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Euler schemes for accretive operators on Banach spaces – Kurzfassung

We look at the Cauchy problem

$$\begin{cases} \dot{u}(t) + Au(t) \ni f(t) \qquad (t \in [0,T]), \\ u(0) = u^0, \end{cases} \tag{CP: f, u^0}$$

where $A \subseteq X \times X$ is an operator on a real Banach space X, $T > 0$, $f \in L^1(0,T;X)$ and $u^0 \in X$. We assume that the operator A is m-accretive of type $\omega \in \mathbb{R}$. We call uniform limits of solutions of implicit Euler schemes corresponding to the Cauchy problem, Euler solutions.

A discretization θ is any triple of the form $(\pi_\theta, f_\theta, u_\theta^0)$, where $\pi_\theta \colon 0 = t_0 < t_1 < \ldots < t_N = T$ is a partition of the interval $[0,T]$ with mesh size $|\pi_\theta| := \max_i |t_{i+1} - t_i|$, $f_\theta \colon [0,T] \to X$ is constant on the intervals $[t_i, t_{i+1})$ and $u_\theta^0 \in X$. For a given operator $A \subseteq X \times X$ and a discretization θ we consider the implicit Euler scheme

$$\begin{cases} \dfrac{u_\theta(t_{i+1}) - u_\theta(t_i)}{t_{i+1} - t_i} + Au_\theta(t_{i+1}) \ni f_\theta(t_i) \text{ for } i \in \{0, \ldots, N-1\}, \\ u_\theta(0) = u_\theta^0. \end{cases} \tag{E_θ}$$

We say that $u_\theta \in C([0,T];X)$ is a solution of the implicit Euler scheme (E_θ), if (E_θ) holds and u_θ is affine on the intervals $[t_i, t_{i+1}]$. Our main result Theorem 1 gives an upper bound for the difference of two solutions of implicit Euler schemes.

Theorem 1. *Let θ and $\hat{\theta}$ be discretizations. Let u_θ be a solution of (E_θ) and let $u_{\hat{\theta}}$ be a solution of $(E_{\hat{\theta}})$. If $(|\pi_\theta| \vee |\pi_{\hat{\theta}}|)\omega \le \frac{1}{2}$, then for all $(u,v) \in A$, $g \in BV(0,T;X)$ and all $t, \hat{t} \in [0,T]$*

$$\begin{aligned} \|u_\theta(t) - u_{\hat{\theta}}(\hat{t})\|_X \le e^{4T\omega^+} \Big(& \|u_\theta^0 - u\|_X + \|u_{\hat{\theta}}^0 - u\|_X + \|f_\theta - g\|_{L^1(0,T;X)} + \|f_{\hat{\theta}} - g\|_{L^1(0,T;X)} \\ & + \sqrt{(|t - \hat{t}| + |\pi_\theta| + |\pi_{\hat{\theta}}|)^2 + |\pi_\theta| t + |\pi_{\hat{\theta}}| \hat{t}} \cdot (\mathrm{essVar}(g) + \|g(0+) - v\|_X) \Big). \end{aligned}$$

Theorem 1 can be used to establish the following results.

Theorem 2 (Wellposedness of the Cauchy problem, Crandall-Evans). *For every $f \in L^1(0,T;X)$ and $u^0 \in \overline{\mathrm{dom}\, A}$ there exists a unique Euler solution of* (CP: f, u^0).

Theorem 3. *If $f \in BV(0,T;X)$ and $u^0 \in \mathrm{dom}\, A$, then the Euler solution u of* (CP: f, u^0) *is Lipschitz continuous and there exists a sequence of discretizations (θ_n) with $|\pi_{\theta_n}| \to 0$ as $n \to \infty$ and a sequence (u_n) of solutions of (E_{θ_n}) with $\|u_n - u\|_X = O(|\pi_{\theta_n}|^{1/2})$.*

Theorem 4 (Interpolation). *Let $0 < \alpha < 1$ and $1 \le p \le \infty$. Let $f \in (L^1(0,T;X), BV(0,T;X))_{\alpha,p}$, $u^0 \in (X, \mathrm{dom}\, A)_{\alpha,p}$ and u be the Euler solution of* (CP: f, u^0).

Then $u \in (C([0,T];X), Lip([0,T];X))_{\alpha,p}$ and there exists a sequence of discretizations (θ_n) with $|\pi_{\theta_n}| \to 0$ as $n \to \infty$ and a sequence (u_n) of solutions of (E_{θ_n}) with $\|u_n - u\|_X = O(|\pi_{\theta_n}|^{\alpha/2})$.

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