TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

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On plastic deformation: From physics over convex analysis to numerical simulation

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Goals of this Graduate Lecture

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- proof some nice results from convex analysis

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2nd lecture (now, live and in color)

- proof some nice results from convex analysis

3rd lecture (next week)

- look into duality of discrete problems and optimization
- choose time/space discretization schemes
- if the time allows it: present a numerical algorithm to solve the problems

Summary from last week

- Stress, Displacement

$$\sigma \in \mathbb{R}^{3\times 3}_{sym}, \quad u \in \mathbb{R}^3$$

- Strain

$$\epsilon = e + p = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \in \mathbb{R}^{3\times 3}_{sym}$$

- Yield function with hardening

$$\Phi(\sigma, \alpha, g) = \phi(\sigma + \alpha) + g \leq 0$$

- Law of equilibrium:

$$-\operatorname{div} \sigma = f$$

- Hooke's law:

$$\exists \mathbf{C} \in \mathbb{R}^{(3\times 3)\times(3\times 3)} : \ \sigma = \mathbf{C}e$$

- Maximum work

$$\dot{p} \in N_E(\sigma)$$

Convex sets and functions

first tiny steps in convexity:

- let X denote a finite vector space
- a set $M \subset X$ is called *convex*, if

$$x, y \in M,\ t \in [0,1] \quad \Rightarrow \quad tx + (1-t)y \in M$$

- a function $f : \Omega \subset X \to \overline{\mathbb{R}}$ is called convex, if

$$\mathrm{epi}(f) := \{(x,y) \in \Omega \times \mathbb{R} : \quad f(x) \leq y\}$$

  is convex

Convex $\neq$ continuous

- functions may jump to ∞ at the the boundary of Ω!
- we call a function *proper*, if

$$f(x) > -\infty \ \forall x \in \Omega, \quad \mathrm{dom}(f) := \{x \in \Omega : \ f(x) < \infty\} \neq \emptyset$$

- and f is *lower semicontinuous* (l.s.c.), if

$$\liminf_{n\to\infty} f(x_n) \geq f(x)$$

  for all sequences $x_n \to x$
- or, equivalently:

$$L(\alpha) := \{x \in \Omega : \ f(x) \leq \alpha\}$$

  is closed for all $\alpha \in \mathbb{R}$

Equivalent definitions

we want to show

$$\liminf_{n\to\infty} f(x_n) \geq f(x) \quad \forall\, x_n \to x \quad \Leftrightarrow \quad L(\alpha) := \{x \in \Omega :\ f(x) \leq \alpha\} \text{ is closed } \forall \alpha \in \mathbb{R}$$

- let $\alpha \in \mathbb{R}$ and $x \in X$, s.t. $f(x) > \alpha$
- from lim inf we know: $\forall \epsilon > 0\ \exists \delta > 0$:

$$f(y) > f(x) - \epsilon \quad \forall y \in B_\delta(x)$$

- for $\epsilon = f(x) - \alpha > 0$, this means

$$f(y) > \alpha\ \forall y \in B_\delta(x)$$

- $\Rightarrow B_\delta(x) \subset L(\alpha)^C$

Lower semicontinuous functions and the epigraph

- our first theorem: for any $f : \Omega \subset X \to \mathbb{R}$ we have

$$\mathrm{epi}(f) \text{ is closed} \quad \Leftrightarrow \quad f \text{ is l.s.c.}$$

proof: "$\Rightarrow$"

- let $\mathrm{epi}(f)^C$ be open
- let $(x, y) \in \mathrm{epi}(f)^C$, i.e., $f(x) > y$
- there is an open neighborhood $(x, y) \in U \times (-\infty, y + \epsilon) \subset \mathrm{epi}(f)^C$
- therefore, $f(z) \geq y + \epsilon$ for $z \in U$
- this means $\liminf_{z \to x} f(z) \geq y + \epsilon$
- choose $y + \epsilon$ maximal: $\liminf_{z \to x} f(z) \geq f(x)$

Lower semicontinuous functions and the epigraph

- our first theorem: for any $f : \Omega \subset X \to \mathbb{R}$ we have

$$\mathrm{epi}(f) \text{ is closed } \Leftrightarrow f \text{ is l.s.c.}$$

proof: "$\Leftarrow$"

- let $(x, y) \in \mathrm{epi}(f)^C$, i.e., $f(x) > y$
- set $\mu = \frac{f(x)+y}{2}$, and therefore, $\mu < f(x)$
- $x \in U := \{z \in \Omega : \; f(z) > \mu\}$ and U is open
- and $U \times (-\infty, \mu) \subset \mathrm{epi}(f)^C$
- therefore, $\mathrm{epi}(f)^C$ is open

Dual spaces

- we define the *dual space* X' of X by

$$X' := \{m : X \to \mathbb{R}, \quad m \text{ is linear and contiuous}\}$$

- usually, elements of X' are denoted by x^*
- the dual pairing is denoted by

$$X' \times X \to \mathbb{R} : \ \langle x^*, x\rangle := x^*(x)$$

- and in case of $X = \mathbb{R}^d$, we can identify $X' = \mathbb{R}^d$:

$$\langle x^*, x\rangle = (x^*)^T x$$

Dual functions?

- if there are dual spaces, why shouldn't we define dual functions?
- so let $f : X \to \overline{\mathbb{R}}$, then

$$f^* : X' \to \overline{\mathbb{R}}, \quad f^*(x^*) := \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}$$

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  is called *polar* or *conjugate* function of f

- interesting: f^* is convex and l.s.c.
- let's proof this!

Proof – convexity

$$f^* : X' \to \overline{\mathbb{R}}, \quad f^*(x^*) := \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}$$

- since x^* is linear by definition, we know that $\langle \cdot, \cdot \rangle$ is bilinear
- therefore, $\langle \cdot, x\rangle - f(x)$ is an affine function

Proof – convexity

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- since x^* is linear by definition, we know that $\langle \cdot, \cdot \rangle$ is bilinear
- therefore, $\langle \cdot, x\rangle - f(x)$ is an affine function
- so we have

$$\begin{aligned} f^*(tx^* + (1-t)y^*) &= \sup_{x \in X}\{t\langle x^*, x\rangle - tf(x) + (1-t)\langle y^*, x\rangle - (1-t)f(x)\} \\ &\leq t \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\} + (1-t)\sup_{x \in X}\{\langle y^*, x\rangle - f(x)\} \end{aligned}$$

Proof – l.s.c.

$$f^* : X' \to \overline{\mathbb{R}}, \quad f^*(x^*) := \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}$$

- we show $\{x^* \in X' : f^*(x^*) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$
- we can conclude

$$\sup_{x \in X} \{\langle x^*, x \rangle - f(x)\} \leq \alpha \Rightarrow \langle x^*, x \rangle - f(x) \leq \alpha \quad \forall x \in X$$

- for each x this is the level set of an affine function
- so it is closed and the intersection of closed sets is closed

Subdifferential

- for a convex function $f : \Omega \subset X \to \overline{\mathbb{R}}$ we call

$$\partial f(x) := \{x^* \in X' : \; f(y) \geq f(x) + \langle x^*, y - x\rangle \; \forall y \in X\}$$

  the *subdifferential* of f at x

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- sure, if f is differentiable:

$$\partial f(x) = \{\nabla f(x)\}$$

- now we have a nice theorem:
- for $f : X \to \overline{\mathbb{R}}$ proper, convex and l.s.c, we have

$$x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*) \quad \forall x \in X, \; x^* \in X'$$

But first, a lemma

- if f is convex and l.s.c., we have $f = f^{**}$ with

$$f^{**} : X \to \overline{\mathbb{R}} : \; f^{**}(x) = \sup_{x^* \in X'} \{\langle x^*, x\rangle - f^*(x^*)\}$$

proof:

- assume f is greater than an affine function: $f(x) > \langle x^*, x\rangle - \alpha$
- in other words: $\alpha > \langle x^*, x\rangle - f(x) \quad \forall x \in X$
- so:

$$\alpha \geq \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\} = f^*(x^*)$$

But second, another lemma

- if f is convex and l.s.c., we have

$$f(x) = \sup_{(x^*,\alpha)\in A} \{\langle x^*, x\rangle - \alpha\}$$

  where

$$(x^*, \alpha) \in A \quad \Leftrightarrow \quad f(x) \geq \langle x^*, x\rangle - \alpha \quad \forall x \in X$$

idea of the proof:

- from the definition we know: $f(x) \geq \sup_{(x^*,\alpha)\in A}\{\langle x^*, x\rangle - \alpha\}$
- by contradiction: assume $f(x_0) > a := \sup_{(x^*,\alpha)\in A}\{\langle x^*, x_0\rangle - \alpha\}$
- epi(f) is closed (f is l.s.c.) $\Rightarrow$ (x, a) can be separated by a linear functional

$$l(x) = \langle z^*, x\rangle - \beta \quad \text{with} \quad a < l(x_0) < f(x_0) \quad ↯$$

Back to the first lemma

- if f is convex and l.s.c., we have $f = f^{**}$ with

$$f^{**} : X \to \overline{\mathbb{R}}(x) : \; f^{**}(x) = \sup_{x^* \in X'} \{\langle x^*, x\rangle - f^*(x^*)\}$$

proof:

- so:

$$\alpha \geq \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\} = f^*(x^*)$$

- now we know:

$$f(x) = \sup_{(x^*,\alpha) \in A} \{\langle x^*, x\rangle - \alpha\} \overset{\text{minimize } \alpha}{=} \sup_{x \in X}\{\langle x^*, x\rangle - f^*(x)\} = f^{**}(x)$$

Theorem Subdifferential

- we wanted to show that for $f : X \to \overline{\mathbb{R}}$ proper, convex and l.s.c, we have

$$x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*) \quad \forall x \in X,\ x^* \in X'$$

now the proof:

- let $x^* \in \partial f(x)$, i.e., $\forall y \in X$:

$$\begin{aligned} f(y) &\geq f(x) + \langle x^*, y - x\rangle \\ &= \langle x^*, y\rangle + \underbrace{f(x) - \langle x^*, x\rangle}_{-\alpha} \end{aligned}$$

- we know that $\alpha = f^*(x^*)$ and therefore, for $x = y$

$$f(x) + f^*(x^*) = \langle x^*, x\rangle$$

Theorem Subdifferential

- we know that $\alpha = f^*(x^*)$ and therefore, for $x = y$

$$f(x) + f^*(x^*) = \langle x^*, x \rangle$$

- we also know that our f satisfies

$$f^{**}(x) = f(x)$$

- so we have

$$f^*(x^*) + f^{**}(x) = \langle x^*, x \rangle$$

- by the definition $f^{**}(x) = \sup_{y^* \in X'} \{\langle y^*, x \rangle - f^*(y^*)\}$ we conclude

$$f^*(x^*) + \langle y^*, x \rangle - f^*(y^*) \leq \langle x^*, x \rangle \quad \forall y^* \in X'$$

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- or, rearranged:

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- and: the part all arguments were equivalences!

How does this correspond to plasticity?

the answer needs some more definitions ☺

- let $M \subset X$ be a set. We define the *indicator* function

$$I_M(x) := \begin{cases} 0 & x \in M \\ \infty & x \notin M \end{cases}$$

- M convex $\Leftrightarrow I_M$ convex
- M closed $\Leftrightarrow I_M$ l.s.c.

Indicator function

$$I_M(x) := \begin{cases} 0 & x \in M \\ \infty & x \notin M \end{cases}$$

let M be convex and closed

- for $x \in \operatorname{int}(M) : \partial I_M(x) = \{0\}$
- for $x \notin M : \partial I_M(x) = \{0\}$
- now for $x \in \partial M$ (boundary):

$$x^* \in \partial I_M(x) \Leftrightarrow I_M(y) \geq I_M(x) + \langle x^*, y - x\rangle \quad \forall y \in X$$

- if $y \in M : \ \langle x^*, y - x\rangle < 0$, i.e., x^* points to the outside
- so, in other notation:

$$x^* \in N_M(x)$$

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- so, in other notation:

$$x^* \in N_M(x)$$

- it follows: $\partial I_M(x) = N_M(x)$ for all $x \in X$!

Last definition: Support function

let $M \subset X$ again convex and closed

- we define the *support function*

$$s_M : X' \to \overline{\mathbb{R}} : \; s_M(x^*) := \sup_{x \in M} \{ \langle x^*, x \rangle \}$$

Last definition: Support function

let $M \subset X$ again convex and closed

- we define the *support function*

$$s_M : X' \to \overline{\mathbb{R}} : \; s_M(x^*) := \sup_{x \in M} \{ \langle x^*, x \rangle \}$$

- fun fact:

$$I_M^*(x^*) = \sup_{x \in X} \{ \langle x^*, x \rangle - I_M(x) \}$$

  will always be attained in M:

$$I_M^*(x^*) = \sup_{x \in M} \{ \langle x^*, x \rangle - I_M(x) \} = s_M(x^*)$$

Last definition: Support function

- now we know

$$I_M^*(x^*) = s_M(x^*)$$

- thus, S_M is convex and l.s.c.
- and, our final result

$$\boxed{x^* \in N_M(x) \Leftrightarrow x^* \in \partial I_M(x) \Leftrightarrow x \in \partial s_M(x^*)}$$

Summary – from the first lecture

- Stress, Displacement

$$\sigma \in \mathbb{R}^{3\times 3}_{sym}, \quad u \in \mathbb{R}^3$$

- Strain

$$\epsilon = e + p = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \in \mathbb{R}^{3\times 3}_{sym}$$

- Yield function with hardening

$$\Phi(\sigma, \alpha, g) = \phi(\sigma + \alpha) + g$$

- Law of equilibrium:

$$-\operatorname{div} \sigma = f$$

- Hooke's law:

$$\exists \mathbf{C} \in \mathbb{R}^{(3\times 3)\times(3\times 3)} : \ \sigma = \mathbf{C}e$$

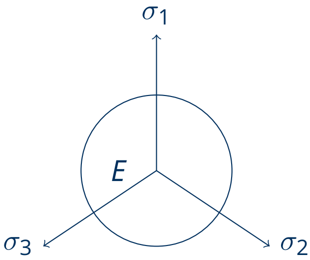
- Maximum work

$$\dot{p} \in N_E(\sigma)$$

Elastic region

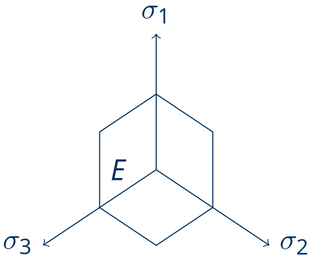
- Elastic behavior if stress σ is located within the *elastic region* $E \subset \mathbb{R}^{3\times 3}_{sym}$

<u>Von Mises</u>



$$\phi(\sigma) = \|\sigma - \frac{1}{3}\operatorname{tr}(\sigma)I\|_F - \sigma_0 \leq 0$$

<u>Tresca</u>



$$\phi(\sigma) = \max_{1\leq i,j\leq 3} |\sigma_i - \sigma_j| - \sigma_0 \leq 0$$

Generalized Maximum Work

- remember: Generalized stress $\Sigma = (\sigma, \alpha, g)$ and plastic strain $P = (p, a, \eta)$

Generalized Maximum Work Principle (von Mises, Taylor, Bishop, Hill)

For $\Phi(\Sigma) = 0$ and $\frac{d}{dt}\Phi(\Sigma) > 0$ we have

$$\Sigma : \dot{P} := \sigma : \dot{p} + \alpha : \dot{a} + g \cdot \dot{\eta} \geq T : \dot{P} \quad \forall T \in \mathcal{E}$$

where $\mathcal{E} = \{\Sigma : \ \Phi(\Sigma) \leq 0\}$.

- in other notation: $\dot{P} \in N_{\mathcal{E}}(\Sigma)$
- note: $\mathcal{E}$ is convex and closed!

What we did last time

- generalized stress $\Sigma = (\sigma, \alpha, g)$ and plastic strain $P = (p, a, \eta)$
- we simply defined a *dissipation function*

$$D(P) := \sup_{\Sigma \in \mathcal{E}} \{P : \Sigma\} \in \mathbb{R} \cup \{\infty\}$$

What we did last time

- generalized stress $\Sigma = (\sigma, \alpha, g)$ and plastic strain $P = (p, a, \eta)$
- we simply defined a *dissipation function*

$$D(P) := \sup_{\Sigma \in \mathcal{E}} \{P : \Sigma\} \in \mathbb{R} \cup \{\infty\}$$

- now we know, this is the support function of $\mathcal{E}$!
- now we know, it is convex, l.s.c., proper
- and so we can tell

$$\dot{P} \in N_{\mathcal{E}}(\Sigma) \quad \Leftrightarrow \quad \Sigma \in \partial D(\dot{P})$$

What will happen next time?

- we are now able to express the evolution of the plastic strain in terms of stress
- ... and the other way around!
- now we are able to state coupled systems of the plasticity problem
- present different approaches of solving the problem

Sources

- plasticity theory:
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