_i \phi_j)^{\frac{1}{2}}]$, we can define a map $C(X) \to M\ \ [C_0(U_{i,j})]_{i,j}$. The properties of the partition of unity ensure that this is a $*$-homomorphism. Furthermore, fixing an enumeration of the set Δ, the composition $C(X) \to M\ \ [C_0(U_{i,j})]_{i,j} \to M_{|\ |}C(X)$ is a rank one inclusion. It is easily shown that it is a bu-equivalence. Similarly, by a short argument, the other composition is a bu-equivalence. Up to some suitable choices, this identifies the algebra $M\ \ [C_0(U_{i,j})]_{i,j}$ as a corner in $M_{|\ |}C(X, x)$ (i.e. there is a projection $p = [(\phi_i \phi_j)^{\frac{1}{2}}] \in M_{|\ |}C(X, x)$ such that $M\ \ [C_0(U_{i,j})]_{i,j} = pM_{|\ |}C(X, x)p$).

It remains to prove that the choice of the cover does not matter for general locally Hausdorff spaces. This is now an easy consequence of the following lemma.

Lemma 4.3.8 *Let* $(X, x, \{U_i, i \in \Delta\})$ *be an object in* $\mathcal{T}$. *Let* $\Delta \cup \{k\}$ *be a disjoint union and let* U_k *be an open and Hausdorff subset of* $X - \{x\}$. *The natural map*

$$(X, x, \{U_i, i \in \Delta \cup \{k\}\}) \to (X, x, \{U_i, i \in \Delta\})$$

which is given by the identity on X *and the inclusion* $\Delta \hookrightarrow \Delta \cup \{k\}$ *induces a* bu-*equivalence after applying the functor* A.

An immediate corollary is the general independence result which we claimed above.

Corollary 4.3.9 *The* bu*-type of the image of a locally Hausdorff, pointed, compact topological space under the functor* A *is independent of the choice of the cover.*

Proof: Given two covers, we consider the union of the covers. The algebra corresponding to the union is equivalent to each of the ones corresponding to the two covers by iterated application of lemma 4.3.8. This shows that the algebras corresponding to the respective covers have to be equivalent as well.

Proof of lemma 4.3.8: Consider the following diagram of extensions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M\ [C_0(U_{k,i,j})]_{i,j} & \longrightarrow & M\ [C_0(U_{i,j})]_{i,j} & \longrightarrow & M\ [C_0(U_{i,j}-U_k)]_{i,j} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M_{\ \cup\{k\}}[C_0(U_{k,i,j})]_{i,j} & \longrightarrow & M_{\ \cup\{k\}}[C_0(U_{i,j})]_{i,j} & \longrightarrow & M\ [C_0(U_{i,j}-U_k)]_{i,j} & \longrightarrow & 0
\end{array}$$

The left vertical arrow induces a bu-equivalence by theorem 4.3.5, since everything happens inside a Hausdorff set. Since the right vertical arrow is an identity, the arrow in the middle has to be a bu-equivalence as well. This proves the lemma.

The outcome of this subsection is, first of all, the extension of some aspects of algebraic topology to locally Hausdorff spaces. We can now compute in a coherent manner the homology and cohomology of a locally Hausdorff space and can interpret its geometric properties. In addition it provides a large class of examples of cell complexes.

Note that, for example, the dimension drop algebras are of this kind. They correspond to the compact pointed locally Hausdorff space which is given as a set by

$$\{*\} \cup (0,1) \cup \{y_1, x_1, y_2, x_2, \ldots, y_n, x_n\}$$

and topologized by requiring that $\lim_{t\to 1} t = x_i$ and $\lim_{t\to 0} t = y_i$ for all $i \in \{1, \ldots, n\}$. The basepoint is chosen to be the disjoint point $*$. The cover by Hausdorff sets is provided by the sets $U_i = (0,1) \cup \{y_i, x_i\}$ for $i \in \{1, \ldots, n\}$.

It would be interesting to study spaces which give rise to special cell complexes as e.g. the cone of the Bott map. Those locally Hausdorff spaces (in fact they cannot be globally Hausdorff) should be of geometrical importance.

Going in a slightly different direction it might be possible to understand the geometric obstructions for a locally Hausdorff space to be be stably homotopy equivalent to a globally Hausdorff space using connective $\mathbf{E}$-theory.

Note that the construction in this section is a special case of the construction of the convolution algebra of a compact orbifold groupoid (see [41] for definitions and further examples). The category of compact orbifold groupoids could serve as another source of examples of cell complexes but we do not develop the connections here.

4.4 Algebraic K-theory of operator algebras

In this subsection we are going to state and prove a theorem which shows that the connective $\mathbf{K}$-groups do not only generalize a geometrically relevant homology theory from CW-complexes to C^*-algebras but also agree with a fundamental object in the study of algebraic properties of a ring, namely the algebraic $\mathbf{K}$-theory. In this section we only want to give a minimal statement. We will generalize that result in section 5.

Theorem 4.4.1 *Let A be a strict cell complex. There is a natural isomorphism $\mathbf{bu}_p(A) \to \mathbf{K}_p^{alg}(A)$ for all $p \leqslant 0$.*

Proof: We are using the universal property of connective $\mathbf{E}$-theory with respect to the properties of homotopy invariance, excision and stability under tensoring with matrix algebras. Excision was proved for negative algebraic $\mathbf{K}$-theory by H. Bass in [5]. Stability under tensoring with matrix algebras is obvious from the general properties of algebraic $\mathbf{K}$-theory. Homotopy invariance is a priori not clear. J. Rosenberg proved in [53] that negative algebraic $\mathbf{K}$-theory is homotopy invariant for commutative C^*-algebras (i.e. that $\mathbf{K}_p^{alg}(C(X)[0,1)) = 0$ for $p \leqslant 0$) which immediately extends to the category of strict cell complexes by an excision argument.

Consider now the homology theory on the category of strict cell complexes which is given by $\mathbf{K}_i^{alg}(?)$ for negative i and $\mathbf{K}_i^{top}(?)$ for non-negative i. Since there is no ambiguity in the definition of $\mathbf{K}_0$, they splice together to a homotopy invariant, excisive homology theory which is stable under tensoring with matrix algebras. We denote this theory by the symbol $\mathbf{K}^{alg,top}$.

Although we do not know whether $\mathbf{K}^{alg,top}$ is homotopy invariant for all separable C^*-algebras we would like to prove the existence of a transformation

$$\mathbf{bu}_*(A) \to \mathbf{K}_*^{alg,top}(A)$$

for strict cell complexes. There is a little argument that has to be carried out. Of course, we would like to apply theorem 3.3.9 but this is a priori not possible. Considering the special case of strict cell complex and following the construction of the transformation along the lines of the proof of theorem 3.2.4 we see that in the end we only need that the homotopy invariance holds for the strict cell complex A and not for all algebras appearing in the extension. This proves the existence of the transformation.

By the concrete computation of the coefficient groups on the right hand side we can conclude that it is an isomorphism on strict cell complexes. This finishes the proof.

In general we can define a map from $\mathbf{bu}$ to some sort of homotopy algebraic $\mathbf{K}$-theory (i.e. the algebraic $\mathbf{K}$-theory of the simplicial ring associated to a C^*-algebra). This map will restrict to the one discussed in the previous theorem. Since at this stage of the thesis we cannot say more about the map than we have already said, we do not discuss this more general case here. There will be a more detailed discussion of this more general construction and its implications in section 5.

4.5 Bivariant homology

4.5.1 Definition and properties

The generator of the coefficient ring of connective **E**-theory is called the Bott map. In this section we want to fix a homomorphism which is a representative of this generator. Furthermore, we want to study the Bott map and want to derive a bivariant homology theory that measures the failure of the Bott map inducing an isomorphism.

The Bott map is concretely given by a $*$-morphism $\sigma : \Sigma \to M_2\Sigma^3$. There is an identification

$$\mathrm{hom}(\Sigma, M_2\Sigma^3) \cong \mathrm{Map}(S^3, \mathrm{hom}(\Sigma, M_2))_+ \cong \mathrm{Map}(S^3, U(2))_+$$

so that

$$[\Sigma, M_2\Sigma^3] \cong \pi_3(U(2)) \cong \mathbb{Z}.$$

The Bott map corresponds to a generator of $\mathbb{Z}$ under this identification. We choose the inclusion of the unit quaternions into $U(2)$.

Denote the class $[\sigma]$ of the Bott map in $\mathbf{bu}$ by $u \in \mathbf{bu}_2(\mathbb{C}, \mathbb{C})$. Let Z be the cone of the Bott map. We want to understand the behavior of the cone of the bott map or rather its fourth desuspension $\Sigma^{-4}(Z)$. Our aim is to show that $\Sigma^{-4}(Z)$ is a monoid object in $\mathbf{bu}$. This then allows to define a bivariant homology theory which is a triangulated module over $\mathbf{bu}$ and measures the defect of the Bott map. (Note that the fourth desuspension may not exist as a C^*-algebra. Although this is not the reason why we introduced desuspensions it turns out to be useful (although not necessary) here.)

There is an exact sequence

$$0 \longrightarrow \Sigma(M_2\Sigma^3) \longrightarrow Z \longrightarrow \Sigma \longrightarrow 0$$

or rather a triangle

$$\Sigma(\Sigma) \xrightarrow{(u)} \Sigma(M_2\Sigma^3) \longrightarrow Z \longrightarrow \Sigma$$

in $\mathbf{bu}$ which gives the following exact sequences.

$$\mathbf{bu}_{n-3}(A,B) \xrightarrow{u} \mathbf{bu}_{n-1}(A,B) \xrightarrow{[\ \otimes \mathrm{id}_B]^*} \mathbf{bu}_n(Z \otimes A, B) \xrightarrow{[\ \otimes \mathrm{id}_B]^*} \mathbf{bu}_{n-4}(A,B)$$

$$\mathbf{bu}_{n+2}(A,B) \xrightarrow{u} \mathbf{bu}_{n+4}(A,B) \xrightarrow{[\ \otimes \mathrm{id}_A]_*} \mathbf{bu}_n(A, Z \otimes B) \xrightarrow{[\ \otimes \mathrm{id}_A]_*} \mathbf{bu}_{n+1}(A,B)$$

The second sequence with $A = B = \mathbb{C}$ allows to conclude that $\mathbf{bu}_{-4}(\mathbb{C}, Z) \cong \mathbb{Z}$ with generator $[\alpha]$ and zero in other dimensions. Using this and the first sequence with $A = \mathbb{C}$ and $B = Z$ we conclude that $\mathbf{bu}_*(Z, Z) = \mathbb{Z}[t]/(t^2)$ with a generator in degree -3. This allows to deduce that $\mathbf{bu}_*(Z, Z)$ carries a $\mathbb{Z}[u]$-module structure with trivial action of the Bott element. This is to say that $u(\mathrm{id}_Z) = 0 \in \mathbf{bu}_2(Z, Z)$.

We can conclude that

$$\mathbf{bu}_4(Z^2, Z) \overset{[\ \otimes \mathrm{id}_Z]^*}{\longrightarrow} \mathbf{bu}_0(Z, Z)$$

is an isomorphism. This indeed follows from $0 = u(\mathrm{id}_Z)^* : \mathbf{bu}_0(Z, Z) \to \mathbf{bu}_2(Z, Z)$ and $0 = [\beta \otimes \mathrm{id}_Z]^* : \mathbf{bu}_3(Z, Z) \to \mathbf{bu}_4(Z^2, Z)$ and the first sequence for $A = B = Z$. We denote the element that is mapped to $[\mathrm{id}_Z]$ by $\eta \in \mathbf{bu}_4(Z^2, Z)$. The corresponding class $\eta \in \mathbf{bu}_0(\Sigma^{-4}(Z)^2, \Sigma^{-4}(Z))$ will define a multiplication with unit $[\alpha]$ for the monoid object $\Sigma^{-4}(Z)$. We have to show associativity and two conditions on the unit (see definition 2.3.14).

First, we want to consider the unit. Note that η satisfies $\eta \circ ([\alpha] \otimes \mathrm{id}_Z) = [\mathrm{id}_Z]$ by definition. Now, we show the second condition that has to be satisfied by the unit of a monoid object. Note that there is an isomorphism $[\alpha]_* : \mathbf{bu}_0(Z, Z) \to \mathbf{bu}_{-4}(\mathbb{C}, Z)$ which assigns

$$\eta \circ ([\alpha] \otimes \mathrm{id}_Z) \mapsto \eta \circ ([\alpha] \otimes \mathrm{id}_Z) \circ [\alpha].$$

The image is $\eta \circ ([\alpha] \otimes [\alpha])$ and therefore agrees with the image of $\eta \circ (\mathrm{id}_Z \otimes [\alpha])$. This shows that

$$\eta \circ (\mathrm{id}_Z \otimes [\alpha]) = [\mathrm{id}_Z].$$

Next we want to show associativity of the multiplication. We compute $\mathbf{bu}_2(Z^2, Z) = 0$ by the second sequence with $A = B = Z$. A computation gives now an isomorphism

$$\mathbf{bu}_8(Z^3, Z) \overset{[\ \otimes \mathrm{id}_Z^2]^*}{\longrightarrow} \mathbf{bu}_4(Z^2, Z).$$

Indeed, by a similar argument as above $0 = u(\mathrm{id}_Z)_* : \mathbf{bu}_4(Z^2, Z) \to \mathbf{bu}_6(Z^2, Z)$ and $0 = [\beta \otimes \mathrm{id}_{Z^2}]^* : \mathbf{bu}_7(Z^2, Z) \to \mathbf{bu}_8(Z^3, Z)$ in the second sequence with $A = Z^2$ and $B = Z$.

For the associativity we have to show that $\eta \circ (\eta \otimes \mathrm{id}_Z) = \eta \circ (\mathrm{id}_Z \otimes \eta)$ as elements in $\mathbf{bu}_8(Z^3, Z)$. The following computation shows that the images under the isomorphisms above agree. At the one hand

$$\eta \circ (\eta \otimes \mathrm{id}_Z) \circ ([\alpha] \otimes \mathrm{id}_Z^2) \circ ([\alpha] \otimes \mathrm{id}_Z) = \eta \circ ([\alpha] \otimes \mathrm{id}_Z) = \mathrm{id}_Z \in \mathbf{bu}_0(Z, Z)$$

and on the other hand

$$\eta \circ (\mathrm{id}_Z \otimes \eta) \circ ([\alpha] \otimes \mathrm{id}_Z^2) \circ ([\alpha] \otimes \mathrm{id}_Z) = \eta \circ (\mathrm{id}_Z \otimes \eta) \circ ([\alpha] \otimes [\alpha] \otimes \mathrm{id}_Z)$$

$$= \eta \circ ([\alpha] \otimes \mathrm{id}_Z) = \mathrm{id}_Z \in \mathbf{bu}_0(Z, Z).$$

This shows the associativity of the multiplication. The preceding results are put together in the following theorem.

Theorem 4.5.1 *The object $\Sigma^{-4}(Z)$ together with the multiplication map $\eta \in \mathbf{bu}_0(\Sigma^{-4}(Z)^2, \Sigma^{-4}(Z))$ and unit $[\alpha] \in \mathbf{bu}_0(\mathbb{C}, \Sigma^{-4}(Z))$ define a monoid object in $\mathbf{bu}$.*

Remark 4.5.2 *The class of* η *defines a canonical splitting* $\Sigma^{-4}(Z)^2 \cong \Sigma^{-4}(Z) \oplus \Sigma^{-1}(Z)$.

The main reason for studying the cone of the Bott map was that it measures the defect of the Bott map. Now, we are going to define a bivariant homology theory that will serve for that purpose.

Definition 4.5.3 *We define* <u>*bivariant homology*</u> $\mathbf{H}$ *to be the bivariant homology theory that corresponds to the monoid object* $\Sigma^{-4}(Z)$.

$$\mathbf{H}_*(A, B) = \mathbf{bu}_*(A, B \otimes \Sigma^{-4}(Z))$$

In particular, this is a bivariant homology theory $\mathbf{H}$ with $\mathbf{H}_*(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$. The theory $\mathbf{H}$ is a triangulated module over $\mathbf{bu}$ and hence inherits all invariance properties of $\mathbf{bu}$. In particular, it satisfies matrix stability (i.e. is invariant under tensoring with matrix algebras or equivalently invariant under Morita equivalence of <u>unital</u> algebras).

Theorem 4.5.4 *The bivariant homology theory* $\mathbf{H}$ *generalizes singular homology with integer coefficients in the sense that we have natural isomorphisms as follows.*

$$\mathbf{H}_n(C(Y, y), C(X, x)) \cong \mathrm{colim}_m [\Sigma^{m+n} X, K(\mathbb{Z})_m \wedge Y]_+$$

(Here $K(\mathbb{Z})_*$ *denotes the Eilenberg-MacLane spectrum corresponding to the integers.)*

This isomorphism is also compatible with the composition product. A proof of this result can be found at the end of section 5.1. Indeed, in remark 5.1.4 we will make precise the relationship between the cone of the Bott map in the world of C^*-algebras and the homotopy fiber of the non-trivial self-map of connective K-theory spectrum <u>bu</u> (see section 5 for more details).

At least the non-bivariant cases follow also by the characterization theorem of singular homology and co-homology by S. Eilenberg and N. Steenrod.

Theorem 4.5.5 *Bivariant homology measures the failure of Bott periodicity on connective* $\mathbf{E}$*-theory in the sense that there is an exact sequence for any* A *and* B *as follows.*

$$\cdots \longrightarrow \mathbf{bu}_{n-2}(A, B) \xrightarrow{u} \mathbf{bu}_n(A, B) \xrightarrow{[\ \otimes \mathrm{id}_B]_*} \mathbf{H}_n(A, B) \xrightarrow{[\ \otimes \mathrm{id}_B]_*} \mathbf{bu}_{n-3}(A, B) \longrightarrow \cdots$$

We refer to this sequence as the fundamental sequence in bivariant homology for C^**-algebras.*

Let us give some first results about the homology for non-commutative spaces. There will be some surprises.

Lemma 4.5.6 *Let A and B be separable C^*-algebras. We have that $\mathbf{H}_*(A, B)$ is zero whenever A or B are stable.*

Proof: Since we have a composition product, it is sufficient to proof that $\mathbf{H}(A, B)$ is zero whenever B is stable. But we know that $\mathbf{bu}(A, B) = \mathbf{E}(A, B)$ for stable B by lemma 4.2.4 and that the Bott map is an isomorphism in $\mathbf{E}$. This implies that there is no failure of the Bott map and hence $\mathbf{H}_*(A, B)$ is equal to zero by the fundamental sequence. This finishes the proof.

I am indebted to Eberhard Kirchberg for pointing out the following far more general result to me.

Lemma 4.5.7 *Let A be a separable C^*-algebra. If there exists a stable sub-algebra E that generates A as an ideal, then A is isomorphic to the zero object in the category $\mathbf{H}$. In particular, the homology and cohomology of A vanishes.*

Proof: The hereditary sub-algebra generated by E is also stable. This follows, since stability is equivalent to the existence of a unital embedding of O_∞ in the multiplier algebra of A such that the sum of the canonical projections in O_∞ converges strongly to the identity element in $M(E)$. Since the inclusion $E \hookrightarrow EAE$ is non-degenerate, we can extend this map uniquely to a map of multiplier algebras. (This extension is of course strongly continuous, being a homomorphism of unital algebras.) This shows that EAE is stable also.

Therefore we can assume without loss of generality that E is hereditary. Let us also assume that it is a split corner. This means that there exists a projection p in the multiplier algebra of A such that $pAp = E$. The projection p is full, since E generates A as an ideal. Let us now look at projections in $M(A \otimes K)$. By Laurence Brown's result (see [10]), $1_A \otimes 1_K$ is Murray-von Neumann equivalent to $p \otimes 1_K$. Since pAp was assumed to be stable, $p \otimes 1_K$ is in turn Murray-von Neumann equivalent to $p \otimes e_{11}$. This implies the existence of a partial isometry v such that $vv^* = p \otimes e_{11}$ and $v^*v = 1_A \otimes 1_K$.

Let s be the unilateral shift. Consider the $*$-homomorphism $\alpha : A \otimes K \to A \otimes K$ which is given by $a \mapsto (e_{11} + svs^*)a(e_{11} + sv^*s^*)$. (Note that $e_{11} + svs^*$ is an isometry.) The image of α is contained in $M_2(A)$ embedded as the upper left corner in $A \otimes K$. Note that the composition $A \to A \otimes K \to M_2(A)$ is just the inclusion of A in the upper left corner of $M_2(A)$. Since this inclusion map factors through something stable, it must be zero in $\mathbf{H}(A, M_2A)$. On the other hand it is the image of id_A under the isomorphism $\mathbf{H}(A, A) \to \mathbf{H}(A, M_2A)$. This implies that $\mathbf{H}(A, A) = 0$ and hence A isomorphic to the zero object in the category $\mathbf{H}$.

Returning to the case of full generality we note that $A \to M_2(A)$ factors through the hereditary sub-algebra of $M_2(A)$ which is generated by $A \otimes e_{11}$ and $E \otimes e_{22}$. This algebra contains E as a split corner and hence has zero homology. This implies that A is isomorphic to the zero object and finishes the proof.

4.5.2 Some computations

In this section we want to provide some computations that allow to get a feeling how connective **K**-theory and bivariant homology behave in easy situations.

Proposition 4.5.8 *Let A and B be cell complexes. Let $a \in \mathbf{bu}(A, B)$ be a morphism in $\mathbf{bu}$ (of degree zero). The homomorphism of graded Abelian groups $\mathbf{H}_*(a) : \mathbf{H}_*(A) \to \mathbf{H}_*(B)$ is an isomorphism if and only if a is a $\mathbf{bu}$-equivalence.*

Proof: Since the $\mathbb{Z}[u]$-module $\mathbf{bu}_*(A)$ is bounded below, the fundamental sequence of bivariant homology is also bounded below. This implies by induction on the dimension and the 5-lemma that $\mathbf{bu}_*(a) : \mathbf{bu}_*(A) \to \mathbf{bu}_*(B)$ is also an isomorphism of graded Abelian groups. This proves our claim by the Yoneda lemma, since $\mathbf{bu}(?, A) \to \mathbf{bu}(?, B)$ is now an equivalence of functors on the category of cell complexes.

Proposition 4.5.9 *The canonical map $A * B \to A \oplus B$ is a $\mathbf{bu}$-equivalence.*

Proof: The proof consists just of an investigation of the proof of the corresponding statement for **K**-theory which was proved by J. Cuntz in [15]. An extension of the result to amalgamated free products with respect to certain split sub-algebras is possible.

The last proposition has a nice interpretation. In an additive category product and co-product coincide almost by definition. What we just showed is that the functor $\mathbf{bu}$ preserves finite co-products. This is a nice feature and was not to be expected. Indeed, it is far from being true in the stable homotopy category of C^*-algebras.

One can go one step further and consider the universal triangulated monoidal category with respect to the property that the functor preserves finite co-products. This will again be a Verdier quotient of the stable homotopy category. It turns out that this category is nothing else but $\mathbf{bu}$. The proof of this characterization relies on the following simple observation.

Proposition 4.5.10 *Consider the evaluation $m : \mathbb{C} * \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$. Its kernel is isomorphic to $c(\iota_2)$ where $\iota_2 : \mathbb{C} \to M_2$ is the canonical inclusion.*

This is, for example, proved by J. Cuntz in [16]. The result follows, since we are requiring the quotient to be monoidal.

Let us now come to some consequences of Bott periodicity.

Example 4.5.11 *Let* $A = \{f \in C([0,\infty], \infty; \mathbf{K}); f(t) \in p_n \mathbf{K} p_n, \forall t < n\}$*. The algebra* A *is strongly Morita equivalent to the contractible algebra* $C([0,\infty),\infty;\mathbf{K})$ *and hence its* $\mathbf{K}$*-theoretic invariants all vanish.*

On the other hand we have an extension.

$$0 \longrightarrow A \longrightarrow A' \longrightarrow \mathbf{K} \longrightarrow 0$$

with $A' = \{f \in C([0,\infty]; \mathbf{K}); f(t) \in p_n \mathbf{K} p_n, \forall t < n\}$*. Since the homology of* $\mathbf{K}$ *is zero and* A' *is homotopy equivalent to* $\mathbb{C}$*, we get that* $\mathbf{H}_*(\mathbb{C}, A) = \mathbb{Z}$*. In particular,* $M_n A$ *is not homotopy equivalent to zero for any* $n \in \mathbb{N}$*.*

Example 4.5.12 *Let* Z *be the cone of the Bott map. It is clear that all* $\mathbf{K}$*-theoretic invariants of this algebra must vanish, since the Bott map is an isomorphism in* $\mathbf{K}$*-theory. Still, one can compute its homology and it is non-trivial as we have seen.*

This observation has an interesting corollary.

Corollary 4.5.13 *The algebra* Z *is not* bu*-equivalent to an algebra of functions on a finite pointed* CW*-complex.*

Proof: The proof will be at the same time a motivation for the next section which contains more subtle considerations of the same kind. A first simple proof of the theorem above is given by noting that $\mathbf{K}_0(Z) = 0$ but $\mathbf{H}_0(Z) = \mathbb{Z}$. This implies that there cannot be a rational isomorphism between the $\mathbf{K}_0(Z)$ and the even cohomology groups of groups of Z. If Z were bu-equivalent to an algebra of functions on a finite pointed CW-complex, then there had to exist a rational isomorphism by the Chern isomorphism theorem. Hence we get a contradiction which finishes the proof.

In the next section we want to study some deeper reasons for the truth of the preceding theorem. In particular, we want to consider what remains of the Chern character and study cohomology operations that detect the genuine non-commutativity of a space.

The next definition contains the notion of matrix homotopy equivalence. This notion is a substantial weakening of the notion of homotopy equivalence. Indeed, the matrix homotopy type seems to be accessible by purely algebraic tools as we will see.

Definition 4.5.14 *Let* A *and* B *be cell complexes. The algebras* A *and* B *are called* *matrix homotopy equivalent, if there are integers* n *and* $m \in \mathbb{N}$ *and homomorphisms* $\alpha : A \to M_n B$ *and* $\beta : B \to M_m A$ *such that*

$$(\beta \otimes id_{M_n}) \circ \alpha : A \to M_{mn} A$$

and

$$(\alpha \otimes \mathrm{id}_{M_m}) \circ \beta : B \to M_{mn}B$$

are homotopic to the inclusions $\mathrm{id}_A \otimes \iota_{nm}$ *and* $\mathrm{id}_B \otimes \iota_{nm}$ *into the upper left corner of the matrix algebras.*

Definition 4.5.15 *Let* (X, x) *and* (Y, y) *be finite pointed* CW*-complexes. The spaces* (X, x) *and* (Y, y) *are called matrix homotopy equivalent if the algebras* $C(X, x)$ *and* $C(Y, y)$ *are matrix homotopy equivalent.*

Note that it is a priori difficult to check whether a map induces a matrix homotopy equivalence, since it always requires the construction of an inverse.

By the work of M. Dădărlat and J. McClure in [24] two finite connected CW-complexes are matrix homotopy equivalent if and only if they are equivalent in **bu**. The following theorem will bring homology into play. In fact, it has to be considered as an easy corollary of the deep work of M. Dădărlat and J. McClure. We state and prove our formulation, since homology is a little easier to deal with than connective **K**-theory.

Theorem 4.5.16 *Let* (X, x) *and* (Y, y) *be connected finite pointed CW-complexes. The pointed spaces* (X, x) *and* (Y, y) *are matrix homotopy equivalent if and only if there exists a map* $\phi : C(X, x) \to M_nC(Y, y)$ *for some integer* $n \in \mathbb{N}$ *that induces an isomorphism in homology.*

Proof: Note that, in particular, $C(X, x)$ and $C(Y, y)$ are cell complexes. By proposition 4.5.8, we get that the class $\mathbf{bu}(a)$ has to be an equivalence. Now, we use M. Dădărlat's result from [24] which implies that (X, x) and (Y, y) are matrix homotopy equivalent.

As an interesting example we can recall that the Poincaré 3-sphere permits a map from the usual 3-sphere which induces an isomorphism in homology. By the preceding theorem, this implies that they are matrix homotopy equivalent. In some sense the passage to the non-commutative context does neglect certain highly non-commutative phenomena as, for example, a perfect fundamental group of a space.

4.5.3 Cohomology operations

In one of the preceding sections we already computed the graded ring $\mathbf{bu}_*(Z, Z)$. It was given by $\mathbb{Z}[t]/(t^2)$ with a generator in degree -3. We denote this ring by $\mathcal{A}$. This is clearly the ring of cohomology operations of bivariant homology with respect to connective **E**-theory. (Note that the ring of cohomology operations with respect to stable homotopy of C^*-algebras might be much more difficult but we do not consider it here.) The $\mathbb{Z}/(2)$-analogue contains also a Bockstein element of degree -1. An easy computation shows that it is given by $(\mathbb{Z}/(2))[t, \beta]/(t^2, \beta^2)$. The structure of this ring is rather surprising. It is much easier to understand than the Steenrod algebra of cohomology operations on $\mathbb{Z}/(2)$-cohomology. It seems natural to consider the Frobenius category of modules over the ring $(\mathbb{Z}/(2))[t, \beta]/(t^2, \beta^2)$ (see [37] for definitions and properties) and to try to classify the modules using the theory of Margolis homology groups (see [37]) with respect to the operations t and β. We do not want to follow this line and rather stick to the integral theory.

The operation t is of subtle nature, since it detects algebras which are not bu-equivalent to commutative algebras. In order to understand this phenomenon let us first state the following theorem from rational homotopy theory.

Theorem 4.5.17 *There are no non-trivial rational stable cohomology operations on singular cohomology.*

This implies that the image of any stable cohomology operation in integral singular cohomology has to be torsion, since it lifts to the zero operation in rational singular cohomology. On the other hand the operation t sometimes survives rationalization in the non-commutative case. The easiest example is the algebra Z. The operation t clearly acts on $\mathbf{bu}_*(Z, Z)$ in a way which remains non-trivial after rationalization.

The following corollary is a consequence of the preceding theorem.

Corollary 4.5.18 *Let A and B be separable C^*-algebras. If A and B are* bu-*equivalent to algebras of functions on pointed CW-complexes that vanish at the basepoint, then the image of $t : \mathbf{bu}_n(A, B) \to \mathbf{bu}_{n-3}(A, B)$ has to be torsion for any $n \in \mathbb{N}$.*

Proof: The image has to vanish after rationalization. Therefore it has to be torsion. This finishes the proof.

In the next section we will see how the operation t appears as a differential in a spectral sequence and obstructs the existence of a rational Chern character.

4.5.4 The Bockstein-Chern spectral sequence

In this section we want to analyze the fundamental sequence of bivariant homology with respect to its algebraic properties. We derive some properties which might be useful in concrete computations. Furthermore, the existence of the Chern character for commutative algebras will be clarified. An analogous object in the non-commutative setting is provided by the Bockstein-Chern spectral sequence. For simplicity we argue only for cell complexes. Although we do not make this explicit we are partially working with $\mathbb{Z}/(2)$-graded bivariant homology theories in this section. Note that any $\mathbb{Z}$-graded bivariant homology theory naturally induces a $\mathbb{Z}/(2)$-graded bivariant homology theory by considering its even and its odd part.

The fundamental sequence of Abelian groups

$$\cdots \longrightarrow \mathbf{bu}_{n-2}(A,B) \xrightarrow{u} \mathbf{bu}_n(A,B) \xrightarrow{[\ \otimes \mathrm{id}_B]_*} \mathbf{H}_n(A,B) \xrightarrow{[\ \otimes \mathrm{id}_B]_*} \mathbf{bu}_{n-3}(A,B) \longrightarrow \cdots$$

can be put into the following triangular form.

$$\begin{array}{ccc} \oplus_s \mathbf{bu}_s(A,B) & \xrightarrow{\quad u \quad} & \oplus_s \mathbf{bu}_s(A,B) \\ & \nwarrow \qquad \swarrow & \\ & \oplus_s \mathbf{H}_s(A,B). & \end{array}$$

Here u is of degree 2, α is of degree zero and β of degree -3. This implies that the kernel and co-kernel of u give rise to an extension

$$0 \longrightarrow \mathrm{coker}(u)_s \longrightarrow \mathbf{H}_s(A,B) \longrightarrow \ker(u)_{s-3} \longrightarrow 0.$$

Since the co-kernel identifies with the s-part of $\mathbf{bu}_*(A,B) \otimes_{\mathbb{Z}[u]} \mathbb{Z}$ and the kernel identifies with $\mathrm{Tor}^{1,s-3}_{\mathbb{Z}[u]}(\mathbf{bu}_*(A,B),\mathbb{Z})$, we end up with the following description (compare [63] for an analogous result in the commutative case).

Theorem 4.5.19 *Let A and B be cell complexes. The following diagram is exact.*

$$0 \longrightarrow \mathbf{bu}_*(A,B) \otimes_{\mathbb{Z}[u]} \mathbb{Z} \longrightarrow \mathbf{H}_*(A,B) \longrightarrow \mathrm{Tor}^{1,*-3}_{\mathbb{Z}[u]}(\mathbf{bu}_*(A,B),\mathbb{Z}) \longrightarrow 0$$

Here $\mathbb{Z}$ is the $\mathbb{Z}[u]$-module on which u acts as zero.

The triangle gives rise to an exact couple and hence we get a spectral sequence. Since u is not of degree zero, the spectral sequence is somewhat distorted. Under the assumption that $\mathbf{bu}_*(A,B)$ is finitely generated as a $\mathbb{Z}[u]$-module no element can be infinitely divisible by u. So the exact couple gives rise to the following isomorphism (by proposition 5.9.10 in [69]).

$$\frac{\mathbf{bu}_n(A,B)}{\cup_{i=1}^{\infty}\ker(u^i : \mathbf{bu}_n(A,B) \to \mathbf{bu}_{n+2i}(A,B)) + u(\mathbf{bu}_{n-2}(A,B))} \longrightarrow E_n^{\infty}(A,B).$$

On the other hand the spectral sequence is clearly bounded below and hence converges to $\mathrm{colim}_n \mathbf{bu}_*(A,B)$ (by theorem 5.9.7 in [69]) where the co-limit is taken with respect to morphisms given by multiplication with u. (Note that the co-limit is naturally a $\mathbb{Z}/(2)$-module.) We can identify the target of the spectral sequence with $\mathbf{bu}_*(A,B) \otimes_{\mathbb{Z}[u]} \mathbb{Z}$ (where $\mathbb{Z}$ is regarded as $\mathbb{Z}/(2)$-graded $\mathbb{Z}[u]$-module with u acting as 1).

Since $\mathbb{Z}$ (with the module structure above) is a flat $\mathbb{Z}[u]$-module, the natural map $\mathbf{bu}_*(A,B) \otimes_{Z[u]} \mathbb{Z} \to \mathbf{E}_*(A,B) \otimes_{Z[u]} \mathbb{Z}$ is a transformation of $\mathbb{Z}/(2)$ bivariant homology theories. It is an isomorphism on coefficients and hence for cell complexes. We use the notation $\mathbf{E}'_*(A,B) = \mathbf{E}_*(A,B) \otimes_{\mathbb{Z}[u]} \mathbb{Z}$. Note that this is the usual (i.e. according to [28]) description of $\mathbf{E}$-theory as a $\mathbb{Z}/(2)$-graded bivariant homology theory.

Note that the differential in the exact couple is given by $\alpha \circ \beta$, i.e. by the action of the cohomology operation $t \in \mathbb{Z}[t]/(t^2) = \mathbf{bu}_*(Z,Z)$. If we specialize to the case where A and B are commutative (i.e. algebras of functions on a finite pointed CW-complex vanishing at the basepoint) the differential vanishes after rationalization by the preceding discussion of rational cohomology operations for singular cohomology. By induction and an inspection of the differentials of the derived couples (see definition 5.9.1 in [69]), it is obvious that also the induced differentials on the derived exact couples have to vanish. Hence the E^∞-term of the rationalized spectral sequence is equal to the $\mathbb{Z}/(2)$-graded bivariant group $\mathbf{H}_*(A,B) \otimes \mathbb{Q}$.

The statement is now that for commutative A and B there is a filtration on $\mathbf{E}'_*(A,B) \otimes \mathbb{Q}$ such that the sub-quotients are isomorphic to $\mathbf{H}_*(A,B) \otimes \mathbb{Q}$. Since we are working over $\mathbb{Q}$, this clearly lifts to a natural isomorphism $\mathbf{E}'_*(A,B) \otimes \mathbb{Q} \to \mathbf{H}_*(A,B) \otimes \mathbb{Q}$.

This is just the statement of the Chern isomorphism theorem and a concrete construction of the isomorphism. In the integral case or for non-commutative algebras the E^∞-term of the spectral sequence does not have to do so much with $\mathbf{H}_*(A,B)$ (apart from being a sub-quotient, of course). The spectral sequence allows the precise analysis of the obstruction to the existence of a Chern character. Since the classical Chern isomorphism theorem is a special case, we want to state it too.

Theorem 4.5.20 (Chern isomorphism theorem) *Let A and B be rationally* bu*-equivalent to algebras of functions on finite pointed CW-complexes that vanish at the basepoint. There is a natural map*

$$\mathbf{E}'_*(A,B) \otimes \mathbb{Q} \to \mathbf{H}_*(A,B) \otimes \mathbb{Q}$$

which is an isomorphism of $\mathbb{Z}/(2)$-graded Abelian groups.

In fact, there is a converse to this statement.

Theorem 4.5.21 *Let A be a cell complex. The following two conditions are equivalent.*

- *There exists an abstract isomorphism of $\mathbb{Z}/(2)$-graded Abelian groups $\mathbf{E}'_*(A)$ and $\mathbf{H}_*(A)$ after tensoring with $\mathbb{Q}$.*
- *The algebra A is rationally* bu*-equivalent to a commutative algebra.*

Proof: We have to prove only one direction, since the other direction is a special case of the Chern isomorphism theorem. Note that $\mathbf{bu} \otimes \mathbb{Q}$ is a triangulated homology theory on the category of separable C^*-algebras. (One either checks the axioms directly or constructs $\mathbf{bu}$ as a Verdier quotient.)

The proof is just a question about the possibilities that can occur. The coefficient ring of rational $\mathbf{bu}$-theory is of dimension 1 so that the UC spectral sequence of rational $\mathbf{bu}$-theory collapses at the E^2-term. This results in the usual UC sequence.

$$0 \longrightarrow \mathrm{Ext}^{*-1}_{\mathbb{Q}[u]}(\mathbf{bu}_*(A) \otimes \mathbb{Q}, \mathbf{bu}_*(B) \otimes \mathbb{Q}) \longrightarrow \mathbf{bu}_*(A, B) \otimes \mathbb{Q} \longrightarrow \hom_{\mathbb{Q}[u]}(\mathbf{bu}_*(A) \otimes \mathbb{Q}, \mathbf{bu}_*(B) \otimes \mathbb{Q}) \longrightarrow 0$$

The exactness of this sequence implies that any abstract isomorphism of $\mathbb{Q}[u]$-modules between, say $\mathbf{bu}_*(A) \otimes \mathbb{Q}$ and $\mathbf{bu}_*(B) \otimes \mathbb{Q}$, can be lifted to a rational $\mathbf{bu}$-equivalence between A and B. It is therefore natural to look at the possible $\mathbb{Q}[u]$-modules that can occur. One convinces oneself very quickly that any finitely generated $\mathbb{Q}[u]$-module decomposes as a finite direct sum of shifted copies of finitely generated free modules and shifted copies of modules of the type $\mathbb{Q}[u]/(u^n)$. The second type of summands are called Bott torsion modules.

All these indecomposable summands are represented as the rational connective $\mathbf{K}$-groups of certain cell complexes. The free modules are represented by suspensions or desuspensions of the complex numbers $\mathbb{C}$. The Bott torsion modules are represented by suitable suspensions or desuspensions of the algebras constructed as the cone of a suitable power of the Bott map.

Therefore by the UC sequence the decomposition of the $\mathbb{Q}[u]$-module $\mathbf{bu}_*(A) \otimes \mathbb{Q}$ can be lifted to a decomposition of A itself considered as an object in $\mathbf{bu} \otimes \mathbb{Q}$ into a sum of algebras representing the different summands of $\mathbf{bu}_*(A)$ in the category $\mathbf{bu} \otimes \mathbb{Q}$.

It is clear how the different summands of A contribute to the homology of the algebra A. For a given summand, we denote by k the dimension of its generator as a $\mathbb{Q}[u]$-module. For the free modules one gets $\mathbb{Q}$ in dimension k and for the Bott torsion module $\mathbb{Q}[u]/(u^n)$ one gets $\mathbb{Q}$ in dimension k and $k + 2n + 1$. This follows from the fundamental sequence of bivariant homology.

The contribution of the summands to rational $\mathbf{K}$-theory is computed by formally inverting the Bott map $[\sigma] = u$. This implies that the free modules contribute $\mathbb{Q}$ in the dimension of the generator modulo 2 and the torsion modules do not contribute at all.

The argument now goes as follows. If there is an abstract isomorphism, then the $\mathbb{Q}[u]$-module $\mathbf{bu}_*(A) \otimes \mathbb{Q}$ has to be free, since the dimensions of the $\mathbb{Z}/(2)$-graded vector spaces $\mathbf{H}_*(A)$ and $\mathbf{K}_*(A)$ do not match, if there are Bott torsion summands. Furthermore, the dimensions of the generators clearly lie in non-negative dimensions. Since the free $\mathbb{Q}[u]$-module with a single generator in dimension $k \geqslant 0$ can be realized by the commutative algebras $C(S^k, +)$, we get an isomorphism of $\mathbb{Q}[u]$-modules $\mathbf{bu}_*(A) \to \mathbf{bu}_*(A')$ where A' is a commutative algebra (indeed a wedge of spheres). We can now use the UC sequence to lift this abstract isomorphism to an element in rational **bu**-theory which has to be an isomorphism. This finishes the proof.

The proof of the last theorem revealed that the structure of rational **bu**-theory is somehow easy to understand. Note that there is the following commutative square for the Lefshetz numbers (see definition 4.2.17). Denote by $\tau_{\mathbb{Q}}$ the concrete trace for $\mathbf{bu} \otimes \mathbb{Q}$.

$$\begin{array}{ccc} \mathbf{bu}_*(A,A) & \longrightarrow & \mathbb{Z}[u] \\ \downarrow & & \downarrow \\ \mathbf{bu}_*(A,A) \otimes \mathbb{Q} & \xrightarrow{\mathbb{Q}} & \mathbb{Q}[u] \end{array}$$

By naturality of the construction of the concrete trace and flatness of $\mathbb{Q}$, the preceding diagram commutes.

In particular, this implies first of all that we can compute the Euler characteristics rather easily.

Theorem 4.5.22 *Let A be a cell complex. Its Euler characteristic satisfies the following equality.*

$$\chi(A) = \mathbf{dim}_{\mathbb{Q}}(\mathbf{K}_0(A) \otimes \mathbb{Q}) - \mathbf{dim}_{\mathbb{Q}}(\mathbf{K}_1(A) \otimes \mathbb{Q})$$

Proof: It is obvious from the definition of the concrete trace that $\chi(\Sigma^k) = (-1)^k$. This implies by the additivity of the trace that the Euler characteristic of Bott torsion modules vanishes. Hence the Euler characteristic does only see the free part of the $\mathbb{Q}[u]$-module. The generators in even degrees contribute $+1$ and the generators in odd degrees -1. This proves our claim.

4.5.5 The Adams spectral sequence

Similarly to the construction of the UC spectral sequence in connective **E**-theory we want to construct an Adams spectral sequences. We start out with the construction of a geometric projective resolution related to homology. We again refer to the work of D. Christensen in [12] for the abstract machinery which makes the following arguments work.

Let A be a separable C^*-algebra with $\mathbf{bu}_*(A)$ finitely generated. For simplicity let us assume that A is a cell complex. A modified result is true in the case of A not being a cell complex.

By definition we have the equality $\mathbf{H}_*(A) = \mathbf{bu}_*(\mathbb{C}, A \otimes \Sigma^{-4}(Z))$. By Poincaré duality (see proposition 4.2.20), Z is isomorphic to $\Sigma^5(D(Z))$ and hence there is a natural isomorphism

$$\mathbf{H}_*(A) = \mathbf{bu}_*(\mathbb{C}, A \otimes \Sigma^{-4}(Z)) \cong \mathbf{bu}_*(\Sigma^{-1}(Z), A).$$

Since $\mathbf{H}_*(A)$ is finitely generated as a $\mathcal{A}$-module, there exist (by the description of homology above) integers $k^1, \cdots, k^n$ and a map $\oplus_{i=1}^n \Sigma^{k^i}(Z) \to A$ which induces a surjection of the free $\mathcal{A}$-module $\mathbf{H}_*(\oplus_{i=1}^n \Sigma^{k_i}(Z))$ onto $\mathbf{H}_*(A)$. Completing this map to a distinguished triangle in $\mathbf{bu}$ yields a triangle

$$\begin{array}{ccc} A_1 & \longleftarrow & B \\ & \searrow & \uparrow \\ & & \oplus_{i=1}^n \Sigma^{k^i}(Z) \end{array}$$

where the dotted arrow is of degree minus one. Again, (like in the proof of the existence of the geometric projective resolution) we can continue this process. We obtain a diagram with $A = A_0$ as follows.

$$\begin{array}{ccccccccccc} \cdots & A_4 & \xleftarrow{3} & A_3 & \xleftarrow{2} & A_2 & \xleftarrow{1} & A_1 & \xleftarrow{0} & A_0 \\ & & \searrow & \uparrow & \searrow & \uparrow & \searrow & \uparrow & \searrow & \uparrow \\ \cdots & & & \oplus_{i=1}^{n_3} \Sigma^{k_3^i}(Z) & & \oplus_{i=1}^{n_2} \Sigma^{k_2^i}(Z) & & \oplus_{i=1}^{n_1} \Sigma^{k_1^i}(Z) & & \oplus_{i=1}^{n_0} \Sigma^{k_0^i}(Z) \end{array}$$

Note that applying $\mathbf{bu}_*(?, B)$ yields an exact couple

$$\begin{array}{ccc} \oplus_{p,q} \mathbf{bu}_{p+q}(A_p, B) & \xrightarrow{\quad i \quad} & \oplus_{p,q} \mathbf{bu}_{p+q}(A_p, B) \\ & {}_{k}\nwarrow \quad \swarrow_{j} & \\ & \oplus_{p,q} \mathbf{bu}_{p+q}(\oplus_{i=1}^{n_p} \Sigma^{k_p^i}(Z), B) & \end{array}$$

with bi-degrees of i,j and k equal to $(-1, 1)$, $(0, 0)$ and $(1, 0)$. Note further that

$$\mathbf{bu}_*(\oplus_{i=1}^{n_p} \Sigma^{k_p^i}(Z), B) = \mathrm{hom}_{\mathcal{A}}(\mathbf{H}_*(\oplus_{i=1}^{n_p} \Sigma^{k_p^i}(Z)), \mathbf{H}_*(B))$$

and that

$$\cdots \longrightarrow \mathbf{H}_{*+2}(\oplus_{i=1}^{n_2} \Sigma^{k_2^i}(Z)) \longrightarrow \mathbf{H}_{*+1}(\oplus_{i=1}^{n_1} \Sigma^{k_1^i}(Z)) \longrightarrow \mathbf{H}_*(\oplus_{i=1}^{n_0} \Sigma^{k_0^i}(Z)) \longrightarrow \mathbf{H}_*(A)$$

is a projective resolution of $\mathbf{H}_*(A)$ as an $\mathcal{A}$-module.

Collecting everything together we get (e.g. again by theorem 2.8 in the book by J. McCleary [39]) a co-homological spectral sequence with

$$E_2^{p,q} = \mathrm{Ext}_{\mathcal{A}}^{p,q}(\mathbf{H}_*(A), \mathbf{H}_*(B)).$$

We show that the spectral sequence is conditionally convergent (see definition 3.18 in [39] or definition 5.10 in [9]), i.e. we have to show (and this is the definition) that

$$\lim_s \mathbf{bu}_*(A_s, B) = \lim_s^1 \mathbf{bu}_*(A_s, B) = 0.$$

The proof requires some preparations.

First of all, we have the following extension

$$0 \longrightarrow \mathbf{H}_{*+1}(A_{l+1}) \longrightarrow \mathbf{H}_*(\oplus_{i=1}^{n_l}\Sigma^{k_l^i}(Z)) \longrightarrow \mathbf{H}_*(A_l) \longrightarrow 0$$

of $\mathcal{A}$-modules for $l \geqslant 0$. Since $\mathbf{H}_*(\oplus_{i=1}^{n_l}\Sigma^{k_l^i}(Z))$ is free as an Abelian group, the sequence implies that $\mathbf{H}_*(A_l)$ is free as an Abelian group for $l \geqslant 1$. Since A_l is a cell complex, the homology $\mathbf{H}_*(A_1)$ is finitely generated as Abelian group an therefore concentrated in a certain range of dimensions. That is, there exists an integer $N \in \mathbb{N}$ such that $\mathbf{H}_k(A_1)$ is zero whenever $k \notin [-N, N]$. The next lemma shows that we can find a certain geometric projective resolution for which much more is true.

Lemma 4.5.23 *There exists a geometric projective resolution in the situation discussed above with the property that the homology of* A_l *is concentrated in dimensions* $[-N-2(l-1), N-2(l-1)]$*, i.e.* $\mathbf{H}_k(A_l) = 0$ *whenever* $k \notin [-N-2(l-1), N-2(l-1)]$.

Proof: We proof the result by induction. The case $l = 1$ is trivially true. Observe that the homology of Z is concentrated in dimensions -1 and -4 and the generators of $\mathbf{H}_k(A_l)$ correspond to maps $\Sigma^{-1-k}(Z) \to A_l$. We take the map $\oplus_{i=1}^{n_l}\Sigma^{k_l^i}(Z) \to A_l$, which corresponds to a <u>minimal</u> set of generators of $\mathbf{H}_*(A_l)$ as an Abelian group, to be the next step in the geometric projective resolution. A generator of $\mathbf{H}_k(A_l)$ contributes with $\mathbb{Z}$ in $\mathbf{H}_k(\oplus_{i=1}^{n_l}\Sigma^{k_l^i}(Z))$ and $\mathbf{H}_{k-3}(\oplus_{i=1}^{n_l}\Sigma^{k_l^i}(Z))$.

Since $\mathbf{H}_k(A_l)$ is free Abelian for all k, the maps

$$\mathbf{H}_{N-2(l-1)-j}(\oplus_{i=1}^{n_l}\Sigma^{k_l^i}(Z)) \to \mathbf{H}_{N-2(l-1)-j}(A_l)$$

are isomorphisms of Abelian groups for $j \in \{0,1,2\}$. Furthermore, the groups $\mathbf{H}_k(\oplus_{i=1}^{n_l}\Sigma^{k_l^i}(Z))$ vanish for $k > N-2(l-1)$.

This proves that $\mathbf{H}_k(A_{l+1})$ vanishes if $k > N-2l$. The vanishing of $\mathbf{H}_k(A_{l+1})$ in dimensions below $-N-2l$ is obvious. This proves the lemma.

We can now proceed with the proof of the conditional convergence of the spectral sequence. We assume that we are working with a geometric resolution which satisfies the properties stated in the lemma above. The vanishing of the higher dimensional homology groups imply that $\mathbf{bu}_k(A_l) \xrightarrow{u} \mathbf{bu}_{k+2}(A_l)$ is an isomorphism whenever $k > N-2(l-1)-3 = N-2l-1$ by the fundamental sequence of bivariant homology.

Picking generators of $\mathbf{bu}_{N-2l}(A_l)$ as Abelian group we get a distinguished triangle in $\mathbf{bu}$ as follows.

$$\Sigma(T_l) \longrightarrow \oplus_{i=1}^{n_l}\Sigma^{-(N-2l)} \longrightarrow A_l \longrightarrow T_l$$

Note that $\mathbf{bu}_m(\oplus_{i=1}^{n_l}\Sigma^{-(N-2l)}) \to \mathbf{bu}_m(A_l)$ is an isomorphism in dimensions $m \geqslant N-2l$. This implies that $\mathbf{bu}_*(T_l)$ is concentrated in dimensions $[-N-2l, N-2l]$.

In order to show conditional convergence we want to show that the groups $\mathbf{bu}_k(A_l, B)$ actually vanish for fixed k if l is big enough. Using excision it remains to show that $\mathbf{bu}_k(\Sigma^{-(N-2l)}, B) = \mathbf{bu}_{k+N-2l}(B)$ and $\mathbf{bu}_k(T_l, B)$ vanish whenever l is big enough. The first group clearly vanishes for big values of l, since the $\mathbb{Z}[u]$-module $\mathbf{bu}_*(B)$ is bounded below, because it is finitely generated.

The second group vanishes by the universal coefficient spectral sequence for big values of l, since the range of dimensions in which $\mathbf{bu}_*(T_l)$ is non-trivial decreases more and more as l grows.

This proves that

$$\lim_s \mathbf{bu}_*(A_s, B) = \lim_s^1 \mathbf{bu}_*(A_s, B) = 0,$$

as claimed above. At the same time it becomes obvious that we have strong convergence, since the image

$$\mathbf{bu}_{s+t}(A_{s+r}, B) \to \mathbf{bu}_{s+t}(A_s, B)$$

is getting trivial as r grows. This proves the following theorem.

Theorem 4.5.24 (Adams spectral sequence) *Let A and B be cell complexes. There exists a co-homological spectral sequence with*

$$E_2^{p,q} = \mathrm{Ext}_{\mathcal{A}}^{p,q}(\mathbf{H}_*(A), \mathbf{H}_*(B))$$

which converges strongly to $\mathbf{bu}_*(A, B)$.

4.5.6 Künneth theorem and UC theorem

In this section we want to provide a Künneth theorem and a universal coefficient theorem for bivariant homology. Furthermore, we want to state that a version of the universal coefficient theorem which was developed by M. Dădărlat and T. Loring in [22] applies as well to bivariant homology.

The statements of the Künneth theorem and UC theorem for bivariant homology are much easier to understand than the corresponding statements for connective $\mathbf{K}$-theory. Although the statements are in direct analogy with the corresponding statements in singular homology and co-homology the proofs are quite different. Only the analysis of the cone of the Bott map allows to construct the geometric resolutions needed.

Theorem 4.5.25 (Künneth and UC theorem for bivariant homology) *Let A and B be cell complexes. There are short exact sequences as follows.*

$$0 \longrightarrow \mathbf{H}_*(A) \otimes \mathbf{H}_*(B) \longrightarrow \mathbf{H}_*(A \otimes B) \longrightarrow \mathrm{Tor}_{\mathbb{Z}}^{*-1}(\mathbf{H}_*(A), \mathbf{H}_*(B)) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}^{*-1}(\mathbf{H}_*(A), \mathbf{H}_*(B)) \longrightarrow \mathbf{H}_*(A, B) \longrightarrow \mathrm{hom}_{\mathbb{Z}}(\mathbf{H}_*(A), \mathbf{H}_*(B)) \longrightarrow 0$$

Furthermore, those sequences are natural in A *and* B *and split unnaturally.*

Proof: This follows by pure homological algebra (as in the other cases) once we have provided a geometric projective resolution of the homology of A as a $\mathbb{Z}$-module. One first constructs the corresponding spectral sequences. Since $\mathbb{Z}$ has dimension one, there are no non-vanishing differentials and therefore the spectral sequences give rise to the extensions above.

In the proof of the existence of the Adams spectral sequence we even constructed a geometric projective resolution (in fact, free resolution) of the homology of A as an $\mathcal{A}$-module which of course remains to be a projective resolution of Abelian groups after forgetting the $\mathcal{A}$-module structure. The naturality of the sequences above is again pure homological algebra (i.e. the fact that $\mathrm{Ext}^{p,q}$ is independent of the choice of the projective resolution plus the comparison theorem 2.2.6 in [69]).

The existence of splits is a general consequence. For the proof one splits A (unnaturally) into $A_0 \oplus A_1$ where the homology of A_0 is concentrated in even degrees and the homology of A_1 is concentrated in odd degrees. This is clearly possible by the UC theorem. Similarly one splits B. The bivariant group $\mathbf{H}_*(A, B)$ then decomposes as $\mathbf{H}_*(A_0, B_0) \oplus \mathbf{H}_*(A_1, B_0) \oplus \mathbf{H}_*(A_0, B_1) \oplus \mathbf{H}_*(A_1, B_1)$. For each of the summands the exact sequences above split trivially degree-wise, since either the left or the right group in the extension vanishes. This finishes the proof.

The theorem above has an immediate corollary.

Corollary 4.5.26 *Any cell complex is* $\mathbf{H}$*-equivalent to a commutative algebra.*

Proof: The proof is obvious, since the commutative algebras exhaust the possible homology groups. Indeed, since everything is finitely generated we only have to provide commutative algebras A_{m,p^n} with the property that $\mathbf{H}_m(A_{m,p^n}) = \mathbb{Z}/(p^n)$ and $\mathbf{H}_k(A_{m,p^n}) = 0$ for $k \neq m$. The (de)suspensions of mod-p^n Moore spaces (e.g. co-fibers of multiplication by p^n) have of course this property. The UC theorem above then shows that an isomorphism of homology groups can be lifted to a morphism in the bivariant homology theory $\mathbf{H}$. This then has to be a $\mathbf{H}$-equivalence by the Yoneda lemma and the naturality of the UC sequence.

There is an interesting article by M. Dădărlat and T. Loring [22] in which they provide a modified version of the UC theorem for $\mathbf{KK}$-theory. They consider all $\mathbf{K}$-groups with coefficients in $\mathbb{Z}$ and in $\mathbb{Z}/(p^n)$ together. There are natural operations of degree minus one between those groups which are the Bockstein operations and the natural mappings of degree zero which are induced by mappings of the coefficients. (See [22] or [58] for a detailed discussion of Bockstein operations in this context.) This structure can be encoded by regarding the various $\mathbf{K}$-groups as modules over a category or equivalently by considering the sum of the various K-groups of an algebra A (they denote it by $\underline{\mathbf{K}}(A)$, but we will rather use $\overline{\mathbf{K}}(A)$ since $\underline{\mathbf{K}}(?)$ is already in use) as a module over a certain ring Λ consisting of the operations. Their main result (1.4 in [22]) is the existence of an isomorphism

$$\mathbf{KK}(A,B) \xrightarrow{=} \hom\ (\overline{\mathbf{K}}(A), \overline{\mathbf{K}}(B))$$

whenever $\mathbf{K}_*(A)$ is finitely generated as an Abelian group and A is in the bootstrap category (see definition 22.3.4 in [7]). This is a complete algebraization of the problem of computing bivariant KK-groups.

A similar result can be proved for bivariant homology groups. In fact, an analogous result holds for any bivariant homology theory for which the ring of coefficients has at most dimension 1. We omit the proof, since it can be taken verbatim from [22]. Denote by $\overline{\mathbf{H}}(A)$ the sum of $\mathbf{H}_*(A)$ and $\mathbf{H}_*(A;\mathbb{Z}/(p^n))$ for all primes p and all natural numbers n. Note that the ring Λ which consists of Bockstein operations and induced mappings of coefficients acts on $\overline{\mathbf{H}}(A)$ and, furthermore, that any class in $\mathbf{H}(A,B)$ induces a mapping of Λ-modules $\overline{\mathbf{H}}(A) \to \overline{\mathbf{H}}(B)$.

Theorem 4.5.27 (Algebraization of bivariant homology) *Let A and B be cell complexes. The natural map*

$$\mathbf{H}(A,B) \to \hom\ (\overline{\mathbf{H}}(A), \overline{\mathbf{H}}(B))$$

is an isomorphism.

This shows on the one hand that bivariant homology is in a certain sense understandable. On the other hand it reveals that a lot of structure is lost by the passage from **bu** to **H**.

5 Modules over ring spectra - some connections

In this section we will show how the relation between connective E-theory and algebraic K-theory, which we stated in section 4.4, gives rise to precise statements of more general type. Along the way we are going to find a resemblance of Gel'fand duality in the notions of duality in the triangulated category of finite module spectra over the connective K-theory spectrum.

In fact, most of what we are doing for connective E-theory in this section can be done for E-theory as well. There is an identification of E-theory on the category of cell complexes with the homotopy category of finite $\underline{\mathrm{BU}}$-module spectra.

5.1 Connective E-theory, module spectra and co-assembly

Consider the functor $\underline{\mathbf{K}}^{\mathrm{H}} : s\mathcal{C} \to \mathrm{Sp}$ from the category of separable C^*-algebras to the strict category of symmetric spectra which assigns to a C^*-algebra the homotopy algebraic K-theory spectrum (see appendix D). For further information about symmetric spectra we refer to the work of M. Hovey, J. Smith and B. Shipley in [33] and appendix B. By the exterior multiplication which is discussed in appendix D, all the spectra in the image are modules over the image of the complex numbers.

Proposition 5.1.1 *The spectrum $\underline{\mathbf{K}}^{\mathrm{H}}(\mathbb{C})$ is equivalent to the connective K-theory spectrum.*

This statement is proved in an article by Ulrike Tillmann [66]. For simplicity we denote from now on $\underline{\mathbf{K}}^{\mathrm{H}}(\mathbb{C})$ by $\underline{\mathrm{bu}}$.

The last proposition shows that all resulting spectra are modules over the connective K-theory spectrum $\underline{\mathrm{bu}}$. All induced maps are maps of module spectra over $\underline{\mathrm{bu}}$. This together implies that we can define a functor from the category of separable C^*-algebras to the homotopy category of module spectra over the connective K-theory spectrum. This is a triangulated category and we denote it by $\mathcal{D}_{\underline{\mathrm{bu}}}$. The strict category of module spectra carries a stable model structure. The construction of this model structure and a proof of its properties can by found in the work of S. Schwede and B. Shipley in [60]. We denote this functor by $\underline{\mathbf{K}}^{\mathrm{H}} : s\mathcal{C} \to \mathcal{D}_{\underline{\mathrm{bu}}}$ also.

Now, we are going to verify the three properties of $\underline{\mathbf{K}}^{\mathrm{H}}$ needed in order to apply theorem 3.3.6. The homotopy invariance follows by the properties of homotopy algebraic K-theory (see appendix D). Furthermore, any short exact sequence gives rise to a triangle by extending the result of A. Suslin and M. Wodzicki (see appendix D) to the case of homotopy algebraic K-theory. This is also discussed in appendix D.

Since homotopy algebraic K-theory is obviously invariant under tensoring with matrix algebras, by theorem 3.3.6, the functor above induces a functor $\mathbf{bu} \to \mathcal{D}_{\underline{\mathrm{bu}}}$. Let us subsume this statement, which is indeed a trivial combination of observations, in the following theorem.

Theorem 5.1.2 *There is a triangulated functor* $\underline{\mathbf{K}}^{\mathrm{H}}$ *from the category* $\mathbf{bu}$ *to the homotopy category of module spectra over the connective* $\mathbf{K}$*-theory spectrum* $\underline{\mathrm{bu}}$ *which sends the generator* $\mathbb{C}$ *to the generator* $\underline{\mathrm{bu}}$ *such that the induced map*

$$\mathbf{bu}_*(\mathbb{C},\mathbb{C}) \to \mathrm{hom}_{\mathcal{D}_{\underline{\mathrm{bu}}}}(\underline{\mathrm{bu}},\underline{\mathrm{bu}})$$

is an isomorphism of graded unital rings.

Proof: The existence of the transformation is clear by the preceding discussion. The map $\mathbf{bu}_*(\mathbb{C},\mathbb{C}) \to \mathrm{hom}_{\mathrm{D}_{\underline{\mathrm{bu}}}}(\underline{\mathrm{bu}},\underline{\mathrm{bu}})$ is a map of graded unital rings by construction.

Note that there is a transformation from algebraic $\mathbf{K}$-theory to the 'algebraic-topological' $\mathbf{K}$-theory (see section 4.4) for strict cell complexes. By the universal property of $\underline{\mathbf{K}}^{\mathrm{H}}$ (see remark D.1.11 in appendix D), this implies the existence of a transformation $\mathbf{K}^{\mathrm{H}}$ to 'algebraic-topological' $\mathbf{K}$-theory for strict cell complexes. (In order to use the universal property of $\underline{\mathbf{K}}^{\mathrm{H}}$, we have to provide a transformation of symmetric spectra from the spectra computing algebraic $\mathbf{K}$-theory to the spectra computing 'algebraic-topological' $\mathbf{K}$-theory. This can be easily constructed using the well known transformation to topological $\mathbf{K}$-theory.)

We have a diagram

$$\begin{array}{ccccc}
\mathbf{bu}_*(\mathbb{C},\mathbb{C}) & \longrightarrow & \mathrm{hom}_{\mathcal{D}_{\underline{\mathrm{bu}}}}(\underline{\mathrm{bu}},\underline{\mathrm{bu}}) & \longrightarrow & \mathbf{K}_*^{\mathrm{alg,top}}(\mathbb{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}[u] & \longrightarrow & \mathbb{Z}[u] & \longrightarrow & \mathbb{Z}[u]
\end{array}$$

of unital rings. Since $\beta \circ \alpha$ is an isomorphism by the main theorem in section 4.4, β has to be surjective and hence is an isomorphism. (Indeed, all surjective homomorphisms $\mathbb{Z}[u] \to \mathbb{Z}[u]$ are isomorphisms.) This implies that α is an isomorphism and hence finishes the proof.

Note that there is an obvious forgetful functor from the strict category of $\underline{\mathrm{bu}}$-module spectra to the category of symmetric spectra. This functor has a left adjoint which can be thought of as a free functor. There is also a natural inclusion of the category of pointed CW-complexes into the category of symmetric spectra which is given by forming the suspension spectrum with the appropriate actions of the symmetric groups. This is a right adjoint functor to taking the zeroth space of the symmetric spectrum. Composing those two functors we can assign to any pointed CW-complex a strict $\underline{\mathrm{bu}}$-module in a covariant functorial way (this is now also a left adjoint being the composition of left adjoint functors). We denote this functor by $\mathrm{F} : CW_+ \to Sp_{\underline{\mathrm{bu}}}$. We may assume that the functor F takes values in fibrant objects.

Note that there is now a certain ambiguity in the computation of connective $\mathbf{K}$-groups for finite CW-complexes. Let (X, x) be a finite pointed CW-complex. We used to compute the connective $\mathbf{K}$-theory as the homotopy groups of $map_{\underline{bu}}(F(X), \underline{bu})$ which is the same as $[X, \underline{bu}_*]$, since F was left adjoint to the forgetful functor. Now there is another way to define the connective $\mathbf{K}$-groups. Just take the homotopy groups of $\underline{\mathbf{K}}^H(C(X, x))$. We better show that these definitions naturally agree. This is a matter of co-assembly as Michael Joachim pointed out to me. The construction of the co-assembly map is analogous to the construction of the assembly map. It goes along the lines of the article of M. Weiss and B. Williams [71] but is dual.

Theorem 5.1.3 *Let (X, x) be a finite pointed CW-complex. There is a natural co-assembly map*

$$\underline{\mathbf{K}}^H(C(X, x)) \to \mathrm{map}_{\underline{bu}}(F(X, x), \underline{bu})$$

in the category $\mathcal{D}_{\underline{bu}}$ and it is an isomorphism.

Proof: Let us first construct the map. Let (X, x) be a finite pointed CW-complex. Note that we have a split extension as follows.

$$0 \longrightarrow C(X, x) \longrightarrow C(X) \xrightarrow{ev_x} \mathbb{C} \longrightarrow 0$$

Since both sides of the equation are excisive, we can reduce the proof to the case of unpointed spaces (or, equivalently, spaces with a disjoint basepoint).

Note that $Sp_{\underline{bu}}$ is a simplicial model category. This implies that we have the notions of homotopy limit and homotopy co-limit. For a precise discussion of these concepts see the transcript of P. Hirschhorn [31].

We are attempting to define a contravariant functor from finite CW-complexes to $\underline{bu}$-modules which is homotopy invariant. First, we include the category of finite CW-complexes into the category of finite pointed CW-complexes by adding a disjoint base point and then apply the functor $\underline{\mathbf{K}}^H$ which sends a pointed space to a $\underline{bu}$ module spectrum. Note that this step is necessary, since the business of assembly (as developed in [71]) and the dual notion of co-assembly do not work for pointed spaces. We denote this composition by $\underline{\mathbf{K}}^H(C(?_+, +)) : CW^{fin} \to Sp_{\underline{bu}}$.

Denote by $sp(X)$ the category whose objects are maps $\Delta^n \to X$ and morphisms $\Delta^m \to \Delta^n$ over X which are induced by monotone injective mappings from $\{0, \dots, m\}$ to $\{0, \dots, n\}$ in the obvious way. Now, we are going to define certain homotopy limits and homotopy co-limits with respect to this category.

Note that we have internal hom's (which we denote by $\mathrm{map}_{\underline{bu}}$) in the category of $\underline{bu}$-module spectra and that $\underline{\mathbf{K}}^H(C(\Delta^0_+, +)) = \underline{bu}$ by definition. We have a diagram in the strict category of $\underline{bu}$-module spectra as follows.

$$
\begin{array}{ccc}
\underline{\mathbf{K}}^{H}(C(X_+,+)) & \longrightarrow & \mathbf{holim}_{sp(X)^{op}}\underline{\mathbf{K}}^{H}(C(\Delta^n_+,+)) \\
 & & \uparrow \\
 & & \mathbf{holim}_{sp(X)^{op}}\underline{\mathbf{K}}^{H}(C(\Delta^0_+,+)) \\
 & & \downarrow \\
 & & \mathbf{holim}_{sp(X)^{op}}\mathbf{map}_{\underline{bu}}(F(\Delta^0),\underline{bu}) \\
 & & \uparrow \\
 & & \mathbf{map}_{\underline{bu}}(\mathbf{hocolim}_{sp(X)}F(\Delta^0),\underline{bu}) \\
 & & \uparrow \\
\mathbf{map}_{\underline{bu}}(F(X),\underline{bu}) & \longleftarrow & \mathbf{map}_{\underline{bu}}(F(|sp(X)|),\underline{bu})
\end{array}
$$

We want to argue that all maps that are appearing in this diagram are in fact well-defined, natural, and induce isomorphisms in $\mathcal{D}_{\underline{bu}}$.

First of all, let us say a word about the map α, since it is the actual co-assembly map. Note that the maps $\Delta^n \to X$ induce a natural maps $\underline{\mathbf{K}}^H(C(X_+,+)) \to \underline{\mathbf{K}}^H(C(\Delta^n_+,+))$ which give a natural transformation from the constant functor $sp(X) \to Sp_{\underline{bu}}$ (which maps everything to $\underline{\mathbf{K}}^H(C(X_+,+))$) and the functor $sp(X) \to Sp_{\underline{bu}}$ which assigns to the object $\Delta^n \to X$ in $sp(X)$ the spectrum $\underline{\mathbf{K}}^H(C(\Delta^n_+,+))$. The homotopy limit of the constant functor is just $\underline{\mathbf{K}}^H(C(X_+,+))$ and α is the map induced by the natural transformation.

The map ν is induced by the map $X \to |sp(X)|$ which is a weak equivalence of CW-complexes and hence is a homotopy equivalence. This implies that it induces a weak equivalence after applying F. The maps β comes from projections $\Delta^n_+ \to \Delta^0_+$ which are homotopy equivalences. Since F takes values in fibrant objects, the induced map of homotopy limits is a weak equivalence also (see [31] theorem 19.4.3). The same is true for the map γ. By theorem 19.1.12 in [31], the map δ induces an equivalence. In fact, this is a triviality once one has a definition of homotopy limit and homotopy co-limit. We restated the needed results from [31] in appendix B.2. The map η induces an equivalence, since realization commutes with homotopy co-limits (see e.g. [31] theorem 19.8.7). Taking everything together this implies the existence of the required map, since all weak-equivalences are inverted in $\mathcal{D}_{\underline{bu}}$.

It remains to show that α (which is the only interesting piece of our co-assembly map) and hence our co-assembly map also induces an isomorphism in $\mathcal{D}_{\underline{bu}}$. This can be seen by noting that both functors satisfy excision and are isomorphic at the one-point space $X = *$ in $\mathcal{D}_{\underline{bu}}$. By excision, this implies that they are weakly homotopy equivalent for finite CW-complexes. This finishes the proof.

Remark 5.1.4 *We defined bivariant homology in section 4 using the cone of the Bott map in the* C^**-algebra setting. By theorem 5.1.2, it has become obvious that the cone of the Bott map in the category of* C^**-algebras indeed corresponds to the homotopy fiber of the Bott map of the connective* $\mathbf{K}$*-theory spectrum. Indeed, one checks that there is an isomorphism* $\underline{\mathbf{K}}^H(\Sigma^{-1}(Z)) \cong \mathbf{K}(\mathbb{Z})$ *of* $\underline{bu}$*-modules in* $\mathcal{D}_{\underline{bu}}$*. Here, we regard* $\mathbf{K}(\mathbb{Z})$ *as a fibrant object in* $Sp_{\underline{bu}}$ *with its natural* $\underline{bu}$*-module structure.*

(Note that in the preceding remark $\mathbf{K}(?)$ denotes the functor which sends an Abelian group to its Eilenberg-MacLane spectrum, where, in distinction, $\underline{\mathbf{K}}(?)$ denotes the functor which sends a ring to is algebraic $\mathbf{K}$-theory spectrum.)

Now, we are able to prove theorem 4.5.4. Indeed, let (X, x) and (Y, y) be finite pointed CW-complexes. we have the following natural isomorphisms as follows.

$$\begin{aligned}
\mathbf{bu}(C(Y,y), C(X,x) \otimes \Sigma^{-4}(Z)) &\cong \mathbf{bu}(C(Y,y) \otimes \Sigma^{-1}(Z), C(X,x)) \\
&\cong \hom_{\mathcal{D}_{\underline{bu}}}(\underline{\mathbf{K}}^H(C(Y,y)), \underline{\mathbf{K}}^H(\Sigma^{-1}(Z) \otimes C(X,x))) \\
&\cong \hom_{\mathcal{D}_{\underline{bu}}}(\underline{\mathbf{K}}^H(C(Y,y)), \underline{\mathbf{K}}^H(\Sigma^{-1}(Z)) \wedge \underline{\mathbf{K}}^H(C(X,x))) \\
&\cong \hom_{\mathcal{D}_{\underline{bu}}}(\mathrm{map}_{\underline{bu}}(F(Y,y), \underline{bu}), \mathbf{K}(\mathbb{Z}) \wedge \mathrm{map}_{\underline{bu}}(F(X,x), \underline{bu})) \\
&\cong \hom_{\mathcal{D}_{\underline{bu}}}(\mathrm{map}_{\underline{bu}}(F(Y,y), \underline{bu}), \mathbf{K}(\mathbb{Z}) \wedge \mathrm{map}_{\underline{bu}}(F(X,x), \underline{bu})) \\
&\cong \hom_{\mathcal{D}_{\underline{bu}}}(F(X,x), \mathbf{K}(\mathbb{Z}) \wedge F(Y,y)) \\
&\cong \mathrm{colim}_n[\Sigma^n(X), \mathbf{K}(\mathbb{Z})_* \wedge Y]_+
\end{aligned}$$

It is obvious that the isomorphisms are compatible with the composition product. This finishes the proof of theorem 4.5.4.

5.2 Gelf'fand duality vs. Spanier-Whitehead duality

This part contains a remark about the intertwining of different notions of duality which has been achieved implicitly in the preceding section.

There is an interesting interpretation of theorem 5.1.3. In some sense it matches Spanier-Whitehead duality and Gel'fand duality. Note that $map_{\underline{bu}}(F(X), \underline{bu})$ is the Spanier-Whitehead dual of $F(X)$. Let

$$D : \mathcal{D}_{\underline{bu}} \to (\mathcal{D}_{\underline{bu}})^{op}$$

be the duality functor.

The following diagram commutes up to a natural equivalence.

$$
\begin{array}{ccc}
CW^{op} & \xrightarrow{G} & s\mathcal{C} \\
\Big\downarrow{\scriptstyle F^{op}} & & \Big\downarrow{\scriptstyle \underline{\mathbf{K}}^{H}} \\
\mathcal{D}_{\underline{bu}}^{op} & \xrightarrow{D^{op}} & \mathcal{D}_{\underline{bu}}
\end{array}
$$

Here, the upper horizontal arrow is given by Gel'fand duality, the lower horizontal arrow is given by Spanier-Whitehead duality. The left vertical arrow is the free functor into $\underline{bu}$-modules and the right vertical arrow is the functor that assign to a separable C^*-algebra its homotopy algebraic $\mathbf{K}$-theory spectrum which is naturally a $\underline{bu}$-module spectrum.

The equivalence which establishes the commutativity of the diagram is given by the co-assembly map which we have constructed in theorem 5.1.3.

A Triangulated categories

A.1 Definitions

A self-contained treatment of triangulated categories can be found in [43]. In this section we want to recall the definition of a triangulated category. We also want to define what we mean by a triangulated module. The original definition of triangulated categories goes back to independent work of D. Puppe [49] and J.L. Verdier [67]. It was designed to reveal the structure of the stable homotopy category and of derived categories of modules over a ring. Later, there were applications in representation theory and non-commutative algebra.

Our applications are in the spirit of stable homotopy theory. The main result in section 5 is that a certain universal triangulated homology theory (which is in particular a triangulated category) on the category of separable C^*-algebras can be identified with the homotopy category of module spectra over the connective $\mathbf{K}$-theory spectrum $\underline{bu}$. The identification is an equivalence of triangulated categories.

This result links two parts of mathematics which are not disjoint but previously have not been considered to be so closely connected.

Definition A.1.1 *Let T be an additive category and let Σ be an additive automorphism. A* <u>*triangle*</u> *in T is a diagram of the form*

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma(A)$$

such that the compositions $v \circ u$, $w \circ v$ and $\Sigma(u) \circ w$ are equal to zero.

If for some reason we are working with Σ^{-1} as the automorphism (and this will be the case for triangulated homology theories for separable C^*-algebras) we are going to use diagrams of the form

$$\Sigma(A) \longrightarrow B \longrightarrow C \longrightarrow A$$

rather than

$$A' \longrightarrow B' \longrightarrow C' \longrightarrow \Sigma^{-1}(A').$$

Note that up to isomorphism of diagrams this does not make any difference.

Definition A.1.2 *A* <u>*triangulated*</u> *category T is an additive category, together with an additive automorphism Σ and a class of distinguished triangles. The following conditions have to be satisfied.*

1. *Any triangle which is isomorphic to a distinguished triangle is distinguished.*

2. *The triangle*

$$A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow \Sigma(A)$$

is distinguished for any $A \in \mathrm{ob}(T)$.

3. *For any morphism* $f : A \to B$ *in* T *there exists a distinguished triangle of the following form.*

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow \Sigma(A)$$

4. *Consider the following two triangles.*

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma(A)$$

$$B \xrightarrow{-v} C \xrightarrow{-w} \Sigma(A) \xrightarrow{-\ (u)} \Sigma(B)$$

If one is distinguished, then so is the other.

5. *For any commutative diagram of the form*

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma(A) \\
\downarrow{\scriptstyle f} & & \downarrow & & & & \downarrow{\scriptstyle (f)} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma(A')
\end{array}$$

where the rows are distinguished triangles there exists a morphism $h : C \to C'$ *which makes the diagram commutative.*

6. *Let* $f : A \to B$ *and* $g : B \to C$ *be morphisms in* T. *Let the following triangles*

$$A \xrightarrow{f} B \longrightarrow D \longrightarrow \Sigma(A)$$

$$A \xrightarrow{g \circ f} C \longrightarrow F \longrightarrow \Sigma(A)$$

$$B \xrightarrow{g} C \longrightarrow E \longrightarrow \Sigma(B)$$

be distinguished. We can complete these to a diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \longrightarrow & D & \longrightarrow & \Sigma(A) \\
\downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle (\mathrm{id}_A)} \\
A & \xrightarrow{g \circ f} & C & \longrightarrow & F & \longrightarrow & \Sigma(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E & \xrightarrow{\mathrm{id}_E} & E & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma(A) & \xrightarrow{(f)} & \Sigma(B) & \longrightarrow & \Sigma(D) & \longrightarrow & \Sigma^2(A)
\end{array}$$

such that the first two rows and the second column are the given triangles, all rows and columns are distinguished triangles and, furthermore, the square

$$\begin{array}{ccc} B & \longrightarrow & D \\ \downarrow{\scriptstyle g} & & \downarrow \\ C & \longrightarrow & F \end{array}$$

is homotopy Cartesian (see definition A.1.3).

Definition A.1.3 *A square*

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{j} & D \end{array}$$

is called <u>*homotopy Cartesian*</u> *if there exists a distinguished triangle*

$$A \xrightarrow{\binom{f}{-g}} C \oplus B \xrightarrow{(j\ h)} D \longrightarrow \Sigma(A)$$

for some <u>*boundary map*</u> δ*.*

Definition A.1.4 *Let* (T', Σ') *and* (T'', Σ'') *be triangulated categories. A triangulated functor* $F : T' \to T''$ *is an additive functor together with natural isomorphisms*

$$\phi_X : F(\Sigma'(X)) \to \Sigma''(F(X))$$

such that for any distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma'(A).$$

in (T', Σ') *the sequence*

$$F(A) \xrightarrow{F(\ )} F(B) \xrightarrow{F(\ )} F(C) \xrightarrow{\quad c \circ F(\ ) \quad} \Sigma''(F(A)).$$

is a distinguished triangle in (T'', Σ'')*.*

There are notions of triangulated monoidal categories. The precise requirements on the compatibility between the distinguished triangles and the tensor product are not clear yet. Our definition follows the most naive notion of compatibility. There are more subtle conditions according to P. May in [38]. They are very natural, since they are satisfied in stable homotopy theories coming from suitable monoidal model categories. Since in our applications there is no model category, we stick to the naive notion.

Definition A.1.5 *Let* T *be a triangulated category which is also monoidal. It is said to be* <u>*triangulated monoidal*</u> *if the following compatibility conditions are satisfied.*

- *The functors* $? \otimes A : T \to T$ *and* $A \otimes ? : T \to T$ *are triangulated functors for any object* $A \in \mathrm{ob}(T)$.
- *There are natural equivalences of functors* $r : \Sigma(?_1 \otimes ?_2) \to ?_1 \otimes \Sigma(?_2)$ *and* $l : \Sigma(?_1 \otimes ?_2) \to \Sigma(?_1) \otimes ?_2$ *such that for any* $A, B \in \mathrm{ob}(T)$ *the following diagram is* ***skew****-commutative.*

$$
\begin{array}{ccc}
\Sigma^2(A \otimes B) & \xrightarrow{(r(A,B))} & \Sigma(A \otimes \Sigma(B)) \\
\Big\downarrow{\scriptstyle (l(A,B))} & & \Big\downarrow{\scriptstyle l(A,\ \ (B))} \\
\Sigma(\Sigma(A) \otimes B) & \xrightarrow[r(\ \ (A),B)]{} & \Sigma(A) \otimes \Sigma(B)
\end{array}
$$

Remark A.1.6

- *The diagram above just recalls the fact that in our context the switch* $s : \Sigma^2 \to \Sigma^2$ *is not the identity but minus the identity up to homotopy. (There might be other contexts where this is not true. In those situations the conventions have to be suitably adapted.) We are going to omit the explicit mentioning of the isomorphisms, since all reasonable diagrams commute if one is careful about switching suspensions.*
- *We abbreviate* $A \otimes B$ *by* AB.

Definition A.1.7 *Let* T *be a triangulated category. Let* A *and* B *be objects in* T. *Define* $T_n(A, B) = \mathrm{hom}_T(\Sigma^n A, B)$.

The preceding definition enriches any triangulated category over $\mathbb{Z}$-graded Abelian groups. (This just means that there is an associated category with morphism between A and B given by the $\mathbb{Z}$-graded Abelian groups $T_*(A, B)$. The old category is obtained by forgetting the non-zero parts in the graduation.) Note that this definition is not in contradiction with definition 2.3.7, since we are working with the inverse automorphism in the setting of triangulated homology theories (see section 2.3). At this stage this might cause some confusion but other conventions would cause even more.

Remark A.1.8 *One important property of triangulated categories is that the* hom-*functor is homological resp. co-homological in the variables. That is to say that for any distinguished triangle*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma(A)$$

and for any D *in* T *there are long exact sequences as follows.*

$$\cdots \longrightarrow T_{n+1}(D, C) \longrightarrow T_n(D, A) \longrightarrow T_n(D, B) \longrightarrow T_n(D, C) \longrightarrow T_{n-1}(D, A) \longrightarrow \cdots$$

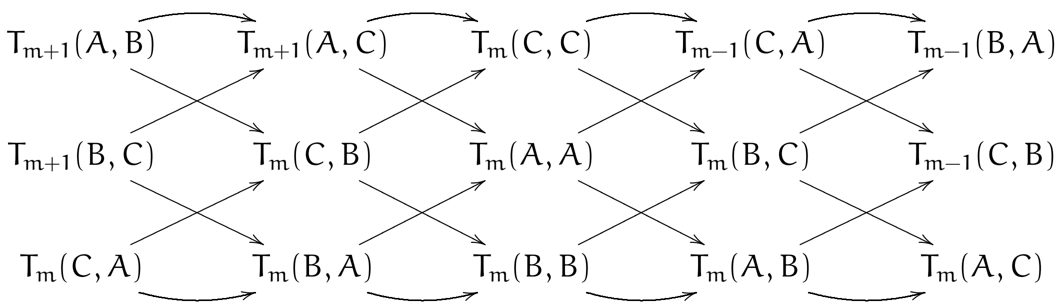
$$\cdots \longrightarrow T_{n+1}(A, D) \longrightarrow T_n(C, D) \longrightarrow T_n(B, D) \longrightarrow T_n(A, D) \longrightarrow T_{n-1}(C, D) \longrightarrow \cdots$$

In the next theorem we prove that several exact sequences one can get out of a distinguished exact triangle assemble to an exact braided sequence.

Theorem A.1.9 *Let*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma(A)$$

be a distinguished triangle. The following braided diagram is exact along the waves and commutes.



Proof: The exactness of the diagram is obvious from the previous observation. Let us prove commutativity. Any square (or deformed square) in this diagram is of the form that the two ways in the square just differ by the order of composition on the left and right. Hence the associativity of the composition in T implies that those squares commute. The hexagons in the diagram commute, since each way one can go gives zero. This proves the claim.

Definition A.1.10 *Let* (T, Σ) *be a triangulated category. An additive category* P *together with a functor* $F : T \to P$ *is called a triangulated module over* (T, Σ) *if the following conditions hold.*

- *The functor* $P(A, F(?)) : T \to Ab$ *is homological for any* $A \in ob(P)$.
- *The functor* $P(F(?), B) : T \to Ab$ *is co-homological for any* $B \in ob(P)$.

Triangulated modules frequently arise. For example, definition 2.3.15 gives plenty of examples. An obvious question under what conditions it is true that a triangulated module sits inside a triangulated category such that the comparison functor becomes triangulated. We do not know the answer to this question.

A.2 Localization

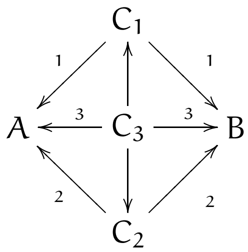
In this section of the appendix we want to discuss localization of classes of morphisms in a triangulated category. It will turn out that there is a nice description of the category obtained and, furthermore, it is again a triangulated category. The result which we want to present are taken from [43]. We want to discuss a description of certain special cases of localization in monoidal triangulated categories.

Definition A.2.1 *Let* $F : (T_1, \Sigma) \to (T_2, \Sigma)$ *be a triangulated functor. We denote by* $\ker(F)$ *the full triangulated sub-category of* (T_1, Σ) *whose objects map to objects isomorphic to* 0 *in* (T_2, Σ).

Theorem A.2.2 *Let* (T, Σ) *be an essentially small triangulated category and* $(R, \Sigma) \subset (T, \Sigma)$ *a triangulated sub-category. Then there exists a triangulated functor* $F : (T, \Sigma) \to (T/R, \Sigma)$ *which is universal with respect to the property that* R *is contained in* $\ker(F)$*, i.e. the category* R *is mapped to objects isomorphic to zero.*

A proof of this theorem can be found in the book by A. Neeman [43]. We are not going to prove it but we want to give the starting point. Note that A. Neeman uses a different notion of category in his book which is quite confusing. As far as localization is concerned we are always working with essentially small categories. Therefore we do not have to think about existence of categories of fractions and so on.

Suppose that (R, Σ) is a full triangulated sub-category of (T, Σ). Let T_R be the sub-category of morphisms with the property that their cones lie inside R. The objects of the category T/R will just be the objects of T. For any pair of objects A and B we define $\Gamma(A, B)$ to be the class of diagrams of the form $A \leftarrow C \to B$ with $\alpha \in \hom_{T_R}(C, A)$ and $\beta \in \hom_T(C, B)$. We use the notation $[\alpha, C, \beta]$ to describe such a diagram. Two triples $[\alpha_1, C_1, \beta_1]$ and $[\alpha_2, C_2, \beta_2]$ are considered to be equivalent, if there exists a triple $[\alpha_3, C_3, \beta_3]$ and morphisms $\delta : C_3 \to C_1$ and $\eta : C_3 \to C_2$ such that the diagram



commutes. (One has to show that this indeed defines an equivalence relation.) The equivalence classes which are denoted by $\Gamma(A, B)$ form a set, since our category was required to be skeletally small. The proof proceeds with showing that $\Gamma(A, B)$ carries an associative composition product. Finally, one has to show that the resulting category is triangulated and has the universal properties that are required. This finishes our outline of the idea of the proof.

Definition A.2.3 *The quotient* T/R *is called the* *<u>Verdier quotient</u>* *of* (T, Σ) *by* (R, Σ) *and the triangulated functor* F *is called the* *<u>Verdier localization map</u>*.

Proposition A.2.4 *Let* (T, Σ) *be an essentially small triangulated category. Let* K *be any class of morphisms in the category* T. *There exists a triangulated functor* $F : (T, \Sigma) \to (T[K^{-1}], \Sigma)$ *with the universal property that the morphisms in* K *map to isomorphisms.*

Proof: Apply theorem A.2.2 to the full triangulated sub-category generated by the cones of the morphisms in K. It is easy to see that the universal properties translate into each other using general properties of triangulated categories.

Remark A.2.5 *Suppose one starts with a triangulated monoidal category and requires the universal property among triangulated monoidal functors into triangulated monoidal categories. The construction of Verdier works also in this context and the resulting category in our outline of the proof of theorem A.2.2 carries indeed a natural monoidal structure.*

The following theorem allows to identify the Verdier quotient of a category (T, Σ) in easy situations with something understandable in terms of the category (T, Σ). In particular, this will be useful in section 4.

Theorem A.2.6 *Let (T, Σ) be a triangulated monoidal category. Let U be the unit of the tensor product and let $f : Q \to U$ be a morphism in T. Suppose that the switch $s : Q^2 \to Q^2$ is equal to $\mathrm{id} \in \mathrm{hom}_T(Q^2, Q^2)$. The localization $T[f^{-1}]$ has the following special form:*

- $\mathrm{ob}(T[f^{-1}]) = \mathrm{ob}(T)$
- $\mathrm{hom}_{T[f^{-1}]}(A, B) = \mathbf{colim}_{n \in \mathbb{N}} \mathrm{hom}_T(Q^n A, B)$

Outline of proof: One either checks that our setting allows an associative composition product and defines a triangulated category with the universal property or goes through the construction in the outline of the proof of theorem A.2.2.

This finishes our discussion of localization. In section 4 we are going to apply the categorical dual of theorem A.2.6.

A.3 Duality

In this part we discuss an intrinsic notion of duality in triangulated categories.

Definition A.3.1 *Let A be an object in a monoidal triangulated category with unit U. It is called dualizable, if there exists an object $D(A)$ and morphisms $\epsilon : U \to D(A) \otimes A$ and $\eta : A \otimes D(A) \to U$ such that the following conditions hold.*

- *The composite $A \overset{\mathrm{id}_A \otimes}{\to} A \otimes D(A) \otimes A \overset{\otimes \mathrm{id}_A}{\to} A$ is the identity.*
- *The composite $D(A) \overset{\otimes \mathrm{id}_{D(A)}}{\to} D(A) \otimes A \otimes D(A) \overset{\mathrm{id}_{D(A)} \otimes}{\to} D(A)$ is the identity.*

Theorem A.3.2 *Let (T, Σ) be a triangulated category. Let A be dualizable, $D(A)$ denote its dual and η and ϵ be the maps discussed in the preceding definition.*

- *The composite*

  $$\mathrm{hom}_T(A \otimes B, C) \to \mathrm{hom}_T(D(A) \otimes A \otimes B, D(A) \otimes C) \stackrel{(\ \otimes \mathrm{id}_B)^*}{\to} \mathrm{hom}_T(B, D(A) \otimes C)$$

  is an isomorphism for all B *and* C.

- *The composite*

  $$\mathrm{hom}_T(B, C \otimes A) \to \mathrm{hom}_T(B \otimes D(A), C \otimes A \otimes D(A)) \stackrel{(\mathrm{id}_C \otimes\ )_*}{\to} \mathrm{hom}_T(B \otimes D(A), C)$$

  is an isomorphism for all B *and* C.

We state a proposition which relates the notion of duality to the notion of distinguished triangles.

Proposition A.3.3 *Let* (T, Σ) *be a triangulated category. Consider a distinguished triangle*

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} \Sigma(A)$$

in T. *If two of the objects are dualizable, then so is the third.*

Furthermore, the induced triangle

$$D(C) \xrightarrow{D(b)} D(B) \xrightarrow{D(a)} D(A) \xrightarrow{D(c)} \Sigma(D(C))$$

is distinguished.

Using the two duality maps ϵ and η we can define a T_*-valued trace.

Definition A.3.4 *Let* $f : A \to A$ *be a self map of* A *(possibly not of degree zero). Consider the following composition.*

$$U \longrightarrow D(A) \otimes A \xrightarrow{\mathrm{id}_{D(A)} \otimes f} D(A) \otimes A \longrightarrow A \otimes D(A) \longrightarrow U$$

We denote the element in $T_*(U, U)$ *which is defined by this composition by* $\tau(f)$.

A treatment of the properties of traces in triangulated categories can be found in [38].

Definition A.3.5 *Let* (T, Σ) *be a triangulated category. A trace* τ *is called* *<u>additive</u>* *if for any self-map of a distinguished triangle*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow (a) \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \end{array}$$

the equality

$$\tau(a) - \tau(b) + \tau(c) = 0$$

holds.

Philosophically (according to P. May [38]) this should be true for any reasonable trace. Nevertheless the proof for the particular trace defined above is difficult, and in [38] it is only carried out in the case, where $(\mathcal{T}, \Sigma)$ is the stable homotopy category of a monoidal model category.

B Spectra and cohomology theories

B.1 Definitions

The foundational reference on the theory of spectra and cohomology theories is the famous book by F. Adams [1]. A more comprehensive introduction to this part of algebraic topology can be found in the book by R.M. Switzer [65]. A comprehensive treatment of spectra and symmetric spectra can be found in [33]. This section will serve as a recollection of results about symmetric spectra. We will stick to the case of spectra of simplicial sets, since some technicalities are easier to deal with in this case. A detailed treatment of the theory of simplicial sets is given in the book by P.G. Goerss and J.F. Jardine [27]. We do not want to repeat the definitions of a simplicial set and the related notions. Those definitions can be found in [27] or any other book about simplicial aspects of algebraic topology.

Denote by Δ the category with objects the natural numbers $\{[n], n \in \mathbb{N}\}$ considered as finite totally ordered sets and morphisms monotone mappings. We denote by $\Delta[n]$ the standard simplicial set $\mathrm{hom}\ ([?], [n])$.

Recall that S^1 is the simplicial set given by $\Delta[1]/\partial\Delta[1]$. We denote its n-fold smash product by $S^n =_{\mathrm{def}} S^1 \wedge \cdots \wedge S^1$. Note that there is a natural action of the symmetric group Σ_n on S^n which permutes the smash factors.

Definition B.1.1 • *A (naive) <u>spectrum</u> is*

1. *A sequence $X_0, X_1, \ldots, X_n, \ldots$ of pointed simplicial sets and*
2. *a pointed map*

$$\sigma_n : S^1 \wedge X_n \to X_{n+1}$$

for any $n \geqslant 0$.

- *A map of (naive) spectra (X_n, σ_n) and (X'_n, σ'_n) is a sequence of pointed maps $f_n : X_n \to X'_n$ such that*

$$\sigma'_{n+1} \circ (\mathrm{id}_{S^1} \wedge f_n) = f_{n+1} \circ \sigma_n : S^1 \wedge X_n \to X'_{n+1}.$$

This defines the category of (naive) spectra and very much (in fact, all the homotopy theory) can be done using this category. It turns out that the homotopy category which can be constructed is symmetric monoidal and it was a long standing question whether a symmetric tensor product could be introduced at the strict level of spectra as well. More modern developments showed that a different notion of spectra was necessary in order to achieve this goal. Those are called symmetric spectra and were defined by M. Hovey, B. Shipley and J. Smith in [33]. There are also other approaches to construct symmetric monoidal categories of spectra. References about those can be found in [33] as well.

Definition B.1.2 • *A* *<u>symmetric spectrum</u>* *is*

1. *A sequence* $X_0, X_1, \ldots, X_n, \ldots$ *of pointed simplicial sets,*
2. *a pointed map*
$$\sigma_n : S^1 \wedge X_n \to X_{n+1}$$
for any $n \geqslant 0$ *and*
3. *a basepoint preserving left action of the symmetric group* Σ_n *on* X_n *for every* $n \in \mathbb{N}$ *such that the composition*
$$\sigma_{n+p-1} \circ (\mathrm{id}_{S^1} \wedge \sigma_{n+p-2}) \circ \cdots \circ (\mathrm{id}_{S^{p-1}} \wedge \sigma_n) : S^p \wedge X_n \to X_{n+p}$$
is $\Sigma_p \times \Sigma_n$*-equivariant for all possible* p *and* n*.*

- *A map of symmetric spectra* (X_n, σ_n) *and* (X'_n, σ'_n) *is a sequence of pointed and* Σ_n*-equivariant maps* $f_n : X_n \to X'_n$ *such that*
$$\sigma'_{n+1} \circ (\mathrm{id}_{S^1} \wedge f_n) = f_{n+1} \circ \sigma_n : S^1 \wedge X_n \to X'_{n+1}.$$

Definition B.1.3 *Denote the category of (naive) spectra by* $Sp^{\mathbb{N}}$ *and the category of symmetric spectra by* Sp *.*

We state only results for the category of symmetric spectra, since they are non-classical although the proofs are in most of the cases as easy or as hard as for the category of (naive) spectra. Proofs of the following results and a much more comprehensive and extensive treatment of them can be found in [33].

Proposition B.1.4 *The category of symmetric spectra* Sp *is bi-complete. Limits and co-limits are formed degree-wise.*

Definition B.1.5 ([33]) *Let* (X_n, σ_n) *and* (X'_n, σ'_n) *be symmetric spectra. The tensor product (or smash product)* $(X_n, \sigma_n) \wedge (X'_n, \sigma'_n)$ *is defined to be*
$$(\vee_{p+q=n} \Sigma_{n+} \wedge_{\ p\times\ q} (X_p \wedge X_q), \sigma''_n)$$
where the σ''_n *are the maps which are naturally induced.*

Theorem B.1.6 *The category of symmetric spectra with the smash product is symmetric monoidal closed (for a precise definition see [36]).*

This theorem is very important for further constructions. It allows to speak about monoids and modules over monoids in the category of symmetric spectra. The notion of monoid and module over a monoid are purely category theoretic notions. For definitions we refer to the standard source in this context [36].

It is an important consequence of the general theory that the categories of modules over a commutative monoid in the category of symmetric spectra is again symmetric monoidal closed.

Definition B.1.7 *Let* R *be a commutative monoid object in the category of symmetric spectra. Denote the category of* R*-module spectra by* Sp_R.

An essential part of structure on the category of symmetric spectra is its model structure. Since we are not going to define the notion of a model category (see [33] for a definition and further references), we are going to take a very informal point of view. Let us start with a definition.

Definition B.1.8 *A map* $f : (X_n, \sigma_n) \to (X'_n, \sigma'_n)$ *of symmetric spectra is called a* <u>*stable equivalence*</u> *if for any generalized cohomology theory* E *(defined on the category of symmetric spectra) the induced map* $E^*(f)$ *is an isomorphism. Denote the class of stable equivalences by* $\mathcal{W}$.

Proposition B.1.9 *Let* $f : (X_n, \sigma_n) \to (X'_n, \sigma'_n)$ *be a map of symmetric spectra. If* f *induces an isomorphism of stable homotopy groups after forgetting the* Σ_n*-actions, then* f *is a stable equivalence of symmetric spectra.*

A proof of the preceding proposition can be found in [33], theorem 3.1.11.

Definition B.1.10 *The homotopy category of symmetric spectra is defined to be the category of fractions* $\mathrm{Sp}\ [\mathcal{W}^{-1}]$ *and denoted by* $\mathrm{Ho}(\mathrm{Sp}\ )$.

Of course, there is a question of existence of this category of fractions. This is where model categories are used. Furthermore, the model structure allows to find a concrete description of the sets of morphisms in the category of fractions in terms of homotopy classes of maps between fibrant and co-fibrant replacements. The usefulness of a model structure of course goes far beyond this implication.

In analogy one can define a homotopy category of module spectra over a certain monoid object in the category of symmetric spectra.

Definition B.1.11 *Let* R *be a commutative monoid in the category of symmetric spectra. The homotopy category of* R*-module spectra is defined to be* $\mathrm{Sp}_R[(\mathcal{W} \cap \mathrm{Sp}_R)^{-1}]$ *and denoted by* $\mathcal{D}_R$.

Again, the definition leaves the question about existence open. A monoidal model structure on the category of module spectra Sp_R is provided in the work of S. Schwede and B. Shipley in [60]. The term 'brave new algebra' is used for homotopy categories of module spectra. This is because the homotopy categories of module spectra over Eilenberg-MacLane spectra identify with the usual derived categories (for a proof see [60]). Therefore the framework of symmetric spectra and module spectra extends the framework of graded Abelian groups and chain complexes of modules over a ring.

B.2 Homotopy limits and homotopy co-limits

In this section we subsume some results about homotopy limits and homotopy co-limits. We do not give proper definitions and proofs and refer to [31] for a more comprehensive treatment. Homotopy limits and homotopy co-limits replace limits and co-limits in homotopy categories, since these do not behave so well. In order to construct homotopy limits and co-limits one usually uses a simplicial model structure (see [31] for a definition).

We need the following abstract properties. Proofs can be found in [31].

Lemma B.2.1 *Let $\mathcal{C}$ be a small category. Let $F : \mathcal{C} \to \mathcal{M}$ be a diagram in a simplicial and monoidal model category. We denote by* map(?, ?) *the internal* hom-*functor. The following natural map*

$$\mathrm{holim}_{\mathcal{C}^{op}} \mathrm{map}(F(c), B) \to \mathrm{map}(\mathrm{hocolim}_{\mathcal{C}} F(c), B)$$

is an isomorphism in the homotopy category of $\mathcal{M}$.

Lemma B.2.2 *Let $F_i : \mathcal{C} \to \mathcal{M}$ for $i = 1, 2$ be diagrams in M that take values in fibrant objects. Let $T : F_1 \to F_2$ be a natural transformation. If T is object-wise a weak equivalence, then the induced map*

$$\mathrm{holim}_{\mathcal{C}} F_1(c) \to \mathrm{holim}_{\mathcal{C}} F_2(c)$$

is a weak equivalence as well.

We do not want to say more about the notions of homotopy limit and homotopy co-limit, since this would go beyond the scope of this thesis.

C C^*-Algebras

In this part of the appendix we want to recall some definitions. We do not intend to give an introduction to the theory of C^*-algebras but rather recollect some notations and results needed.

C.1 Definitions

Definition C.1.1
- *A complex Banach algebra is a $\mathbb{C}$-algebra which (with the same $\mathbb{C}$-linear structure) is a complete normed vector space such that the norm satisfies*

$$\forall x, y \in A : \|xy\| \leqslant \|x\|\|y\|.$$

- *A* <u>*C^*-algebra*</u> *is a complex Banach algebra A with an involutive anti-automorphism which we denote by $x \mapsto x^*$ such that the following is true.*

$$\forall x \in A : \|x^* x\| = \|x\|^2$$

- *Denote the category of C^*-algebras with $*$-preserving homomorphisms by $\mathcal{C}$. We denote the set of $*$-preserving homomorphisms between two C^*-algebras A and B by $\hom(A, B)$.*

Definition C.1.2
- *A C^*-algebra is called* <u>*separable*</u> *if it has a countable dense subset.*
- *Denote the full sub-category of $\mathcal{C}$ whose objects are separable C^*-algebras by $s\mathcal{C}$.*

Remark C.1.3
- *Note that the category $s\mathcal{C}$ is essentially small, since any separable C^*-algebra can be embedded into the algebra of bounded operators on a Hilbert space of countable Hilbert space dimension. We may as well assume that we are working with an equivalent small category and we will do so without further mentioning.*
- *There are several choices of the tensor product on the category $\mathcal{C}$. We will always take the maximal tensor product and recall that it makes the category $\mathcal{C}$ into a symmetric monoidal category (See appendix C.4 for definitions and notation.).*
- *We will carelessly make no distinction between the word $*$-homomorphism, homomorphism and morphism. As long as we are talking about C^*-algebras these will always mean $*$-preserving homomorphism.*

Definition C.1.4 *Let X be a locally compact topological space.*

- *We denote the Alexandrov compactification (one-point compactification) of* X *by* X_+.
- *Let* Y *be locally compact. The homotopy classes of maps from* X *to* Y *are denoted by* $[X, Y]$.
- *Assume that* (X, x) *and* (Y, y) *are pointed spaces. The (pointed) homotopy classes of pointed maps from* X *to* Y *are denoted by* $[(X, x), (Y, y)]_+$.

Definition C.1.5 *Let* A *be a* C^**-algebra and* (X, x) *be a pointed compact Hausdorff space.*

- *For any pointed compact Hausdorff space* (X, x) *denote the* C^**-algebra of complex valued functions on* X *vanishing at* $x \in X$ *by* $C(X, x)$ *or* $C_0(X - \{x\})$.
- *Denote the algebra of* A*-valued continuous functions on* X *by* $C(X; A)$.
- *Denote the algebra of* A*-valued continuous functions on* X *vanishing at* $x \in X$ *by* $C(X, x; A)$ *or* $C_0(X - \{x\}; A)$.
- *We denote the algebra of bounded functions on* $[0, 1)$ *with values in* A *by* $A_b[0, 1)$. *The sub-algebra of functions vanishing in* 0 *is denoted by* $A_b([0, 1), 0)$ *which is sometimes abbreviated by* A_b.

Definition C.1.6 *We are going to use the following abbreviations. Let* A *be a* C^**-algebra.*

- *Denote by* $\Sigma(A)$ *the algebra* $C(S^1, 1; A)$.
- *Denote by* $c(A)$ *the algebra* $C([0, 1], 0; A)$. *The cone comes with a canonical evaluation at* 1 *which we denote by* $ev_1^A : c(A) \to A$. *We use the convenient abbreviation* $A(0, 1]$.
- *Let* $f : A \to B$ *be a* $*$*-homomorphism. Its cone* $c(f)$ *is defined as the pull-back* $c(B) \oplus_B A$ *along the evaluation map. It comes with an evaluation map to* A *which we denote by* ev_1.
- *Let* $f : A \to B$ *be a* $*$*-homomorphism. Its cylinder* $cyl(f)$ *is defined as the pull-back* $B[0, 1] \oplus_B A$ *along the evaluation map at* 1. *It comes with an evaluation map to* A *which we denote by* ev_1.

Definition C.1.7 *Let* A *and* B *be* C^**-algebras. Two* $*$*-homomorphisms* $f, g : A \to B$ *are called homotopic, if there exists a homotopy* $H : A \to B[0, 1]$ *such that* $ev_0 \circ H = f$ *and* $ev_1 \circ H = g$. *Homotopy is an equivalence relation and we denote the set of homotopy classes of* $*$*-homomorphisms from* A *to* B *by* $[A, B]$. *We denote the homotopy class of a* $*$*-homomorphism* $f : A \to B$ *by* $[f] \in [A, B]$.

Remark C.1.8 *Note that the algebras* $C(X, x; A)$ *identify with the algebras* $A \otimes C(X, x)$ *in a canonical way.*

Definition C.1.9
- *We denote the algebra of complex* $n \times n$ *matrices by* M_n*. We abbreviate the algebraic tensor product* $A \otimes M_n$ *by* $M_n A$*.*
- *We denote the algebra of bounded compact linear operators on the standard separable Hilbert space* $l^2(\mathbb{N})$ *by* K*.*

Theorem C.1.10 (I. Gel'fand and M.A. Naĭmark)
- *The category of commutative* C^**-algebras and* $*$*-homomorphisms is contravariantly equivalent to the category of pointed compact Hausdorff spaces and pointed continuous maps.*
- *This contravariant equivalence restricts to a contravariant equivalence between the full sub-categories of separable* C^**-algebras and metrizable compact pointed Hausdorff spaces.*

Definition C.1.11 *Let* A *and* B *be* C^**-algebras. We denote their co-product in the category of* C^**-algebras (which is commonly called 'free product') by* $A * B$*. It is given by the universal completion of the algebraic free product considered as a sub-algebra of the of algebra of bounded operators on a Hilbert space.*

We finish this section of the appendix by collecting some theorems and lemmas which we need at some point. The first lemma is a well known fact from linear algebra.

Lemma C.1.12 *We have a canonical homeomorphism*

$$PSU(n)_+ \to \hom(M_n, M_n)$$

such that $PSU(n) \ni A \mapsto \{B \mapsto ABA^*\}$ *and* $* \mapsto \{B \mapsto 0\}$*.*

The following lemma is taken from the work of G. Segal in [61].

Lemma C.1.13 *Let* (X, x) *and* (Y, y) *be finite pointed CW-complexes. The canonical map*

$$\mathrm{colim}_n[C(Y, y), M_n(C(X, x))] \to [C(Y, y), C(X, x) \otimes \mathbf{K}]$$

is an isomorphism.

This is just proposition 1.2 in [61]. The proof only requires that Y is locally contractible at $y \in Y$.

The following theorem has become a standard result in the theory of Banach spaces. It becomes important to us because it allows to split extensions of C^*-algebras at least in the category of topological spaces.

Theorem C.1.14 (R. Bartle and L. Graves in [4]) *Let* A *and* B *be Banach spaces. Let* $f : A \to B$ *be a bounded linear surjection. Then there exists a continuous and bounded (not necessarily linear) split* $s : B \to A$ *(i.e.* $f \circ s = id_B$*).*

C.2 Extensions

We will uses the terms 'extension' and 'short exact sequence' for the same things, namely diagrams

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

where $A \to B$ is a kernel of $B \to C$ and $B \to C$ is a co-kernel of $A \to B$. I.e. we require that $A \to B$ is injective, that the image of $A \to B$ is a closed 2-sided ideal, that the composition $A \to C$ is zero and that the induced map $B/A \to C$ is an isomorphism of C^*-algebras.

Definition C.2.1 *Let*

$$0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$$

and

$$0 \longrightarrow A \longrightarrow C' \longrightarrow B \longrightarrow 0$$

be extensions of B *by* A. *The extensions are called* <u>*isomorphic*</u>, *if there is a commutative diagram as follows in which the left and right most arrows are identities.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & C' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

By the work of R. Busby in [11] it follows that the set of isomorphism classes of extensions of A by B is in bijection with $\hom(A, Q(B))$ where $Q(B)$ denotes the Calkin algebra (i.e. the quotient of the multiplier algebra $M(A)$ by A). The identification goes as follows.

Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an extension. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \vdots & & \\ 0 & \longrightarrow & A & \longrightarrow & M(A) & \longrightarrow & Q(A) & \longrightarrow & 0 \end{array}$$

where the dotted arrow exists by the universal property of quotients. Conversely given a map $C \to Q(A)$ we can complete it to a diagram as above by taking B to be the pull-back $M(A) \oplus_{Q(A)} C$. This gives an extension by an easy argument. One easily shows that these assignments are inverse to each other.

An important observation is that the assignment $(A, B) \mapsto \hom(A, Q(B))$ is functorial in A, but <u>not</u> functorial in B. We are not using the description above in our arguments and only wanted to state the result because it is the classical description of extensions.

In particular, the description above implies that the isomorphism classes of extensions form a set for any C^*-algebras A and B.

C.3 The extension category

In this part we introduce the notion of 'extension category'. The possibility of its construction is not particular to the category of separable C^*-algebras. It gives a general procedure of constructing a concrete functor into a triangulated category with a prescribed class of distinguished triangles. We already noted that the existence of such a functor might be easy to proof using a naive stable homotopy category and localization as described at the end of section 3.3. The important achievement is that there is a concrete description of the set of morphisms using extensions and that there are lots of motivating examples of morphisms arising obviously that way.

Note that J. Cuntz considers a related construction in [18]. However, his approach uses special extension, like a Toeplitz extension and a certain universal extension, which need not exist in more general situations. There were also earlier attempts to understand **KK**-theory on the basis of n-step extensions, see [74].

Definition C.3.1

- *Let A and B be C^*-algebras. Denote by $\mathrm{ext}^n(A,B)$ the class of n-step extensions of B by A, i.e. diagrams of the form*

$$0 \longrightarrow A \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n \longrightarrow B \longrightarrow 0$$

for which there exists a decomposition into extensions

$$0 \longrightarrow A \longrightarrow E_1 \longrightarrow F_1 \longrightarrow 0$$

$$0 \longrightarrow F_1 \longrightarrow E_2 \longrightarrow F_2 \longrightarrow 0$$

$$\cdots$$

$$0 \longrightarrow F_{n-1} \longrightarrow E_n \longrightarrow B \longrightarrow 0.$$

- *The n-step extensions form a category with morphisms commutative diagrams of the form*

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

The composition is given by vertical composition of diagrams.

- *Two* n*-step extensions of* B *by* A *are called* *congruent* *if they lie in the same component of the category of* n*-step extensions (i.e. two* n*-step extensions are congruent, if there exists a chain of morphisms of* n*-step extensions connecting them).*

- *Two* n*-step extensions of* B *by* A *are called* *homotopic**, if there is an extension of* B *by* $A[0,1]$ *and a commutative diagram in which all vertical arrows are surjections as follows.*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle ev_1} & & \uparrow & & & & \uparrow & & \| & & \\
0 & \longrightarrow & A[0,1] & \longrightarrow & H_1 & \longrightarrow & \cdots & \longrightarrow & H_n & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ev_0} & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

In order to show that homotopy is an equivalence relation we need a lemma.

Lemma C.3.2 *Consider a commutative diagram of extensions as follows.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

Furthermore, assume that f *is surjective. Then the induced sequence*

$$0 \longrightarrow A' \oplus_A A'' \longrightarrow B' \oplus_B B'' \longrightarrow C' \oplus_C C'' \longrightarrow 0$$

is exact.

Proof: The map $B' \oplus_B B'' \to C' \oplus_C C''$ is surjective. Indeed, given a pair $(c', c'') \in C' \oplus_C C''$ we find $b' \in B'$ that maps to c' and $b'' \in B''$ that maps to c''. The difference of the images of b' and b'' in B maps to 0 in C by assumption on c' and c''. Hence this difference, say $a \in B$, lies in the image of A. By assumption on $f : A' \to A$, we can lift a to $a' \in A' \subset B'$. The pair $(b' - a', b'')$ is an element of $B' \oplus_B B''$ and maps to (c', c''). This proves the surjectivity.

The kernel of this surjection obviously coincides with $A' \oplus A'' \cap B' \oplus_B B'' = A' \oplus_B A'' = A' \oplus_A A''$. This finishes the proof.

Proposition C.3.3 *Homotopy of extensions is an equivalence relation.*

Proof: The relation is clearly symmetric and reflexive by similar arguments as in proposition 2.2.4. In order to show transitivity consider two composable homotopies. They give rise to a diagram with five rows. We proceed by taking the pull-back in the middle. The resulting pull-back sequence defines a homotopy. Indeed, by lemma C.3.2 it is exact. Furthermore, all vertical arrows are surjective, since pull-backs of surjections are surjective. This finishes the proof.

Proposition C.3.4 *Congruent extensions are homotopic.*

Proof: Since we showed that homotopy is an equivalence relation, we only have to show that a morphism of extensions gives rise to a homotopy. Let the diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & B & \longrightarrow & 0 \\
 & & \| & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

be a morphism of extensions. We consider the diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A[0,1] & \longrightarrow & F_1[0,1] & \longrightarrow & \cdots & \longrightarrow & F_n[0,1] & \longrightarrow & B[0,1] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ev_1^A} & & \downarrow{\scriptstyle ev_1^{F_1}} & & & & \downarrow{\scriptstyle ev_1^{F_n}} & & \downarrow{\scriptstyle ev_1^B} & & \\
0 & \longrightarrow & A & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & B & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle f_1} & & & & \uparrow{\scriptstyle f_n} & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

Since the morphisms downwards pointing are surjections, the pull-back of this diagram gives a n-step extension

$$0 \longrightarrow A[0,1] \longrightarrow cyl(f_1) \longrightarrow \cdots \longrightarrow cyl(f_n) \longrightarrow B[0,1] \longrightarrow 0.$$

This n-step extensions comes together with a diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_n & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle ev_1^A} & & \uparrow & & & & \uparrow & & \uparrow{\scriptstyle ev_1^B} & & \\
0 & \longrightarrow & A[0,1] & \longrightarrow & cyl(f_1) & \longrightarrow & \cdots & \longrightarrow & cyl(f_n) & \longrightarrow & B[0,1] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ev_0^A} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle ev_0^B} & & \\
0 & \longrightarrow & A & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

in which all vertical morphisms are epimorphisms. Taking the pull-back of the middle extension along the inclusion of the constant functions

$$B \xrightarrow{(B)} B[0,1]$$

gives the required homotopy and finishes the proof.

Definition C.3.5 *Denote by* $\mathrm{Ext}^n(A,B)$ *the set of homotopy classes of* n*-step extensions of* A *by* B. *It is a fact, that the homotopy classes form a set, but we do not want to prove it here.*

Proposition C.3.6 *There is an associative composition map* $\mathrm{Ext}^n(A,B) \times \mathrm{Ext}^m(B,C) \to \mathrm{Ext}^{n+m}(A,C)$ *which is given by the Yoneda product. We denote the product by* $\mathrm{Ext}^n(A,B) \times \mathrm{Ext}^m(B,C) \ni (a,b) \mapsto a \cdot b \in \mathrm{Ext}^{n+m}(A,C)$

Proof: One easily checks that the Yoneda product is compatible with the homotopy relation.

There is also an action of the category $s\mathcal{C}$ on the set of extensions as will become obvious in the following definition and the proposition below.

Definition C.3.7 *Consider the diagram*

$$\begin{array}{ccccccccccccc} & & & & & & & & & & C & & \\ & & & & & & & & & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & A & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

in which the horizontal sequence is a n*-step extension* $a \in \mathrm{Ext}^n(A,B)$. *The pull-back extension*

$$0 \longrightarrow A \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_{n-1} \longrightarrow F_n \oplus_B C \longrightarrow C \longrightarrow 0$$

is denoted by $a \circ f \in \mathrm{Ext}^n(A,C)$. *(This clearly well defined up to homotopy.)*

Proposition C.3.8 *Let* $a \in \mathrm{Ext}^n(A,B)$, $b \in \mathrm{Ext}^m(B,C)$ *be extensions and* $f : D \to C$, $g : E \to D$ *be* $*$*-homomorphisms. The following associativity rules hold.*

$$(a \cdot b) \circ f = a \cdot (b \circ f) \in \mathrm{Ext}^{n+m}(A,D)$$
$$(b \circ f) \circ g = b \circ (fg) \in \mathrm{Ext}^m(B,E)$$

Let us continue with another definition.

Definition C.3.9

- *Let* A *be a* C^**-algebra. We denote the cone extension*

$$0 \longrightarrow \Sigma(A) \longrightarrow c(A) \xrightarrow{ev_1^A} A \longrightarrow 0$$

by $C_A \in \mathrm{Ext}^1(\Sigma(A),A)$.

- *Let* A *and* B *be* C^**-algebras and let* $f : A \to B$ *be a* $*$*-homomorphism. We denote the cone extension of* f

$$0 \longrightarrow \Sigma(B) \longrightarrow c(f) \xrightarrow{ev_1^A} A \longrightarrow 0$$

by $C_f \in \mathrm{Ext}^1(\Sigma(B), A)$. *Note that according to this definitions the equality* $C_A = C_{\mathrm{id}_A}$ *holds.*

We want to extend a definition which we already gave in section 2.2 to n-step extensions.

Definition C.3.10
- *Denote by* $t : \Sigma \to \Sigma$ *the twist of the suspension (i.e. the map which is induced by complex conjugation on* $S^1 \subset \mathbb{C}$*).*

- *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an extension $a \in \mathrm{Ext}^1(A, C)$. *We define the suspended extension* $\Sigma(a) \in \mathrm{Ext}^1(\Sigma(A), \Sigma(C))$ *to be the following extension.*

$$0 \longrightarrow \Sigma(A) \xrightarrow{t\ \ (f)} \Sigma(B) \xrightarrow{(g)} \Sigma(C) \longrightarrow 0$$

- *The suspension of* n*-step extensions is defined by decomposing into* 1*-step extensions and suspending those. This rule ensures that* $\Sigma(a \cdot b) = \Sigma(a) \cdot \Sigma(b)$.

Lemma C.3.11 *The two maps* $\mathrm{Ext}^n(A, B) \to \mathrm{Ext}^{n+1}(\Sigma(A), B)$ *which are given by*

$$\mathrm{Ext}^n(A, B) \ni a \mapsto C_A \cdot a \in \mathrm{Ext}^{n+1}(\Sigma(A), B)$$

and

$$\mathrm{Ext}^n(A, B) \ni a \mapsto \Sigma(a) \cdot C_B \in \mathrm{Ext}^{n+1}(\Sigma(A), B)$$

are equal.

Proof: It suffices to prove the result in the case $n = 1$, since the general result follows by decomposing the n-step extension into 1-step extensions and an induction argument.

Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be the given extension.

It suffices to find a congruence between the following two extensions.

$$0 \longrightarrow \Sigma(A) \longrightarrow c(A) \longrightarrow B \xrightarrow{g} C \longrightarrow 0$$

$$0 \longrightarrow \Sigma(A) \xrightarrow{t\ \ (f)} \Sigma(B) \xrightarrow{(g)} c(C) \longrightarrow C \longrightarrow 0$$

Consider the following diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Sigma(A) & \longrightarrow & c(A) & \longrightarrow & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Sigma(A) & \longrightarrow & c(B) & \longrightarrow & cyl(g) & \xrightarrow{ev_0} & C & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \Sigma(A) & \xrightarrow{t\ \ (f)} & \Sigma(B) & \xrightarrow{(g)} & c(C) & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

with $\alpha : c(B) \to cyl(f)$ and $\beta : \Sigma(B) \to c(B)$ switching the direction of the interval. The commutativity is easily checked. This finishes the proof. Note that it was crucial for the commutativity that we defined the suspension of an extension with the switch of the interval.

Lemma C.3.12 *The extensions* $\Sigma(c_A)$ *and* $c_{\ (A)} \in Ext^1(\Sigma^2(A), \Sigma(A))$ *are homotopic.*

Proof: Consider the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & c(\Sigma(A)) & \xrightarrow{c(\ )} & c(c(A)) & \longrightarrow & c_0(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ev_1^{\ (A)}} & & \downarrow{\scriptstyle ev_1^{c(A)}} & & \downarrow{\scriptstyle ev_1^{A}} & & \\
0 & \longrightarrow & \Sigma(A) & \longrightarrow & c(A) & \longrightarrow & A & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle ev_1^{A}} & & \\
0 & \longrightarrow & 0 & \longrightarrow & c(A) & \longrightarrow & c(A) & \longrightarrow & 0
\end{array}$$

The pull-back of this diagram is given by the extension

$$0 \longrightarrow \Sigma(\Sigma(A)) \longrightarrow c(c(A)) \longrightarrow c(A) \oplus_A c(A). \longrightarrow 0$$

Denote (only until the end of the proof) $c(A) \oplus_A c(A)$ by Q. The interval over this extension comes with the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma(\Sigma(A))[0,1] & \longrightarrow & c(c(A)[0,1] & \longrightarrow & Q[0,1] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ev_0 \oplus ev_1} & & \downarrow{\scriptstyle ev_0 \oplus ev_1} & & \downarrow{\scriptstyle ev_0 \oplus ev_1} & & \\
0 & \longrightarrow & \Sigma(\Sigma(A)) \oplus \Sigma(\Sigma(A)) & \longrightarrow & c(c(A)) \oplus c(c(A)) & \longrightarrow & Q \oplus Q & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle (\ ,0)\oplus(0,\ )} & & \\
0 & \longrightarrow & \Sigma(\Sigma(A)) \oplus \Sigma(\Sigma(A)) & \longrightarrow & c(\Sigma(A)) \oplus \Sigma(c(A)) & \longrightarrow & \Sigma(A) \oplus \Sigma(A) & \longrightarrow & 0
\end{array}$$

The pull-back of this diagram is again an extension. It comes with two natural evaluations which map onto the extensions $\Sigma(c_A)$ and $c_{\ (A)}$. The source of the homotopy is $\Sigma(\Sigma(A)[0,1]$. We still have to fix the target. If we can construct a map from $\Sigma(A)$ into the target of the pull-back such that both evaluations on $\Sigma(A)$ are splittings, then we can take the pull-back of the constructed extension along this map and obtain the desired homotopy. The required maps are easily constructed.

Remark C.3.13 *Note that in the last lemma we cannot replace 'homotopic' by 'congruent'. It is therefore much harder to construct a strict extension category rather than an extension category up to homotopy. We do not want to follow this line here.*

Now we are going to proceed by defining a category of extensions in which the cone extension is inverted.

Definition C.3.14 *Let A and B be C^*-algebras. Define sets*

$$S'(A, B) = \mathrm{colim}_n \mathrm{Ext}^n(\Sigma^n(A), B)$$

where the co-limit is taken with respect to the canonical maps given by the cone extensions

$$\mathrm{Ext}^n(\Sigma^n(A), B) \ni a \mapsto C_{\ n(A)} \cdot a \in \mathrm{Ext}^{n+1}(\Sigma^{n+1}(A), B).$$

Theorem C.3.15 *There is an associative product $S'(A, B) \times S'(B, C) \to S'(A, C)$ extending the Yoneda-product*

Proof: Given classes $[a] \in S'(A, B)$ and $[b] \in S'(B, C)$, we find $n, m \in \mathbb{N}$ and representatives $a \in \mathrm{Ext}^n(\Sigma^n(A), B)$ and $b \in \mathrm{Ext}^m(\Sigma^m(B), C)$. Define the composition product $[a] \cdot [b]$ to be the image of $\Sigma^m(a) \cdot b$ in $S'(A, C)$. We have to show that this is well-defined. It suffices to show that one can replace a by $C_{\ n(A)} \cdot a$ and b by $C_{\ m(B)} \cdot b$.

Obviously $[\Sigma^m(C_{\ n(A)} \cdot a) \cdot b] = [\Sigma^m(C_{\ n(A)}) \cdot \Sigma^m(a) \cdot b]$ By lemma C.3.12, the last term is equal to $[C_{\ m+n(A)} \cdot \Sigma^m(a) \cdot b] = [\Sigma^m(a) \cdot b]$.

The other case is proved using lemma C.3.11 and similar arguments. This finishes the proof.

Definition C.3.16 *We define the category S' to be the category with objects separable C^*-algebras and morphisms between algebras A and B the set $S'(A, B)$. The identity morphism of an object A is given by the image of C_A in $S'(A, A)$.*

Lemma C.3.17 *There is a functor $s_{\mathrm{ext}} : s\mathcal{C} \to S'^{\mathrm{op}}$ with $s_{\mathrm{ext}}(f) = C_f$ (see definition C.3.9 for notations).*

Proof: In order to show the functoriality we consider a composable pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$. Let $C_f \in \mathrm{Ext}^1(\Sigma(B), A)$, $C_g \in \mathrm{Ext}^1(\Sigma(C), B)$ and $C_{g \circ f} \in \mathrm{Ext}^1(\Sigma(C), A)$ be the corresponding cones. It suffices to construct a congruence between $\Sigma(C_g) \cdot C_f \in \mathrm{Ext}^2(\Sigma^2(C), A)$ and $\Sigma(C_C) \cdot C_{g \circ f}$.

Consider the diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma(C) & \longrightarrow & c(f) & \longrightarrow & B & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow g & & \\
0 & \longrightarrow & \Sigma(C) & \longrightarrow & c(C) & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma(B) & \longrightarrow & c(g) & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (g)} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Sigma(C) & \longrightarrow & c(g \circ f) & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Composing the suspension of the first diagram with the second diagram gives the required congruence. Thence the proof is finished.

C.4 Monoidal structure

Interesting enough there are different monoidal structures on the category of C^*-algebras. The most significant ones are given by the minimal and the maximal tensor product. We only want to define and consider the maximal tensor product, since it serves best for our purposes. We do not want to give proofs of the theorems, since they are well known (see e.g. the book by G. Murphy [42] or the one by B. Blackadar [7]).

Definition C.4.1 *Let A and B be C^*-algebras. The maximal tensor product is defined to be the universal C^*-completion of the algebraic tensor product of A and B. We denote the maximal tensor product of A and B by $A \otimes B$.*

The next series of theorems subsumes the most important properties of the maximal tensor product.

Theorem C.4.2 *The category $\mathcal{C}$ of C^*-algebras and the category $s\mathcal{C}$ of separable C^*-algebras are symmetric monoidal categories with respect to the maximal tensor product.*

Theorem C.4.3 *The maximal tensor product is universal with respect to the property that whenever there are two commuting representations $\phi : A \to C$ and $\psi : B \to C$ of A and B in C, then there is a unique morphism $A \otimes B \to C$.*

Theorem C.4.4 *The maximal tensor product is exact in the sense that the functor $? \otimes A$ maps short exact sequences to short exact sequences.*

D Algebraic K-theory

D.1 Definitions

We do not intend to give an introduction to algebraic K-theory in this section. It only serves as a place of recollections. For an introduction and a detailed list of references we refer to [3].

The higher algebraic K-groups are best understood as the homotopy groups of a spectrum. There are several fundamental constructions in algebraic topology that were necessary to give a proper definition of higher algebraic K-theory. The main contributions are by D. Quillen [50]. The definition of negative algebraic K-groups goes back to H. Bass [5] and was done before. They can be defined in a purely algebraic manner. Shortly after Quillen's constructions there began a process of 'spacification' in algebraic K-theory which resulted in a number of spectra which were actually computing both, the negative and the positive algebraic K-groups, as their homotopy groups (e.g. the work of E. Pedersen and C. Weibel [46]). Later, the different spectra were shown to be equivalent in a homotopy category of spectra.

Denote the category of rings by $\mathcal{Rng}$. Since we do not want to go through these constructions, we will assume the existence of a functor

$$\underline{\mathbf{K}} : \mathcal{Rng} \longrightarrow \mathcal{Sp}$$

from the category of rings to the category of symmetric spectra which has the properties listed below. The existence of a functor with the properties below is a non-trivial fact. A suitable candidate for a functor is provided in the thesis of M. Schlichting [56], section 7. The work of M. Schlichting [56] is to our knowledge the only reference which deals with the non-connective case. The precise result which is required is not stated in [56]. A precise proof of the result needed is work in progress of J. Hornbostel and M. Schlichting, see [32]. Therefore this section will lack a definite mathematical accuracy.

- The i-th homotopy group of $\underline{\mathbf{K}}(R)$ coincides with the i-th algebraic K-group of the ring R for $i \in \mathbb{Z}$.
- Let $f : R \to Q$ be a map of rings. The induced map on homotopy groups of K-theory spectra is the usual map in algebraic K-theory.
- There is a transformation

  $$\underline{\mathbf{K}}(R) \wedge \underline{\mathbf{K}}(Q) \longrightarrow \underline{\mathbf{K}}(R \otimes Q)$$

  such that
- the transformation is compatible with the associativity and commutativity isomorphism of the respective monoidal structures.

The third property is the one which is not explicitly stated in the thesis of M. Schlichting [56] but actually contained in the proof of lemma 7.7 in [56].

Proposition D.1.1 *The properties of $\underline{\mathbf{K}}$ imply that whenever Q is a R-module, then the map above induces a $\underline{\mathbf{K}}(R)$-module structure on $\underline{\mathbf{K}}(Q)$. Furthermore, for any commutative ring R the functor $\underline{\mathbf{K}}$ associates to R a commutative ring spectrum.*

First of all, we want to extend $\underline{\mathbf{K}}$ to the category of algebras (i.e. 'non-unital rings'). This is done by a standard procedure. Denote by $\mathrm{hf}(f)$ the homotopy fiber of a map of symmetric spectra. The homotopy fiber is a certain homotopy limit. We do not want to discuss its precise definition here.

Definition D.1.2 *Let R be an algebra. Denote by R^+ its unitalization (this is of course a ring). There is a natural exact sequence in the category of algebras as follows.*

$$0 \longrightarrow R \longrightarrow R^+ \xrightarrow{u} \mathbb{Z} \longrightarrow 0$$

We define $\underline{\mathbf{K}'}(R) = \mathrm{hf}(\underline{\mathbf{K}}(R^+) \to \underline{\mathbf{K}}(\mathbb{Z}))$.

If R is unital, then R^+ splits in the category of rings as $R \oplus \mathbb{Z}$. This shows that in this case the natural map $\underline{\mathbf{K}'}(R) \to \underline{\mathbf{K}}(R)$ is a weak equivalence. Hence $\underline{\mathbf{K}'}$ is a natural extension of $\underline{\mathbf{K}}$. Note that the properties of the functor $\underline{\mathbf{K}}$ extend also.

Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an extension of algebras.

It is an important question to find conditions which ensure that the natural map $\underline{\mathbf{K}}(A) \to \mathrm{hf}(\underline{\mathbf{K}}(B) \to \underline{\mathbf{K}}(C))$ is an equivalence of symmetric spectra.

Definition D.1.3 *If an algebra A satisfies the property above for all extensions in which it appears as the ideal, then the algebra A is said to satisfy* <u>*excision*</u> *in algebraic $\mathbf{K}$-theory.*

There is an amazing result by A. Suslin and M. Wodzicki [64] which entirely solves the problem in the case of C^*-algebras.

Theorem D.1.4 (A. Suslin, M. Wodzicki in [64]) *If A is C^*-algebra, then it satisfies excision.*

Indeed, much more is true but we do not want to go into details. The proof and the precise statements can be found in [64].

For our purposes, the algebraic $\mathbf{K}$-theory functor lacks one important property which is homotopy invariance. This is not surprising, since we usually compute the algebraic

K-theory after forgetting the topology. We have to bring the topology back into play. This requires the extension of the algebraic K-theory functor to simplicial algebras and a way to encode the topology of the rings into the simplicial structure. This is a standard way to make functors homotopy invariant. A more extensive discussion of the methods needed and the constructions made can be found in the work of M. Paluch [45].

Definition D.1.5 *Let $A_\bullet$ be a simplicial algebra. The geometric realization of the simplicial spectrum $\underline{\mathbf{K}}(A_\bullet)$ is called the* <u>*homotopy algebraic* K*-theory spectrum*</u> *of the simplicial ring $A_\bullet$.*

Again, this extends the domain of definition of algebraic K-theory. Note that the homotopy algebraic K-theory spectrum of the constant simplicial algebra associated to an algebra R is naturally equivalent to the algebraic K-theory spectrum. There is a standard simplicial algebra which is assigned to any topological algebra and which encodes its topological structure to some extent.

Definition D.1.6 *Let A be a topological algebra. Denote by $A(\Delta^n)$ the A-valued continuous functions on Δ^n. We call $A(\Delta^\bullet)$ the simplicial algebra which is associated to the topological algebra A. We denote the homotopy algebraic* K*-theory spectrum of $A(\Delta^\bullet)$ by $\underline{\mathbf{K}}^H(A)$.*

Definition D.1.7 *The homotopy algebraic* K*-groups are defined to be the homotopy groups of the homotopy algebraic* K*-theory spectrum.*

Proposition D.1.8 *Let A and B be topological algebras. Furthermore, assume that A is a B-module. Then $\underline{\mathbf{K}}^H(A)$ is naturally a $\underline{\mathbf{K}}^H(B)$-module in the category of symmetric spectra.*

Proof: Note that the conditions of the theorem imply that $A(\Delta^n)$ is a $B(\Delta^n)$-module in a way which is coherent with the boundary maps etc. This implies that there is a simplicial diagram of module structures. We can realize the diagram object-wise and get the required statement, since the smash product commutes with realization. This finishes the proof.

Proposition D.1.9 *Homotopy algebraic* K*-theory is homotopy invariant.*

Proof: This is a standard argument. It can be found in [70].

Since there is an inclusion of the constant functions into the continuous functions on a simplex, we get a natural transformation $\underline{\mathbf{K}} \to \underline{\mathbf{K}}^H$. Although the homotopy algebraic K-groups are constructed in order to simplify matters for topological rings it is important to have a device that allows to transport properties from the algebraic setting to the homotopy algebraic setting. This is done via a spectral sequence.

Theorem D.1.10 *There is a right half-plane multiplicative spectral sequence with*

$$E^1_{p,q} = \underline{\mathbf{K}}_q(A(\Delta^p))$$

which converges to $\underline{\mathbf{K}}^H_{p+q}(A)$. *The edge map* $\underline{\mathbf{K}}_p(A) \to \underline{\mathbf{K}}^H_p(A)$ *is just the one induced by the natural transformation mentioned above.*

Proof: This is a standard spectral sequence which allows the computation of homotopy groups of a simplicial space or a simplicial spectrum. The proof goes along the lines of the proof of the existence an analogous spectral sequence in a more algebraic setting, [70]. Another discussion of the proof can be found in [45].

Remark D.1.11 *The preceding spectral sequence reveals also that* $\underline{\mathbf{K}}^H$ *is the universal homotopy invariant extension of algebraic* $\mathbf{K}$*-theory. Indeed, let* $\underline{\mathbf{K}} \to F$ *be a transformation (at the level of symmetric spectra) to a functor which is homotopy invariant functor after taking homotopy groups. We get a transformation* $\underline{\mathbf{K}}^H \to F^H$. *By the homotopy invariance, the spectral sequence for* F *collapses and the edge map* $F(A) \to F^H(A)$ *is an isomorphism. This shows the universal property.*

There is also a relative variant of this spectral sequence which we do not need but which makes apparent that the excision properties extend to homotopy algebraic $\mathbf{K}$-theory. In particular, all C^*-algebras satisfy excision in homotopy algebraic $\mathbf{K}$-theory.

For a detailed discussion of the statements above we refer to [45].

References

[1] J.F. Adams, Stable homotopy and generalized homology, *Chicago Lectures in Mathematics*, University of Chicago Press, Chicago (1974)

[2] J.F. Adams, On the groups $J(X)$ IV, *Topology* 5 (1966), 21-71

[3] D. Arlettaz, Algebraic K-theory of rings from a topological viewpoint, *Publicacions Matemátiques* 44 (2001), no. 1, 3-84

[4] R.G. Bartle and L.M. Graves, Mappings between function spaces, *Transactions of the American Mathematical Society*, **72** (1952), 400-413

[5] H. Bass, Algebraic K-theory *W. A. Benjamin Inc.*, New York-Amsterdam (1968)

[6] J.C. Becker and D.H. Gottlieb, A history of duality in algebraic topology, *History of topology*, North-Holland, Amsterdam (1999), 725-745

[7] B. Blackadar, K-theory for operator algebras, *Mathematical Sciences Research Institute Publications 5. Springer-Verlag*, New York (1986)

[8] B. Blackadar, Matricial and ultramatricial topology, *Operator algebras, mathematical physics, and low-dimensional topology*, Istanbul (1991), 11-38

[9] J.M. Boardman, Conditionally convergent spectral sequences, *Homotopy invariant algebraic structures*, Baltimore (1998), 49-84

[10] L.G. Brown, Stable isomorphism of hereditary subalgebras of C^*-algebras, *Pacific Journal of Mathematics* 71 (1977), no. 2, 335-348

[11] R.C. Busby, Extensions in certain topological algebraic categories, *Transactions of the American Mathematical Society*, **159** (1971), 41-56

[12] D.J. Christensen, Ideals in triangulated categories: phantoms, ghosts and skeleta. *Advances in Mathematics*, **136** (1998), no. 2, 284-339

[13] A. Connes and N. Higson, Déformations, morphismes asymptotiques et K-théorie bivariante, *C. R. Acad. Sci. Paris Sr. I Math.* **311** (1990), no. 2, 101–106

[14] A. Connes, Non-commutative Geometry, *Academic Press* (1994)

[15] J. Cuntz, The K-groups of free products of C^*-algebras, *Operator algebras and applications, Part I*, Kingston, Ontario (1980), 81-84

[16] J. Cuntz, A new look at KK-theory, K-*theory* 1 (1987), 31-51

[17] J. Cuntz, Generalized homomorphisms between C^*-algebras and KK-theory, *Dynamics and processes, 1981*, LNM **1031**, Springer (1983), 31-45,

[18] J. Cuntz, Bivariante kk-Theorie für lokalkonvexe Algebren, *Documenta Mathematica* 2 (1997), 139-182

[19] J. Cuntz and G. Skandalis, Mapping cones and exact sequences in KK-theory, *Journal of Operator Theory* **15** (1986), 163-180

[20] M. Dădărlat, A Note on Asymptotic Homomorphism, K-*theory* 8 (1994), no.5, 465-482

[21] M. Dădărlat, Shape theory and asymptotic morphisms for C^*-algebras, *Duke Mathematical Journal* **73** (1994), no. 3, 687-711

[22] M. Dădărlat and T. Loring, a universal multicoefficient theorem for the Kasparov groups, *Duke Mathematical Journal* 84, no.2 (1996), 355-377

[23] M. Dădărlat and A. Nemethi, Shape theory and (connective) K-theory, *Journal of Operator Algebras* **23** (1990), no.2, 207-291

[24] M. Dădărlat and J. McClure, When are two commutative C^*-algebras stably homotopy equivalent ? *Mathematische Zeitschrift* **235** (2000), no.3, 499-523

[25] J. Dixmier, Points séparés dans le spectre d'une C^*-algébre, *Acta Sci. Math. Szeged* **22** (1961), 115-128

[26] R. Exel and T.A. Loring, Extending cellular cohomology to C^*-algebras, *Transactions of the American Mathematical Society* **329** (1992), no. 1, 141-160

[27] P.G. Goerss and J.F. Jardine, Simplicial Homotopy Theory, *Birkäuser Verlag* (1999)

[28] E. Guentner, N. Higson and J. Trout, Equivariant E-theory for C^*-algebras, *Memoirs of the American Mathematical Society* **148** (2000), no. 703

[29] N. Higson, A characterization of KK-theory, *Pacific Journal of Mathematics* **126** (1987), no. 2, 253-276.

[30] N. Higson, Categories of fractions and excision in KK-theory, *Journal of Pure and Applied Algebra* **65** (1990), no. 2, 119-138.

[31] P.S. Hirschhorn, Model categories and their localizations, *Mathematical Surveys and Monographs, AMS* **99**, Providence (2003)

[32] J. Hornbostel and M. Schlichting, Localization in hermitian K-theory, to appear

[33] M. Hovey, B. Shipley and J. Smith, Symmetric spectra, *Journal of the American Mathematical Society* **13** (2000), no. 1, 149-208

[34] T.G. Houghton-Larsen and K. Thomsen, Universal (Co)Homology Theories, *K-theory* **16** (1999), 1-27

[35] G.G. Kasparov, The operator K-functor and extensions of C^*-algebras, *Math. USSR Izv.* **16**, (1981), 513-572

[36] S. MacLane, Category theory for the working mathematician, *Graduate texts in mathematics* **5**, Springer-Verlag (1971)

[37] H.R. Margolis, Spectra and the Steenrod Algebra, *North-Holland Mathematical Library* (1983)

[38] P. May, On the additivity of traces, *Advances in Mathematics* **163** (2001), no. 1, 34-73

[39] J. McCleary, A user's guide to spectral sequences, *Cambridge studies in advanced mathematics* **58** (2001)

[40] R. Meyer, The Mayer-Vietoris Sequence for Generalized Homology Theories, wwwmath.uni-muenster.de/u/ramayer

[41] I. Moerdijk, Orbifolds as groupoids: an introduction. *Orbifolds in mathematics and physics*, Madison (2001), 205-222

[42] G.J. Murphy, C^*-algebras and operator theory, *Academic Press, Inc.*, Boston (1990)

[43] A. Neeman, Triangulated categories, *Annals of Mathematics Studies*, Princeton University Press Nr. **148** (2001)

[44] M. Paluch, Topology on $S^{-1}S$ for Banach algebras, *Topology* **35** (1996), no. 4, 887-900

[45] M. Paluch, Algebraic K-theory and topological spaces, K-*theory pre-print archive*, no. 0471

[46] E.K. Pedersen and C.Weibel, A nonconnective delooping of algebraic K-theory, *Algebraic and geometric topology*, New Brunswick (1983), 166-181

[47] G. Pedersen, C^*-algebras and their Automorphism Groups *London Mathematical Society Monographs* **14**, Academic Press, London-New York (1979)

[48] N.C. Phillips, Recursively sub-homogenous algebras, *arXiv:math:OA* **0101156**

[49] D. Puppe, On the structure of the stable homotopy category, *Colloquium on algebraic topology, Aarhus Universitet Matematisk Institut* (1962), 65-71

[50] D. Quillen, Higher algebraic K-theory, *Proceedings of the International Congress of Mathematicians* 1, Vancouver (1974), 171-176

[51] J. Rosenberg, The role of K-theory in noncommutative algebraic topology, *Operator algebras and K-theory*, San Francisco (1981), 155-182,

[52] J. Rosenberg, K and KK: topology and operator algebras, *Operator theory: operator algebras and applications, Part 1*, Durham (1988), 445-480

[53] J. Rosenberg, The Algebraic K-theory of Operator Algebras, *K-Theory* 12 (1997), 75-99

[54] J. Rosenberg and C. Schochet, The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor, *Duke Mathematical Jourmal* 55 (1987), 431-474

[55] S. Sakai, C^*-algebras and W^*-algebras, *Ergebnisse der Mathematik und ihrer Grenzgebiete* 60 (1971)

[56] M. Schlichting, Delooping the K-theory of exact categories and negative K-groups, www.math.uiuc.edu/~mschlich/research.html

[57] C. Schochet, Topological Methods for C^*-algebras, III, *Pacific Journal of Mathematics* 114 (1984), no. 2, 399-445

[58] C. Schochet, Topological Methods for C^*-algebras, IV, *Pacific Journal of Mathematics* 114 (1984), no. 2, 447-468

[59] H. Schubert, Categories, *Springer Verlag* (1972)

[60] S. Schwede and B.E. Shipley, Algebras and modules in monoidal model categories, *Proceedings of the London Mathematical Society*, 80 (2000), no. 2, 491-511

[61] G. Segal, K-homology theory and algebraic K-theory, *Springer Lecture Notes in Mathematics* 575 (1975), 113-127

[62] G. Skandalis, Le bifoncteur de Kasparov n'est pas exact, *C. R. Acad. Sci. Paris Sr. I Math.* 313 (1991), no. 13, 939-941

[63] L. Smith, On the relation of connective K-theory to homology, *Proceedings of the Cambridge Philosophical Society* 68 (1970), 637-639

[64] A. Suslin and M. Wodzicki, Excision in algebraic K-theory, *Annals of Mathematics* 136 (1992), no. 1, 51-122

[65] R. M. Switzer, Algebraic topology - homology and homotopy, *Classics in Mathematics*, Springer-Verlag (1975)

[66] U. Tillmann, K-theory of fine topological algebras, Chern character, and assembly. K-*Theory* 6 (1992), no. 1, 57-86.

[67] J.L. Verdier, Des catégories dérivées des catégories abeliennes, *Asterisque* 239 (1996)

[68] D. Voiculescu, Asymptotically commuting finite rank unitary operators without commuting approximants, *Acta Sci. Math.* 45, Szeged (1983), no. 1-4, 429-431

[69] C. Weibel, An introduction to homological algebra *Cambridge studies in advanced mathematics* 38 (1994)

[70] C. Weibel, Homotopy algebraic K-theory, *Algebraic* K-*theory and algebraic number theory*, Honolulu (1987), 461-488

[71] M. Weiss and B. Williams, Assembly, *Novikov conjectures, index theorems and rigidity*, Vol. 2, Oberwolfach (1993), 332-352,

[72] G.W. Whitehead, Elements of Homotopy Theory, *Springer Verlag* (1978)

[73] J. Wolbert, Classifying modules over K-theory spectra, *Journal of Pure and Applied Algebra* 124 (1998), no. 1-3, 289-323

[74] R. Zekri, A new description of Kasparov's theory of C^*-algebra extensions, *Journal of Functional Analysis* 84 (1989), no. 2, 441-471

Lebenslauf Andreas Berthold Thom

Andreas Berthold Thom;
geboren am 10.12.1977 in Dresden;

Eltern: Hans-Joachim Thom
Monika Thom, geb. Thimmig;

Schulbildung:	1984-91 81. Polytechnische Oberschule "Willy Ermer" in Dresden-Dölzschen; 1991-96 Gymnasium in Dresden-Plauen;
Hochschulreife (Abitur):	am 30.6.1996 in Dresden;
Studium:	Diplomstudiengang Physik (Nebenfach Chemie) an der Technischen Universität Dresden im WS 1996, nach dem Zivildienst fortgesetzt als Diplomstudiengang Mathematik (Nebenfach Physik) an der TU Dresden von SS 1998 bis SS 1999; Erreichen des 'Vordiploms'; Fortsetzung des Studiums an der University of Cambridge in England im WS 1999 und SS 2000;
Prüfungen:	Abschluss des 'Certificate of Advanced Studies in Mathematics' an der University of Cambridge in England mit dem Prädikat 'distinction' im Juli 2000;
Tätigkeiten:	1997-1999 Studentische Hilfskraft, Universität Dresden; seit Januar 2001 wissenschaftlicher Mitarbeiter am Sonderforschungsbereich 478, Universität Münster;
Beginn der Dissertation:	1.10.2000 am Mathematischen Institut, Betreuer: Prof. Dr. J. Cuntz