Heat kernel estimates for the fractional Laplacian and related non-local operators

The fractional Laplacian $\Delta^{\alpha/2}$ in a d-dimensional Euclidean space is a non-local integral operator with the singular kernel of the form $|x-y|^{-d-\alpha}$, $0<\alpha<2$. As in the case of the standard Laplacian Δ, the fundamental solution of the corresponding heat equation is called the *heat kernel*. The heat kernel has a probabilistic interpretation: it represents the transition densities $p(t,x,y)$ from point x to point y in time t for the corresponding strong Markov process – the isotropic α-stable process.

Unlike the Laplacian, where the heat kernel is given by an explicit formula (the Gaussian kernel), the closed-form expression for the heat kernel of the fractional Laplacian remains unknown. Therefore, the best one can hope for are sharp two-sided estimates.

I will begin by introducing the basic properties of the fractional Laplacian, including its connection to the heat semigroup, the associated Dirichlet form, and the fundamental heat kernel estimates. A key aspect of these estimates is how they arise naturally from exact scaling properties.

Next, I will discuss various formulations of the fractional Laplacian on open subsets of Euclidean space, such as the *reflected*, *restricted*, and *censored* fractional Laplacians. For each case, I will explain the corresponding Markov process and show the relevant heat kernel estimates. All these operators share the feature that their singular kernels are either equal or comparable to $|x-y|^{-d-\alpha}$ —the so-called *uniformly elliptic case.* The analysis relies on a combination of functional-analytic and probabilistic techniques.

I will conclude with some very recent results for scenarios where the singular kernel may vanish at the boundary. I will also motivate the study of such degenerate cases. Surprisingly, the resulting heat kernel estimates exhibit qualitatively different behavior, highlighting new phenomena in this setting.