UPPER FUNCTIONS FOR SAMPLE PATHS OF LÉVY(-TYPE) PROCESSES

FRANZISKA KÜHN

Abstract. We study the small-time asymptotics of sample paths of Lévy processes and Lévy-type processes. Namely, we investigate under which conditions the limit

$$\limsup_{t\to 0} \frac{1}{f(t)}|X_t - X_0|$$

is finite resp. infinite with probability 1. We establish integral criteria in terms of the infinitesimal characteristics and the symbol of the process. Our results apply to a wide class of processes, including solutions to Lévy-driven SDEs and stable-like processes. For the particular case of Lévy processes, we recover and extend earlier results from the literature. Moreover, we present a new maximal inequality for Lévy-type processes, which is of independent interest.

1. Introduction

A mapping $f:[0,1]\to[0,\infty)$ is called an upper function for a stochastic process $(X_t)_{t\geq 0}$ if

$$\limsup_{t\to 0} \frac{1}{f(t)}|X_t - X_0| \leq 1 \quad \text{almost surely,}$$

i.e. a typical sample path $t\mapsto X_t(\omega)$ grows asymptotically at most as fast as $f(t)$. In this article, we are interested in upper functions for Lévy and Lévy-type processes. Our aim is to establish integral criteria in terms of the characteristics and the symbol of the process – see Section 3 for definitions – which characterize whether f is an upper function.

For Lévy processes, the study of upper functions was initiated by Khintchine [20]. He showed that any one-dimensional Lévy process satisfies the following law of iterated logarithm (LIL):

$$-\liminf_{t\to 0} \frac{X_t}{\sqrt{2t\log\log\frac{1}{t}}} = \limsup_{t\to 0} \frac{X_t}{\sqrt{2t\log\log\frac{1}{t}}} = \sigma \quad \text{a.s.,}$$

where $\sigma \geq 0$ is the diffusion coefficient. In consequence, the small-time asymptotics of a Lévy process is governed by the Gaussian part if $\sigma \neq 0$. For this reason our focus is on processes with vanishing diffusion part. Khintchine [20] also showed – under some mild assumptions – that f is an upper function for a Lévy process $(X_t)_{t\geq 0}$ if

$$\int_0^1 \frac{1}{t}\mathbb{P}(|X_t| \geq cf(t))\,dt < \infty$$

for a suitable constant $c>0$. In practice, it is often difficult to check whether the latter integral is finite. There is a more tractable criterion in terms of the Lévy measure ν. Namely, it holds for a wide class of functions f that

$$\limsup_{t\to 0} \frac{1}{f(t)}|X_t| = \begin{Bmatrix} 0 \\ \infty \end{Bmatrix} \text{ a.s.} \iff \int_0^1 \nu(\{y\in\mathbb{R}^d;\,|y|\geq f(t)\})\,dt \begin{Bmatrix} <\infty \\ =\infty \end{Bmatrix}; \tag{1}$$

this characterization is classical for stable Lévy processes, see e.g. [12], and has been extended to general one-dimensional Lévy processes by Wee & Kim [41]. For some processes, (1) breaks down, and it may happen that $\limsup_{t\to 0}\frac{1}{f(t)}|X_t| \in (0,\infty)$ almost surely, see [4, 36, 41] for details. A number of further characterizations for upper functions are collected in Theorem 2.1. We require only mild assumptions on the Lévy process $(X_t)_{t\geq 0}$ and the mapping f; thus generalizing earlier results in the literature. For power functions $f(t)=t^\kappa$, there is a close connection to the Blumenthal–Getoor index, cf. Corollary 2.4.

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The second part of our results is about the small-time asymptotics of Lévy-type processes. Intuitively, a Lévy-type process behaves locally like a Lévy process but the Lévy triplet depends on the current position of the process, see Section 3 below for the precise definition. Important examples include solutions to Lévy-driven stochastic differential equations (SDEs), processes of variable order and random time changes of Lévy processes, just to mention a few. Studying the sample path behaviour of Lévy-type processes is much more delicate than in the Lévy case because the processes are no longer homogeneous in space, see [8, Chapter 5] for a survey on results for the closely related class of Feller processes. Schilling [37] introduced generalized Blumenthal–Getoor index and obtained a criterion for a power function $f(t) = t^{\kappa}$ to be an upper function of a Lévy-type process, see also [8, Theorem 5.16]. A recent paper by Reker [34] studies the small-time asymptotics of solutions to SDEs driven by jump processes. Moreover, there are LIL-type results for Lévy-type processes and other classes of jump processes available in the literature, see [10, 22, 25] and the references therein. Our contribution in this paper is two-fold. Firstly, we establish sufficient conditions in terms of the characteristics and the symbol, which ensure that a given mapping f is an upper function for a Lévy-type process, cf. Theorem 2.6. On the way, we obtain new results on upper functions for Markov processes, cf. Section 5. Secondly, we prove a criterion for a given function f *not* to be an upper function, cf. Theorem 2.10. The key ingredients for the proofs are a new maximal inequality for Lévy-type processes, cf. Section 4, and a conditional Borel–Cantelli lemma for backward filtrations.

2. Main results

This section is divided into two parts: First, we present our results for Lévy processes and then, in the second part, we state the results which apply for the wider class of Lévy-type processes. See Section 3 below for definitions and notation. The following is our first main result.

2.1. **Theorem.** *Let $(X_t)_{t\geq 0}$ be a Lévy process with Lévy triplet $(b,0,\nu)$ and characteristic exponent ψ satisfying the sector condition, i.e. $|\operatorname{Im}\psi(\xi)| \leq C \operatorname{Re}\psi(\xi)$, $\xi \in \mathbb{R}^d$, for some constant $C>0$. Let $f:[0,1]\to(0,\infty)$ be non-decreasing, and assume that one of the following conditions holds.*

(A1) *The Lévy measure ν satisfies*

$$\limsup_{r\to 0} \frac{\int_{|y|\leq r} |y|^2\,\nu(dy)}{r^2 \nu(\{|y|>r\})} < \infty.$$

(A2) *There is a constant $M>0$ such that*

$$\int_{r<f(t)} \frac{1}{f(t)^2}\,dt \leq M\frac{f^{-1}(r)}{r^2}, \qquad r\in(0,1).$$

The following statements are equivalent.

(L1) $\int_0^1 \nu(\{|y|\geq cf(t)\})\,dt<\infty$ *for some $c>0$,*
(L2) $\int_0^1 \sup_{|\xi|\leq 1/(\varepsilon f(t))} |\psi(\xi)|\,dt<\infty$ *for some (all) $\varepsilon>0$,*
(L3) $\int_0^1 \mathbb{P}\left(\sup_{s\leq t}|X_s|\geq \varepsilon f(t)\right)\frac{1}{t}\,dt<\infty$ *for all $\varepsilon>0$,*
(L4) $\int_0^1 \mathbb{P}(|X_t|\geq \varepsilon f(t))\frac{1}{t}\,dt<\infty$ *for all $\varepsilon>0$,*
(L5) $\limsup_{t\to 0}\frac{1}{f(t)}\sup_{s\leq t}|X_s| = 0$ *almost surely,*
(L6) $\limsup_{t\to 0}\frac{1}{f(t)}|X_t| = 0$ *almost surely,*
(L7) $\limsup_{t\to 0}\frac{1}{f(t)}|X_t| < \infty$ *almost surely.*

In particular,

$$\limsup_{t\to 0}\frac{1}{f(t)}|X_t| \in \{0,\infty\} \quad a.s.$$

In (L5)-(L7) *we may replace 'almost surely' by 'with positive probability'.*

Theorem 2.1 generalizes [12, Corollary 11.3] for stable processes and the results from [3, Section III.4] for subordinators. Savov [36] proved the equivalence of (L1) and (L6) under (a bit stronger condition than) (A2) and the assumption that $(X_t)_{t\geq 0}$ has paths of unbounded variation. The equivalence of (L4) and (L6) goes back to Khintchine [20], see also [39, Appendix, Theorem 2]. The proof of Theorem 2.1 will be presented in Section 6.

2.2. *Remark.* (i) The proof of Theorem 2.1 shows that the implications

$$(L2) \implies (L3) \implies (L4) \implies (L5) \implies (L6) \implies (L7) \implies (L1)$$

hold without the sector condition. The sector condition is only needed to relate the integrals in (L1) and (L2) to each other. In fact, the key for the proof of (L1) $\implies$ (L2) is the implication

$$\exists c > 0 : \int_0^1 \nu(\{|y| \geq cf(t)\})\, dt < \infty \implies \forall \varepsilon > 0 : \int_0^1 \sup_{|\xi| \leq 1/(\varepsilon f(t))} \operatorname{Re} \psi(\xi)\, dt < \infty, \tag{2}$$

which does not require the sector condition, see Lemma 6.1 below; the sector condition is then used to replace $\operatorname{Re}\psi$ by $|\psi|$ in the integral on the right-hand side.
(ii) For the equivalences to hold, it is crucial that one of the assumptions (A1), (A2) is satisfied; if both assumptions are violated, then the equivalences break down in general and it may happen that

$$0 < \limsup_{t\to 0} \frac{1}{f(t)} |X_t| < \infty \quad \text{a.s.},$$

see [4, 36, 41] and Example 2.5 below.
(iii) Condition (A2) holds for any continuous function $f : [0,1] \to [0,\infty)$ satisfying $\frac{f(t)}{t} \uparrow \infty$ as $t \downarrow 0$ and $\frac{f(t)}{t^\alpha} \downarrow 0$ as $t \downarrow 0$ for some $\alpha > \frac{1}{2}$, cf. [36, Proof of Corollary 2.1]. While this criterion is useful in many cases, it is too restrictive in some situations. For instance, if $(X_t)_{t\geq 0}$ is an isotropic α-stable Lévy process, then $f(t) = t^{1/(\alpha-\varepsilon)}$ is an upper function, cf. Example 2.3 below, but clearly $\frac{f(t)}{t} \uparrow \infty$ as $t \to 0$ fails to hold if $\alpha < 1$. On the other hand, a straightforward calculation shows that the Lévy measure of the isotropic α-stable Lévy process satisfies (A1), and therefore Theorem 2.1 applies in this case without any additional growth assumptions on f. For further comments on (A1) and equivalent formulations, we refer to Remark 6.2.

Let us illustrate Theorem 2.1 with an example.

2.3. **Example.** Let $(X_t)_{t\geq 0}$ be an α-stable pure-jump Lévy process, $\alpha \in (0,2)$, that is, a Lévy process with Lévy triplet $(0,0,\nu)$ where the Lévy measure ν is of the form

$$\nu(A) = \int_0^\infty \int_{\mathbb{S}^{d-1}} \mathbb{1}_A(r\theta) \frac{1}{r^{1+\alpha}}\, \mu(d\theta)\, dr$$

for a measure μ on the sphere $\mathbb{S}^{d-1}$ satisfying $\mu(\mathbb{S}^{d-1}) > 0$, see [35] for a thorough discussion of stable processes. Theorem 2.1 shows that

$$\limsup_{t\to 0} \frac{1}{f(t)} |X_t| = \begin{Bmatrix} 0 \\ \infty \end{Bmatrix} \text{ a.s.} \iff \int_0^1 |f(t)|^{-\alpha}\, dt \begin{Bmatrix} < \infty \\ = \infty \end{Bmatrix} \tag{3}$$

for any non-decreasing function $f : [0,1] \to [0,\infty)$, and so we recover the classical characterization for upper functions of sample paths of stable Lévy processes, see e.g. [12, Corollary 11.3].

For power functions $f(t) = t^\kappa$, the finiteness of $\limsup_{t\to 0} \frac{1}{f(t)} |X_t|$ can be characterized in terms of the Blumenthal–Getoor index

$$\beta := \inf \left\{ \alpha > 0; \int_{|y|<1} |y|^\alpha\, \nu(dy) < \infty \right\} \in [0,2],$$

which was introduced in [7]. The following result is immediate from Theorem 2.1.

2.4. **Corollary.** *Let $(X_t)_{t\geq 0}$ be a Lévy process with Lévy triplet $(b,0,\nu)$ and assume that the characteristic exponent satisfies the sector condition. Then*

$$\limsup_{t\to 0} \frac{1}{t^\kappa} |X_t| = \begin{Bmatrix} 0 \\ \infty \end{Bmatrix} \textit{ a.s.} \iff \int_{|y|<1} |y|^{1/\kappa}\, \nu(dy) \begin{Bmatrix} < \infty \\ = \infty \end{Bmatrix} \tag{4}$$

for every $\kappa \in (\frac{1}{2}, \infty)$, and

$$\limsup_{t\to 0} \frac{1}{t^\kappa} |X_t| = \begin{Bmatrix} 0 \\ \infty \end{Bmatrix} \textit{ a.s.} \quad \textit{according as} \quad \begin{Bmatrix} \kappa < 1/\beta \\ \kappa > 1/\beta \end{Bmatrix} \tag{5}$$

for all $\kappa > 0$. If (A1) *from Theorem 2.1 is satisfied, then*

$$\limsup_{t\to 0} \frac{1}{\sqrt{t}} |X_t| = 0 \quad \textit{a.s.}$$

The characterization (5) goes back to Pruitt [33] and Blumenthal & Getoor [7], see also [35, Proposition 47.24]. Note that the critical case $\kappa = 1/\beta$ is excluded in (5); one has to check by hand whether the integral $\int_{|y|<1} |y|^\beta \, \nu(dy)$ is finite. In (4) the critical case is $\kappa = \frac{1}{2}$; this is due to the fact that $\int_{|y|<1} |y|^2 \, \nu(dy) < \infty$ is always satisfied but at the same time there are pure-jump Lévy processes with

$$0 < \limsup_{t\to 0} \frac{1}{\sqrt{t}} |X_t| < \infty \quad \text{a.s.}$$

cf. [4, 41]. Consequently, (4) fails, in general, for $\kappa = \frac{1}{2}$. Let us give an example of such a process and explain why this is not a contradiction to Theorem 2.1.

2.5. **Example.** Let $(X_t)_{t\geq 0}$ be a one-dimensional Lévy process with Lévy triplet $(0,0,\nu)$ and Lévy measure

$$\nu(dy) = \frac{1}{2}\frac{1}{|y|^2} \varphi'(|y|) \mathbb{1}_{(0,1/e^e)}(|y|)\, dy$$

for $\varphi(r) = 1/\log\log\frac{1}{r}$. Note that ν is indeed a Lévy measure, i.e. $\int \min\{|y|^2, 1\}\, \nu(dy) < \infty$. As

$$\int_{|y|\leq r} |y|^2 \, \nu(dy) = \varphi(r), \tag{6}$$

it follows from [4, Theorem 2.2] that

$$\limsup_{t\to 0} \frac{1}{\sqrt{t}} |X_t| = \sqrt{2} \quad \text{a.s.}$$

In particular, (4) breaks down for $\kappa = \frac{1}{2}$ and the equivalences in Theorem 2.1 fail to hold for $f(t) = \sqrt{t}$. This is not, however, a contradiction to Theorem 2.1 because the assumptions (A1) and (A2) in Theorem 2.1 are both violated. It is straightforward to check that (A2) fails for $f(t) = \sqrt{t}$; to see that (A1) fails we note that, by the Karamata's Tauberian theorem, see e.g. [6],

$$\nu(\{|y| > r\}) = \int_r^{1/e^e} \frac{1}{y^2} \varphi'(y)\, dy \approx \frac{1}{2} r^{-2} \frac{1}{\log\frac{1}{r}\left(\log\log\frac{1}{r}\right)^2} \quad \text{as } r \to 0,$$

and thus, by (6) and the definition of φ,

$$\lim_{r\to 0} \frac{\int_{|y|\leq r} |y|^2\, \nu(dy)}{r^2 \nu(\{|y| > r\})} = \infty.$$

Next we present our results for the wider class of Lévy-type processes, see Section 3 below for the definition. The following theorem gives sufficient conditions for an increasing function $f : [0,1] \to [0,\infty)$ to be an upper function of a Lévy-type process.

2.6. **Theorem.** *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with characteristics $(b(x), 0, \nu(x, dy))$ and symbol q satisfying the sector condition. Let $x \in \mathbb{R}^d$ and $R > 0$ such that*

$$M := \limsup_{r\to 0} \sup_{|z-x|\leq R} \frac{\int_{|y|\leq r} |y|^2\, \nu(z, dy)}{r^2 \nu(z, \{|y| > r\})} < \infty. \tag{A1'}$$

Then the implications

$$\text{(LTP1)} \implies \text{(LTP2)} \implies \text{(LTP3)} \implies \text{(LTP4)}$$

hold for any non-decreasing function $f : [0,1] \to (0,\infty)$, where

(LTP1) $\int_0^1 \sup_{|z-x|\leq f(t)} \nu(z, \{|y| \geq cf(t)\})\, dt < \infty$ *for some $c > 0$,*
(LTP2) $\int_0^1 \sup_{|z-x|\leq f(t)} \sup_{|\xi|\leq 1/(\varepsilon f(t))} |q(z,\xi)|\, dt < \infty$ *for some (all) $\varepsilon > 0$,*
(LTP3) $\int_0^1 \sup_{|z-x|\leq f(t)} \mathbb{P}^z \left(\sup_{s\leq t} |X_s - z| \geq \varepsilon f(t)\right) \frac{1}{t}\, dt < \infty$ *for all $\varepsilon > 0$,*
(LTP4) $\limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t} |X_s - x| = 0$ $\mathbb{P}^x$*-almost surely.*

2.7. *Remark.* (i) The implications (LTP2) $\implies$ (LTP3) $\implies$ (LTP4) hold for *any* Lévy-type process, i.e. all the additional assumptions are only needed for the proof of the implication (LTP1) $\implies$ (LTP2).
(ii) The implication (LTP3) $\implies$ (LTP4) holds for any strong Markov process, see Theorem 5.1.

(iii) In Theorem 2.6 we assume that the symbol q satisfies the sector condition, cf. (19). A close look at the proof shows that we actually only need a local sector condition, in the sense that, for fixed $x \in \mathbb{R}^d$ there are constants $r > 0$ and $C > 0$ such that

$$|\operatorname{Im} q(z,\xi)| \le C \operatorname{Re} q(z,\xi) \qquad \text{for all } \ \xi \in \mathbb{R}^d, |z-x| \le r.$$

The same is true for the Proposition 2.9 and Theorem 2.10 below.
(iv) As already mentioned, assumption (A1') is crucial for the proof of the implication (LTP1) $\Longrightarrow$ (LTP2). One might ask whether this implication also holds under assumption (A2) from Theorem 2.1. We did not manage to prove this, but there is the following slightly weaker result. Consider

$$\int_0^1 \sup_{|z-x|\le R} \nu(z,\{|y| \ge cf(t)\})\,dt < \infty \quad \text{for some } c>0,\ R>0. \tag{LTP1'}$$

Note that (LTP1') is a bit stronger than (LTP1). If (A2) holds and

$$\forall r \in (0,R)\,\exists x_0 \in \overline{B(x,r)}\ \forall \xi \in \mathbb{R}^d, |\xi| \ge 1 : \sup_{|z-x|\le r} |q(z,\xi)| \le |q(x_0,\xi)|, \tag{7}$$

then (LTP1') $\Longrightarrow$ (LTP2). Because of the majorization condition (7), the proof of this assertion is analogous to the case of Lévy processes, see the proof of Theorem 2.1 and Lemma 6.1.

For the particular case that $(X_t)_{t\ge0}$ is a Lévy process, Theorem 2.6 yields the implications (L1) $\Longrightarrow$ (L2) $\Longrightarrow$ (L3) $\Longrightarrow$ (L5) from Theorem 2.1. In this sense, Theorem 2.6 is a natural extension of Theorem 2.1. Unlike in the Lévy case, it is not to be expected that the conditions (LTP1)-(LTP4) in Theorem 2.6 are equivalent for a general Lévy-type process. However, there is the following partial converse.

2.8. **Proposition.** *Let $(X_t)_{t\ge0}$ be a Lévy-type process with characteristics $(b(x),0,\nu(x,dy))$, and let $f:[0,1]\to[0,\infty)$ be non-decreasing.*

(i) *If $x \in \mathbb{R}^d$ is such that*

$$\int_0^1 \sup_{|z-x|\le f(t)} \mathbb{P}^z\left(\sup_{s\le t}|X_s - z| \ge f(t)\right)\frac{1}{t}\,dt < \infty,$$

then

$$\int_0^1 \inf_{|z-x|\le 10f(t)} \nu(z;\{|y| > 10f(t)\})\,dt < \infty.$$

(ii) *Assume that* (A1') *from Theorem 2.6 holds for some $R>0$ and $x\in\mathbb{R}^d$. If*

$$\int_0^1 \inf_{|z-x|\le Cf(t)} \nu(z;\{|y| > Cf(t)\})\,dt < \infty$$

for a constant $C>0$, then

$$\int_0^1 \inf_{|z-x|\le Cf(t)} \sup_{|\xi|\le c/f(t)} \operatorname{Re} q(z,\xi)\,dt < \infty$$

for all $c>0$.

The next result gives a lower bound for the growth of the sample paths of a Lévy-type process.

2.9. **Proposition.** *Let $(X_t)_{t\ge0}$ be a Lévy-type process with symbol q satisfying the sector condition. Let $x\in\mathbb{R}^d$. If $f:[0,1]\to(0,\infty)$ is a function such that*

$$\limsup_{t\to0} t\cdot \inf_{|z-x|\le Rf(t)} \sup_{|\xi|\le 1/(Cf(t))} \operatorname{Re} q(z,\xi) = \infty, \tag{8}$$

for every $R\ge1$ and some constant $C=C(R)>0$, then

$$\limsup_{t\to0} \frac{1}{f(t)} \sup_{s\le t}|X_s - x| = \infty \quad \mathbb{P}^x\text{-a.s.} \tag{9}$$

Moreover:

(i) *If additionally f is non-decreasing, then*

$$\limsup_{t\to0} \frac{1}{f(t)}|X_t - x| = \infty \quad \mathbb{P}^x\text{-a.s.}$$

(ii) *If f is regularly varying at zero, then* (9) *holds under the milder assumption that* (8) *is satisfied for* $R = 1$.

In [37, Theorem 4.3], this result was shown for power functions $f(t) = t^\kappa$ but in fact the proof goes through for arbitrary functions f, see Section 7. The following statement is an immediate consequence of Proposition 2.9: If f is a non-negative function and q the symbol of a Lévy-type process $(X_t)_{t\geq 0}$ such that

$$\limsup_{t\to 0} t^{-\beta/\alpha} f(t) < \infty \quad \text{and} \quad \inf_{|z-x|\leq r} \operatorname{Re} q(z,\xi) \geq c|\xi|^\alpha \quad \text{for all } |\xi| \gg 1$$

for some $x \in \mathbb{R}^d$, $r > 0$, $\alpha > 0$, $c > 0$ and $\beta > 1$, then

$$\limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t} |X_s - x| = \infty \quad \mathbb{P}^x\text{-a.s.}$$

Our final main result gives an integral criterion for a function f *not* to be an upper function of a Lévy-type process.

2.10. **Theorem.** *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with characteristics $(b(x), 0, \nu(x,dy))$ and symbol q, and let $f : [0,1] \to (0,\infty)$ be non-decreasing. Let $x \in \mathbb{R}^d$ be such that one of the following conditions holds.*

(C1) *q satisfies the sector condition and there is $\kappa \in [0,1)$ such that*

$$\sup_{|z-x|\leq f(t)} \sup_{|\xi|\leq 1/f(t)} |q(z,\xi)| \leq ct^{-\kappa} \inf_{|z-x|\leq Rf(t)} \sup_{|\xi|\leq 1/f(t)} |q(z,\xi)|, \qquad t \in (0,1), \tag{10}$$

for every $R \geq 1$ and some constant $c = c(R) > 0$.

(C2) *There are constants $\alpha \in (0,2]$, $r > 0$ and $c > 0$ such that*

$$\sup_{|z-x|\leq r} |q(z,\xi)| \leq c(1 + |\xi|^\alpha), \qquad |\xi| \gg 1, \tag{11}$$

and $\liminf_{t\to 0} t^{-2/\alpha} f(t) = \infty$.

Then:

(i) *If*

$$\int_0^1 \inf_{|z-x|\leq Cf(t)} \nu(z, \{|y| \geq Cf(t)\})\, dt = \infty \tag{12}$$

for some constant $C > 0$, then

$$\limsup_{t\to 0} \frac{1}{f(t)} |X_t - x| \geq \frac{C}{5} \quad \mathbb{P}^x\text{-a.s.}$$

(ii) *Assume that* (A1') *from Theorem 2.6 is satisfied for some $R > 0$. If*

$$\int_0^1 \inf_{|z-x|\leq Cf(t)} \sup_{|\xi|\leq 1/f(t)} |q(z,\xi)|\, dt = \infty \tag{13}$$

for some constant $C > 0$, then

$$\limsup_{t\to 0} \frac{1}{f(t)} |X_t - x| \geq \frac{C}{5} \quad \mathbb{P}^x\text{-a.s.}$$

2.11. *Remark.* (i) If the constant C in (12) resp. (13) can be chosen arbitrarily large, then

$$\limsup_{t\to 0} \frac{1}{f(t)} |X_t - x| = \infty \quad \mathbb{P}^x\text{-a.s.}$$

(ii) The growth condition (11) on the symbol holds automatically for $\alpha = 2$, cf. (18).
(iii) If f is regularly varying, then it suffices to check (10) in (C1) for $R = 1$. Moreover, we note that (10) is trivially satisfied if the symbol q does not depend on z, i.e. if $(X_t)_{t\geq 0}$ is a Lévy process.
(iv) For the particular case of Lévy processes, Theorem 2.10 is known – see Theorem 2.1 and the references below it – but Theorem 2.10 seems to be the first result in this direction which applies for the much wider class of Lévy-type processes. Let us comment on the differences in the proofs. For Lévy processes, the standard approach to prove an assertion of the form

$$\limsup_{t\to 0} \frac{1}{f(t)} |X_t| \geq C \quad \text{a.s.}$$

is to construct a suitable sequence $(A_n)_{n\in\mathbb{N}}$ of sets which involves only the increments of $(X_t)_{t\geq 0}$ and which satisfies

$$\limsup_{n\to\infty} A_n \subseteq \left\{ \limsup_{t\to 0} \frac{1}{f(t)} |X_t| \geq C \right\},$$

and to use (the difficult direction of) the Borel–Cantelli lemma to deduce from $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$ that $\mathbb{P}(\limsup_n A_n) = 1$. This approach relies heavily on the independence of the increments – ensuring the sets $(A_n)_{n\in\mathbb{N}}$ are independent – and so it fails to work in the more general framework of Lévy-type processes. We fix this issue by using a conditional Borel-Cantelli lemma for backward filtrations, cf. Proposition A.1. Moreover, our proof uses a new maximal inequality for Lévy-type processes which is of independent interest, cf. Section 4.

We close this section with some illustrating examples.

2.12. **Example** (Process of variable order). Let $(X_t)_{t\geq 0}$ be a Lévy-type process with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$ for $\alpha : \mathbb{R}^d \to (0,2)$ continuous; a sufficient condition for the existence of such a process is that α is Hölder continuous and bounded away from zero, see e.g. [1, 23, 26] for details. Let us mention that Negoro [32] was one of the first to study the small-time asymptotics of processes of variable order. If we set $\alpha^*(x,r) := \sup_{|z-x|\leq r} \alpha(z)$ and $\alpha_*(x,r) := \inf_{|z-x|\leq r} \alpha(z)$, then our results show that

$$\int_0^1 |f(t)|^{-\alpha^*(x,r)}\, dt < \infty \text{ for some } r > 0 \implies \limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t} |X_s - x| = 0 \quad \mathbb{P}^x\text{-a.s.} \tag{14}$$

and

$$\int_0^1 |f(t)|^{-\alpha_*(x,r)}\, dt = \infty \text{ for some } r > 0 \implies \limsup_{t\to 0} \frac{1}{f(t)} |X_t - x| = \infty \quad \mathbb{P}^x\text{-a.s.} \tag{15}$$

for any $f : [0,1] \to [0,\infty)$ non-decreasing. By the continuity of α, this entails that

$$\int_0^1 |f(t)|^{-\beta}\, dt < \infty \text{ for some } \beta > \alpha(x) \implies \limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t} |X_s - x| = 0 \quad \mathbb{P}^x\text{-a.s.}$$

and

$$\int_0^1 |f(t)|^{-\beta}\, dt = \infty \text{ for some } \beta < \alpha(x) \implies \limsup_{t\to 0} \frac{1}{f(t)} |X_t - x| = \infty \quad \mathbb{P}^x\text{-a.s.}$$

In particular, this generalizes [32, Theorem 2.1], which is about the particular case that f is a power function. If α has a local maximum at x, then (14) yields

$$\int_0^1 |f(t)|^{-\alpha(x)}\, dt < \infty \implies \limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t} |X_s - x| = 0 \quad \mathbb{P}^x\text{-a.s.}$$

This holds, in particular, if $\alpha(x) = \alpha$ is constant, i.e. if $(X_t)_{t\geq 0}$ is an isotropic α-stable Lévy process. An analogous consideration works for (15) if α has a local minimum at x. In particular, we recover the classical criterion for isotropic α-stable Lévy processes, cf. Example 2.3.

2.13. **Example** (Stable-type process). Consider a Lévy-type process $(X_t)_{t\geq 0}$ with characteristics $(0,0,\nu(x,dy))$, where

$$\nu(x,dy) = \kappa(x,y) \frac{1}{|y|^{d+\alpha}}\, dy$$

for some $\alpha \in (0,2)$ and a mapping $\kappa : \mathbb{R}^d \times \mathbb{R} \to (0,\infty)$ which is symmetric in the y-variable and satisfies $0 < \inf_{x,y} \kappa(x,y) \leq \sup_{x,y} \kappa(x,y) < \infty$, see e.g. [2, 26] for the existence of such processes. Since

$$\frac{1}{M} r^{-\alpha} \leq \nu(x, \{|y| \geq r\}) \leq M r^{-\alpha}, \qquad r > 0,$$

for a constant $M > 0$ not depending on $x \in \mathbb{R}^d$, it follows from Theorem 2.6 and Theorem 2.10 that

$$\limsup_{t\to 0} \frac{1}{f(t)} |X_t - x| = \begin{Bmatrix} 0 \\ \infty \end{Bmatrix} \mathbb{P}^x\text{-a.s.} \quad \text{according as} \quad \int_0^1 |f(t)|^{-\alpha}\, dt \begin{Bmatrix} < \infty \\ = \infty \end{Bmatrix}$$

for any non-decreasing function $f : [0,1] \to [0,\infty)$.

2.14. **Example** (Lévy-driven SDE). Let $(L_t)_{t\geq 0}$ be a pure-jump Lévy process with characteristic exponent ψ satisfying the sector condition, and assume that the Lévy measure ν_L satisfies (A1) from Theorem 2.1. Let $(X_t)_{t\geq 0}$ be the unique weak solution to an SDE

$$dX_t = \sigma(X_{t-})\,dL_t, \qquad X_0 = x,$$

for a bounded continuous function $\sigma : \mathbb{R} \to \mathbb{R}$. Then $(X_t)_{t\geq 0}$ is a Lévy-type process with symbol $q(x,\xi) := \psi(\sigma(x)\xi)$, cf. [31, 27, 38]. Fix $x \in \mathbb{R}$ such that $\sigma(x) \neq 0$. By Theorem 2.6 and Theorem 2.10, the following statements hold for any non-decreasing function $f : [0,1] \to [0,\infty)$.

(i) If there exists a constant $c > 0$ such that

$$\int_0^1 \nu_L(\{|y| \geq cf(t)\})\,dt < \infty,$$

then

$$\limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t} |X_s - x| = 0 \quad \text{and} \quad \limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t} |L_s| = 0.$$

(ii) Assume that $\psi^*(r) := \sup_{|\xi|\leq r} |\psi(\xi)|$ satisfies the following weak scaling condition (at zero): There are constants $\alpha > 0$ and $C > 0$ such that

$$\psi^*(\lambda r) \geq C\lambda^\alpha \psi^*(r) \qquad \text{for all } \ r > 0,\ \lambda \in (0,1).$$

If

$$\int_0^1 \nu_L(\{|y| \geq cf(t)\})\,dt = \infty$$

for some constant $c > 0$, then

$$\limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t} |X_s - x| > 0 \quad \text{and} \quad \limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t} |L_s| > 0.$$

The remainder of the article is organized as follows. After introducing basic definitions and notation in Section 3, we establish a new maximal inequality for Lévy-type processes in Section 4 and study some of its consequences. In Section 5 we obtain integral criteria for upper functions of Markov processes. They are the key for the proofs of Theorem 2.1 and Theorem 2.6, which are presented in Section 6. Finally, in Section 7, we give the proofs of Proposition 2.8, Proposition 2.9 and Theorem 2.10.

3. Basic definitions and notation

We consider the Euclidean space $\mathbb{R}^d$ with the canonical scalar product $x \cdot y := \sum_{j=1}^d x_j y_j$ and the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ generated by the open balls $B(x,r) := \{y \in \mathbb{R}^d; |y-x| < r\}$. For a real-valued function f, we denote by ∇f the gradient and by $\nabla^2 f$ the Hessian of f. If ν is a measure, say on $\mathbb{R}^d$, we use the short-hand $\nu(\{|y| > r\})$ for $\nu(\{y \in \mathbb{R}^d; |y| > r\})$.
An operator A defined on the space $C_c^\infty(\mathbb{R}^d)$ of compactly supported smooth functions is a *Lévy-type operator* if it has a representation of the form

$$\begin{aligned} Af(x) &= b(x)\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}(Q(x)\cdot\nabla^2 f(x)) \\ &\quad + \int_{y\neq 0} (f(x+y) - f(x) - y\cdot\nabla f(x)\mathbb{1}_{(0,1)}(|y|))\,\nu(x,dy), \qquad f \in C_c^\infty(\mathbb{R}^d), \end{aligned} \tag{16}$$

where $b(x) \in \mathbb{R}^d$ is a vector, $Q(x) \in \mathbb{R}^{d\times d}$ is a positive semi-definite matrix and $\nu(x,dy)$ is a measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int_{y\neq 0} \min\{1,|y|^2\}\,\nu(x,dy) < \infty$ for each $x \in \mathbb{R}^d$. The family $(b(x),Q(x),\nu(x,dy))$, $x \in \mathbb{R}^d$, is the *(infinitesimal) characteristics* of A. Equivalently, A can be written as a pseudo-differential operator

$$Af(x) = -\int_{\mathbb{R}^d} q(x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi, \qquad f \in C_c^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

with *symbol*

$$q(x,\xi) := -ib(x)\cdot\xi + \frac{1}{2}\xi\cdot Q(x)\xi + \int_{y\neq 0}\left(1 - e^{iy\cdot\xi} + iy\cdot\xi\mathbb{1}_{(0,1)}(|y|)\right)\nu(x,dy), \qquad x,\xi \in \mathbb{R}^d.$$

For each fixed $x \in \mathbb{R}^d$, the mapping $\xi \mapsto q(x,\xi)$ is continuous and negative definite (in the sense of Schoenberg). In consequence, $\xi \mapsto \sqrt{|q(x,\xi)|}$ is subadditive, i.e.

$$\sqrt{|q(x,\xi+\eta)|} \le \sqrt{|q(x,\xi)|} + \sqrt{|q(x,\eta)|}, \qquad x,\xi,\eta \in \mathbb{R}^d, \tag{17}$$

which implies

$$|q(x,\xi)| \le 2 \sup_{|\eta|\le 1} |q(x,\eta)|(1+|\xi|^2), \qquad x,\xi \in \mathbb{R}^d,$$

see e.g. [21, Theorem 6.2]. In particular, $(x,\xi) \mapsto q(x,\xi)$ is locally bounded if, and only if, there is for every compact set $K \subseteq \mathbb{R}^d$ some constant $C > 0$ such that

$$|q(x,\xi)| \le C(1+|\xi|^2), \qquad \xi \in \mathbb{R}^d,\ x \in K. \tag{18}$$

The local boundedness of q can also be characterized in terms of the characteristics; namely, q is locally bounded if, and only if,

$$\forall K \subseteq \mathbb{R}^d \text{ compact} : \sup_{x\in K} \left(|b(x)| + |Q(x)| + \int_{y\neq 0} \min\{1,|y|^2\} \nu(x,dy) \right) < \infty.$$

Applying Taylor's formula in (16) shows that the local boundedness of the symbol q of the Lévy-type operator A implies $\|Af\|_\infty < \infty$ for every $f \in C_c^\infty(\mathbb{R}^d)$. We say that q satisfies the *sector condition* if there is a constant $C > 0$ such that

$$|\operatorname{Im} q(x,\xi)| \le C \operatorname{Re} q(x,\xi) \qquad \text{for all } \ x,\xi \in \mathbb{R}^d. \tag{19}$$

Next we introduce the probabilistic objects. Let A be a Lévy-type operator and $(\Omega,\mathcal{A},\mathbb{P})$ a probability space. A stochastic process $X_t : \Omega \to \mathbb{R}^d$, $t \ge 0$, with càdlàg sample paths is a *solution to the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem with initial distribution* μ if $\mathbb{P}(X_0 \in \cdot) = \mu(\cdot)$ and

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Af(X_s)\, ds, \qquad t \ge 0,$$

is a martingale with respect to the canonical filtration $\mathcal{F}_t := \sigma(X_s; s \le t)$ for every $f \in C_c^\infty(\mathbb{R}^d)$. A tuple $(X_t, t \ge 0; \mathbb{P}^x, x \in \mathbb{R}^d)$ consisting of a family of probability measures $\mathbb{P}^x$, $x \in \mathbb{R}^d$, on a measurable space $(\Omega,\mathcal{A})$ and a stochastic process $X_t : \Omega \to \mathbb{R}^d$ with càdlàg sample paths is called a *Lévy-type process with symbol* q if

(i) $(X_t, t \ge 0; \mathbb{P}^x, x \in \mathbb{R}^d)$ is a strong Markov process;
(ii) $(X_t)_{t\ge 0}$ solves the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem for the Lévy-type operator A with symbol q. More precisely, for each $x \in \mathbb{R}^d$, the stochastic process $(X_t)_{t\ge0}$ considered on the probability space $(\Omega,\mathcal{A},\mathbb{P}^x)$ is a solution to the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem with initial distribution $\mu = \delta_x$;
(iii) q is locally bounded.

Note that (iii) entails that Af is bounded for every $f \in C_c^\infty(\mathbb{R}^d)$, and so the integral $\int_0^t Af(X_s)\, ds$ appearing in the definition of the martingale problem is well-defined. If the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem is *well-posed*, i.e. there exists a unique solution to the martingale problem for any initial distribution μ, then the strong Markov property (i) is automatically satisfied, cf. [11, Theorem 4.4.2]. Well-posedness is, however, not necessary for the existence of strongly Markovian solutions to martingale problems; one can use so-called Markovian selections to construct such solutions, see [11, Section 4.5] and [29]. For a thorough discussion of martingale problems associated with Lévy-type operators, we refer to [8, 16, 18]. The following classes of stochastic processes are examples of Lévy-type processes:

- Lévy processes: A *Lévy process* is a stochastic process $(X_t)_{t\ge0}$ with stationary and independent increments and càdlàg sample paths. It is uniquely determined (in distribution) by its Lévy triplet (b,Q,ν) and its characteristic exponent ψ, cf. [35, 21]. Any Lévy process is a Lévy-type process in the sense of the above definition; the corresponding operator A is the pseudo-differential operator with symbol $q(x,\xi) := \psi(\xi)$ and characteristics (b,Q,ν).
- Feller processes: If $(X_t)_{t\ge0}$ is a Feller process whose infinitesimal generator $(A,\mathcal{D}(A))$ satisfies $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$, then $(X_t)_{t\ge0}$ is a Lévy-type operator; this follows from a result by Courrège & von Waldenfels, see [8, 18, 28] for further information.

- solutions to Lévy-driven SDEs: Let $(L_t)_{t\geq 0}$ be a Lévy process with characteristic exponent ψ. If σ is a bounded continuous function and the stochastic differential equation (SDE)
$$dX_t = \sigma(X_{t-})dL_t, \qquad X_0 = x,$$
has a unique weak solution $(X_t)_{t\geq 0}$, then $(X_t)_{t\geq 0}$ is a Lévy-type process with symbol $q(x,\xi) = \psi(\sigma(x)^T\xi)$, cf. [11, 38]. The assumptions on σ can be relaxed, cf. [27, 29].

4. A maximal inequality for Lévy-type processes

In this section, we establish a new maximal inequality for Lévy-type processes and present some consequences of this inequality. Before we start, we recall the following maximal inequality, which will be used frequently in this paper.

4.1. **Proposition.** *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with symbol q. There is an absolute constant $c > 0$ such that*
$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s - x| \geq r\right) \leq ct \sup_{|z-x|\leq r}\sup_{|\xi|\leq 1/r}|q(z,\xi)|, \qquad x \in \mathbb{R}^d,\ t > 0,\ r > 0. \tag{20}$$
This maximal inequality goes back to Schilling [37], see also [8, Theorem 5.1] and [29, Proposition 2.8]. Let us mention two variants of this inequality: a version for random times, cf. [26, Lemma 1.29], and a localized version, cf. [30, Lemma 4.1]. If we denote by $\tau_r^x = \inf\{t \geq 0; |X_t - x| \geq r\}$ the first exit time of $(X_t)_{t\geq 0}$ from the open ball $B(x,r)$, then (20) can be equivalently formulated as follows:
$$\mathbb{P}^x(\tau_r^x \leq t) \leq ct \sup_{|z-x|\leq r}\sup_{|\xi|\leq 1/r}|q(z,\xi)|, \qquad x \in \mathbb{R}^d,\ t > 0,\ r > 0.$$
For the proofs of our main results, we need an upper bound for the probability
$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s - x| < r\right).$$
The following maximal inequality allows us to derive suitable bounds and is of independent interest.

4.2. **Proposition.** *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with characteristics $(b(x), Q(x), \nu(x,dy))$, $x \in \mathbb{R}^d$, and denote by τ_r^x the first exit time from $B(x,r)$. Then*
$$\mathbb{P}^x(\tau_r^x \geq t) \leq \frac{1}{1 + tG(x,2r)} \quad \text{with} \quad G(x,r) := \inf_{|z-x|\leq r}\nu(z,\{|y| > r\})$$
for all $x \in \mathbb{R}^d$ and $r > 0$.

Note that Proposition 4.2 implies that
$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s - x| < r\right) \leq \frac{1}{1 + tG(x,2r)}, \qquad x \in \mathbb{R}^d,\ r > 0.$$
Intuitively, $G(x,r) = \inf_{|z-x|\leq r}\nu(z,\{|y| > r\})$ quantifies the likelihood of a jump of modulus $> r$ while the process is close to its starting point x. The idea behind our estimate is that the process leaves immediately the ball $B(x,r)$ if a jump of modulus $> 2r$ occurs. Other means of leaving the ball, e.g. due to a drift or diffusion part, are not taken into account. In consequence, Proposition 4.2 does well for pure-jump processes but less so e.g. for processes with a non-vanishing diffusion part. There is a related estimate in [8, Theorem 5.5], see also [37, Lemma 6.3], giving an upper bound for $\mathbb{P}^x(\tau_r^x \geq t)$ in terms of the symbol[1]; for the particular that q satisfies the sector condition (19), it reads
$$\mathbb{P}^x(\tau_r^x \geq t) \leq c\frac{1}{1 + th(x,r)} \quad \text{with} \quad h(x,r) := \sup_{|\xi|\leq 1/(2r)}\inf_{|z-x|\leq r}\operatorname{Re} q(z,\xi) \tag{21}$$
for some constant $c > 0$. In some situations, (21) gives better estimates than Proposition 4.2 – e.g. if there is a diffusion part – but our result has its advantages e.g. if the sector condition is not satisfied. For instance, for $q(x,\xi) = i\xi + \sqrt{|\xi|}$, we get $\mathbb{P}^x(\tau_r^x \geq t) \leq 1/(1 + ctr^{-1/2}) \leq c' r^{1/2}t^{-1}$ while [8, Theorem 5.5] gives only $\mathbb{P}^x(\tau_r^x \geq t) \leq c'' r^{1/3}t^{-1}$; note that the estimates are of interest only if the right-hand sides are less or equal than 1, i.e. for $r > 0$ small.

[1] Beware that there is a typo in the definition of $k(x,r)$ in [8, Theorem 5.5]; the two suprema should be infima.

Proof of Proposition 4.2. For fixed $x \in \mathbb{R}^d$, $\varepsilon > 0$ and $r > 0$, pick $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\mathbb{1}_{B(x,r)} \leq \chi \leq \mathbb{1}_{B(x,r+\varepsilon)}$. As $\chi(X_t) = 1$ on $\{t < \tau_r^x\}$, it follows from Dynkin's formula that

$$\mathbb{P}^x(\tau_r^x > t) = \mathbb{E}^x(\chi(X_t)\mathbb{1}_{\{\tau_r^x > t\}}) \leq \mathbb{E}^x(\chi(X_{t\wedge\tau_r^x})) = 1 + \mathbb{E}^x\left(\int_{(0,t\wedge\tau_r^x)} A\chi(X_s)\,ds\right), \tag{22}$$

where A is the Lévy-type operator associated with the family of triplets $(b(x), Q(x), \nu(x,dy))$, see (16). For $z \in B(x,r)$, we have $\chi(z) = 1$, $\nabla\chi(z) = 0$ and $\nabla^2\chi(z) = 0$, and so

$$A\chi(z) = \int_{y\neq 0}(\chi(z+y) - 1)\,\nu(z,dy).$$

Using that $0 \leq \chi \leq 1$ on $\mathbb{R}^d$ and $\chi = 0$ outside $B(x, r+\varepsilon)$, we find that

$$A\chi(z) \leq \int_{|(z+y)-x|\geq r+\varepsilon}(\chi(z+y)-1)\,\nu(z,dy) \leq -\int_{|y|\geq 2r+\varepsilon}\nu(z,dy)$$

for all $z \in B(x,r)$. Since $X_s \in B(x,r)$ for $s < \tau_r^x$, it follows from (22) that

$$\mathbb{P}^x(\tau_r^x > t) \leq 1 - \mathbb{E}^x\left(\int_{(0,t\wedge\tau_r^x)}\nu(X_s, \{|y| \geq 2r+\varepsilon\})\,ds\right)$$

for all $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ using the dominated convergence theorem, we arrive at

$$\begin{aligned}\mathbb{P}^x(\tau_r^x > t) &\leq 1 - \mathbb{E}^x\left(\int_{(0,t\wedge\tau_r^x)}\nu(X_s,\{|y|>2r\})\,ds\right)\\ &\leq 1 - \mathbb{E}^x(\tau_r^x \wedge t)\inf_{|z-x|\leq r}\int_{|y|>2r}\nu(z,dy).\end{aligned} \tag{23}$$

The elementary estimate $\mathbb{E}^x(\tau_r^x \wedge t) \geq t\mathbb{P}^x(\tau_r^x > t)$ now gives

$$\mathbb{P}^x(\tau_r^x > t) \leq 1 - t\mathbb{P}^x(\tau_r^x > t)\inf_{|z-x|\leq r}\int_{|y|>2r}\nu(z,dy),$$

i.e.

$$\mathbb{P}^x(\tau_r^x > t) \leq \frac{1}{1 + t\inf_{|z-x|\leq r}\nu(z,\{|y|>2r\})}.$$

Thus,

$$\mathbb{P}^x(\tau_r^x \geq t) = \lim_{\varepsilon\to 0}\mathbb{P}^x(\tau_r^x > t-\varepsilon) \leq \frac{1}{1 + t\inf_{|z-x|\leq r}\nu(z,\{|y|>2r\})}.$$ □

4.3. **Corollary.** *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with characteristics $(b(x), Q(x), \nu(x,dy))$, and denote by τ_r^x the exit time from the ball $B(x,r)$. Then*

$$\mathbb{E}^x\tau_r^x \leq \frac{1}{G(x,2r)} \tag{24}$$

and

$$\mathbb{P}^x(\tau_r^x \geq t) \leq C_0\exp(-C_1 tG(x,2r)) \tag{25}$$

for all $x \in \mathbb{R}^d$, $t > 0$ and $r > 0$, where $C_0, C_1 < \infty$ are uniform constants and $G(x,r)$ is the mapping defined in Proposition 4.2.

Proof. By (23),

$$\mathbb{E}^x(\tau_r^x \wedge t) \leq \frac{1 - \mathbb{P}^x(\tau_r^x > t)}{G(x,2r)} \leq \frac{1}{G(x,2r)}.$$

Letting $t \to \infty$ using Fatou's lemma, proves the first assertion. The second inequality is obtained from Proposition 4.2 by an iteration argument using the Markov property; it is the same reasoning as in [8, Proof of Theorem 5.9]. □

4.4. *Remark.* (i) For the particular case that $(X_t)_{t\geq 0}$ is a Lévy process, we know that $N_t := \sharp\{s \leq t; |\Delta X_s| > 2r\}$ is a Poisson process with intensity $\lambda = \nu(\{|y| > 2r\})$, where ν is the Lévy measure, and so

$$\mathbb{P}^x(\tau_r^x \geq t) = \lim_{\varepsilon\to 0}\mathbb{P}^x(\tau_r^x > t+\varepsilon) \leq \lim_{\varepsilon\to 0}\mathbb{P}^x(N_{t+\varepsilon} = 0) = e^{-\lambda t} = e^{-t\nu(\{|y|>2r\})},$$

which is (25) with $C_0 = C_1 = 1$. If $(X_t)_{t\geq 0}$ is a general Lévy-type process $(X_t)_{t\geq 0}$, then $(N_t)_{t\geq 0}$ is no longer a Poisson process but our result shows that we can still get an analogous estimate

in terms of the jump intensity $G(x,2r)$. This fits well to the intuition that a Lévy-type process behaves locally like a Lévy process.
(ii) The estimate (25) is optimal for a wide family of jump processes. However, our approach incorporates only the tails of the Lévy measures and therefore some information may be lost, leading to non-optimal estimates for certain processes. This is best seen for the particular case of stable Lévy processes, for which Taylor [40] derived upper and lower bounds for $\mathbb{P}(\tau_r \geq t)$ (i.e. $x=0$). He shows for $r>0$ small that

$$\mathbb{P}(\tau_r \geq t) \asymp e^{-ctr^{-\alpha}} \qquad \text{for stable processes of type A}$$
$$\mathbb{P}(\tau_r \geq t) \asymp e^{-ctr^{-\alpha/(1-\alpha)}} \qquad \text{for stable processes of type B, } \alpha \in (0,1),$$

where the constants c in the lower and upper bound may differ. Here, 'type B' means essentially that the process has a projection which is a subordinator – formally, the Lévy measure is concentrated on a hemisphere $\{y \in \mathbb{R}^d; y_j \geq 0\}$ for some $j \in \{1,\dots,d\}$ – and all other stable processes are of type A. While our estimate (25) yields the correct upper bound for stable processes of type A, we only get the (sub-optimal) upper bound $e^{-ctr^{-\alpha}}$ for processes of type B.

As a direct consequence of Proposition 4.2, we also obtain the following corollary.

4.5. **Corollary.** *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with characteristics $(b(x),Q(x),\nu(x,dy))$, and let $c\in[0,1]$. If $x\in\mathbb{R}^d$, $t>0$ and $r>0$ are such that*

$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s-x|>r\right)\leq c,$$

then

$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s-x|>r\right)\geq (1-c)tG(x,2r)$$

for $G(x,r)$ defined in Proposition 4.2.

As an immediate consequence, we see that

$$\limsup_{t\to 0}\mathbb{P}^x\left(\sup_{s\leq t}|X_s-x|>r(t)\right)<1$$

implies

$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s-x|>r(t)\right)\geq CtG(x,2r(t))$$

for small $t>0$ and some constant $C>0$, which will be useful lateron.

Proof of Corollary 4.5. By Proposition 4.2,

$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s-x|\leq r\right)\leq \frac{1}{1+tG(x,2r)},$$

which is equivalent to

$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s-x|\leq r\right)\leq 1-tG(x,2r)\mathbb{P}^x\left(\sup_{s\leq t}|X_s-x|\leq r\right).$$

Hence,

$$\begin{aligned}\mathbb{P}^x\left(\sup_{s\leq t}|X_s-x|>r\right)&\geq tG(x,2r)\mathbb{P}^x\left(\sup_{s\leq t}|X_s-x|\leq r\right)\\ &= tG(x,2r)\left[1-\mathbb{P}^x\left(\sup_{s\leq t}|X_s-x|>r\right)\right],\end{aligned}$$

which proves the assertion. □

Let us illustrate the results from this section with an example.

4.6. **Example.** Let $(X_t)_{t\geq 0}$ be a process of variable order, i.e. a Lévy-type process with symbol $q(x,\xi)=|\xi|^{\alpha(x)}$ for a continuous mapping $\alpha:\mathbb{R}^d\to(0,2]$. Denote by τ_r^x the first exit time of $(X_t)_{t\geq 0}$ from the ball $B(x,r)$ and set $\alpha_*(x,r):=\inf_{|z-x|\leq r}\alpha(z)$. The following estimates hold for uniform constants $c_0,\dots,c_4\in(0,\infty)$:

(i) $\mathbb{P}^x(\tau_r^x \ge t) \le 1/(1+c_0 t r^{-\alpha_*(x,r)})$ and $\mathbb{P}^x(\tau_r^x \ge t) \le c_1 \exp(-c_2 t r^{-\alpha_*(x,r)})$,
(ii) $\mathbb{E}^x(\tau_r^x) \le c_3 r^{\alpha_*(x,r)}$,
(iii) $\mathbb{P}^x\left(\sup_{s\le t}|X_s - x| \ge r\right) \ge c_4 t r^{-\alpha_*(x,r)}$ for $t = t(r) > 0$ small.

5. Integral criteria for upper functions

Let $(X_t)_{t\ge0}$ be a Markov process and $f : [0,1] \to [0,\infty)$ a non-decreasing function. The aim of this section is to derive sufficient conditions for

$$\limsup_{t\to0} \frac{1}{f(t)} \sup_{s\le t}|X_s - x| \le c \quad \mathbb{P}^x\text{-a.s.} \tag{26}$$

in terms of certain integrals. Our first main result is the following theorem.

5.1. **Theorem.** *Let $(X_t)_{t\ge0}$ be a Markov process with càdlàg sample paths and $f : [0,1] \to [0,\infty)$ a non-decreasing function. If*

$$\int_0^1 \frac{1}{t} \sup_{|z-x|\le f(t)} \mathbb{P}^z\left(\sup_{s\le t}|X_s - z| \ge f(t)\right) dt < \infty \tag{27}$$

for some $x \in \mathbb{R}^d$, then

$$\limsup_{t\to0} \frac{1}{f(t)} \sup_{s\le t}|X_s - x| \le 4 \quad \mathbb{P}^x\text{-}a.s.$$

Proof. **1°** Claim:

$$\mathbb{P}^x\left(\sup_{s\le 2t}|X_s - x| \ge 2r\right) \le 3 \sup_{|z-x|\le r} \mathbb{P}^z\left(\sup_{s\le t}|X_s - z| \ge r\right), \qquad x \in \mathbb{R}^d,\ r>0,\ t>0. \tag{28}$$

To prove this, we note that

$$\mathbb{P}^x\left(\sup_{s\le 2t}|X_s - x| \ge 2r\right) \le \mathbb{P}^x\left(\sup_{s\le t}|X_s - x| \ge 2r\right) + \mathbb{P}^x\left(\sup_{s\le t}|X_{s+t} - x| \ge 2r\right),$$

and, by the Markov property,

$$\begin{aligned}\mathbb{P}^x\left(\sup_{s\le t}|X_{s+t} - x| \ge 2r\right) &= \mathbb{E}^x\left(\mathbb{P}^z\left[\sup_{s\le t}|X_s - x| \ge 2r\right]\Big|_{z=X_t}\right)\\ &\le \mathbb{P}^x(|X_t - x| \ge r) + \sup_{|z-x|\le r} \mathbb{P}^z\left(\sup_{s\le t}|X_s - z| \ge r\right).\end{aligned}$$

2° This part of the proof uses an idea from Khintchine [20]. Fix $x \in \mathbb{R}^d$ such that (27) holds. Since f is monotone, we have

$$p_n := \mathbb{P}^x\left(\sup_{2^{-(n+1)}\le s\le 2^{-n}} \frac{1}{f(s)} \sup_{r\le s}|X_r - x| \ge 4\right) \le \mathbb{P}^x\left(\sup_{s\le 2^{-n}}|X_s - x| \ge 4f(2^{-(n+1)})\right)$$

for every $n \in \mathbb{N}$. Take any $\theta_n \in [2^{-n}, 2^{-(n-1)}]$, then $\theta_n/2 \le 2^{-n}$ and using 1° and the monotonicity of f, we get

$$p_n \le \mathbb{P}^x\left(\sup_{s\le\theta_n}|X_s - x| \ge 4f(\theta_n/4)\right) \le 9 \sup_{|z-x|\le 3f(\theta_n/4)} \mathbb{P}^z\left(\sup_{s\le\theta_n/4}|X_r - z| \ge f(\theta_n/4)\right).$$

Writing $\theta_n = 2^{-u}$ for $u \in [n-1,n]$ and integrating with respect to $u \in [n-1,n]$, it follows that

$$p_n \le 9\int_{n-1}^n \sup_{|z-x|\le 3f(2^{-u-2})} \mathbb{P}^z\left(\sup_{s\le 2^{-u-2}}|X_r - z| \ge f(2^{-u-2})\right) du.$$

By a change of variables ($t = 2^{-u-2}$),

$$p_n \le \frac{9}{|\log 2|}\int_{2^{-(n+2)}}^{2^{-(n+1)}} \frac{1}{t} \sup_{|z-x|\le 3f(t)} \mathbb{P}^z\left(\sup_{r\le t}|X_r - z| \ge f(t)\right) dt,$$

and so (27) yields $\sum_{n\in\mathbb{N}} p_n < \infty$. Applying the Borel–Cantelli lemma, we conclude that

$$\limsup_{n\to\infty} \sup_{2^{-(n+1)}\le s\le 2^{-n}} \frac{1}{f(s)} \sup_{r\le s}|X_r - x| \le 4 \quad \mathbb{P}^x\text{-a.s.}$$

□

It is natural to ask whether the two suprema in (27) are needed, i.e. if upper functions can also be characterized in terms of the integral $\int_0^1 \frac{1}{t}\mathbb{P}^x(|X_t - x| \geq Cf(t))\,dt$. Our next result shows that this is possible under some additional assumptions.

5.2. **Proposition.** *Let $(X_t)_{t\geq 0}$ be a strong Markov process with càdlàg sample paths. Let $f : [0,1] \to [0,\infty)$ be a non-decreasing function such that*[2]

$$C := \operatorname{essinf}\left\{\limsup_{n\to\infty} \frac{f(s^n)}{f(s^{n+1})}; s \in (0,1)\right\} < \infty. \tag{29}$$

Assume that the following conditions are satisfied for some constants $\varrho, \kappa > 0$ and a function $R : [0,1] \to (0,\infty]$:

$$\limsup_{t\to 0} \sup_{|z-x|\leq R(t)} \mathbb{P}^z(|X_t - z| \geq \varrho f(t)) < 1 \tag{30}$$

$$\sum_{n\geq 1} \mathbb{P}^x\left(\sup_{u\leq s^n} |X_u - x| > R(s^n)\right) < \infty \quad \text{for a.e. } s \in (0,1) \tag{31}$$

$$\int_0^1 \frac{1}{t}\mathbb{P}^x(|X_t - x| > \kappa f(t))\,dt < \infty. \tag{32}$$

Then

$$\limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t} |X_s - x| \leq C(\varrho + \kappa) \quad \mathbb{P}^x\text{-a.s.}$$

5.3. *Remark.* (i) Since f is non-decreasing, the constant C in (29) is greater or equal than 1. If f is regularly varying at zero, i.e. if the limit

$$L(a) := \lim_{t\to 0} \frac{f(at)}{f(t)}$$

exists for all $a > 0$, then $C = 1$; this follows from the fact that, by Karamata's characterization theorem, see e.g. [6], the limit L is of the form $L(a) = a^\varrho$ for some $\varrho \geq 0$.
(ii) There is a trade-off between (30) and (31) regarding the choice of R; e.g. for $R \equiv \infty$, condition (31) is trivially satisfied but a uniform bound for $z \in \mathbb{R}^d$ is needed in (30).
(iii) If $(X_t)_{t\geq 0}$ is a Lévy-type process, then the maximal inequality (20) shows that (31) is automatically satisfied for $R(t) \equiv R$ constant.
(iv) It is not hard to check that

$$\int_{(0,1)} \frac{1}{t}\mathbb{P}^x\left(\sup_{s\leq t} |X_s - x| > \kappa f(t)\right) dt < \infty \tag{33}$$

implies that (31) and (32) hold with $R(t) = \kappa f(t)$.

For the proof of Proposition 5.2 we use the following Ottaviani-type inequality; for $R = \infty$ this is the classical Ottaviani inequality for Markov processes, see e.g. [13, p. 420] or [17, p. 125].

5.4. **Lemma.** *Let $(X_t)_{t\geq 0}$ be a strong Markov process with càdlàg sample paths. Then*

$$\mathbb{P}^x\left(\sup_{s\leq t} |X_s - x| > u + v\right) \leq \frac{1}{1 - \alpha_{R,x}(t,u)}\left[\mathbb{P}^x(|X_t - x| > v) + \mathbb{P}^x\left(\sup_{s\leq t} |X_s - x| > R\right)\right]$$

for all $x \in \mathbb{R}^d$, $u, v > 0$ and $R \in (0,\infty]$, where

$$\alpha_{R,x}(t,u) := \sup_{s\leq t} \sup_{|z-x|\leq R} \mathbb{P}^z(|X_s - z| \geq u).$$

Proof. Denote by τ_r^x the first exit time of $(X_t)_{t\geq 0}$ from the closed ball $\overline{B(x,r)}$ and set $\sigma := \tau_{u+v}^x$ for fixed $u, v > 0$. We have

$$\begin{aligned}\mathbb{P}^x\left(\sup_{s\leq t} |X_s - x| > u + v\right) &= \mathbb{P}^x(\sigma \leq t) \\ &\leq \mathbb{P}^x(|X_t - x| > v) + \mathbb{P}^x\left(|X_t - x| \leq v, \sigma \leq t, \tau_R^x > t\right) + \mathbb{P}^x(\tau_R^x \leq t).\end{aligned}$$

[2]Here, essinf denotes the essential infimum with respect to Lebesgue measure.

By the strong Markov property,

$$\begin{aligned}\mathbb{P}^x\left(|X_t - x| \le v, \sigma \le t, \tau_R^x > t\right) &\le \mathbb{P}^x\left(|X_t - X_\sigma| \ge u, \sigma \le t, |X_\sigma - x| \le R\right) \\ &= \mathbb{E}^x\left[\mathbb{1}_{\{\sigma \le t\}}\mathbb{1}_{\{|X_\sigma - x| \le R\}}\mathbb{P}^z(|X_{t-s} - z| \ge u)\big|_{z=X_\sigma, s=\sigma}\right] \\ &\le \alpha_{R,x}(t,u)\mathbb{P}^x(\sigma \le t),\end{aligned}$$

and so

$$\begin{aligned}\mathbb{P}^x\left(\sup_{s\le t}|X_s - x| > u + v\right)(1 - \alpha_{R,x}(t,u)) &\le \mathbb{P}^x(|X_t - x| > v) + \mathbb{P}^x(\tau_R^x \le t) \\ &= \mathbb{P}^x(|X_t - x| > v) + \mathbb{P}^x\left(\sup_{s\le t}|X_s - x| > R\right). \qquad \square\end{aligned}$$

Proof of Proposition 5.2. **1°** Claim:

$$\sum_{n\in\mathbb{N}}\int_0^1 \mathbb{P}^x(|X_{s^n} - x| > \kappa f(s^n))\log\frac{1}{s}\,ds < \infty. \tag{34}$$

By a change of variables, $t = s^n$, $dt = nt^{(n-1)/n}\,ds$, we find that

$$\int_0^1 \mathbb{P}(|X_{s^n} - x| > \kappa f(s^n))\log\frac{1}{s}\,ds = \int_0^1 \frac{1}{n^2}t^{1/n}\log\frac{1}{t}\mathbb{P}^x(|X_t - x| > \kappa f(t))\frac{1}{t}\,dt.$$

As

$$\sum_{n\in\mathbb{N}}\frac{1}{n^2}t^{1/n}\log\frac{1}{t} \le 2, \qquad t \in (0,1),$$

cf. Lemma A.2, the monotone convergence theorem yields

$$\sum_{n\in\mathbb{N}}\int_0^1 \mathbb{P}(|X_{s^n} - x| > \kappa f(s^n))\log\frac{1}{s}\,ds \le 2\int_0^1 \mathbb{P}^x(|X_t - x| > \kappa f(t))\frac{1}{t}\,dt,$$

and the latter integral is finite by (32). This proves (34). In particular, there is a Lebesgue null set $N \subseteq (0,1)$ such that

$$\sum_{n\in\mathbb{N}}\mathbb{P}^x(|X_{s^n} - x| > \kappa f(s^n)) < \infty \qquad \text{for all } \; s \in (0,1) \setminus N. \tag{35}$$

2° Fix $\varepsilon > 0$, and take $s \in (0,1) \setminus N$ such that $\limsup_{n\to\infty} f(s^n)/f(s^{n+1}) \le (C+\varepsilon)$ for the constant C defined in (29). By Lemma 5.4, we have

$$\begin{aligned}&\mathbb{P}^x\left(\sup_{r\le s^n}|X_r - x| > (\kappa + \varrho)f(s^n)\right) \\ &\le \frac{1}{1 - \alpha_{R,x}(s^n, \varrho f(s^n))}\left[\mathbb{P}^x(|X_{s^n} - x| > \kappa f(s^n)) + \mathbb{P}^x\left(\sup_{u\le s^n}|X_u - x| > R(s^n)\right)\right],\end{aligned}$$

where $\alpha_{R,x}(t,r) := \sup_{u\le t}\sup_{|z-x|\le R(t)}\mathbb{P}^z(|X_u - z| \ge r)$. From (30) and the monotonicity of f, we see that there exists some $\delta \in (0,1)$ such that

$$\alpha_{R,x}(s^n, \varrho f(s^n)) \le \sup_{r\le s^n}\sup_{|z-x|\le R(s^n)}\mathbb{P}^z(|X_r - z| \ge \varrho f(r)) \le 1 - \delta \tag{36}$$

for $n \gg 1$. Thus,

$$\mathbb{P}^x\left(\sup_{r\le s^n}|X_r - x| > (\kappa + \varrho)f(s^n)\right) \le \frac{1}{\delta}\left[\mathbb{P}^x(|X_{s^n} - x| > \kappa f(s^n)) + \mathbb{P}^x\left(\sup_{u\le s^n}|X_u - x| > R(s^n)\right)\right],$$

which implies, by (35) and (31),

$$\sum_{n\in\mathbb{N}}\mathbb{P}^x\left(\sup_{r\le s^n}|X_r - x| > (\kappa + \varrho)f(s^n)\right) < \infty.$$

Applying the Borel-Cantelli lemma gives

$$\limsup_{n\to\infty}\frac{1}{f(s^n)}\sup_{r\le s^n}|X_r - x| \le \varrho + \kappa \quad \mathbb{P}^x\text{-a.s.}$$

If $t \in [s^{n+1}, s^n)$ for some $n \gg 1$, then

$$\frac{1}{f(t)} \sup_{r \leq t} |X_r - x| \leq \frac{1}{f(s^{n+1})} \sup_{r \leq s^n} |X_r - x| = \frac{f(s^n)}{f(s^{n+1})} \frac{1}{f(s^n)} \sup_{r \leq s^n} |X_r - x|,$$

and so

$$\limsup_{t \to 0} \frac{1}{f(t)} \sup_{r \leq t} |X_r - x| \leq \limsup_{n \to \infty} \left(\frac{f(s^n)}{f(s^{n+1})} \frac{1}{f(s^n)} \sup_{r \leq s^n} |X_r - x| \right) \leq (C + \varepsilon)(\kappa + \varrho)$$

$\mathbb{P}^x$-almost surely. Since $\varepsilon > 0$ is arbitrary, this finishes the proof. □

Combining Theorem 5.1 with the maximal inequality (20), we get the following criterion; see [25, Proposition 9] for a closely related result.

5.5. **Corollary.** *Let $(X_t)_{t \geq 0}$ be a Lévy-type process with symbol q. If $f : [0,1] \to [0, \infty)$ is a non-decreasing function such that*

$$\int_0^1 \sup_{|z-x| \leq f(t)} \sup_{|\xi| \leq 1/(Cf(t))} |q(z, \xi)| \, dt < \infty \tag{37}$$

for some constant $C > 0$, then

$$\lim_{t \to 0} \frac{1}{f(t)} \sup_{s \leq t} |X_s - x| = 0 \quad \mathbb{P}^x\text{-a.s.}$$

Proof. If the integral in (37) is finite some $C > 0$, then it is finite for arbitrary small $C > 0$. *Indeed:* Since $\xi \mapsto \sqrt{|q(z, \xi)|}$ is subadditive, we have

$$|q(z, 2\xi)| = |q(z, \xi + \xi)| \leq \left(\sqrt{|q(z, \xi)|} + \sqrt{|q(z, \xi)|} \right)^2 = 4|q(z, \xi)|,$$

which implies that

$$\int_0^1 \sup_{|z-x| \leq f(t)} \sup_{|\xi| \leq 1/(2^{-n}Cf(t))} |q(z, \xi)| \, dt \leq 4^n \int_0^1 \sup_{|z-x| \leq f(t)} \sup_{|\xi| \leq 1/(Cf(t))} |q(z, \xi)| \, dt < \infty$$

for every $n \in \mathbb{N}$. Applying the maximal inequality (20) and Theorem 5.1 yields

$$\limsup_{t \to 0} \frac{1}{f(t)} \sup_{s \leq t} |X_s - x| \leq 4C2^{-n} \quad \mathbb{P}^x\text{-a.s.}$$

Letting $n \to \infty$ proves the assertion. □

We conclude this section with the following result on the growth of sample paths of Lévy-type processes.

5.6. **Proposition.** *Let $(X_t)_{t \geq 0}$ be a Lévy-type process with symbol q. Then:*

(i) $\limsup_{t \to 0} t^{-\kappa} \sup_{s \leq t} |X_s - x| = 0$ *$\mathbb{P}$-a.s. for any $\kappa < \frac{1}{2}$.*
(ii) *If $x \in \mathbb{R}^d$ is such that*

$$\sup_{|z-x| \leq R} \sup_{|\xi| \leq r} |q(z, \xi)| \leq c \frac{r^2}{|\log r|^{1+\varepsilon}}, \qquad r \gg 1, \tag{38}$$

for some constants $R > 0$, $c > 0$ and $\varepsilon > 0$, then

$$\limsup_{t \to 0} \frac{1}{\sqrt{t \log |\log t|}} \sup_{s \leq t} |X_s - x| = 0 \quad \mathbb{P}^x\text{-a.s.}$$

Khintchine [20] (see also [39, Appendix, Theorem 4]) showed that any Lévy process without Gaussian component satisfies

$$\limsup_{t \to 0} \frac{|X_t|}{\sqrt{t \log |\log t|}} = 0 \quad \text{a.s.}$$

One might expect that an analogous result holds for Lévy-type processes but this does not seem to follow from our results; note that (38) is stronger than assuming that $(X_t)_{t \geq 0}$ has no Gaussian component, cf. [30, Lemma A.3].

Proof of Proposition 5.6. **1°** Because of the subadditivity of $\xi \mapsto \sqrt{|q(x,\xi|}$, it holds that

$$|q(x,\xi)| \le \sup_{|\eta|\le 1} |q(x,\eta)|(1+|\xi|^2),$$

cf. [21, Theorem 6.2], and so

$$\sup_{|z-x|\le 1} \sup_{|\xi|\le r} |q(z,\xi)| \le c'(1+r^2)$$

for some constant $c' > 0$. Hence,

$$\int_0^1 \sup_{|z-x|\le 1} \sup_{|\xi|\le 1/(Ct^\kappa)} |q(z,\xi)|\, dt < \infty$$

for any $\kappa \in (0,\frac{1}{2})$ and $C > 0$. By Corollary 5.5, this proves (i).

2° Set $f(t) := \sqrt{t \log\log \frac{1}{t}}$, then, by (38),

$$\int_0^{1/e^e} \sup_{|z-x|\le R} \sup_{|\xi|\le 1/(Cf(t))} |q(z,\xi)|\, dt \le \frac{c}{C^2} \int_0^{1/e^e} \frac{1}{t\log\log\frac{1}{t}} \frac{1}{|\log\sqrt{C^2 t \log\log\frac{1}{t}}|^{1+\varepsilon}}\, dt$$

for every $C > 0$, and the latter integral is finite. Corollary 5.5 gives the assertion. □

In the remainder of the article, we prove the results announced in Section 2.

6. Proofs of Theorem 2.1 and Theorem 2.6

For the proof of Theorem 2.1 and Theorem 2.6, we need the following result which links two of our integral conditions.

6.1. **Lemma.** *Let $\psi : \mathbb{R}^d \to \mathbb{C}$ be a continuous negative definite function with Lévy triplet $(b,0,\nu)$, and set*

$$\psi^*(r) := \sup_{|\xi|\le r} \operatorname{Re}\psi(\xi), \qquad r > 0.$$

If $f : [0,1] \to [0,\infty)$ is a non-decreasing function, then the implication

$$\int_0^1 \nu(\{|y| \ge f(t)\})\, dt < \infty \implies \int_0^1 \psi^*\left(\frac{1}{f(t)}\right) dt < \infty$$

holds in each of the following two cases.

(A1) *The Lévy measure ν satisfies*

$$\limsup_{r\to 0} \frac{\int_{|y|\le r} |y|^2\, \nu(dy)}{r^2 \nu(\{|y| > r\})} < \infty.$$

(A2) *There is a constant $c > 0$ such that*

$$\int_{r<f(t)} \frac{1}{f(t)^2}\, dt \le c\frac{f^{-1}(r)}{r^2}, \qquad r \in (0,1).$$

6.2. *Remark.* (i) There are several equivalent formulations of condition (A1) in terms of so-called concentration functions. If we define, following [33],

$$G(r) := \nu(\{|y| > r\}) \quad \text{and} \quad K(r) := \frac{1}{r^2}\int_{|y|\le r} |y|^2\, \nu(dy),$$

then (A1) can be stated equivalently in the following way:

$$\limsup_{r\to 0} \frac{K(r)}{G(r)} < \infty.$$

Set

$$h(r) := \int_{y\ne 0} \min\left\{1, \frac{|y|^2}{r^2}\right\} \nu(dy) = K(r) + G(r),$$

then we see that

$$\text{(A1)} \iff \liminf_{r\to 0} \frac{G(r)}{h(r)} > 0. \tag{39}$$

Since

$$\frac{1}{c}h(r) \le \psi^*\left(\frac{1}{r}\right) \le ch(r), \qquad r > 0,$$

for some constant $c > 0$, depending only on the dimension d, see e.g. [37, Lemma 5.1 and p. 595] or [14, Lemma 4], it follows that

$$\text{(A1)} \iff \liminf_{r\to 0} \frac{G(r)}{\psi^*(1/r)} = \liminf_{r\to 0} \frac{\nu(\{|y| > r\})}{\psi^*(1/r)} > 0. \tag{40}$$

Moreover, there is a sufficient condition for (A1) in terms of the function

$$I(r) := \int_{y\neq 0} \min\{r^2, |y|^2\}\, \nu(dy) = r^2 h(r);$$

namely, if

$$\liminf_{r\to 0} \frac{I(2r)}{I(r)} > 1, \tag{41}$$

then (A1) holds. *Indeed:* By Tonelli's theorem,

$$\begin{aligned} I(r) = \int_{y\neq 0} \int_0^{\min\{r^2,|y|^2\}} dz\, \nu(dy) &= \int_{\mathbb{R}} \int_{y\neq 0} \mathbb{1}_{\{|z|<r^2\}} \mathbb{1}_{\{|z|<|y|^2\}}\, \nu(dy)\, dz \\ &= \int_0^{r^2} \nu(\{|y| > \sqrt{z}\})\, dz. \end{aligned}$$

Thus,

$$I(2r) = I(r) + \int_{r^2}^{4r^2} \nu(\{|y| > \sqrt{z}\})\, dz \le I(r) + 3r^2 \nu(\{|y| > r\}).$$

Consequently, (41) implies that

$$1 < \liminf_{r\to 0} \frac{I(2r)}{I(r)} \le 1 + \liminf_{r\to 0} \frac{3r^2 \nu(\{|y| > r\})}{I(r)}.$$

As $I(r) = r^2 h(r)$, this is equivalent to (39) and hence to (A1). Let us mention that a condition similar to (41) appears in the monograph [3] by Bertoin in the study of upper functions for sample paths of subordinators.

(ii) If ν is the Lévy measure of a one-dimensional Lévy process and $\nu(\{|y| \ge r\})$ grows faster than $\log r$ as $r \to 0$, then (A1) implies that (the law of) X_t has a smooth density $p_t \in C_b^\infty(\mathbb{R})$ for every $t > 0$, see [19, Section 5] and also [24, p. 127].

Proof of Lemma 6.1. **1°** Suppose that (A1) holds. Then there is some constant $c > 0$ such that

$$\psi^*\left(\frac{1}{r}\right) = \sup_{|\xi|\le 1/r} \operatorname{Re}\psi(\xi) \le c\nu(\{|y| > r\})$$

for $r > 0$ small, cf. (40). Since we may assume without loss of generality that $f(t) \to 0$ as $t \downarrow 0$, we find that

$$\int_0^\delta \psi^*\left(\frac{1}{f(t)}\right) dt \le c \int_0^\delta \nu(\{|y| > f(t)\})\, dt$$

for some $\delta > 0$. As ψ is bounded on compact sets, this proves the assertion.

2° Suppose that (A2) holds. From

$$\psi^*(r) = \sup_{|\xi|\le r} \operatorname{Re}\psi(\xi) \le 2 \int_{y\neq 0} \min\{1, |y|^2 r^2\}\, \nu(dy),$$

we get

$$\int_0^1 \psi^*\left(\frac{1}{f(t)}\right) dt \le 2 \int_0^1 \frac{1}{f(t)^2} \int_{|y|\le f(t)} y^2\, \nu(dy)\, dt + 2 \int_0^1 \nu(\{|y| > f(t)\})\, dt. \tag{42}$$

The second integral on the right-hand side of (42) is finite by assumption, and so it suffices to show that the first integral

$$J := \int_0^1 \frac{1}{f(t)^2} \int_{|y|\le f(t)} y^2\, \nu(dy)\, dt$$

is finite. By Tonelli's theorem and (A2), we have

$$J = \int_{y\neq 0} |y|^2 \int_{f(t)\geq |y|} \frac{1}{f(t)^2}\, dt\, \nu(dy) \leq c \int_{y\neq 0} f^{-1}(|y|)\, \nu(dy).$$

Since f is non-decreasing, we find by another application of Tonelli's theorem that

$$J \leq c \int_{y\neq 0} \int_{t\leq f^{-1}(|y|)} dt\, \nu(dy) \leq c \int_{y\neq 0} \int_{f(t)\leq |y|} dt\, \nu(dy) = c \int_0^1 \nu(\{|y| \geq f(t)\})\, dt < \infty. \qquad \square$$

Proof of Theorem 2.1. (L1) $\implies$ (L2): If $\int_0^1 \nu(\{|y| \geq cf(t)\})\, dt < \infty$, then it follows from Lemma 6.1 and the sector condition that

$$\int_0^1 \sup_{|\xi|\leq 1/(cf(t))} |\psi(\xi)|\, dt < \infty.$$

By the subadditivity of $\xi \mapsto \sqrt{|\psi(\xi)|}$, this implies that

$$\int_0^1 \sup_{|\xi|\leq 1/(2^{-n}cf(t))} |\psi(\xi)|\, dt < \infty.$$

for all $n \in \mathbb{N}$, see the proof of Corollary 5.5. Since the integral expression is monotone w.r.t. c, we conclude that

$$\int_0^1 \sup_{|\xi|\leq 1/(cf(t))} |\psi(\xi)|\, dt < \infty \qquad \text{for all} \quad c > 0.$$

(L2) $\implies$ (L3): This is clear from the maximal inequality, cf. Proposition 4.1.
(L3) $\iff$ (L4): The implication (L3) $\implies$ (L4) is obvious. The other direction is immediate from Etemadi's inequality, see e.g. [5, Theorem 22.5] or [15, Theorem 7.6], which shows that

$$\mathbb{P}\left(\sup_{s\leq t} |X_s| \geq 3r\right) \leq 3\mathbb{P}(|X_t| \geq r), \qquad r > 0,\ t > 0.$$

(L3) $\implies$ (L5): This is immediate from Theorem 5.1; note that the supremum in (27) breaks down because Lévy processes are homogenous in space.
(L5) $\implies$ (L6) $\implies$ (L7): Obvious.
(L7) $\implies$ (L1): In dimension $d = 1$, this follows from [4, Proposition 4.2]. The following reasoning works in any dimension $d \geq 1$. By Blumenthal's 0-1 law, there exists a constant $C > 0$ such that

$$\limsup_{t\to 0} \frac{1}{f(t)} |X_t| \leq \frac{C}{2} \quad \text{almost surely.} \tag{43}$$

Suppose that $\int_0^1 \nu(\{|y| \geq 2Cf(t)\})\, dt$ is infinite. As f is non-decreasing, the series test yields

$$\sum_{n=2}^{\infty} \nu(\{|y| \geq 2Cf(1/n)\}) \left(\frac{1}{n-1} - \frac{1}{n}\right) = \infty. \tag{44}$$

The random variables

$$N_{s,t}^{(r)} := \sharp\{u \in (s,t]; |\Delta X_u| \geq r\}, \qquad 0 \leq s < t,\ r > 0,$$

are Poisson distributed with parameter $(t-s)\nu(\{|y| \geq r\})$, and so $Y_n := N_{1/(n+1),1/n}^{2Cf(1/n)}$ are Poisson distributed with parameter $\lambda_n := \nu(\{|y| \geq 2Cf(1/n)\}) \left(\frac{1}{n} - \frac{1}{n+1}\right)$. Using the elementary estimate $1 - e^{-x} \geq x/(1+x)$, we get

$$\sum_{n\in\mathbb{N}} \mathbb{P}(Y_n \geq 1) = \sum_{n\in\mathbb{N}} \left(1 - e^{-\lambda_n}\right) \geq \sum_{n\in\mathbb{N}} \frac{\lambda_n}{1+\lambda_n} \geq \sum_{n\in\mathbb{N}} \min\left\{\lambda_n, \frac{1}{2}\right\} = \infty;$$

here we use that (44) implies $\sum_{n\in\mathbb{N}} \lambda_n = \infty$ because $\frac{1}{n-1} - \frac{1}{n} \approx \frac{1}{n^2} \approx \frac{1}{n+1} - \frac{1}{n}$ for $n \gg 1$. Since the random variables Y_n, $n \in \mathbb{N}$, are independent, the Borel–Cantelli lemma shows that the event $\{Y_n \geq 1 \text{ infinitely often}\}$ has probability 1. Thus, with probability 1 there are infinitely many $n \in \mathbb{N}$ such that $|\Delta X_u| \geq 2Cf(1/n)$ for some $u \in [\frac{1}{n+1}, \frac{1}{n}]$. Since either $|X_u| \geq Cf(1/n) \geq Cf(u)$ or $|X_{u-}| \geq Cf(1/n) \geq Cf(u-)$ for any such $u \in [\frac{1}{n+1}, \frac{1}{n}]$, we conclude that

$$\limsup_{t\to 0} \frac{1}{f(t)} |X_t| \geq C \quad \text{almost surely,}$$

which contradicts (43). Hence, $\int_0^1 \nu(\{|y| \geq 2Cf(t)\})\,dt < \infty$. See (the proof of) Theorem 2.10 for an alternative reasoning.
The random variables $\limsup_{t\to 0} \frac{1}{f(t)}|X_t|$ and $\limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t}|X_s|$ are $\mathcal{F}_{0+}$-measurable, and therefore Blumenthal's 0-1-law shows that the events in (L5)-(L7) have probability 0 or 1. Consequently, 'almost surely' may be replaced by 'with positive probability' in each of the statements. □

Proof of Theorem 2.6. (LTP1) $\implies$ (LTP2): Without loss of generality, $f(t) \to 0 = f(0)$ as $t \downarrow 0$; otherwise the assertion is immediate from the local boundedness of the symbol, cf. (18). It follows from (A1') that

$$\liminf_{r\to 0} \inf_{|z-x|\leq R} \frac{\nu(z, \{|y| > r\})}{\sup_{|\xi|\leq 1/r} \operatorname{Re} q(z,\xi)} > 0,$$

see Remark 6.2(i). Since the sector condition holds (with a constant not depending on $z \in \overline{B(x,R)}$), we find that

$$\liminf_{r\to 0} \inf_{|z-x|\leq R} \frac{\nu(z, \{|y| > r\})}{\sup_{|\xi|\leq 1/r} |q(z,\xi)|} > 0,$$

i.e. there are constants $K > 0$ and $\delta > 0$ such that

$$\sup_{|\xi|\leq 1/r} |q(z,\xi)| \leq K\nu(z, \{|y| > r\}), \qquad z \in \overline{B(x,R)},$$

for $r \leq \delta$. As $f(t) \to 0$ as $t \downarrow 0$, this implies

$$\sup_{|z-x|\leq f(t)} \sup_{|\xi|\leq 1/(cf(t))} |q(z,\xi)| \leq K \sup_{|z-x|\leq f(t)} \nu(z, \{|y| > cf(t)\})$$

for $t > 0$ small. Integrating with respect to t and using the local boundedness of q, we conclude that

$$\int_0^1 \sup_{|z-x|\leq f(t)} \sup_{|\xi|\leq 1/(cf(t))} |q(z,\xi)|\,dt < \infty.$$

(LTP2) $\implies$ (LTP3): If the integral in (LTP2) is finite for some $\varepsilon > 0$, then it is finite for all $\varepsilon > 0$; this follows from the subadditivity of $\xi \mapsto \sqrt{|q(z,\xi)|}$, see the proof of Corollary 5.5. The implication (LTP2) $\implies$ (LTP3) is now immediate from the maximal inequality (20).
(LTP3) $\implies$ (LTP4): cf. Theorem 5.1. □

7. Proof of the converse and the lower growth bounds

In this section, we present the proofs of Proposition 2.8, Proposition 2.9 and Theorem 2.10.

Proof of Proposition 2.8. **1°** If

$$\int_0^1 \sup_{|z-x|\leq f(t)} \mathbb{P}^z\left(\sup_{s\leq t}|X_s - z| \geq f(t)\right) \frac{1}{t}\,dt < \infty,$$

then Theorem 5.1 shows that $\limsup_{t\to 0} \frac{1}{f(t)} \sup_{s\leq t} |X_s - x| \leq 4$ $\mathbb{P}^x$-almost surely. Consequently, $\mathbb{P}^x(A_k) \to 1$ for $A_k := \{\forall t \leq 1/k : \frac{1}{f(t)} \sup_{s\leq t} |X_s - x| < 5\}$. Hence,

$$\sup_{t\leq 1/k} \mathbb{P}^x\left(\sup_{s\leq t}|X_s - x| \geq 5f(t)\right) \leq \mathbb{P}^x(A_k^c) \xrightarrow[k\to\infty]{} 0.$$

By Corollary 4.5, this implies

$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s - x| \geq 5f(t)\right) \geq \frac{1}{2}tG(x, 10f(t)), \qquad t \leq \frac{1}{k},$$

for $k \gg 1$ sufficiently large, where $G(x,r) := \inf_{|z-x|\leq r} \nu(z, \{|y| > r\})$. Dividing both sides by t and integrating over $t \in (0,1)$ yields $\int_0^1 G(x, 10f(t))\,dt < \infty$, which proves (i).
2° Suppose that

$$\int_0^1 G(x, Cf(t))\,dt < \infty$$

for some $C > 0$ and $G(x,r)$ as in 1°, and assume that (A1') holds for some $R > 0$. It follows from Remark 6.2 that there is some constant $\gamma > 0$ such that

$$\liminf_{r\to 0} \inf_{|z-x|\leq R} \frac{\nu(z,\{|y|>r\})}{\sup_{|\xi|\leq 1/r} \operatorname{Re} q(z,\xi)} \geq \gamma.$$

Thus,

$$\inf_{|z-x|\leq Cf(t)} \sup_{|\xi|\leq 1/(Cf(t))} \operatorname{Re} q(z,\xi) \leq \frac{1}{\gamma} G(x, Cf(t))$$

for $t > 0$ small. Since the symbol q is bounded on compact sets, integration with respect to t gives

$$\int_0^1 \inf_{|z-x|\leq Cf(t)} \sup_{|\xi|\leq 1/(Cf(t))} \operatorname{Re} q(z,\xi)\, dt < \infty.$$

Because of the subadditivity of the mapping $\xi \mapsto \sqrt{\operatorname{Re} q(z,\xi)}$, we may replace $1/(Cf(t))$ by $c/f(t)$ for any $c > 0$, compare the proof of Corollary 5.5. □

Proof of Proposition 2.9. Let $f : [0,1] \to [0,\infty)$ be such that

$$\limsup_{t\to 0} t \inf_{|z-x|\leq Rf(t)} \sup_{|\xi|\leq 1/(Cf(t))} \operatorname{Re} q(z,\xi) = \infty, \tag{45}$$

for all $R \geq 1$ and some constant $C = C(R) > 0$.

1° Claim: the convergence in (45) holds for any $C > 0$. *Indeed*: Clearly, it suffices to show that (45) holds with C replaced by $C2^n$, $n \in \mathbb{N}$. Because of the subadditivity of $\xi \mapsto \sqrt{\operatorname{Re} q(z,\xi)}$, we have $\operatorname{Re} q(z, 2\xi) \leq 4 \operatorname{Re} q(z,\xi)$ for all $\xi, z \in \mathbb{R}^d$ implying

$$\sup_{|\xi|\leq r} \operatorname{Re} q(z,\xi) \geq \frac{1}{4} \sup_{|\xi|\leq 2r} \operatorname{Re} q(z,\xi) \geq \ldots \geq \frac{1}{4^n} \sup_{|\xi|\leq 2^n r} \operatorname{Re} q(z,\xi)$$

for all $r > 0$. Using this estimate for $r = 1/(2^n C f(t))$, we see that (45) holds with C replaced by $C2^n$.

2° The idea for this part of the proof is from [37]. For fixed $R \geq 1$, pick $(t_k)_{k\in\mathbb{N}} \subseteq (0,1)$ with $t_k \downarrow 0$ and

$$\lim_{k\to\infty} t_k \inf_{|z-x|\leq Rf(t_k)} \sup_{|\xi|\leq 1/(Rf(t_k))} \operatorname{Re} q(z,\xi) = \infty.$$

Then the maximal inequality (21) shows that

$$\mathbb{P}^x\left(\sup_{s\leq t_k} |X_s - x| < Rf(t_k)\right) \xrightarrow{k\to\infty} 0,$$

and so, by Fatou's lemma,

$$\begin{aligned}
\mathbb{P}^x\left(\limsup_{k\to\infty} \left\{\sup_{s\leq t_k} |X_s - x| \geq Rf(t_k)\right\}\right) &\geq \limsup_{k\to\infty} \mathbb{P}^x\left(\sup_{s\leq t_k} |X_s - x| \geq Rf(t_k)\right) \\
&= 1 - \liminf_{k\to\infty} \mathbb{P}^x\left(\sup_{s\leq t_k} |X_s - x| < Rf(t_k)\right) \\
&= 1.
\end{aligned}$$

Consequently, there is a measurable set Ω_0 with $\mathbb{P}^x(\Omega_0) = 1$ such that $\sup_{s\leq t_k} |X_{t_k}(\omega) - x| \geq Rf(t_k)$ infinitely often for every $\omega \in \Omega_0$. In particular,

$$\limsup_{k\to\infty} \frac{1}{f(t_k)} \sup_{s\leq t_k} |X_s(\omega) - x| \geq R, \qquad \omega \in \Omega_0.$$

3° Now assume additionally that f is non-decreasing. For $\omega \in \Omega_0$, let $s_k = s_k(\omega) \in [0, t_k]$ be such that

$$|X_{s_k}(\omega) - x| \geq \frac{1}{2} \sup_{s\leq t_k} |X_s(\omega) - x|.$$

By the monotonicity, we have $f(s_k) \leq f(t_k)$, and so

$$\limsup_{t\to 0} \frac{1}{f(t)} |X_t(\omega) - x| \geq \limsup_{k\to\infty} \frac{1}{f(s_k)} |X_{s_k}(\omega) - x| \geq \frac{1}{2} \limsup_{k\to\infty} \frac{1}{f(t_k)} \sup_{s\leq t_k} |X_s(\omega) - x| \geq \frac{R}{2}.$$

As $R \geq 1$ is arbitrary, this proves (i).

4° It remains to prove (ii). To this end, we show that if f is regularly varying at zero, i.e.

$$\exists \beta > 0 \ \forall \lambda > 0 \ : \ \lim_{t\to 0} \frac{f(\lambda t)}{f(t)} = \lambda^\beta,$$

then (45) for $R = 1$ implies (45) for all $R \geq 1$. The desired lower bound for the growth of the sample paths then follows from the first part of this proof. Let $C > 0$ be such that (45) holds with $R = 1$. As we have seen in 1°, it follows that (45) holds with $R = 1$ for any $C > 0$. Since f is regularly varying at zero, there is $\lambda > 0$ such that $f(\lambda t)/f(t) \geq R$ for $t > 0$ small. Thus,

$$\begin{aligned}\limsup_{t\to 0} t \inf_{|z-x|\leq Rf(t)} \sup_{|\xi|\leq 1/(Cf(t))} \operatorname{Re} q(z,\xi) &\geq \limsup_{t\to 0} t \inf_{|z-x|\leq f(\lambda t)} \sup_{|\xi|\leq 1/(Cf(t))} \operatorname{Re} q(z,\xi) \\ &= \frac{1}{\lambda} \limsup_{t\to 0} t \inf_{|z-x|\leq f(t)} \sup_{|\xi|\leq 1/(Cf(t/\lambda))} \operatorname{Re} q(z,\xi).\end{aligned}$$

Using once more that f is regularly varying, we find that $f(t/\lambda) \geq \frac{1}{2}\frac{1}{\lambda^\beta} f(t) =: \gamma f(t)$ for $t > 0$ small. Hence,

$$\limsup_{t\to 0} t \inf_{|z-x|\leq Rf(t)} \sup_{|\xi|\leq 1/(Cf(t))} \operatorname{Re} q(z,\xi) \geq \frac{1}{\lambda} \limsup_{t\to 0} t \inf_{|z-x|\leq f(t)} \sup_{|\xi|\leq 1/(\gamma Cf(t))} \operatorname{Re} q(z,\xi) = \infty. \qquad \square$$

The key for the proof of our final main result, Theorem 2.10, is the following proposition.

7.1. **Proposition.** *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with characteristics $(b(x), 0, \nu(x,dy))$ and symbol q. Let $f : [0,1] \to [0,\infty)$ be a non-decreasing function. If*

$$\limsup_{n\to\infty} \frac{1}{n^2} \sup_{|z-x|\leq 3f(1/n)} \sup_{|\xi|\leq c/f(1/n)} |q(z,\xi)| < 1 \tag{46}$$

and

$$\int_0^1 \inf_{|z-x|\leq 5f(t)} \nu(z, \{|y| > 5f(t)\})\, dt = \infty, \tag{47}$$

for some $c > 0$ and $x \in \mathbb{R}^d$, then

$$\limsup_{t\to 0} \frac{1}{f(t)} |X_t - x| \geq 1 \quad \mathbb{P}^x\text{-a.s.}$$

7.2. *Remark.* (i) Replacing f by $C\cdot f$ for $C > 0$, we obtain immediately a sufficient condition for

$$\limsup_{t\to 0} \frac{1}{f(t)} |X_t - x| \geq C \quad \mathbb{P}^x\text{-a.s.}$$

(ii) By the local boundedness of q, there is a finite constant $c = c(R,x)$ such that $|q(z,\xi)| \leq c(1+|\xi|^2)$ for all $\xi \in \mathbb{R}^d$ and $|z-x| \leq R$, cf. (18). Thus, $\liminf_{t\downarrow 0} f(t)/t = \infty$ is a sufficient condition for (46); let us mention that this growth condition on f also appears in the study of upper functions for sample paths of Lévy processes, cf. [36]. More generally, if $\sup_{|z-x|\leq R} |q(z,\xi)| \leq c(1+|\xi|^\alpha)$ for some $\alpha \in (0,2]$, then (46) holds for any function f satisfying $\liminf_{t\downarrow 0} f(t)/t^{2/\alpha} = \infty$.

Proof of Proposition 7.1. Let $x \in \mathbb{R}^d$ and $c > 0$ be such that (46) and (47) hold, and set

$$G(x,r) := \inf_{|z-x|\leq r} \nu(z, \{|y| > r\}).$$

Using the subadditivity of $\xi \mapsto \sqrt{|q(z,\xi)|}$, we see that (46) actually holds for *any* $c > 0$.

1° By the monotonicity of f and $r \mapsto G(x,r)$, it follows from (47) that

$$\sum_{n=2}^\infty \left(\frac{1}{n-1} - \frac{1}{n}\right) G(x, 5f(1/n)) \geq \int_0^1 G(x, 5f(t))\, dt = \infty,$$

and so

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} G(x, 5f(1/n)) = \infty. \tag{48}$$

Moreover, we note that (47) implies that $f(t) \to 0$ as $t \downarrow 0$.

2° Denote by $(\mathcal{F}_t)_{t\geq 0}$ the canonical filtration of $(X_t)_{t\geq 0}$. We claim that

$$\sum_{n\in\mathbb{N}} \mathbb{E}^x(\mathbb{1}_{A_n} \mid \mathcal{F}_{1/(n+1)}) = \infty \quad \mathbb{P}^x\text{-a.s.} \tag{49}$$

for

$$A_n := \left\{ \frac{1}{f(1/n)} \sup_{\frac{1}{n+1} \le r < \frac{1}{n}} |X_r - x| \ge 1 \right\}.$$

To prove (49) we fix $n \in \mathbb{N}$ and note that, by the Markov property,

$$\mathbb{E}^x(\mathbb{1}_{A_n} \mid \mathcal{F}_{1/(n+1)}) = u(X_{1/(n+1)})$$

where

$$u(z) := \mathbb{P}^z \left(\sup_{r \le \frac{1}{n(n+1)}} |X_r - x| \ge f(1/n) \right), \qquad z \in \mathbb{R}^d.$$

We need a lower bound for the mapping u. If $z \notin B(x, f(1/n)$, then $|X_0 - x| \ge f(1/n)$ $\mathbb{P}^z$-a.s. which gives $u(z) = 1$. Next we consider the case $z \in B(x, f(1/n))$. By the triangle inequality,

$$u(z) \ge \mathbb{P}^z \left(\sup_{r \le \frac{1}{n(n+1)}} |X_r - z| \ge 2f(1/n) \right) =: U(z).$$

The maximal inequality (20) shows that

$$U(z) \le c' \frac{1}{n(n+1)} \sup_{|z-y| \le 2f(1/n)} \sup_{|\xi| \le 1/(2f(1/n))} |q(y,\xi)|$$

for some absolute constant $c' > 0$. Since $|z - x| \le f(1/n)$ and $\sqrt{|q(y,\cdot)|}$ is subadditive, we get

$$U(z) \le 4c' \frac{1}{n(n+1)} \sup_{|y-x| \le 3f(1/n)} \sup_{|\xi| \le 1/f(1/n)} |q(y,\xi)|,$$

see the proof of Corollary 5.5. Thus, by (46), $U(z) \le 1 - \varepsilon$ for $n \gg 1$ and some $\varepsilon \in (0,1)$. Applying Corollary 4.5 and using $|z - x| \le f(1/n)$, we find that

$$u(z) \ge U(z) \ge \varepsilon \frac{1}{n(n+1)} G(z, 4f(1/n)) \ge \varepsilon \frac{1}{n(n+1)} G(x, 5f(1/n)), \qquad z \in B(x, f(1/n)),$$

for $n \gg 1$. In summary,

$$\mathbb{E}^x(\mathbb{1}_{A_n} \mid \mathcal{F}_{1/(n+1)}) \ge \min \left\{ \varepsilon \frac{1}{n(n+1)} G(x, 5f(1/n)), 1 \right\}$$

for $n \gg 1$. Thus, by (48), $\sum_{n \in \mathbb{N}} \mathbb{E}^x(\mathbb{1}_{A_n} \mid \mathcal{F}_{1/(n+1)}) = \infty$ $\mathbb{P}^x$-a.s.

3° The almost sure divergence of the series implies by the conditional Borel-Cantelli lemma for backward filtrations, cf. Proposition A.1, that

$$\mathbb{P}^x \left(\limsup_{n \to \infty} A_n \right) = 1,$$

and so there is a measurable set $\tilde{\Omega}$ with $\mathbb{P}^x(\tilde{\Omega}) = 1$ such that

$$\forall \omega \in \tilde{\Omega} \; \forall n \gg 1 \; \exists t_n = t_n(\omega) \in \left[\frac{1}{n+1}, \frac{1}{n} \right) \; : \; \frac{1}{f(1/n)} |X_{t_n}(\omega) - x| \ge 1.$$

Using the monotonicity of f, we conclude that

$$\limsup_{t \to 0} \frac{1}{f(t)} |X_t(\omega) - x| \ge \limsup_{n \to \infty} \frac{1}{f(t_n)} |X_{t_n}(\omega) - x| \ge 1, \qquad \omega \in \tilde{\Omega}. \qquad \square$$

Proof of Theorem 2.10. First we prove (i). Let $f \ge 0$ be non-decreasing and $c > 0$ such that $\int_0^1 \inf_{|z-x| \le cf(t)} \nu(z; \{|y| > cf(t)\}) \, dt = \infty$. We consider separately the cases that (C1) resp. (C2) holds.

1° Assume that (C1) holds. If for every $R \ge 1$ the limit

$$\limsup_{t \to 0} t \inf_{|z-x| \le Rf(t)} \sup_{|\xi| \le 1/(Cf(t))} \operatorname{Re} q(z,\xi) \tag{50}$$

is infinite for some constant $C = C(R)$, then Proposition 2.9 yields

$$\limsup_{t \to 0} \frac{1}{f(t)} |X_t - x| = \infty \quad \mathbb{P}^x\text{-a.s.}$$

On the other hand, if (50) is finite for some $R \geq 1$ and all $C > 0$, then

$$\limsup_{t\to 0} t^2 \sup_{|z-x|\leq Cf(t)} \sup_{|\xi|\leq 1/f(t)} \operatorname{Re} q(z,\xi) \leq c' \limsup_{t\to 0} t \frac{\sup_{|z-x|\leq Cf(t)} \sup_{|\xi|\leq 1/f(t)} |q(z,\xi)|}{\inf_{|z-x|\leq Rf(t)} \sup_{|\xi|\leq 1/f(t)} |q(z,\xi)|}$$

for some constant $c' > 0$, and the latter limit is zero by (C1). Hence, (46) holds. Applying Proposition 7.1 proves the assertion.

2° If (C2) holds, then the assertion is immediate from Proposition 7.1 and Remark 7.2(ii).

It remains to show (ii). To this end, assume additionally that (A1') holds for some $R > 0$ and let f be non-decreasing with $\int_0^1 \inf_{|z-x|\leq cf(t)} \sup_{|\xi|\leq 1/f(t)} |q(z,\xi)| \, dt = \infty$ for some $c > 0$. Then Proposition 2.8(ii) yields $\int_0^1 \inf_{|z-x|\leq cf(t)} \nu(z, \{|y| > cf(t)\}) \, dt = \infty$, and applying (i) finishes the proof. □

Appendix A.

In the proof of Proposition 7.1 we used the following conditional Borel-Cantelli lemma for backward filtrations.

A.1. **Proposition.** *Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of decreasing σ-algebras. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of non-negative random variables such that X_n is $\mathcal{F}_n$-measurable for each $n \in \mathbb{N}$. If $Y := \sup_{n\in\mathbb{N}} X_n$ is integrable, then there is a $\mathbb{P}$-null set N such that*

$$\left\{ \sum_{n\in\mathbb{N}} \mathbb{E}(X_n \mid \mathcal{F}_{n+1}) = \infty \right\} \subseteq \left\{ \sum_{n\in\mathbb{N}} X_n = \infty \right\} \cup N.$$

The idea for our proof is from Chen [9].

Proof. Set $M_n := \mathbb{E}(X_n \mid \mathcal{F}_{n+1})$ and $R_n := \sum_{i=n}^{\infty} X_i$. Since R_{n+1} is $\mathcal{F}_{n+1}$-measurable, we find using the tower property

$$\begin{aligned}\mathbb{E}\left(\frac{1}{(1+R_1)^2} \sum_{n=1}^k M_n \right) = \mathbb{E}\left(\mathbb{1}_{\{R_1<\infty\}} \frac{1}{(1+R_1)^2} \sum_{n=1}^k M_n \right) &\leq \mathbb{E}\left(\sum_{n=1}^k \frac{M_n}{(1+R_{n+1})^2} \mathbb{1}_{\{R_{n+1}<\infty\}} \right) \\ &= \mathbb{E}\left(\sum_{n=1}^k \frac{X_n}{(1+R_{n+1})^2} \mathbb{1}_{\{R_{n+1}<\infty\}} \right)\end{aligned}$$

for all $k \in \mathbb{N}$. Since

$$\frac{X_n}{(1+R_{n+1})^2} = \frac{(1+R_n)-(1+R_{n+1})}{(1+R_n)(1+R_{n+1})} \frac{1+R_n}{1+R_{n+1}} \leq \left(\frac{1}{1+R_{n+1}} - \frac{1}{1+R_n} \right)(1+Y)$$

on $\{R_{n+1} < \infty\}$, we have

$$\sum_{n=1}^k \frac{X_n}{(1+R_{n+1})^2} \leq (1+Y)\left(\frac{1}{1+R_k} - \frac{1}{1+R_1} \right) \leq 1+Y,$$

and so

$$\mathbb{E}\left(\frac{1}{(1+R_1)^2} \sum_{n=1}^k M_n \right) \leq \mathbb{E}(1+Y) < \infty.$$

Letting $k \to \infty$ using monotone convergence, we see that $\frac{1}{(1+R_1)^2} \sum_{n\in\mathbb{N}} M_n < \infty$ almost surely, which proves the assertion. □

The following estimate was needed in the proof of Proposition 5.2.

A.2. **Lemma.**

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} t^{1/n} \log \frac{1}{t} \leq 2 \qquad \textit{for all } \; t \in (0,1).$$

Proof. Fix $t \in (0,1)$. By the fundamental theorem of calculus, we have for every $n \in \mathbb{N}$

$$0 \leq t^{1/(n+1)} - t^{1/n} = -\int_{1/(n+1)}^{1/n} t^r \log t \, dr \geq \log(t^{-1}) t^{1/n} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \log(t^{-1}) t^{1/n} \frac{1}{n(n+1)}.$$

Thus,

$$\sum_{n\in\mathbb{N}} \frac{1}{n(n+1)} t^{1/n} \log(t^{-1}) \leq \sum_{n\in\mathbb{N}} \left(t^{1/(n+1)} - t^{1/n} \right) = \lim_{N\to\infty} t^{1/N} - t = 1 - t.$$

As $(n+1)/n \leq 2$ for all $n \in \mathbb{N}$, this proves the assertion. □

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(F. Kühn) TU Dresden, Fakultät Mathematik, Institut für Mathematische Stochastik, 01062 Dresden, Germany. E-Mail: franziska.kuehn1@tu-dresden.de