TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

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On plastic deformation: From physics over convex analysis to numerical simulation

Lecture 1 // Dresden, June 18, 2019

Goals of this Graduate Lecture

Why are we here and what will we learn?

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1st lecture:

- develop (an easy) physical model of elasticity and plasticity
- consider engineering approaches for typical materials (e.g. steel)

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Why are we here and what will we learn?

1st lecture:

- develop (an easy) physical model of elasticity and plasticity
- consider engineering approaches for typical materials (e.g. steel)

2nd/3rd: lecture:

- proof some nice results from convex analysis
- choose time/space discretization schemes
- look into duality of discrete problems and optimization
- if the time allows it: numerical algorithms to solve the problems

Slides online

All slides will be available at

`https://www.math.tu-dresden.de/~jaap/`

Position and Displacement

Let's start with defining some basic properties:

- initial state of the body: $\Omega_0 \subset \mathbb{R}^3$ bounded and closed
- for each particle x in Ω_0 we denote the position at time t by

$$\boxed{y(x,t) \quad \text{with} \quad y(x,0) = x}$$

- displacement is the difference of the current to the initial position:

$$\boxed{u(x,t) := y(x,t) - y(x,0) = y(x,t) - x}$$

Strain – a measure for the local deformation

Next step: we need a measure for local deformation with the following properties:

- derivative of the displacement: (3×3)-matrix
- symmetric
- invariant under rigid motions

Ansatz:

- consider local changes in angles between 3 points

Strain

- consider 3 points

$$x,\ x+a,\ x+b \in \Omega_0$$

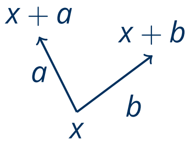
  with $|a| = |b| = 1$
- after deformation at time t:

$$y(x,t),\ y(x+a,t),\ y(x+b,t)$$

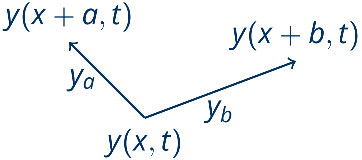
- then, we consider

$$\lim_{h\to 0} \frac{\langle y_{ha}, y_{hb}\rangle - \langle ha, hb\rangle}{h^2}$$

Before deformation:



After deformation:



Strain

$$\boxed{\lim_{h\to 0} \frac{\langle y_{ha}, y_{hb}\rangle - \langle ha, hb\rangle}{h^2}}$$

with some Taylor around x:

- $y_{ha} = y(x+ha,t) - y(x,t) = ha^T \nabla y(x) + \mathcal{O}(h^2)$
- $y_{hb} = y(x+hb,t) - y(x,t) = hb^T \nabla y(x) + \mathcal{O}(h^2)$

and using $y(x,t) = u(x,t) + x$ yields $\nabla y = \nabla u + I$

- $y_{ha} = ha^T \nabla u(x) + ha + \mathcal{O}(h^2)$
- $y_{hb} = hb^T \nabla u(x) + hb + \mathcal{O}(h^2)$

Strain

$$\boxed{\lim_{h\to 0} \frac{\langle y_{ha}, y_{hb}\rangle - \langle ha, hb\rangle}{h^2}}$$

- $y_{ha} = ha^T \nabla u(x) + ha + \mathcal{O}(h^2)$
- $y_{hb} = hb^T \nabla u(x) + hb + \mathcal{O}(h^2)$

Therefore, we have

$$\begin{aligned}\lim_{h\to 0} \frac{\langle y_{ha}, y_{hb}\rangle - \langle ha, hb\rangle}{h^2} &= a^T \nabla u(x) b + b^T \nabla u(x) a + a^T \nabla u(x) \nabla u(x)^T b \\ &= a^T \left(\nabla u(x) + \nabla u(x)^T + \nabla u(x) \nabla u(x)^T \right) b\end{aligned}$$

Strain

$$\lim_{h \to 0} \frac{\langle y_{ha}, y_{hb} \rangle - \langle ha, hb \rangle}{h^2} = a^T \left(\nabla u(x) + \nabla u(x)^T + \nabla u(x) \nabla u(x)^T \right) b$$

we define the *Strain tensor*

$$\eta(u) := \frac{1}{2} \left(\nabla u(x) + \nabla u(x)^T + \nabla u(x) \nabla u(x)^T \right)$$

Strain

$$\lim_{h \to 0} \frac{\langle y_{ha}, y_{hb} \rangle - \langle ha, hb \rangle}{h^2} = a^T \left(\nabla u(x) + \nabla u(x)^T + \nabla u(x) \nabla u(x)^T \right) b$$

we define the *Strain tensor*

$$\eta(u) := \frac{1}{2} \left(\nabla u(x) + \nabla u(x)^T + \nabla u(x) \nabla u(x)^T \right)$$

and the linearized (small-) *Strain tensor*

$$\boxed{\epsilon(u) := \tfrac{1}{2} \left(\nabla u(x) + \nabla u(x)^T \right)}$$

- we assume $|\nabla u|$ is small: $\eta(u) \approx \epsilon(u)$

Stress – a measure for the internal forces

Let $\Omega' \subseteq \Omega_0$. Newton told us that

$$F = m \cdot a = \rho \cdot \ddot{u}$$

$\Rightarrow$ in our case

$$\underbrace{\int_{\Omega'} f \, dx}_{\text{volume forces}} + \underbrace{\int_{\partial\Omega'} s_n \, dS}_{\text{boundary forces}} = \int_{\Omega'} \rho \cdot \ddot{u} \, dx$$

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Cauchy told us: $\exists \, \sigma(x,t) \in \mathbb{R}^{3\times 3}_{sym}$, with

$$s_n = \sigma \cdot n$$

Stress

together:

$$\int_{\Omega'} f\, dx + \int_{\partial\Omega'} \sigma \cdot n\, dS = \int_{\Omega'} \rho \cdot \ddot{u}\, dx$$

now Gauss told us that this is

$$\int_{\Omega'} f + \mathrm{div}(\sigma)\, dx = \int_{\Omega'} \rho \cdot \ddot{u}\, dx$$

and therefore

$$f + \mathrm{div}(\sigma) = \rho \cdot \ddot{u}$$

Stress

together:

$$\int_{\Omega'} f\,dx + \int_{\partial\Omega'} \sigma \cdot n\,dS = \int_{\Omega'} \rho \cdot \ddot{u}\,dx$$

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and therefore

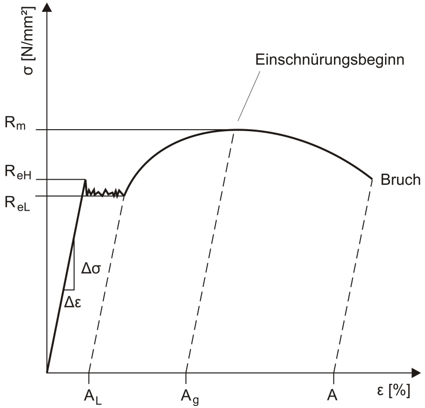
$$f + \operatorname{div}(\sigma) = \rho \cdot \ddot{u}$$

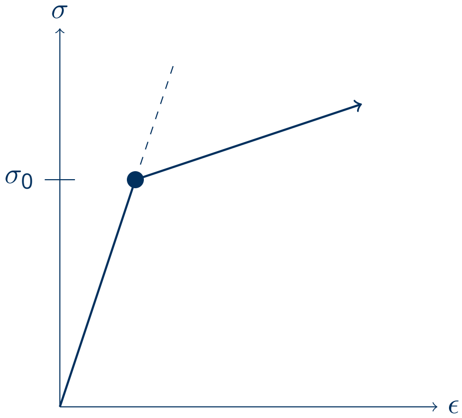
we assume a static process:

$$\boxed{-\operatorname{div}(\sigma) = f}$$

Stress vs Strain

Relationship of Stress and Stain for steel-like materials:





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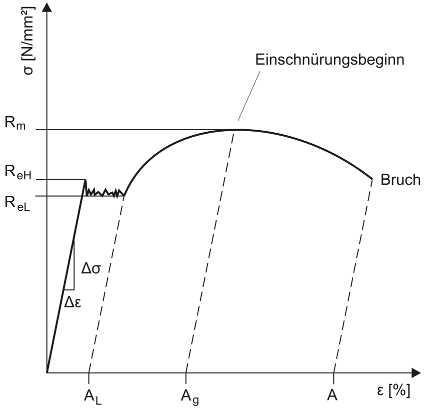
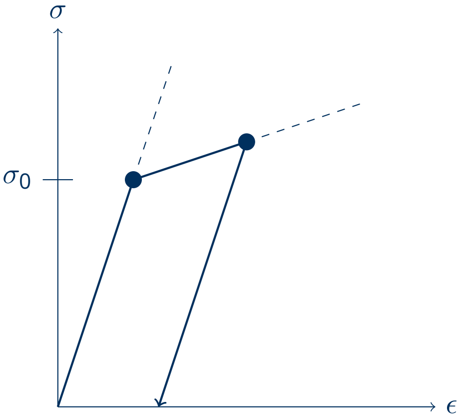




Linear Elasticity

- we observe a linear relation between stress and strain
- at least for small stresses:

$$\exists\, \mathbf{C} \in \mathbb{R}^{(3\times 3)\times(3\times 3)} : \ \sigma = \mathbf{C}\epsilon$$

- this is called *Hooke's law*

What are small stresses?

- we collect all *small* stresses in a set called *elastic region E*
- it is described by a *yield function* $\phi : \mathbb{R}^{3\times3} \to \mathbb{R}$:

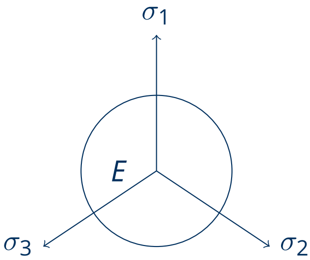
$$E := \{\sigma \in \mathbb{R}^{3\times3}_{sym} : \ \phi(\sigma) \leq 0\}$$

- there is no *natural* choice for ϕ, it relies on engineers' observations

Elastic region – smooth example

- there are different approaches in engineering

Von Mises



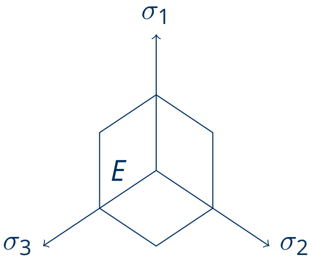
- easiest model in literature
- we take the Frobenius norm of the *deviatoric* stress
- in principle axis, it forms a cylinder

$$\phi(\sigma) = \|\sigma - \frac{1}{3}\operatorname{tr}(\sigma)I\|_F - \sigma_0 \leq 0$$

Elastic region – nonsmooth example

- we don't consider the whole deviatoric stress, but the difference in the eigenvalues
- this is also called *shear stress*
- it forms a hexagonal prisma with same diameter as Von Mises

Tresca

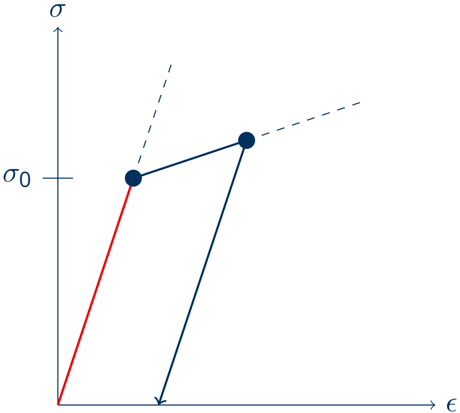


$$\phi(\sigma) = \max_{1 \leq i,j \leq 3} |\sigma_i - \sigma_j| - \sigma_0 \leq 0$$

Stress vs Strain

Back to the model

- now we can describe the red part:
- linear elastic theory stops working at σ_0
- now we start with *plastic behavior*



Evolution of a plastic strain

- we had the definition of strain:

$$\epsilon(u) := \frac{1}{2}\left(\nabla u(x) + \nabla u(x)^T\right)$$

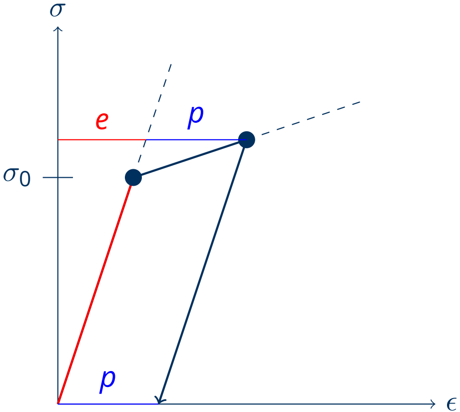
- we decompose it into elastic and plastic parts:

$$\epsilon = e + p$$

- now we have for all stresses

$$\sigma = \mathbf{C}e$$

- usual assumption: p is trace-free



Evolution of a plastic strain

- from the observations, we conclude

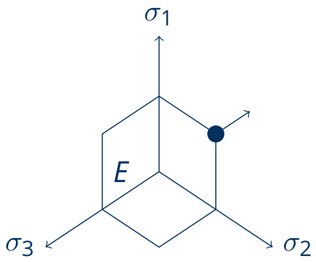
$$\phi(\sigma) < 0 \Rightarrow \dot{p} = 0$$

- at the boundary (pointing inside)

$$\left(\phi(\sigma) = 0 \quad \text{and} \quad \frac{d}{dt}\phi(\sigma) \leq 0\right) \Rightarrow \dot{p} = 0$$

- at the boundary (pointing outside)

$$\left(\phi(\sigma) = 0 \quad \text{and} \quad \frac{d}{dt}\phi(\sigma) > 0\right) \Rightarrow \dot{p} \neq 0$$

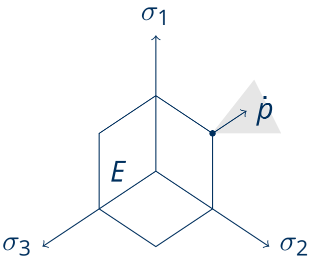


Evolution of a plastic strain

Maximum Work Principle
(von Mises, Taylor, Bishop, Hill)

For $\phi(\sigma) = 0$ and $\frac{d}{dt}\phi(\sigma) > 0$ we have

$$\sigma : \dot{p} \geq \tau : \dot{p} \quad \forall \tau \in E$$

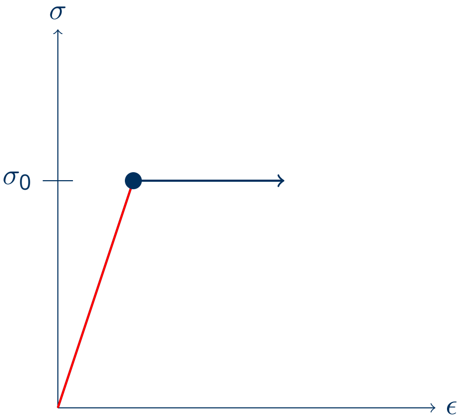


- since E is convex by definition, $\dot{p}$ is located in the *normal cone* of E at σ
- formally written:

$$\boxed{\dot{p} \in N_E(\sigma)}$$

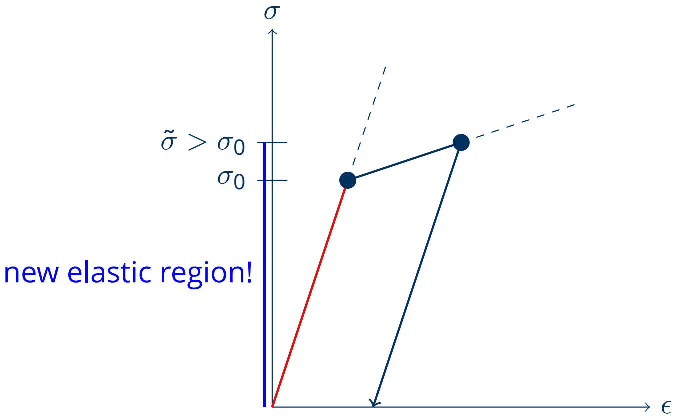
Perfect plasticity

- now we have an evolution of p
- but σ can't escape E!



Look at the elastic region!

- at the second point, we ware outside the initial elastic region
- E changes!



Hardening

- if plastic strain p evolves, the elastic region E may change
- this is called *hardening*
- in general two different approaches:

1. Kinematic hardening: E moves in direction p
2. Isotropic hardening: E grows depending on p

Hardening

- if plastic strain p evolves, the elastic region E may change
- this is called *hardening*
- in general two different approaches:

1. Kinematic hardening: E moves in direction p
2. Isotropic hardening: E grows depending on p

$\Rightarrow$ if no hardening: <u>perfect plasticity</u>

Hardening – formal

Kinematic hardening:

- E moves in direction α
- simply include it:

$$\Phi_{kin}(\sigma, \alpha) = \phi(\sigma + \alpha)$$

- usually, α depends linearly on p (more later)

Isotropic hardening:

- E grows by some scalar g
- so we modify the yield stress:

$$\Phi_{iso}(\sigma, g) = \phi(\sigma) + g$$

- g depends somehow on p (again more later)

$\Rightarrow$ ... or a combination of both: $\Phi(\sigma, \alpha, g) = \phi(\sigma + \alpha) + g$

Summary – so far

- Stress, Displacement

$$\sigma \in \mathbb{R}^{3\times 3}_{sym}, \quad u \in \mathbb{R}^3$$

- Strain

$$\epsilon = e + p = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \in \mathbb{R}^{3\times 3}_{sym}$$

- Yield function with hardening

$$\Phi(\sigma, \alpha, g) = \phi(\sigma + \alpha) + g$$

- Law of equilibrium:

$$-\operatorname{div} \sigma = f$$

- Hooke's law:

$$\exists \mathbf{C} \in \mathbb{R}^{(3\times 3)\times(3\times 3)} : \ \sigma = \mathbf{C}e$$

- Maximum work

$$\dot{p} \in N_E(\sigma)$$

Generalized Maximum Work

$$\Phi(\sigma, \alpha, g) = \phi(\sigma + \alpha) - g$$

- formally, for the new stress variables α and g, we define counterparts in the plastic strain space
- we denote the *generalized* stres

$$\Sigma = (\sigma, \alpha, g)$$

- and define the *generalized* plastic strain

$$P = (p, a, \eta)$$

Generalized Maximum Work

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- we denote the *generalized* stres

$$\Sigma = (\sigma, \alpha, g)$$

- and define the *generalized* plastic strain

$$P = (p, a, \eta)$$

- Why so complicated??

Generalized Maximum Work

- answer: We need this notation for the *generalized maximum work principle*!
- remember: Generalized stress $\Sigma = (\sigma, \alpha, g)$ and plastic strain $P = (p, a, \eta)$

Generalized Maximum Work Principle (von Mises, Taylor, Bishop, Hill)

For $\Phi(\Sigma) = 0$ and $\frac{d}{dt}\Phi(\Sigma) > 0$ we have

$$\Sigma : \dot{P} := \sigma : \dot{p} + \alpha : \dot{a} + g \cdot \dot{\eta} \geq T : \dot{P} \quad \forall T \in \mathcal{E}$$

where $\mathcal{E} = \{\Sigma : \ \Phi(\Sigma) \leq 0\}$.

- in other notation: $\dot{P} \in N_{\mathcal{E}}(\Sigma)$

Generalized Maximum Work

- $\dot{P} \in N_{\mathcal{E}}(\Sigma)$ and the definition $\mathcal{E} = \{\Sigma : \ \Phi(\Sigma) \leq 0\}$ leads to

$$\exists \, \lambda \in \mathbb{R} : \ \dot{P} = \lambda \nabla \Phi(\Sigma)$$

(assuming that we can derive at Σ)

Generalized Maximum Work

- $\dot{P} \in N_{\mathcal{E}}(\Sigma)$ and the definition $\mathcal{E} = \{\Sigma : \ \Phi(\Sigma) \leq 0\}$ leads to

$$\exists\, \lambda \in \mathbb{R} : \ \dot{P} = \lambda \nabla \Phi(\Sigma)$$

  (assuming that we can derive at Σ)

- since $\Phi(\sigma, \alpha, g) = \phi(\sigma + \alpha) + g$:

$$\begin{pmatrix} \dot{p} \\ \dot{a} \\ \dot{\eta} \end{pmatrix} = \lambda \begin{pmatrix} \nabla \phi(\sigma + \alpha) \\ \nabla \phi(\sigma + \alpha) \\ 1 \end{pmatrix}$$

- $\Rightarrow a = p$, so we can skip a

Hardening variables

- Generalized stress $\Sigma = (\sigma, \alpha, g)$ and plastic strain $P = (p, a, \eta)$
- kinematic hardening, we usually find

$$\alpha = -k_1 p$$

  with $k_1 > 0$
- isotropic hardening,

$$g = g(\eta)$$

  is a monotone decreasing, negative function, e.g. $g = -k_2\eta$
- η is often referred to the accumulated plastic evolution

$$\eta = c \cdot \int_0^t \|\dot{p}\| \, ds$$

Dual $\leftrightarrow$ Primal plasticity

- the generalized result

$$\dot{P} \in N_{\mathcal{E}}(\Sigma)$$

  is the basis of the so-called *dual* formulation of plasticity
- there, Σ is unknown and P serves as a couple term

Dual $\leftrightarrow$ Primal plasticity

- the generalized result

$$\dot{P} \in N_{\mathcal{E}}(\Sigma)$$

  is the basis of the so-called *dual* formulation of plasticity
- there, Σ is unknown and P serves as a couple term
- before we consider solutions strategies (next lectures), let's consider the *primal* formulation of plasticity first
- there, P is the unknown and Σ serves as couple term

Primal plasticity

- generalized stress $\Sigma = (\sigma, \alpha, g)$ and plastic strain $P = (p, a, \eta)$
- we simply define a *dissipation function*

$$D(P) := \sup_{\Sigma \in \mathcal{E}} \{P : \Sigma\} \in \mathbb{R} \cup \{\infty\}$$

Primal plasticity

- generalized stress $\Sigma = (\sigma, \alpha, g)$ and plastic strain $P = (p, a, \eta)$
- we simply define a *dissipation function*

$$D(P) := \sup_{\Sigma \in \mathcal{E}} \{P : \Sigma\} \in \mathbb{R} \cup \{\infty\}$$

- D has interesting properties: convex, positively homogeneous, lower semicontinuous and proper.
- this will be explained and proofed in the next lecture

Primal plasticity

$$D(P) := \sup_{\Sigma \in \mathcal{E}} \{P : \Sigma\} \in \mathbb{R} \cup \{\infty\}$$

- we can give an explicit expression for our two flow rules
- Von Mises:

$$D(P) = D(p, \eta) = \begin{cases} \sigma_0 \|p\|_F, & \|p\| \leq \eta \\ \infty, & \text{else} \end{cases}$$

- Tresca:

$$D(P) = D(p, \eta) = \begin{cases} \sigma_0 \|p\|_2, & \|p\| \leq \eta \\ \infty, & \text{else} \end{cases}$$

Proof – von Mises case

$$D(P) := \sup_{\Sigma \in \mathcal{E}} \{P : \Sigma\} \in \mathbb{R} \cup \{\infty\}$$

$$\Phi(\sigma, \alpha, g) = \phi(\sigma + \alpha) + g = \|\sigma + \alpha - \frac{1}{3}\operatorname{tr}(\sigma + \alpha)\, I\|_F + g - \sigma_0 \le 0$$

- 1st: case without isotropic hardening, set $\beta = \sigma + \alpha$, $\operatorname{dev}(\beta) = \beta - \frac{1}{3}\operatorname{tr}(\beta)\, I$

$$\Phi(\sigma, \alpha) = \|\operatorname{dev}(\beta)\|_F - \sigma_0 \le 0$$

- we have $a = p$ and p is trace-free:

$$\begin{aligned} D(P) = D(p, a) = \sup_{(\sigma,\alpha)\in\mathcal{E}} \{\sigma : p + \alpha : a\} &= \sup_{(\sigma,\alpha)\in\mathcal{E}} \{\beta : p\} \\ &= \sup_{(\sigma,\alpha)\in\mathcal{E}} \{\operatorname{dev}(\beta) : p\} \end{aligned}$$

Proof – von Mises case

$$D(P) = \sup_{(\sigma,\alpha)\in\mathcal{E}} \{\mathrm{dev}(\beta) : p\}$$

- we can choose β to be a multiple of p, then $\mathrm{dev}(\beta) = \beta$ and

$$D(P) = \sup_{(\sigma,\alpha)\in\mathcal{E}} \{\mathrm{dev}(\beta) : p\} = \sup_{(\sigma,\alpha)\in\mathcal{E}} \{\|\mathrm{dev}(\beta)\|_F \|p\|_F\}$$

- and since

$$\Phi(\sigma, \alpha) = \|\mathrm{dev}(\beta)\|_F - \sigma_0 \leq 0$$

  we choose

$$D(P) = \sigma_0 \|p\|_F$$

Proof – von Mises case

- 2nd: with isotropic hardening (fast version)
- if g and η are included in the supremum, we will get

$$D(p,\eta) = \sup_{g\leq 0}\{\sigma_0\|p\|_F + g\cdot(\eta - \|p\|_F)\}$$

- and therefore

$$D(p,\eta) = \begin{cases} \sigma_0\|p\|_F, & \|p\| \leq \eta \\ \infty, & \text{else} \end{cases}$$

Why should we use the dissipation?

- in the next lecture we will proof

$$\Sigma \in \partial D(\dot{P})$$

  is equivalent to

$$\dot{P} \in N_{\mathcal{E}}(\Sigma)$$

- this is the desired swap of Σ and $\dot{P}$ for the primal formulation

Summary – 2nd and last try

$$\sigma \in \mathbb{R}^{3\times 3}_{sym}, \quad u \in \mathbb{R}^3$$

$$\epsilon = e + p = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

$$\Phi(\sigma, \alpha, g) = \phi(\sigma + \alpha) + g \leq 0$$

$$\alpha = -k_1 p$$

$$\eta = c \cdot \int_0^t \|\dot{p}\| \, ds$$

$$g = g(\eta)$$

$$-\operatorname{div} \sigma = f$$

$$\sigma = \mathbf{C} e$$

$$\dot{P} \in N_{\mathcal{E}}(\Sigma) \Leftrightarrow \Sigma \in \partial D(\dot{P})$$

Outlook for the next lecture

- we will couple there equations
- need some convex analysis to proof some transformations
- state some closed forms we can solve

- Slides can be found here:

  `https://www.math.tu-dresden.de/~jaap/`

- contact me for the simulations, programs, etc. but you won't get the aluminum plate ☺

Source

- The complete first lecture is based on

  Weimin Han & B. Daya Reddy, Plasticity, Mathematical Theory and Numerical Analysis, Second Edition, Springer Science+Business Media, LLC 2013