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Dresdner Mathematisches Seminar

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A compactness property in general algebra

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Leitung: Prof. Dr. Manuel Bodirsky

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A compactness property in general algebra

General (or "universal") algebra has several strands, some of which have been strongly represented at TU Dresden for years. In this lecture I will explore a strand that grew out of work in the Soviet Union during the 1960s and 70s, and which later fell under the gaze of Ralph McKenzie and his school in the 1980s and 90s.

The strand considers equational classes of algebraic structures such as groups, rings, semigroups, etc. Here an *equational class* (a.k.a. "*variety*") is any class of algebraic structures that can be axiomatized by a set of universally quantified equations (called *identities* or *laws*); think of the associative and commutative laws as two examples of identities, and the class of their models (all commutative semigroups) as an example of an equational class.

Within any equational class, certain "irreducible" models play an important role. These are the models which have no proper representations by "simpler" models, in the following sense: there exists a pair (a, b) in the model so that if h is a homomorphism from the model to any other model and $h(a) \neq h(b)$, then h must be injective. In an equational class of groups, the irreducible models are precisely the groups which are "monolithic," i.e., which have a unique minimal nontrivial normal subgroup which is contained in every other nontrivial normal subgroup.

In 1969, Aleksandr Ol'shanskiĭ proved that every equational class $\mathcal{K}$ of groups has the following property: if $\mathcal{K}$ contains finite irreducible models of arbitrarily large cardinalities, then it contains an infinite irreducible model. From the point of view of mathematical logic, this is a kind of "compactness" property of the class $\mathcal{K}_{irred}$ of irreducible models in $\mathcal{K}$. In the years following, many similar results for other classes of classical algebra were proved.

From the point of view of general algebra, one wants to know how broadly Ol'shanskiĭ's theorem can be generalized. In this lecture I will first sketch a proof of Ol'shanskiĭ's theorem; next I will explain the landscape of the world of all equational classes, to give a sense of the sort of generalization we seek; and finally I will describe a recent result establishing the compactness property for all "finite-signature congruence modular" equational classes, answering a question of Freese and McKenzie from 1981.

I will aim to make this talk accessible to anyone who knows a little bit about finite groups and first-order logic. This is joint work with Keith Kearnes and Ágnes Szendrei.