Cyclic and duplicial objects in computer science: summary

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The starting point of the present PhD thesis is the article 'Cyclic homology arising from adjunctions' by Krähmer, Kowalzig and Slevin (in the following abbreviated by [KKS15]). The aim of this work is to apply the ideas presented in this article to monads and comonads from computer science. This summary gives an overview of the content and results contained in the four main chapters.

Chapter 2: Preliminaries and background

This chapter repeats definitions and known results used in this thesis. The most important notion is the term bi(co)monad:

Definition. A *monad* $\mathbb{T}$ on a category $\mathcal{C}$ consists of a functor $T\colon \mathcal{C} \to \mathcal{C}$ together with a unit and multiplication, given by natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \to T$ and $\mu\colon T^2 \to T$ such that $\mu \circ T\mu = \mu \circ \mu T\colon T^3 \to T$ and $\mu \circ \eta T = \mu \circ T\eta = \mathrm{Id}\colon T \to T$. A *comonad* on a category $\mathcal{C}$ is a monad on the category $\mathcal{C}^{\mathrm{op}}$.

If on a category $\mathcal{C}$ a product $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbb{1}$ can be defined for which besides some coherence conditions also for all objects X, YZ in $\mathcal{C}$ we have $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ and $X \otimes \mathbb{1} \cong \mathbb{1} \cong X$, then we call $(\mathcal{C}, \otimes, \mathbb{1})$ a *monoidal category*. For a monoidal category $\mathcal{C}$ we can ask if an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ can be equipped with a monoidal or comonoidal structure. We call a comonoidal monad *bimonad* and a monoidal comonad *bicomonad*.

The concept of monads is used in many areas. In this thesis we refer to homology theory and computer science. There are also applications in probability theory.

A distributive law of a monad $\mathbb{T}_1$ over a monad $\mathbb{T}_2$ is a natural transformation $\theta\colon \mathbb{T}_1\mathbb{T}_2 \to \mathbb{T}_2\mathbb{T}_1$, subject to several commutativity conditions. Monads and comonads can interact with each other via distributive laws, e. g. they can be composed in presence of such a natural transformation.

Chapter 3: (Co)-monad transformers from mixed distributive laws

An important starting point for the considerations of [KKS15] is Theorem 2.9:

Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be categories such that $\mathbb{S}$ is a comonad on $\mathcal{A}$ and $\mathbb{T}$ is a monad on $\mathcal{A}$ with adjunction $L^T \dashv R^T$ from $\mathcal{A}$ to $\mathcal{C}$.*

1. *Every lift of the comonad $\mathbb{S}$ through the adjunction $L^T \dashv R^T$ induces a mixed distributive law of the monad $\mathbb{T}$ over the comonad $\mathbb{S}$ in $\mathcal{A}$ and a distributive law of comonads in $\mathcal{B}$.*

In this chapter we were able to extend this result with a second part:

Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be categories such that $\mathbb{S}$ is a comonad on $\mathcal{A}$ with adjunction $L^S \dashv R^S$ from $\mathcal{A}$ to $\mathcal{B}$ and $\mathbb{T}$ is a monad on $\mathcal{A}$.*

2. *Every lift of the monad $\mathbb{T}$ through the adjunction $L^S \dashv R^S$ induces a mixed distributive law of the monad $\mathbb{T}$ over the comonad $\mathbb{S}$ in $\mathcal{A}$ and a distributive law of monads in $\mathcal{C}$.*

We further used both parts of this theorem to investigate the relationship between monad and comonad transformers and distributive laws. Both are concepts to compose either two monads or two comonads.

The initial situation of a monad or comonad transformer is the following. Consider a monad $\mathbb{T}$ on a category $\mathcal{A}$ and $L \dashv R$ an adjunction, where $L\colon \mathcal{B} \to \mathcal{A}$, $R\colon \mathcal{B} \to \mathcal{A}$, i. e. there is comonad structure for LR on $\mathcal{A}$ and a monad structure RL on $\mathcal{B}$. Consequently there is a monad structure on RTL, known to be the monad $\mathbb{T}$ transformed by the adjunction $L \dashv R$.

We have shown that if there exists a lift of the monad $\mathbb{T}$ to a monad $\tilde{\mathbb{T}}$ on the category $\mathcal{B}$, then there is a decomposition of the monad for RTL which involves a distributive law of monads on $\mathcal{B}$, lifted from a mixed distributive law on a category $\mathcal{A}$. The dual situation holds in case of a lift of a comonad through an adjunction for a monad.

Chapter 4: Function monads and their algebras in semicartesian categories

Szlachányi suggested to consider function monads as a general example of a bimonad. To understand function monads we need to briefly discuss the concept of a monoidal closed category. A monoidal category $(\mathcal{C}, \otimes)$ is called closed if for each object X in $\mathcal{C}$ the functor $- \otimes X : \mathcal{C} \to \mathcal{C}$ has a right adjoint termed internal hom, written $\hom_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathcal{C}$. A function monad is a monad $\mathbb{T} = (T, \mu, \eta)$ on a monoidal closed category $\mathcal{C}$ that is internally represented, i. e. $T = \hom_{\mathcal{C}}(N, -)$ for a comonoid object $(N, \Delta_N, 1_N)$ together with the monad structure obtained from Δ_N and 1_N. To the best of our knowledge, except for the example of rectangular bands, the algebras of this monad are still not known today. In this work we have proved the following generalisation of Kimuras' classification of rectangular bands:

Theorem. *Let $\mathbb{T} = (T = hom_{\mathcal{C}}(N, -), \mu, \eta)$ denote a function monad in a semicartesian closed category $\mathcal{C}$ with products and (X, ξ) a pointed object in $\mathcal{C}$ such that N is semi-simple for an algebra $(X, \alpha) \in \operatorname{Ob} \mathcal{C}^{\mathbb{T}}$. Then the $\mathbb{T}$-algebra (X, α) is isomorphic to the product of projection algebras $\prod_{\nu \in C(N)} (X_\nu, p_\nu)$. This construction is unique up to isomorphism in the way that it is independent of the choice of $\xi \in Hom_{\mathcal{C}}(\mathbb{1}, X)$.*

Note that the condition we call semi-simple applies to every object in the category (Set), i. e. to every set. Thus we have proven that the function monad algebras in (Set) can be decomposed into a product of projection algebras.

Chapter 5: Applications of distributive laws for Haskell monads and comonads

In the already mentioned article from Kowalzig, Krähmer and Slevin so-called χ-coalgebras are introduced to serve as coefficient in homology theory. We distinguish further between left and right χ-coalgebras. For a better understanding of these coefficients it is useful to give as many examples as possible from different areas. In particular, examples which do not arise from bialgebras are of interest.

With the non-empty list bimonad Kowalzig, Krähmer and Slevin already consider such an example. The list bimonad is given by a monad and a comonad structure on the list functor, where both structures are connected via a mixed distributive law of the list monad over the list comonad. In this chapter a calculation error made by Kowalzig, Krähmer and Slevin is corrected. Further this example is supplemented by adding left and right χ-coalgebra structures such that we can compute a duplicial operator.

As a second example we consider the double negation bimonad in the category (Set). This monad is a special continuation monad known to functional programmers and also plays a role in categorical logics. It turns out that its χ-coalgebras and the duplicial operator are trivial.

The third example uses a slightly different approach. We combine the Maybe monad with the product comonad using a mixed distributive law. Note that both are Hopf structures but for different functors. We where able to calculate many different left as well as right χ-coalgebras for this example.