Boundary value problems for the Willmore and the Helfrich functional for surfaces of revolution

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This talk gives an overview of joint works with A. Dall'Acqua, K. Deckelnick, M. Doemeland, S. Eichmann, and S. Okabe.

A special form of the Helfrich energy for a sufficiently smooth (two dimensional) surface $S \subset \mathbb{R}^3$ (with or without boundary) is defined by

$$\mathscr{H}_\varepsilon(S) := \int_S H^2\, dS + \varepsilon \int_S dS,$$

where $H = (\kappa_1 + \kappa_2)/2$ denotes the mean curvature of S. The first integral may be considered as a bending energy and the second as surface (stretching) energy. $\mathscr{W}(S) := \mathscr{H}_0(S)$ is called the Willmore functional. We consider surfaces of revolution S

$$(x, \varphi) \mapsto \big(x, u(x)\cos\varphi, u(x)\sin\varphi\big), \quad x \in [-1, 1],\ \varphi \in [0, 2\pi],$$

with smooth strictly positive profile curve u subject to Dirichlet boundary conditions

$$u(-1) = \alpha, \quad u(1) = \beta, \quad u'(\pm 1) = 0$$

and aim at minimising $\mathscr{H}_\varepsilon$. Thanks to these boundary conditions the Gauss curvature integral $\int_S K\, dS$ becomes a constant and needs not to be considered.

In the first part of the lecture I shall consider the Willmore case, i.e. $\varepsilon = 0$. After discussing minimisation in the symmetric case $\alpha = \beta$ I shall indicate how much more complicated the problem becomes for $\alpha \neq \beta$. Only when α and β do not differ too much, the profile curve will remain a graph while in general it will become a nonprojectable curve.

In the second part, $\mathscr{H}_\varepsilon$ is considered for $\varepsilon \in [0, \infty)$, but again in the symmetric setting $\alpha = \beta$. We study "relatively large" boundary data $\alpha \geq \alpha_m = c_m \cosh(\frac{1}{c_m}) \approx 1.895$ with $c_m \approx 1.564$ the unique solution of the equation $\frac{2}{c} = 1 + e^{-2/c}$, when one has a catenoid v_α which globally minimises the surface energy. In this case we find minimisers u_ε for any $\varepsilon \geq 0$ and show uniform and locally smooth convergence $u_\varepsilon \to v_\alpha$ under the singular limit $\varepsilon \to \infty$.

At the end I shall present some recent work on obstacle problems.