Remarks on the analysis of finite element methods on a Shishkin mesh: are Scott-Zhang interpolants applicable?

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Abstract

In the first part of the paper we discuss minimal smoothness assumptions for the components of the solution decomposition which allow to prove robust convergence results in the energy norm for linear or bilinear finite elements on Shishkin meshes applied to convection-diffusion problems with exponential boundary layers. In the corresponding derivation the standard Lagrange interpolant is used, in general. In the second part we discuss the question whether or not it is possible to use the Scott-Zhang interpolant.

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1 Introduction

We shall examine the finite element method for the numerical solution of the singularly perturbed linear elliptic boundary value problem

$$Lu \equiv -\varepsilon \triangle u - b \cdot \nabla u + cu = f \qquad \text{in } \Omega = (0,1) \times (0,1) \tag{1a}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1b}$$

where ε is a small positive parameter, b and c are smooth and f is given with $f \in L_2(\Omega)$. Assuming

$$b = (b_1, b_2) > (\beta_1, \beta_2) > 0 \qquad \text{on } \bar{\Omega} \tag{2}$$

with constants β_1, β_2, the solution of (1) typically has exponential boundary layers at $x = 0$ and $y = 0$. Without loss of generality we can as well assume

$$c + \frac{1}{2}\operatorname{div} b \geq \alpha_0 > 0. \tag{3}$$

In [8] we find the first analysis of the finite element method for bilinear elements on a tensor product Shishkin mesh (see the next section for details of the mesh). Let us introduce the ε-weighted H^1 norm by

$$\|v\|_\varepsilon := \varepsilon^{1/2}|v|_1 + \|v\|_0.$$

The proof of the error estimate

$$\| u - u^N \|_\varepsilon \leq CN^{-1} \ln N \tag{4}$$

in the ε-weighted H^1 norm is based on the decomposition of the solution u into a smooth part S and layer components ($u = S + E_1 + E_2 + E_{12}$) which satisfy the pointwise estimates

$$\left| \frac{\partial^{i+j} S}{\partial x^i \partial y^j}(x,y) \right| \leq C\,, \tag{5a}$$

$$\left| \frac{\partial^{i+j} E_1}{\partial x^i \partial y^j}(x,y) \right| \leq C\varepsilon^{-i} e^{-\beta_1 x/\varepsilon}\,, \tag{5b}$$

$$\left| \frac{\partial^{i+j} E_2}{\partial x^i \partial y^j}(x,y) \right| \leq C\varepsilon^{-j} e^{-\beta_2 y/\varepsilon}\,, \tag{5c}$$

$$\left| \frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j}(x,y) \right| \leq C\varepsilon^{-(i+j)} e^{-\beta_1 x/\varepsilon} e^{-\beta_2 y/\varepsilon}\,, \tag{5d}$$

for all $(x,y) \in \overline{\Omega}$ and $0 \leq i+j \leq 2$.

Throughout the paper C denotes a generic constant that is independent of ε and of the mesh.

In the paper [3] the authors simplified the proof of the interpolation error estimates of [8] and extended the analysis to linear elements.

The validity of (5) is the crucial part of the analysis, see [5, 6] for a discussion of sufficient conditions for (5). Let us only remark that $u \in C^2(\bar{\Omega})$ requires additional compatibility conditions which restricts the class of problems considered.

In Section 2 and 3 we will show that it is possible to replace the conditions (5a)-(5d) by weaker conditions which allow us nevertheless to prove the estimate (4). In particular, the assumption $u \in C^2(\bar{\Omega})$ is not necessary which avoids requiring compatibility conditions for f.

In the proofs of Section 2 and 3 it is standard to use the Lagrange interpolant. This is possible because our assumptions include $u \in C(\bar{\Omega})$. But sometimes it would be desirable to use an interpolant with better properties than the Lagrange interpolant. What are its disadvantageous?

- The Lagrange interpolant is not L_2 stable.
- For problems with mixed boundary conditions the solution can be of poor regularity such that $u \notin W^{s,2}$ for any $s > 3/2$. Then, the Lagrange interpolant is not defined. This is also the case for discontinuous Dirichlet boundary conditions.
- In 3D the estimate (see [2])

  $$|u - u^I|_{1,p} \le C \sum_{|\alpha|=1} h^\alpha |D^\alpha u|_{1,p}$$

  is only valid for $p > 2$, but $p = 2$ is the natural choice in the finite element analysis.

Therefore we ask (as a first step for treating the problems just mentioned) in Section 4: Is it possible to replace in the finite element analysis on a Shishkin mesh the Lagrange interpolant by some quasi-interpolant that is defined for non-smooth functions?

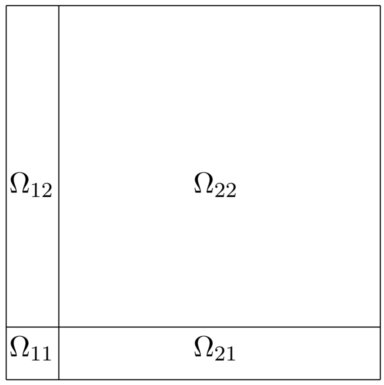
2 The mesh, the analysis of Stynes and O'Riordan and its first modification

Let us define

$$\lambda_x := \min(q, \frac{\sigma\varepsilon}{\beta_1} \ln N) \quad \text{and} \quad \lambda_y = \min(q, \frac{\sigma\varepsilon}{\beta_2} \ln N)$$

with $\sigma > 0$ (which will be fixed later) and $q \in (0, 1)$ arbitrary. Divide the domain Ω as in Figure 1: $\bar{\Omega} = \Omega_{11} \cup \Omega_{12} \cup \Omega_{21} \cup \Omega_{22}$.

The nodes of our rectangular mesh are obtained from the tensor product of a set of N_x points in the x-direction and N_y points in the y-direction. A one-dimensional Shishkin mesh is characterized by an equidistant mesh size h in $[0, \lambda_x]$ and H in $[\lambda_x, 1]$, at the transition point λ_x the mesh switches from coarse to fine. For simplicity, we assume $\beta_1 = \beta_2$ and $\lambda_x := (\sigma\varepsilon \ln N)/\beta_1$, resulting in the definition



$$\Omega_{11} := [0, \lambda_x] \times [0, \lambda_y]$$
$$\Omega_{12} := [0, \lambda_x] \times [\lambda_y, 1]$$
$$\Omega_{21} := [\lambda_x, 1] \times [0, \lambda_y]$$
$$\Omega_{22} := [\lambda_x, 1] \times [\lambda_y, 1]$$

Figure 1: Subregions of Ω

$$h = 2\lambda_x / N \quad \text{and} \qquad H = 2(1 - \lambda_x)/N \tag{6}$$

and the mesh points

$$x_i = y_i = ih \quad \text{for} \quad i = 0, 1, \cdots, N/2 \tag{7a}$$
$$x_i = y_i = \lambda_x + H(i - \frac{N}{2}) \quad \text{for} \quad i = N/2 + 1, \cdots, N. \tag{7b}$$

With linear or bilinear finite elements and the corresponding finite element space $V^N \subset H_0^1(\Omega)$, the finite element method reads:
Find $u^N \in V^N$ with

$$a(u^N, v) = (f, v) \quad \forall v \in V^N. \tag{8}$$

The bilinear form $a(.,.)$ is given by

$$a(w, v) := \varepsilon(\nabla w, \nabla v) + (-b \cdot \nabla w + cw, v);$$

due to property (3) the bilinear form is uniformly V-elliptic with respect to the ε-weighted H^1 norm: one has

$$a(w, w) \geq \alpha \|w\|_\varepsilon^2 \quad \text{for all } w \in H_0^1(\Omega)$$

with some positive α independent of ε.

With the standard Lagrange interpolant $u^I \in V^N$ of u (its existence is not a problem if the validity of (5) for $i + j = 0$ is required) we introduce the splitting of the error into the components

$$\eta = u^I - u, \quad \chi = u^I - u^N \tag{9}$$

and start the error estimate from

$$\alpha\|u^I - u^N\|_\varepsilon^2 \le a(u^I - u^N, u^I - u^N) = a(u^I - u, u^I - u^N) = a(\eta, \chi). \quad (10)$$

To estimate the right-hand side of (10) Stynes and O'Riordan use integration by parts for the convective term:

$$a(\eta, \chi) = \varepsilon(\nabla\eta, \nabla\chi) + (b\eta, \nabla\chi) + ((c + div\, b)\eta, \chi).$$

To estimate the first and the third term one can simply use Cauchy-Schwarz. For the crucial convection term the following techniques are applied:

(i) the inverse inequality on Ω_{22}

$$|(b\eta, \nabla\chi)| \le C\frac{1}{H}\|\eta\|_0\, \|\chi\|_0 \le C\frac{1}{H}\|\eta\|_0\|\chi\|_\varepsilon$$

(this is sufficient because one expects $\|\eta\|_0 \le CH^2$ to be valid on the coarse mesh)

(ii) on $\Omega \setminus \Omega_{22}$:

$$\begin{aligned} |(b\eta, \nabla\chi)| &\le C\|\eta\|_\infty\|\nabla\chi\|_{L_1} \\ &\le C\|\eta\|_\infty(meas(\Omega \setminus \Omega_{22}))^{1/2}\varepsilon^{-1/2}\|\chi\|_\varepsilon\,. \end{aligned}$$

This is sufficient because $meas(\Omega \setminus \Omega_{22}) \le C\varepsilon \ln N$. Equivalently, one could estimate as follows:

$$|(b\eta, \nabla\chi)| \le C\|\eta\|_0\|\nabla\chi\|_0 \le C\|\eta\|_\infty(meas(\Omega \setminus \Omega_{22}))^{1/2}\|\nabla\chi\|_0$$

To estimate $\|\eta\|_\infty$ the pointwise estimates (5) for the second order derivatives are used.

Before we study interpolation error estimates in detail we modify the error estimation technique of Stynes and O'Riordan. First, we split the interpolation error into two parts: $\eta = \eta_S + \eta_E$ with $\eta_S := S - S^I$, for instance. Moreover, the convective term is treated differently. For the smooth part we estimate directly

$$|(b \cdot \nabla\eta_S, \chi)| \le C|\eta_S|_1\|\chi\|_0 \le C|\eta_S|_1\|\chi\|_\varepsilon. \quad (11)$$

For the layer part, we use again integration by parts (χ is zero at the boundary). Let us study, for instance, the layer part E_1. We use

(i*) on Ω_{22} as before an inverse inequality (ii*) on $\Omega \setminus \Omega_{22}$:

$$|(b\eta_{E_1}, \nabla\chi)_{\Omega\setminus\Omega_{22}}| \le C\|\eta_{E_1}\|_{0,\Omega\setminus\Omega_{22}}\varepsilon^{-\frac{1}{2}}\|\chi\|_{\varepsilon,\Omega\setminus\Omega_{22}}$$

in combination with

$$\|\eta_{E_1}\|_{0,\Omega\setminus\Omega_{22}} \le C\varepsilon^{\frac{1}{2}}(N^{-1}\ln N)^2 .$$

For the smooth part S the desired estimates for the interpolation error are available for $S \in H^2(\Omega)$. We have still to investigate sufficient conditions for the validity of the estimates

$$\varepsilon^{1/2}|E - E^I|_1 \le CN^{-1}\ln N \tag{12a}$$

$$\|E - E^I\|_{0,\Omega_{22}} \le CN^{-2}, \quad \|E - E^I\|_{0,\Omega\setminus\Omega_{22}} \le C\varepsilon^{1/2}(N^{-1}\ln N)^2 \tag{12b}$$

and shall do that in the next section.

3 The interpolation error for Lagrange interpolation

For $w \in H^2(\Omega)$, the linear or bilinear interpolant on our tensor product mesh satisfies on each element e the estimates

$$\|w - w^I\|_{0,e} \le C\sum_{|\alpha|=2} h^\alpha\|D^\alpha w\|_{0,e} \tag{13a}$$

$$\|(w - w^I)_x\|_{0,e} \le C\sum_{|\alpha|=1} h^\alpha\|D^\alpha w_x\|_{0,e} \tag{13b}$$

$$\|(w - w^I)_y\|_{0,e} \le C\sum_{|\alpha|=1} h^\alpha \| D^\alpha w_y \|_{0,e} , \tag{13c}$$

here e has the lengths of the orthogonal sides h_x, h_y and $h^\alpha = h_x^{\alpha_1}h_y^{\alpha_2}$, $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| = \alpha_1 + \alpha_2$.

For the smooth part S the desired interpolation error estimates follow immediately. Let us now consider the layer part E_1, for instance.
To estimate the L_2 error, we estimate as follows:

- on Ω_{11}: using $|E_1|_2 \le C\varepsilon^{-3/2}$ we conclude

$$\|E_1 - E_1^I\|_0 \le Ch^2|E_1|_2 \le C\varepsilon^{\frac{1}{2}}(N^{-1}\ln N)^2.$$

- on Ω_{12}: (13a) results in

$$\begin{aligned}\|E_1 - E_1^I\|_0 &\leq C(h^2\|(E_1)_{xx}\|_0 + Hh\|(E_1)_{xy}\|_0 + H^2\|(E_1)_{yy}\|_0) \\ &\leq C\varepsilon^{1/2}(N^{-1}\ln N)^2\end{aligned}$$

  assuming

$$\|(E_1)_{xx}\|_0 \leq C\varepsilon^{-3/2}, \quad \|(E_1)_{xy}\|_0 \leq C\varepsilon^{-1/2}; \quad \|(E_1)_{yy}\| \leq C\varepsilon^{1/2}.$$

- on $\Omega_{22} \cup \Omega_{21}$ one hopes to use the smallness of E_1:

$$\|E_1 - E_1^I\|_0 \leq \|E_1\|_0 + \|E_1^I\|_0.$$

But, unfortunately, the Lagrange interpolant is not L_2 stable. Therefore, we introduce the stronger L_∞ norm and choose the parameter σ of the mesh in such a way that

$$\|E_1^I\|_0 \leq \|E_1^I\|_\infty \leq \|E_1\|_\infty \leq CN^{-2} \text{ for } x \geq \lambda_x. \tag{14a}$$

Then, it follows as well

$$\|E_1 - E_1^I\|_{0,\Omega_{21}} \leq \|E_1 - E_1^I\|_{\infty,\Omega_{21}}(meas\Omega_{21})^{1/2} \leq C\varepsilon^{1/2}N^{-2}(\ln N)^{1/2}. \tag{14b}$$

Next we study $\|(E_1 - E_1^I)_x\|$, the decisive part of the H^1 semi-norm of E_1:

- on Ω_{11}:

$$\|(E_1 - E_1^I)_x\|_0 \leq Ch|E_1|_2 \leq C\varepsilon^{-1/2}N^{-1}\ln N$$

  based on $|E_1|_2 \leq C\varepsilon^{-3/2}$.

- on Ω_{12}: (13b) leads to

$$\|(E_1 - E_1^I)_x\|_0 \leq C(\,h\|(E_1)_{xx}\|_0 + H\|(E_1)_{xy}\|_0) \leq C\varepsilon^{-1/2}N^{-1}\ln N$$

- on $\Omega_{21} \cup \Omega_{22}$: an inverse inequality results in

$$\|(E_1 - E_1^I)_x\|_0 \leq \|(E_1)_x\|_0 + CH^{-1}\|E_1^I\|_0$$

  Thus, (14b) and $\|(E_1)_x\|_0 \leq C\varepsilon^{-1/2}N^{-1}$ on $\Omega_{21} \cup \Omega_{22}$ are sufficient for the desired estimate (12a).

Concerning the corner layer one expects no difficulties at all but if one wants to estimate the interpolation error of E_{12} with respect to the x-derivative in the subdomain Ω_{12}, it is not optimal to use the anisotropic estimate

$$\|(E_{12} - E_{12}^I)_x\|_0 \leq C(\, h\|(E_{12})_{xx}\|_0 + H\|(E_{12})_{xy}\|_0)$$

because the mixed derivative yields the factor ε^{-1} and not the desired $\varepsilon^{-1/2}$. Instead, the estimate

$$\|(E_{12} - E_{12}^I)_x\|_{0,\Omega_{12}} \leq \|(E_{12})_x\|_{0,\Omega_{12}} + \frac{C}{h}(meas\Omega_{12})^{1/2}\|E_{12}\|_{\infty,\Omega_{12}}$$

works.

Summarizing: In the derivation of the interpolation error estimates we never used pointwise estimates for the second-order derivatives but only

$$\|(E_1)_{xx}\|_0 \leq C\varepsilon^{-3/2},\ \|(E_1)_{xy}\|_0 \leq C\varepsilon^{-1/2},\ \|(E_1)_{yy}\|_0 \leq C\varepsilon^{1/2}. \tag{15}$$

Moreover, it is as well sufficient to have for the first-order derivatives of the layer components

$$\|(E_1)_x\|_{0,\Omega_{21}\cup\Omega_{22}} \leq C\varepsilon^{-1/2}N^{-1} \tag{16}$$

and the corresponding estimates for E_2 and E_{12}. It seems possible to prove (16) directly without using pointwise information on the first-order derivatives (with (5) for $i + j = 1$ the estimate (16) follows easily).

Consequently, we can formulate the following sufficient conditions for proving (4).

Theorem 1 *: Let us assume that u allows a decomposition into a smooth part $S \in H^2(\Omega)$ and layer components E_1, E_2, E_{12} which satisfy the pointwise estimates (5) only for $i + j = 0$ and the estimates (15) and (16) (and the corresponding estimates for E_2 and E_{12}). Then,*

$$\|u - u^N\|_\varepsilon \leq CN^{-1}\ln N. \tag{17}$$

If one wants to prove supercloseness results, more smoothness of the solution is required, see [7].

4 The Scott-Zhang interpolant on Shishkin meshes

In many textbooks on finite elements only standard interpolants are discussed; detailed representations of interpolants for non-smooth functions are rare. A good source is the book of Ern and Guermond [4] with a discussion of the interpolants of Clément, Scott-Zhang and some projectors. But most references discuss these interpolants only on shape-regular meshes.

In [1], however, the reader can find precise information on the validity of certain estimates on anisotropic meshes. In particular Table 17.2 of the book compares the validity of the stability estimate

$$(S) \qquad \|Q_h u\|_{W^{m,q}(e)} \le C(meas\, e)^{\frac{1}{q}-\frac{1}{p}} \sum_{|\alpha|\le l-m} h^\alpha |D^\alpha u|_{W^{m,p}(S_e)}$$

and the approximation error estimate

$$(A) \quad \|u - Q_h u\|_{W^{m,q}(e)} \le C(meas\, e)^{\frac{1}{q}-\frac{1}{p}} \sum_{|\alpha|= l-m} h^\alpha |D^\alpha u|_{W^{m,p}(S_e)}$$

on tensor-product meshes for several quasi-interpolants $Q_h u$ of u. Here S_e denotes some macroelement around the given element e which is typical for generalized interpolants.

While for Lagrange interpolation with $p = q = 2$ the case $m = l$, in particular $m = l = 0$ is not allowed, the Clément interpolant satisfies (S) and (A) for $m = 0$ and $0 \le l \le 2$ with $p, q \in [1, \infty]$. But for the Clément interpolant $m = 1$ is not allowed on anisotropic meshes, thus it makes no sense to use Clément interpolation on two-dimensional Shishkin meshes.

However, there are interpolants of Scott-Zhang type for which (S) and (A) are valid for $0 \le m \le l-1, 1 \le l \le 2$ and $(p, q) \in [1, \infty]$ even for certain anisotropic meshes. Therefore we ask the question: can Scott-Zhang interpolants be used on Shishkin meshes?

A basic difficulty lies in the fact that most authors who discuss these interpolants assume the mesh to be locally uniform, i.e., they assume *that there is no abrupt change in the element size* (this simplifies the notation significantly, because the error on some given element is estimated by some norm on a macroelement which contains certain adjacent elements as well and for all these elements the same mesh width h can then be used). Consequently, on our Shishkin mesh we have to look very carefully at elements with $x \in [\lambda_x - h, \lambda_x + H]$ or $y \in [\lambda_y - h, \lambda_y + H]$ because such elements do have neighbours with very different element sizes.

A Scott-Zhang operator allows several possibilities for defining an edge σ_i responsible for the value in a vertex P_i (the value is then the L_2 projection on the finite element space restricted to σ_i). In [1] for the validity of anisotropic estimates the edges are defined in the following way: the operator S_h introduced in [1] uses small sides, the operator L_h large sides with a geometric projection property.

For simplicity, we only consider bilinear elements on our rectangular Shishkin mesh. Then a careful study of the proofs in [1] for getting anisotropic estimates shows:
On a our Shishkin mesh it is useful (and may be necessary) to define the L_2 projections on edges which are all parallel! (with exception of some edges on the boundary)

If we do that, consequently, some defining edges are small, some large; we assume that all defining edges are parallel to the x-axis. Every patch now consists only of two elements. We use the notation $S_e = e_1 \cup e_2$ with $e = e_k$ for $k = 1$ or $k = 2$. The patch sizes on S_e are denoted by $h_{S,1}$ and $h_{S,2}$. Now we additionally define the following rule: the function values in some vertex are defined based on the edge to the *left* of that vertex. (see Figure 2)

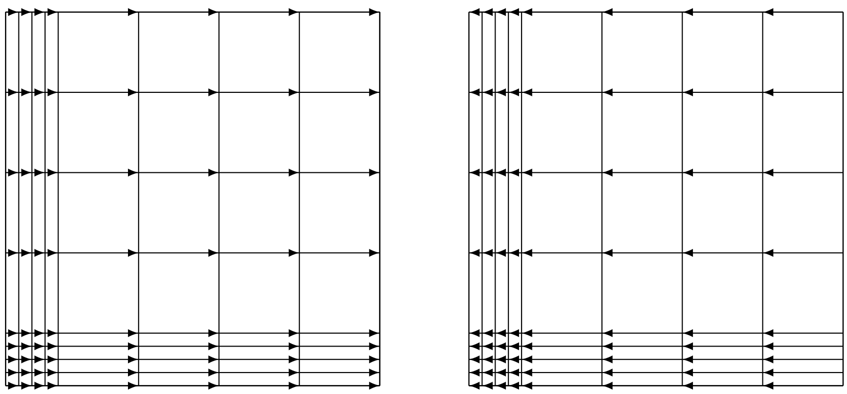


Figure 2: left defining and right defining edges

Consequently, the patch of some element contains the element and its left neighbor. That means, small elements have small neighbors, and only the large elements in the strip $[\lambda_x, \lambda_x + H]$ have small neighbors belonging to the patch S_e. Why do we use the left neighbors and not the right ones? In the derivation of the estimates (S) and (A) for $m = 1$ one needs the property that the quantities

$$\frac{h_{S,1}}{h_{e,1}} \quad \text{and} \quad \frac{h_{S,2}}{h_{e,2}}$$

are uniformly bounded, therefore for small elements also the mesh sizes of the patch have to be small. With our choice now the estimates (A) and (S) on a rectangular Shishkin mesh look as follows:

$$(S_m) \qquad \|Q_h u\|_{W^{m,q}(e)} \le C \frac{|e|^{1/q}}{\min_k |e_k|^{1/p}} \sum_{|\alpha|\le l-m} h_S^\alpha |D^\alpha u|_{W^{m,p}(S_e)}$$

and

$$(A_m) \quad \|u - Q_h u\|_{W^{m,q}(e)} \le C \frac{|e|^{1/q}}{\min_k |e_k|^{1/p}} \sum_{|\alpha|= l-m} h_S^\alpha |D^\alpha u|_{W^{m,p}(S_e)}.$$

We need $m = 0,1$; assume $l = 1,2$, then $p,q \in [1,\infty]$ have to satisfy $W^{l,p}(e) \hookrightarrow W^{m,q}(e)$.

Based on the approximation error estimate (A_m) we now try to get estimates for the interpolation error on Shishkin meshes using Scott-Zhang that are similar to those obtained previously for the Lagrange interpolant. For the Lagrange interpolant we got for the smooth part $S \in H^2(\Omega)$ without any problems

$$\|S - S^I\|_0 \le C\,N^{-2}, \quad |S - S^I|_1 \le C\,N^{-1}.$$

However, with our Scott-Zhang interpolant $Q_N S$ we have difficulties for $p = q = 2$ in the strip $[\lambda_x, \lambda_x + H]$ because the small neighbors of large elements generate the factor $1/(\varepsilon \ln N)^{1/2}$.
While we can live with the estimate

$$\varepsilon^{1/2} |S - Q_N S|_1 \le C\,N^{-1},$$

for handling the convection term we see no alternative using

$$\|S - Q_N S\|_0 \le C\,N^{-2} \quad \text{or} \quad \|S - Q_N S\|_\infty \le C\,N^{-2}.$$

But if we assume more smoothness, i.e., $S \in W^{2,\infty}(\Omega)$, we get the desired L_∞ estimate and the L_2 estimate follows.

For each layer term, for instance E_1, we want to use its smallness for $x \ge \lambda_x$. If $E_1 \in W^{1,1} \cap L_\infty$, we can use

$$\|Q_N E_1\|_\infty \le C\,\|E_1\|_\infty. \tag{18}$$

The stability estimate (18) follows from the representation

$$(Q_N w)(P_i) = \int_{\sigma_i} w\,\psi_i$$

for the values of our Scott-Zhang interpolant at some vertex P_i. Here for the function ψ_i we have $\psi_i \in V^N|_{\sigma_i}$ and with the nodal basis φ_j in the finite element space one has

$$\int_{\sigma_i} \psi_i \varphi_j = \delta_{ij}.$$

In the region $x \leq \lambda_x$ we can use (A_m) to estimate the interpolation error of E_1 in the same way as for the Lagrange interpolant.

Summarizing, we observed the following:

- The property that the Shishkin mesh is not locally uniform leads to difficulties proving for the smooth part of the solution

$$\|S - Q_N S\|_0 \leq C\, N^{-2} \quad \text{for} \quad S \in H^2(\Omega).$$

  Alternatively, one could assume more smoothness of S or introduce a "smoothing" region to introduce a modified mesh which is locally uniform; then the number of mesh points used would depend slightly on ε.

- The smallness of the Scott-Zhang interpolant of layer functions in regions where the layer terms are small can be shown; together with the anisotropic interpolation error estimates the estimate of the influence of the layer terms causes less trouble than the smooth part of the solution.

It is an interesting problem to study in the future whether or not the finite element approximation of the solution of problems with singularities, for instance, hidden in the corner layer and leading to non-smooth solutions, can be analyzed based on Scott-Zhang interpolants.

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