Exponential functionals of Markov additive processes and a Markov modulated extension of the generalized Ornstein-Uhlenbeck process

(Kurzzusammenfassung der Dissertation)

Apostolos Sideris,
supervisor: Prof. Dr. Anita Behme,
Technische Universität Dresden,
Institut für Mathematische Stochastik,
apostolos.sideris@tu-dresden.de

16.12.2021

In the last century the *classical Ornstein-Uhlenbeck process* has been proved to be a very useful and powerful tool for modelling questions from physics, insurance and finance or biology. Its first description in 1930 by its name patrons Leonard Ornstein and George Uhlenbeck for the modelling of the velocity of a free particle in a fluid taking the friction into account, was long before stochastic integration or stochastic differential equations were introduced. Since then this stochastic process has made an impressive career in theory and application, which also inspired each other and is nowadays a basis for more complex models.
Meanwhile, in the second half of the 20th century *Markov modulated models*, roughly spoken, models with a random change of its parameters, became also prominent in applications due to their flexibility and analytical tractability.

The goal of this thesis is to combine these elements to define a *Markov modulated generalized Ornstein-Uhlenbeck process* and to study its properties!

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete stochastic basis and S an at most countable set. A d+1-dimensional continuous time Markov process $(X_t, J_t)_{t\geq 0}$ on $\mathbb{R}^d \times S$ is called a *(d-dimensional) Markov additive process w.r.t. to* $\mathbb{F}$ *($\mathbb{F}$-MAP)*, if for all $s, t \geq 0$ given $J_t = i$, the increments $(X_{t+s} - X_t, J_t)$ is independent of $\mathcal{F}_s$ and equal in distribution to (X_s, J_0) given $J_0 = i$. The marginal process J is a Markov process with state space S and is therefore called *Markovian component* and models the randomly changing environment.
The process X is typically called the *additive component*, and in practice this is the process of interest. Note that X is not Markovian in general.
Roughly spoken, the additive component has two types of behaviour. As long J remains in some state, it behaves in law like a Lévy process, whose characteristic triplet depends on the current state of J. Whenever J switches its state, that triggers an additional jump for X, whose jump size distribution depends on the source and target state of J, but is independent of any other object.
While we summarize our main results of the thesis in chapter 1, in chapter 2 we will investigate

some new ideas and results, concerning fluctuation and moments of MAPs. We do also study stochastic integrals w.r.t. MAPs and present sufficient conditions for the existence of their moments, as well as concrete formulae for the mean of such stochastic integrals.
In chapter 3 we analyse the exponential integral of Markov additive processes, i.e. we consider for a bivariate MAP $((\xi,\eta),J)$ the stochastic process $\mathfrak{E}_{(\xi,\eta)}$ given by

$$\mathfrak{E}_{(\xi,\eta)}(t) := \int_{(0,t]} \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s, \qquad t \geq 0$$

and present necessary and sufficient conditions for almost sure convergence and convergence in probability. In particular these conditions consist of a fluctuation condition for ξ and an integrability condition. The failure of one of these two conditions will imply either divergence in probability or that the exponential integral is degenerated, in the sense that the process $\mathfrak{E}_{(\xi,\eta)}$ can be represented independently of η. As the integrability condition is hard to check, we give also sufficient conditions, where we strengthen the fluctuation condition but can relax the integrability condition.
In chapter 4 we derive a Markov modulated version of the generalized Ornstein-Uhlenbeck process as the unique continuous time process V satisfying for all times $s, t \geq 0$ a certain Markov modulated random recurrence equation. We show that this process satisfies a stochastic differential equation of the form $\mathrm{d}V_t = V_{t-}\mathrm{d}U_t + \mathrm{d}L_t$ for a bivariate MAP $((U,L),J)$. Moreover we show that there exists another additive component (ξ,η) such that we can represent V as follows

$$V_t = \mathrm{e}^{-\xi_t}\left(V_0 + \int_{(0,t]} \mathrm{e}^{\xi_{s-}} \mathrm{d}\eta_s\right), \quad t \geq 0$$

with some initial random variable V_0 conditional independent of (ξ,η) given J. Due to its familiar form we call the process V *Markov modulated generalized Ornstein-Uhlenbeck (MMGOU) process.*
Further we give answer to the natural question, under which circumstances this process is strictly stationary and show that this is the case, when the exponential integral of $((-\xi^*, L^*), J^*)$ converges in probability, where the process $((-\xi^*, L^*), J^*)$ is the dual (or time-reversed) process of $((-\xi, L), J)$. However the convergence criteria are still given in terms of (ξ, L). Moreover if V is strictly stationary, then its distribution is given by the limit in probability of $\mathfrak{E}_{(-\xi^*,L^*)}(t)$ for $t \to \infty$.
In chapter 5 we restrict to the case, where the state space of the Markovian component J is finite and study second order properties of the MMGOU process. In particular we give sufficient conditions for the existence of its moments and give also closed representations of the first two moments for the stationary and the non-stationary case as well. Further we give the autocovariance structure, which is of exponential decay or increase, according to the asymptotic behaviour of U.
In chapter 6 we illustrate applications of the MMGOU process. E.g. We consider a Markov modulated Cramér-Lundberg model with stochastic reinvestment, i.e. we have a spectrally negative MAP, that models the capital of an insurance company and a reinvestment of this capital in some market modelled by some certain stochastic differential equation. Combining these two elements leads to MMGOU process, that models the capital influenced by inflation. Given some positive initial capital, we provide a formula for the ruin probability, i.e. the probability that the resulting MMGOU process becomes non-positive in terms of the distribution of some occurring exponential integral.

References

1. A. Behme, A. Sideris, Exponential functionals of Markov additive processes, *Electronic Journal of Probability*, 25:1-25, 2020

2. A. Behme, A. Sideris, The Markov modulated generalized Ornstein-Uhlenbeck process and an application in risk theory, *Bernoulli* (accepted), 2021+

Further References

1. K. B. Erickson, R. Maller, Generalised Ornstein-Uhlenbeck Processes and the convergence of Lévy Integrals. In M. Émery, M. Ledoux, and M. Yor, editors, *Séminaire des Probabilités XXXVIII*, pages 70-94, Springer, 2005

2. A. Lindner, R. Maller, Lévy Integrals and the Stationarity of Generalised Ornstein-Uhlenbeck Processes, *Stochastic Processes and their Applications*, 115:1701-1722,2005