Geometric finite elements for director shell models

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joint work with Lisa Nebel

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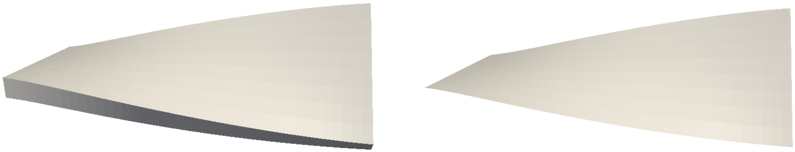
TECHNISCHE UNIVERSITÄT DRESDEN

Nonlinear shell models

Dimensional reduction

- Object is virtually 2-dimensional $\Rightarrow$ model it by 2d equation



Director theories:

- 1 director: Shearing
- 3 directors: Shearing and torsion

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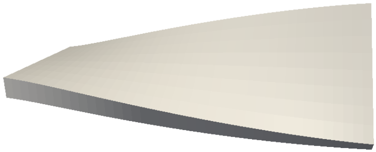

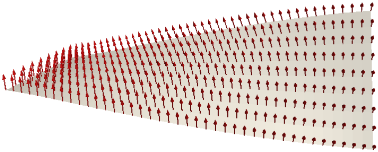

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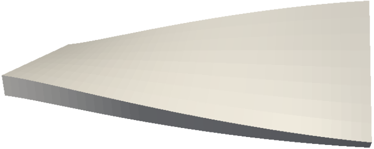

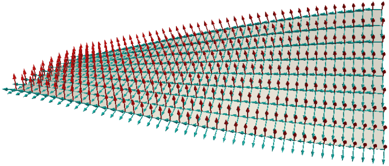

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Manifold-valued boundary value problems

Partial differential equations for functions

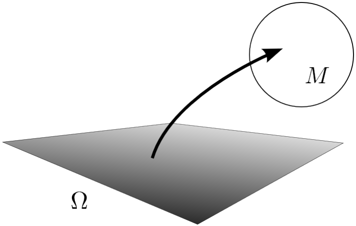
$$\phi : \Omega \to M$$

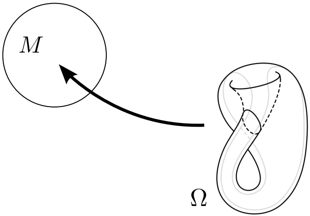
Domain Ω:

- … can be Euclidean or non-Euclidean.

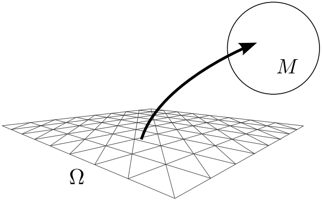
Codomain M:

- Riemannian manifold





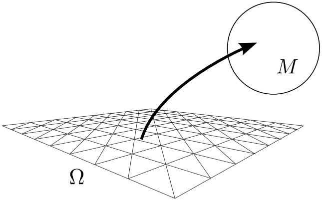
Finite elements for manifold-Valued problems



The challenges:

- Nonlinear function spaces
- No norms, no scalar products, no linear functional analysis
- No polynomials
- No finite elements

Finite elements for manifold-Valued problems



Find a discretization that:

- is geometrically conforming
- is H^1-conforming
- works for any reasonable Riemannian manifold M
- allows for high approximation orders
- preserves frame indifference

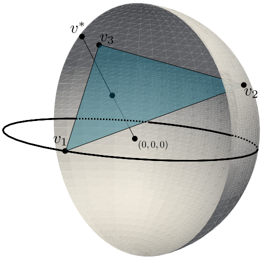
Main problem: Generalize polynomial interpolation

Construction 1: Projection-based FE

Work with an embedding:

- Interpolate in $\mathbb{R}^N$
- Project back onto M

$$I^{\text{proj}}(v, \xi) := P_M\Big(\sum_{i=1}^m v_i \varphi_i(\xi)\Big)$$



Properties:

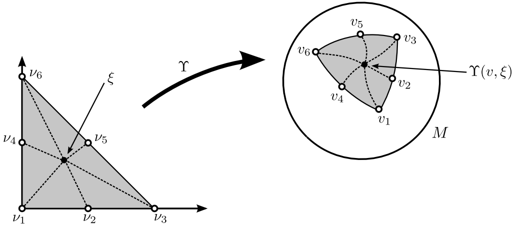
- Well-defined and differentiable
- Nested spaces
- Affine family of finite elements
- Equivariant under isometries Q of M

$$QI^{\text{proj}}(v, \xi) = I^{\text{proj}}(Qv, \xi)$$

only if Q can be extended to an isometry of $\mathbb{R}^N$.

Construction 2: Geodesic interpolation

Intrinsic definition?



Lagrange interpolation revisited:

- Values $v_1, \dots, v_m \in M$ given at the Lagrange nodes
- If M is a vector space, interpolation between the v_i can be written as

$$\sum_{i=1}^{m} v_i \varphi_i(\xi) = \arg\min_{q \in M} \sum_{i=1}^{m} \varphi_i(\xi) \|v_i - q\|^2.$$

Construction 2: Geodesic interpolation

Idea: Replace

$$\|v_i - q\|^2$$

by the Riemannian distance on M

$$\operatorname{dist}(v_i, q)^2$$

Define: Geodesic interpolation [S. (2011, 2013), Grohs (2012)]

$$I^{\text{geo}}(v, \xi) := \arg\min_{q \in M} \sum_{i=1}^{m} \varphi_i(\xi) \operatorname{dist}(v_i, q)^2$$

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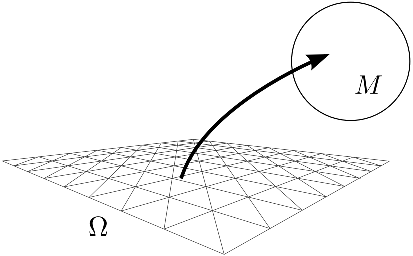
Properties:

- Well-posed under reasonable conditions
- Differentiable in v and ξ
- Equivariant under isometries of M
- $\implies$ Discretizations of frame-indifferent models are frame-indifferent.

Global finite element spaces



Definition (Geometric finite elements)

A geometric finite element function is a continuous function $v_h : G \to M$ such that for each element T of G, $v_h|_T$ is given by a geometric interpolation rule on T.

Discretization error measurements

Minimize harmonic energy:

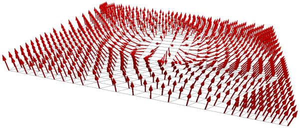
$$\phi : \Omega \to S^2, \qquad E(\phi) = \int_\Omega \|\nabla \phi\|^2 \, dx$$

Lemma

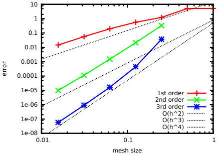
The inverse stereographic map minimizes E in its homotopy class.

Setup

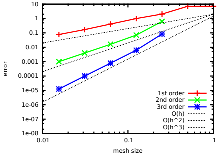
- Domain $\Omega = [-5, 5]^2$
- Dirichlet boundary conditions
- Discretization error for orders $p = 1, 2, 3$:



solution field



L^2-error over mesh size



H^1-error over mesh size

Evaluation of geodesic finite elements

Definition:

$$I^{\text{geo}}(v;\xi) = \operatorname*{arg\,min}_{q \in M} \sum_{i=1}^{m} \varphi_i(\xi)\,\operatorname{dist}(v_i, q)^2$$

Values:

Minimize

$$f_\xi(q) := \sum_{i=1}^{m} \varphi_i(\xi)\,\operatorname{dist}(v_i, q)^2$$

by a Newton-type method on M

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Derivatives wrt. ξ:

Total derivative of $F(\xi, q) := \frac{\partial f_\xi}{\partial q} = 0$ yields

$$\frac{\partial F(\xi, q)}{\partial q} \cdot \frac{\partial I^{\text{geo}}}{\partial \xi} = -\frac{\partial F(\xi, q)}{\partial \xi}$$

1. Evaluate $q := I^{\text{geo}}(v;\xi)$
2. Solve a small linear system

Higher derivatives

Newton-type PDE solvers require:

- First and second derivatives of $I^{\text{geo}}(v_1, \dots, v_m; \xi)$ wrt. to the v_i
- First and second derivatives of $\frac{\partial}{\partial \xi} I^{\text{geo}}(v_1, \dots, v_m; \xi)$ wrt. to the v_i

Total derivative again:

$$\frac{\partial F}{\partial q} \cdot \frac{\partial^2 I^{\text{geo}}}{\partial v_i \, \partial \xi} = - \frac{\partial^2 F}{\partial v \, \partial q} \cdot \frac{\partial q}{\partial \xi} - \frac{\partial^2 F}{\partial q^2} \cdot \frac{\partial q}{\partial v} \cdot \frac{\partial q}{\partial \xi} - \frac{\partial^2 F}{\partial v_i \, \partial \xi} - \frac{\partial^2 F}{\partial q \, \partial \xi} \cdot \frac{\partial q}{\partial v_i}.$$

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Algorithmic differentiation:

- More convenient
- Less error-prone
- I use ADOL-C [Griewank, Walther]

Projection-based FE: Algorithmic issues

The sphere S^d

- Interpolation: $I^{\text{proj}}(v, \xi) = \dfrac{\sum_{i=1}^{m} \varphi_i(\xi) v_i}{\|\sum_{i=1}^{m} \varphi_i(\xi) v_i\|}$
- Gradient $\nabla I^{\text{proj}} := \frac{\partial I^{\text{proj}}}{\partial \xi}$: Available in closed form.

Projection-based FE: Algorithmic issues

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The orthogonal matrices $\mathbf{SO}(3)$

- Projection onto O(3) is the polar decomposition
- Iterative construction: [Higham; Gawlik, Leok (2018)]

$$Q_0 := A \qquad Q_{k+1} := \frac{1}{2}(Q_k + Q_k^{-T})$$

- Similar construction for ∇I^{proj}

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The symmetric positive definite matrices:

- No projection!

Rigorous discretization error bounds

H^1 errors:

- Ellipticity + Céa's lemma + the four conditions
- [technical stuff]

Theorem ([Hardering, S (2018)])

Optimal errors!

Rigorous discretization error bounds

H^1 errors:

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Theorem ([Hardering, S (2018)])

Optimal errors!

L^2 errors:

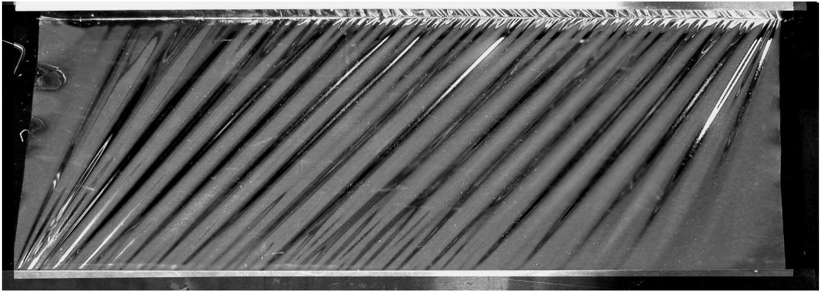
- Redo Aubin–Nitsche trick
- For "predominantly quadratic" energies only
- Generalize Galerkin orthogonality

Theorem ([Hardering, S (2018)])

Optimal errors!

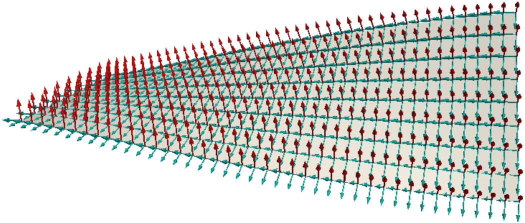
Wrinkling of polyimide sheets

Wong, Pellegrino 2006:



- Shearing of a rectangular plastic sheet
- 380 mm x 128 mm x 25 μm
- $E = 71240\,\mathrm{N/mm^2}$, $\nu = 0.31$
- Prescribed displacement at horizontal edges
- 3 mm shear

Geometrically nonlinear Cosserat plates



Kinematics:

- $\Omega \subset \mathbb{R}^2$
- Midsurface deformation: $m : \Omega \to \mathbb{R}^3$
- Microrotation field: $R : \Omega \to SO(3)$

Strain measures:

- Deformation gradient: $F := (\nabla m | R_3) \in \mathbb{M}^{3\times 3}$
- Translational strain: $U := R^T F$
- Rotational strain: $\mathfrak{K} := R^T \nabla R$

Geometrically nonlinear Cosserat plates

Hyperelastic material law: [Neff] ($h =$ shell thickness)

$$J(m, R) = \int_\Omega \Big[hW_{\text{memb}}(U) + \frac{h^3}{12} W_{\text{bend}}(\mathfrak{K}) + hW_{\text{curv}}(\mathfrak{K}) \Big] \, dx$$

Membrane energy:

$$W_{\text{memb}}(U) = \mu \|\operatorname{sym}(U - I)\|^2 + \mu_c \|\operatorname{skew}(U - I)\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \frac{1}{2} \Big((\det U - 1)^2 + \big(\frac{1}{\det U} - 1\big)^2 \Big)$$

Bending energy:

$$W_{\text{bend}}(\mathfrak{K}_{\mathfrak{b}}) = \mu \|\operatorname{sym}(\mathfrak{K}_b)\|^2 + \mu_c \|\operatorname{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr}[\operatorname{sym}(\mathfrak{K}_b)]^2$$

Curvature energy:

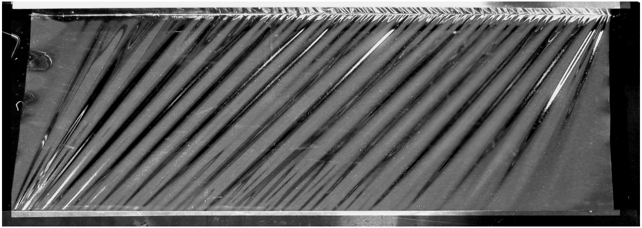
$$W_{\text{curv}}(\mathfrak{K}) = \mu L_c^{1+p} \|\mathfrak{K}\|^{1+p}$$

Theorem ([Neff et al.])

Under suitable conditions, the functional J has minimizers in $H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p}(\Omega, SO(3))$.

Wrinkling of plastic sheets

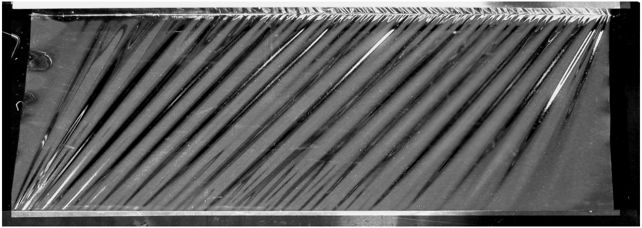
Experiment:



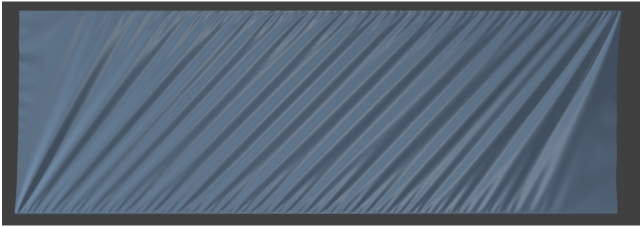
Wrinkling of plastic sheets

Distributed and Unified Numerics Environment

Experiment:

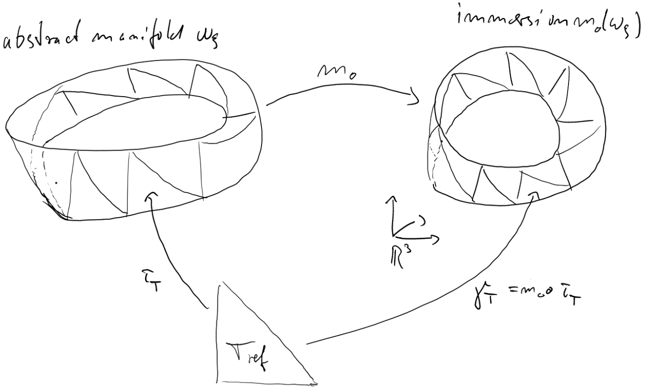


Simulation:



Nonplanar reference configurations

With P. Neff, M. Bîrsan



- Stress-free configuration has curvature
- Nontrivial topologies

Model

Kinematics:

- Abstract two-dimensional manifold ω
- Midsurface deformation: $m : \omega \to \mathbb{R}^3$
- Microrotation field: $R : \omega \to \mathrm{SO}(3)$

Hyperelastic energy:

- Similar to planar case, but with h^5-terms
- Involves curvature of stress-free embedding

Theorem (Ghiba, Bîrsan, Lewintan, Neff (2020))

The shell energy has a minimizer in $H^1(\omega, \mathbb{R}^3 \times SO(3))$.

Proof: Direct method in the calculus of variations.

Existence of discrete solutions

Theorem (Nebel, S. (2022))

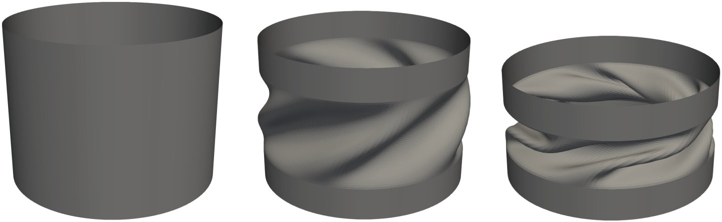
For any grid $\mathcal{G}$ of ω and polynomial order k, the shell energy has a minimizer in the GFE spaces $V_{h,p}^{proj}$ and $V_{h,p}^{geo}$.

Proof:

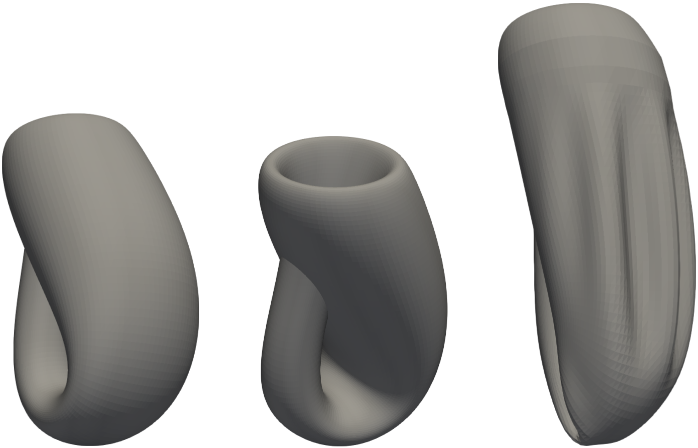
- GFE spaces are conforming.
- Coercivity and convexity properties of shell energy can be used.
- GFE-Function spaces are weakly H^1-complete.
- Direct method in the calculus of variations.

Buckling of a cylinder

from Hunt, Ario (2004)



Nonorientable surfaces



Literature

Geometric finite elements:

- H. Hardering and O. Sander: *Geometric Finite Elements*, in: *Handbook of Variational Methods for Nonlinear Geometric Data*, Springer (2020)

Application to shells and rods:

- O. Sander, P. Neff and M. Bîrsan: *Numerical Treatment of a Geometrically Nonlinear Planar Cosserat Shell Model*, Comp. Mech. (2016), **57(5)**, 817–841
- O. Sander: *Geodesic Finite Elements for Cosserat Rods*, Int. J. Num. Meth. Eng. (2010), **82(13)**, 1645–1670
- A. Müller and M. Bischoff: *A Consistent Finite Element Formulation of the Geometrically Non-linear Reissner–Mindlin Shell Model*, Arch. Computat. Methods Eng. (2022), DOI: 10.1007/s11831-021-09702-7