On the domain of fractional Laplacians and related generators of Feller processes

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Abstract

In this paper we study the domain of stable processes, stable-like processes and more general pseudo- and integro-differential operators which naturally arise both in analysis and as infinitesimal generators of Lévy- and Lévy-type (Feller) operators. In particular we obtain conditions on the symbol of the operator ensuring that certain (variable order) Hölder and Hölder-Zygmund spaces are in the domain. We use tools from probability theory to investigate the small-time asymptotics of the generalized moments of a Lévy or Lévy-type process $(X_t)_{t\geq 0}$,

$$\lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}^x f(X_t)-f(x)\right), \quad x\in\mathbb{R}^d,$$

for functions f which are not necessarily bounded or differentiable. The pointwise limit exists for fixed $x\in\mathbb{R}^d$ if f satisfies a Hölder condition at x. Moreover, we give sufficient conditions which ensure that the limit exists uniformly in the space of continuous functions vanishing at infinity. As an application we prove that the domain of the generator of $(X_t)_{t\geq 0}$ contains certain Hölder spaces of variable order. Our results apply, in particular, to stable-like processes, relativistic stable-like processes, solutions of Lévy-driven SDEs and Lévy processes.

Keywords: Lévy-type processes, Blumenthal–Getoor index, infinitesimal generator, fractional Laplacian, small-time asymptotics, Hölder space of variable order.

2010 Mathematics Subject Classification: Primary: 60J75. Secondary: 60G51, 60J25, 60J35.

1 Introduction

Since the pioneering work of Caffarelli and Silvestre on fractional powers of the Laplacian, see [30, 9], a lot of work has been devoted to fractional powers of the Laplacian from the analytical point of view, we refer to [7, 8, 10, 16, 24] to mention but a few.

The fractional power of the Laplacian is also the generator of a stochastic process with stationary and independent increments (a Lévy process), which allows us to use probabilistic methods for its investigation. In fact, fractional powers of the Laplacian are just a special case of generators of Lévy processes and – if one allows for generators with variable coefficients – of the more general class of Feller processes, the classic result is [13], see [6] for a recent survey. Over the past two and a half decades these operators have been studied from both the analytical community but most of all the probability community, see [5, 11, 12, 17, 20, 26, 32].

Of particular importance is a good understanding of the domain of these operators which, in general, have a representation as pseudo-differential as well as integro-differential operator.

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This is partly due to the fact that for elements in their domains we can construct interesting martingales.

In this paper we study in great detail the domains of rather general generators of Feller processes and, by using probabilistic techniques in combination with analytic techniques, we succeed in finding precise conditions in terms of (variable-order) Hölder and Lipschitz function spaces to belong to these domains, see Theorem 4.1 (for Lévy processes and generators with constant coefficients) and Theorem 4.5 (for Feller processes and generators with variable coefficients). As far as we are aware, these results extend known results for fractional powers of the Laplacian (including those of variable order of differentiability).

For a d-dimensional Lévy process $(L_t)_{t\geq 0}$ with Lévy triplet (b,Q,ν) the family of measures $(p_t)_{t>0}$ on $(\mathbb{R}^d\setminus\{0\},\mathcal{B}(\mathbb{R}^d\setminus\{0\})$ defined by

$$p_t(B) := \frac{1}{t}\mathbb{P}(L_t\in B), \qquad t>0,\ B\in\mathcal{B}(\mathbb{R}^d\setminus\{0\})$$

converges vaguely to the Lévy measure ν, i. e.

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}f(L_t) = \int_{\mathbb{R}^d\setminus\{0\}} f(y)\,\nu(dy) \tag{1}$$

holds for any continuous function f with compact support in $\mathbb{R}^d\setminus\{0\}$, cf. [6, Lemma 2.16] or [3, Proposition 18.2]. By the portmanteau theorem, this implies the following small-time asymptotics

$$\lim_{t\to 0}\frac{1}{t}\mathbb{P}(L_t\in B) = \nu(B) \tag{2}$$

for any Borel set $B\in\mathcal{B}(\mathbb{R}^d\setminus\{0\})$ such that $0\notin\bar{B}$ and the topological boundary ∂B is a ν-null set. Jacod [18] proved that the small-time asymptotics (1) extends to continuous bounded functions $f:\mathbb{R}^d\to\mathbb{R}$ with $f(0)=0$ which satisfy a Hölder condition at $x=0$,

$$|f(x)-f(0)|\leq |x|^\alpha \qquad \text{for all}\quad |x|\leq 1$$

where $\alpha\in(0,2)$ is a suitable constant depending on the Lévy triplet (b,Q,ν), see [18] or [15, p. 2] for details. More recently, Figueroa–López [15] showed that the assumption on the boundedness of f can be replaced by a much weaker integrability condition which basically ensures that the expectation $\mathbb{E}f(L_t)$ is exists for any $t>0$.

In the first part of this paper, Section 3, we establish similar results for the class of Lévy-type processes which includes, in particular, Lévy processes, affine processes, solutions of Lévy-driven stochastic differential equations, and stable-like processes. We will show that any Lévy-type process $(X_t)_{t\geq 0}$ with rich domain and characteristics $(b(x),Q(x),\nu(x,dy))$ satisfies

$$\lim_{t\to 0}\frac{1}{t}\mathbb{P}^x(X_t-x\in B) = \nu(x,B) \qquad \text{for all}\quad x\in\mathbb{R}^d \tag{3}$$

which is the analogue of (2), cf. Corollary 3.3; again $B\in\mathcal{B}(\mathbb{R}^d\setminus\{0\})$ is a Borel set such that $0\notin\bar{B}$ and $\nu(x,\partial B)=0$. Because of the small-time asymptotics (3), we have for fixed $x\in\mathbb{R}^d$

$$\lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}^x f(X_t)-f(x)\right) = \int_{\mathbb{R}^d\setminus\{0\}}\left(f(x+y)-f(x)\right)\nu(x,dy)$$

for any continuous function f with compact support in $\mathbb{R}^d\setminus\{x\}$. Using a localized version of a maximal inequality, cf. Lemma 3.1, we will show that for a rich Lévy-type process $(X_t)_{t\geq 0}$ and fixed $x\in\mathbb{R}^d$ the pointwise limit

$$\lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}^x f(X_t)-f(x)\right) \tag{4}$$

exists for a much larger class of functions. More precisely, we will establish the small-time asymptotics (4) for functions $f:\mathbb{R}^d\to\mathbb{R}$ which satisfy a Hölder condition at x, cf. Theorem 3.4 and 3.5, and need not be bounded, see Theorem 3.7.

In the second part, Section 4, we turn to the question under which assumptions on $f \in C_\infty(\mathbb{R}^d)$ the limit

$$\lim_{t\to 0} \frac{1}{t} \left(\mathbb{E}^x f(X_t) - f(x)\right) \tag{5}$$

exists uniformly (in x) for a rich Lévy-type process $(X_t)_{t\geq 0}$ with bounded coefficients. This is equivalent to asking for sufficient conditions which ensure that a function $f \in C_\infty(\mathbb{R}^d)$ is contained in the domain $\mathcal{D}(A)$ of the generator A of $(X_t)_{t\geq 0}$. The main results in Section 4 are Corollary 4.7 and Corollary 4.8 which state that $\mathcal{D}(A)$ contains certain Hölder spaces of variable order. Our results apply, in particular, to Lévy processes, cf. Theorem 4.1; for instance, if $(L_t)_{t\geq 0}$ is an isotropic α-stable Lévy process, $\alpha \in (0,1)$, then the Hölder space

$$\mathcal{C}_\infty^\beta := \left\{ f \in C_\infty(\mathbb{R}^d);\ \sup_{x,y\in\mathbb{R}^d} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty \right\}$$

is contained in the domain of the generator A of $(L_t)_{t\geq 0}$ for any $\beta \in (\alpha, 1]$ and we have

$$Af(x) = \int_{\mathbb{R}^d \setminus \{0\}} (f(x+y) - f(x))\, \nu(dy), \qquad f \in \mathcal{C}_\infty^\beta,\ x \in \mathbb{R}^d.$$

At the end of Section 4 we discuss several examples, including stable-like dominated processes (Example 4.10), solutions of Lévy-driven SDEs (Example 4.12), stable-like processes and relativistic stable-like processes (Example 4.11).

2 Basic definitions and notation

We consider the Euclidean space $\mathbb{R}^d$ with the canonical scalar product $x \cdot y := \sum_{j=1}^d x_j y_j$ and the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ which is generated by the open balls $B(x,r) := \{y \in \mathbb{R}^d; |y-x| < r\}$ and closed balls $B[x,r] := \{y \in \mathbb{R}^d; |y-x| \leq r\}$. The smooth functions with compact support are denoted by $C_c^\infty(\mathbb{R}^d)$, and $C_\infty(\mathbb{R}^d)$ is the space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ vanishing at infinity. Superscripts $k \in \mathbb{N}$ are used to denote the order of differentiability, e. g. $f \in C_\infty^k(\mathbb{R}^d)$ means that f and its derivatives up to order k are $C_\infty(\mathbb{R}^d)$-functions. We write $\operatorname{supp} f$ for the support of a function $f : \mathbb{R}^n \to \mathbb{R}^d$ and $\{f \in B\} = f^{-1}(B)$ denotes the preimage of a set $B \subseteq \mathbb{R}^d$ under f. For a set $B \subseteq \mathbb{R}^d$ we use ∂B to denote the topological boundary of B. We use $\fint$ and $\fint_B$ as a shorthand for $\int_{\mathbb{R}^d \setminus \{0\}}$ and $\int_{B \setminus \{0\}}$, respectively.

Throughout, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes a probability space. A stochastic process $(L_t)_{t\geq 0}$ is called a *Lévy process* if it has stationary and independent increments, $L_0 = 0$ almost surely and the sample paths $t \mapsto L_t(\omega)$ are càdlàg (right-continuous with finite left-hand limits) for almost all $\omega \in \Omega$. By the Lévy-Khintchine formula, every Lévy process can be uniquely characterized by its *characteristic exponent* $\psi(\xi) := -\log \mathbb{E} e^{i\xi\cdot X_1}$,

$$\psi(\xi) = -ib\cdot\xi + \frac{1}{2}\xi\cdot Q\xi + \fint \left(1 - e^{iy\cdot\xi} + iy\cdot\xi \mathbb{1}_{(0,1)}(|y|)\right) \nu(dy), \quad \xi \in \mathbb{R}^d, \tag{6}$$

where (b, Q, ν) is the *Lévy triplet* consisting of the drift $b \in \mathbb{R}^d$, the symmetric positive semidefinite diffusion matrix $Q \in \mathbb{R}^{d\times d}$ and the Lévy measure ν on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$ satisfying $\fint \min\{|y|^2, 1\}\, \nu(dy) < \infty$. A function $\psi : \mathbb{R}^d \to \mathbb{C}$ with $\psi(0) = 0$ is called *continuous negative definite* if it admits a Lévy–Khintchine representation of the form (6).

A *Lévy-type process* is a Markov process whose transition semigroup is a Feller semigroup; for further details see e. g. [6]. Without loss of generality, we may assume that the sample paths of a Lévy-type process are càdlàg. If $C_c^\infty(\mathbb{R}^d)$ is contained in the domain $\mathcal{D}(A)$ of the generator A of a Lévy-type process $(X_t)_{t\geq 0}$, then we call $(X_t)_{t\geq 0}$ a *rich* Lévy-type process. Lévy-type processes are also known as Feller processes, and we will use both terms synonymously. Our main reference for Feller processes is the monograph [6]. If $(X_t)_{t\geq 0}$ is a rich Lévy-type process with generator A, then $A|_{C_c^\infty(\mathbb{R}^d)}$ is a *pseudo-differential operator*,

$$Af(x) = -q(x,D)f(x) := -\int_{\mathbb{R}^d} e^{i\,x\cdot\xi} q(x,\xi) \widehat{f}(\xi)\, d\xi, \qquad f \in C_c^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d$$

where $\widehat{f}(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx$ denotes the Fourier transform of f and

$$q(x,\xi) = q(x,0) - ib(x)\cdot\xi + \frac{1}{2}\xi\cdot Q(x)\xi + \int\!\!\!\!\!\!\not\,\,\, \left(1 - e^{iy\cdot\xi} + iy\cdot\xi \mathbb{1}_{(0,1)}(|y|)\right) \nu(x,dy) \tag{7}$$

is the negative definite *symbol*, cf. [6, Theorem 2.21]. For simplicity, we assume that $q(x,0) = 0$. For each fixed $x \in \mathbb{R}^d$ the tuple $(b(x), Q(x), \nu(x,dy))$ is a Lévy triplet. We call the family $(b(x), Q(x), \nu(x,dy))_{x\in\mathbb{R}^d}$ the *characteristics* of q and use (b,Q,ν) as a shorthand. It is not difficult to see that

$$Af(x) = b(x)\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}\left(Q(x)\cdot\nabla^2 f(x)\right) + \int\!\!\!\!\!\!\not\,\,\, \left(f(x+y) - f(x) - \nabla f(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right) \nu(x,dy)$$

for any $f \in C_c^\infty(\mathbb{R}^d)$, see e. g. [6, Theorem 2.21], where $\nabla^2 f$ denotes the Hessian and $\operatorname{tr} A$ the trace of a matrix A. By [6, Theorem 2.30], $q(x,0) = 0$ implies that the mapping $x \mapsto q(x,\xi)$ is continuous for all $\xi \in \mathbb{R}^d$. We say that a rich Lévy-type process $(X_t)_{t\geq 0}$ has *bounded coefficients* if its symbol q has bounded coefficients, i. e. there exists a constant $c > 0$ such that $|q(x,\xi)| \leq c(1+|\xi|^2)$ for all $x,\xi \in \mathbb{R}^d$. We will frequently use the following result from [6, Proposition 2.27(d), Theorem 2.31].

2.1 Theorem *Let q be given by (7) such that $q(x,0) = 0$. For any compact set $K \subseteq \mathbb{R}^d$:*

(i). $C_K := \sup_{x\in K}\sup_{|\xi|\leq 1} |q(x,\xi)| < \infty$,

(ii). $\sup_{x\in K} |q(x,\xi)| \leq 2C_K(1+|\xi|^2)$ *for all* $\xi \in \mathbb{R}^d$,

(iii). $\sup_{x\in K}(|b(x)| + |Q(x)| + \int\!\!\!\!\!\!\not\,\,\, (|y|^2 \wedge 1)\, \nu(x,dy)) < \infty$.

If q has bounded coefficients, then the statements also hold for $K = \mathbb{R}^d$.

The following result can be found in [6, Theorem 2.44].

2.2 Theorem *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with symbol q and characteristics (b,Q,ν). Then $(X_t)_{t\geq 0}$ is a semimartingale and its semimartingale characteristics (B,C,μ) relative to the truncation function $y\mathbb{1}_{(0,1)}(|y|)$ are given by*

$$B_t = \int_0^t b(X_s)\, ds, \qquad C_t = \int_0^t Q(X_s)\, ds, \qquad \mu(\cdot, ds, dy) = \nu(X_s, dy)\, ds. \tag{8}$$

3 Pointwise limits

In this section we investigate the small-time asymptotics of generalized moments, i. e. we study limits of the form

$$\lim_{t\to 0} \frac{1}{t}\left(\mathbb{E}^x f(X_t) - f(x)\right) \tag{9}$$

for a rich Feller process $(X_t)_{t\geq 0}$ and any fixed $x \in \mathbb{R}^d$. Recall that a function f is contained in the domain $\mathcal{D}(A) \subseteq C_\infty(\mathbb{R}^d)$ of the generator A, if the limit exists *uniformly* in $C_\infty(\mathbb{R}^d)$, i. e.

$$\mathcal{D}(A) := \left\{ f \in C_\infty(\mathbb{R}^d); \exists g \in C_\infty(\mathbb{R}^d) : \lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\left|\frac{1}{t}\left(\mathbb{E}^x f(X_t) - f(x)\right) - g(x)\right| = 0 \right\},$$
$$Af(x) := \lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}^x f(X_t) - f(x)\right).$$

It is, in general, a non-trivial task to check whether a function $f \in C_\infty(\mathbb{R}^d)$ is in the domain of the generator; typically, this requires assumptions on the smoothness, e. g. $f \in C_\infty^2(\mathbb{R}^d)$ if $(X_t)_{t\geq 0}$ has bounded coefficients, cf. [6, Theorem 2.37(h)].

We are interested in proving the existence of the limit (9) (and also determining it) for functions f which are not necessarily bounded or differentiable. Intuitively, there are two issues which we have to consider:

(i). We have to ensure that the expectation $\mathbb{E}^x f(X_t)$ exists; therefore, we need an assumption on the growth of f at infinity.

(ii). For the existence of the limit (9) for a fixed $x \in \mathbb{R}^d$ the behaviour of f close to $x \in \mathbb{R}^d$ is crucial. For instance, if $X_t := t$ is a deterministic drift process, then the limit (9) exists if, and only if, f is differentiable at x. This means that we have to make an assumption on the local regularity of f at x, typically Hölder continuity or differentiability.

In a first step we consider the particular case that f vanishes at infinity and satisfies $f|_{B(x,\delta)} = 0$ for some $\delta > 0$; for such functions f we show in Theorem 3.2

$$\lim_{t\to 0} \frac{1}{t} \left(\mathbb{E}^x f(X_t) - f(x)\right) = \int_{y \neq 0} \left(f(x+y) - f(x)\right) \nu(x, dy).$$

This implies, in particular, that $t^{-1}\mathbb{P}^x(X_t - x \in \cdot)$ converges vaguely to $\nu(x,\cdot)$ as $t \to 0$, cf. Corollary 3.3, and so,

$$\lim_{t\to 0} \frac{1}{t} \mathbb{P}^x(X_t - x \in A) = \nu(x, A)$$

for any $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ such that $0 \notin \bar{A}$ and $\nu(x, \partial A) = 0$. In Theorem 3.4 and Theorem 3.5 we show that the assumption $f|_{B(x,\delta)} = 0$ on the regularity of f at x can be replaced by a local Hölder or differentiability condition. The required regularity can be expressed in terms of fractional moments of $\nu(x,\cdot)$ or in terms of the generalized Blumenthal–Getoor index at infinity,

$$\beta_\infty^x := \inf\left\{\gamma > 0; \lim_{r\to\infty} \frac{1}{r^\gamma} \sup_{|\xi|\leq r} |q(x,\xi)| < \infty\right\}.$$

Finally, in Theorem 3.7, we extend Theorem 3.4 to functions f which are not necessarily bounded.

The following upper bound for the small-time asymptotics of $\mathbb{P}(|X_t - x| \geq r)$ will be one of our main tools.

3.1 Lemma *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with symbol q. For any $x \in \mathbb{R}^d$ there exists a constant $c = c(x) > 0$ such that*

$$\limsup_{t\to 0} \frac{1}{t} \mathbb{P}^x\left(|X_t - x| \geq r\right) \leq \limsup_{t\to 0} \frac{1}{t} \mathbb{P}^x\Big(\sup_{s\leq t} |X_s - x| \geq r\Big) \leq c(x) \sup_{|\xi|\leq r^{-1}} |q(x,\xi)|$$

for all $r > 0$. Moreover, $c(K) := \sup_{x\in K} c(x) < \infty$ for any compact set $K \subseteq \mathbb{R}^d$.

Lemma 3.1 is a localized variant of a known maximal inequality, cf. [6, Corollary 5.2]; for the readers' convenience we include a full proof.

Proof of Lemma 3.1. For fixed $x \in \mathbb{R}^d$ and $r > 0$ denote by $\tau_r^x := \inf\{t \geq 0; X_t \notin B(x,r)\}$ the exit time from the ball $B(x,r)$. As

$$\{|X_t - x| \geq r\} \subseteq \left\{\sup_{s\leq t} |X_s - x| \geq r\right\} \subseteq \{\tau_r^x \leq t\},$$

it suffices to show that

$$\limsup_{t\to 0} \frac{1}{t} \mathbb{P}^x(\tau_r^x \leq t) \leq c \sup_{|\xi|\leq r^{-1}} |q(x,\xi)| \tag{$\star$}$$

for some constant $c > 0$. To this end, fix $x \in \mathbb{R}^d$, $r > 0$ and pick $u \in C_c^\infty(\mathbb{R}^d)$ such that $u(0) = 1$, $\operatorname{supp} u \subseteq B(0,1)$ and $0 \leq u \leq 1$. If we set $u_r^x(y) := u((y-x)/r)$, then $u_r^x \in C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$, and an application of Dynkin's formula gives

$$\mathbb{E}^x u_r^x(X_{t\wedge\tau_r^x}) - 1 = \mathbb{E}^x \left(\int_{[0,t\wedge\tau_r^x)} A u_r^x(X_s)\, ds\right)$$

where A denotes the generator of $(X_t)_{t\geq 0}$. Thus,

$$\begin{aligned}\mathbb{P}^x\left(\tau_r^x \leq t\right) \leq \mathbb{E}^x\left(1 - u_r^x(X_{t\wedge\tau_r^x})\right) &= -\mathbb{E}^x\left(\int_{[0,t\wedge\tau_r^x)} Au_r^x(X_s)\,ds\right)\\ &= -\mathbb{E}^x\left(\int_{[0,t\wedge\tau_r^x)} \mathbb{1}_{\{|X_s - x|<r\}} Au_r^x(X_s)\,ds\right).\end{aligned}$$

Since

$$\begin{aligned}-Au_r^x(y) = \int_{\mathbb{R}^d} e^{iy\cdot\xi} q(y,\xi)\widehat{u_r^x}(\xi)\,d\xi &= e^{-ix\cdot\xi} r^d \int_{\mathbb{R}^d} e^{iy\cdot\xi} q(y,\xi)\widehat{u}(r\xi)\,d\xi\\ &= e^{-ix\cdot\xi}\int_{\mathbb{R}^d} e^{iy\cdot\xi} q(y,r^{-1}\xi)\widehat{u}(\xi)\,d\xi\end{aligned}$$

for all $y \in \mathbb{R}^d$, we get

$$\mathbb{P}^x\left(\tau_r^x \leq t\right) \leq t\,\mathbb{E}^x\left(\int_{\mathbb{R}^d} \sup_{s<t\wedge\tau_r^x} |q(X_s, r^{-1}\xi)|\,|\widehat{u}(\xi)|\,d\xi\right).$$

As $X_s \in B(x,r)$ for all $s < t\wedge\tau_r^x$, there exists by Theorem 2.1 a constant $C = C(r,x)$ such that

$$\sup_{s<t\wedge\tau_r^x} |q(X_s, r^{-1}\xi)|\,|\widehat{u}(\xi)| \leq C(1+|\xi|^2)|\widehat{u}(\xi)| \in L^1(d\xi)$$

for all $t \geq 0$. On the other hand, $q(x,0) = 0$ implies that $x \mapsto q(x,\xi)$ is continuous for all $\xi \in \mathbb{R}^d$, see [6, Theorem 2.30]), and therefore

$$\sup_{s<t\wedge\tau_r^x} |q(X_s, r^{-1}\xi)|\,|\widehat{u}(\xi)| \xrightarrow{t\to 0} |q(x,r^{-1}\xi)|\,|\widehat{u}(\xi)| \quad \text{for almost all } \xi \in \mathbb{R}^d.$$

Applying the dominated convergence theorem yields

$$\limsup_{t\to 0}\frac{1}{t}\mathbb{P}^x\left(\tau_r^x \leq t\right) \leq \int_{\mathbb{R}^d} |q(x,r^{-1}\xi)|\,|\widehat{u}(\xi)|\,d\xi.$$

Now $(\star)$ follows using the estimate from Theorem 2.1

$$|q(x,r^{-1}\xi)| \leq 2\sup_{|\eta|\leq r^{-1}} |q(x,\eta)|(1+|\xi|^2) \qquad \text{for all } \ \xi \in \mathbb{R}^d,\ r > 0. \qquad \square$$

The next result is well known for Lévy processes, see [3, Proposition 18.2] or [6, Lemma 2.16].

3.2 Theorem *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with symbol q and characteristics (b,Q,ν). Let $f \in C_\infty(\mathbb{R}^d)$ and suppose that $f|_{B(x_0,\delta)} = 0$ for some $x_0 \in \mathbb{R}^d$ and $\delta > 0$. Then*

$$\frac{1}{t}\mathbb{E}^x f(X_t) \xrightarrow{t\to 0} \int\!\!\!\!\!\!\not\;\; f(x+y)\,\nu(x,dy)$$

uniformly in a neighbourhood of x_0. In particular, $x \mapsto \int f(x+y)\,\nu(x,dy)$ is continuous at $x = x_0$.

Proof. For fixed $\varepsilon > 0$ choose $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\|f-\chi\|_\infty \leq \varepsilon$. Without loss of generality, we may assume that $\chi|_{B(x_0,\delta)} = 0$. Obviously,

$$\begin{aligned}\left|\frac{1}{t}\mathbb{E}^x f(X_t) - \int\!\!\!\!\!\!\not\;\; f(x+y)\,\nu(x,dy)\right| &\leq \left|\frac{1}{t}\mathbb{E}^x(f-\chi)(X_t)\right| + \int\!\!\!\!\!\!\not\;\; |f(x+y)-\chi(x+y)|\,\nu(x,dy)\\ &\quad + \left|\frac{1}{t}\mathbb{E}^x\chi(X_t) - \int\!\!\!\!\!\!\not\;\; \chi(x+y)\,\nu(x,dy)\right|\\ &=: I_1 + I_2 + I_3.\end{aligned}$$

We estimate the terms separately. Using that $\chi(x) = 0$, $\nabla\chi(x) = 0$ and $\nabla^2\chi(x) = 0$ for all $x \in B(x_0,\delta/4)$, we find for all $x \in B(x_0,\delta/4)$

$$I_3 = \left|\frac{1}{t}(\mathbb{E}^x\chi(X_t) - \chi(x)) - A\chi(x)\right| \leq \sup_{x\in\mathbb{R}^d}\left|\frac{1}{t}(\mathbb{E}^x\chi(X_t) - \chi(x)) - A\chi(x)\right| \xrightarrow{t\to 0} 0$$

as $\chi \in C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$. For I_2 we note that for any $x \in B(x_0, \delta/4)$

$$I_2 \le \int_{|y| \ge \delta/4} |f(x+y) - \chi(x+y)| \, \nu(x, dy) \le \varepsilon \sup_{x \in B(x_0, \delta/4)} \nu(x, \mathbb{R}^d \setminus B(0, \delta/4)).$$

Note that the constant on the right-hand side is finite, see e. g. [6, Theorem 2.30(d)], and $\delta > 0$ is a fixed constant which does not depend on ε. Since

$$I_1 \le \frac{\varepsilon}{t} \mathbb{P}^x \left(|X_t - x_0| \ge \frac{\delta}{2} \right) \le \frac{\varepsilon}{t} \mathbb{P}^x \left(|X_t - x| \ge \frac{\delta}{4} \right)$$

for all $x \in B(x_0, \delta/4)$, it follows from Lemma 3.1 that there exists a constant $C > 0$ such that

$$\limsup_{t \to 0} \frac{1}{t} I_1 \le C \varepsilon \sup_{x \in B(x_0, \delta/4)} \sup_{|\xi| \le 4\delta^{-1}} |q(x, \xi)|.$$

The above estimates show

$$\begin{aligned} \limsup_{t \to 0} \left| \frac{1}{t} \mathbb{E}^x f(X_t) - \fint f(x+y) \, \nu(x, dy) \right| & \\ \le \varepsilon \Big(\sup_{x \in B(x_0, \delta/4)} \nu(x, \mathbb{R}^d \setminus B(0, \delta/4)) + C & \sup_{x \in B(x_0, \delta/4)} \sup_{|\xi| \le 2\delta^{-1}} |q(x, \xi)| \Big) \xrightarrow{\varepsilon \to 0} 0. \end{aligned}$$

The assertion on the continuity follows directly from the local uniform convergence and the fact that $x \mapsto \mathbb{E}^x f(X_t)$ is continuous as $(X_t)_{t \ge 0}$ is a Feller process. □

If we use Theorem 3.2 for the shifted function $f(\cdot - x_0)$ for a fixed $x_0 \in \mathbb{R}^d$, we get:

3.3 Corollary *Let $(X_t)_{t \ge 0}$ be a rich Lévy-type process with symbol q and characteristics (b, Q, ν). If $f \in C_\infty(\mathbb{R}^d)$ and $f|_{B(0,\delta)} = 0$ for some $\delta > 0$, then*

$$\lim_{t \to 0} \frac{1}{t} \mathbb{E}^x f(X_t - x) = \fint f(y) \, \nu(x, dy) \qquad \text{for all } \; x \in \mathbb{R}^d.$$

Corollary 3.3 shows that the family of measures $p_t(dy) := t^{-1} \mathbb{P}^x(X_t - x \in dy)$, $t > 0$, on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$ converges vaguely to $\nu(x, dy)$ for each fixed $x \in \mathbb{R}^d$. By the portmanteau theorem, Corollary 3.3 implies

$$\lim_{t \to 0} \frac{1}{t} \mathbb{P}^x(X_t - x \in A) = \nu(x, A) \tag{10}$$

for any Borel set $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ such that $0 \notin \bar{A}$ and $\nu(x, \partial A) = 0$.

The next step is to relax the assumption "$f|_{B(x_0,\delta)} = 0$" in Theorem 3.2. To this end we define, following [27], for fixed $x_0 \in \mathbb{R}^d$ the generalized Blumenthal–Getoor index at ∞

$$\beta_\infty^{x_0} := \inf \left\{ \gamma > 0; \lim_{r \to \infty} \frac{1}{r^\gamma} \sup_{|\xi| \le r} |q(x_0, \xi)| < \infty \right\}.$$

Since any continuous negative definite function grows at most quadratically at infinity, we have $\beta_\infty^{x_0} \in [0, 2]$ for any $x_0 \in \mathbb{R}^d$; moreover,

$$\fint_{|y| \le 1} |y|^\beta \, \nu(x_0, dy) < \infty \qquad \text{for all } \; \beta > \beta_{x_0}^\infty. \tag{11}$$

If $q(x_0, \cdot)$ has no diffusion part, i. e. $Q(x_0) = 0$, and satisfies the sector condition, i. e. if there exists a constant $C > 0$ such that $|\operatorname{Im} q(x_0, \xi)| \le C \operatorname{Re} q(x_0, \xi)$ for all $\xi \in \mathbb{R}^d$, then

$$\fint_{|y| \le 1} |y|^\beta \, \nu(x_0, dy) < \infty \implies \beta_{x_0}^\infty \le \beta. \tag{12}$$

In this case, the Blumenthal–Getoor index can be equivalently characterized in terms of fractional moments of the Lévy measure

$$\beta_\infty^{x_0} = \inf \left\{ \gamma > 0; \fint_{|y| \le 1} |y|^\gamma \, \nu(x_0, dy) < \infty \right\};$$

this is a special case of [27, Proposition 5.4], see also [4].

3.4 Theorem (Regularity at x_0) *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with symbol q and characteristics (b,Q,ν). Suppose that $f\in C_\infty(\mathbb{R}^d)$ satisfies one of the following conditions for some fixed $x_0\in\mathbb{R}^d$.*

(A1) *There exist constants $\alpha>\beta_\infty^{x_0}$ and $C>0$ such that*

$$|f(x)-f(x_0)|\leq C|x-x_0|^\alpha \qquad \text{for all } \; x\in B(x_0,1).$$

(A2) *f is differentiable at $x=x_0$ and there exist $\alpha>\beta_\infty^{x_0}$ and $C>0$ such that*

$$|f(x)-f(x_0)-\nabla f(x_0)\cdot(x-x_0)|\leq C|x-x_0|^\alpha \qquad \text{for all } \; x\in B(x_0,1).$$

(A3) *f is twice continuously differentiable in a neighbourhood of x_0.*

Then the limit

$$\lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}^{x_0}f(X_t)-f(x_0)\right)$$

exists and takes the value

(A1) $Lf(x_0):=\fint\left(f(x_0+y)-f(x_0)\right)\nu(x_0,dy)$,

(A2) $Lf(x_0):=b(x_0)\cdot\nabla f(x_0)+\fint\big(f(x_0+y)-f(x_0)-\nabla f(x_0)\cdot y\mathbb{1}_{(0,1)}(|y|)\big)\,\nu(x_0,dy)$,

(A3) $Lf(x_0):=b(x_0)\cdot\nabla f(x_0)+\frac{1}{2}\operatorname{tr}\big(Q(x_0)\cdot\nabla^2 f(x_0)\big)$
$\qquad +\fint\big(f(x_0+y)-f(x_0)-\nabla f(x_0)\cdot y\mathbb{1}_{(0,1)}(|y|)\big)\,\nu(x_0,dy)$,

depending on which of the conditions (A1)-(A3) *is satisfied.*

Proof. Pick a cut-off function $\chi\in C_c^\infty(\mathbb{R}^d)$, $0\leq\chi\leq 1$, such that $\chi|_{B(x_0,1)}=1$, $\chi|_{B^c(x_0,2)}=0$ and set $\chi_\delta(x):=\chi(\delta^{-1}x)$ for $\delta>0$.

(A1) Without loss of generality, we may assume $f(x_0)=0$, otherwise we consider the shifted function $x\mapsto f(x)-f(x_0)$. As $\alpha>\beta_\infty^{x_0}$, we have

$$\fint |f(x_0+y)|\,\nu(x_0,dy)\leq C\fint_{|y|\leq 1}|y|^\alpha\,\nu(x_0,dy)+\|f\|_\infty\nu(x_0,\mathbb{R}^d\setminus B(0,1))<\infty,$$

and therefore it follows from Theorem 3.2 and the dominated convergence theorem that

$$\begin{aligned}\frac{1}{t}\mathbb{E}^{x_0}([f(1-\chi_\delta)](X_t))&\xrightarrow{t\to 0}\fint f(x_0+y)(1-\chi_\delta(x_0+y))\,\nu(x_0,dy)\\&\xrightarrow{\delta\to 0}\fint f(x_0+y)\,\nu(x_0,dy).\end{aligned}$$

On the other hand, if we set $C_\delta:=\sup_{|y-x_0|\leq 2\delta}|f(y)|$, then $C_\delta\to 0$ as $\delta\to 0$ and

$$\begin{aligned}|\mathbb{E}^{x_0}([f\chi_\delta](X_t))|&\leq\int_0^{C_\delta}\mathbb{P}^{x_0}\big(|f(X_t)|\geq r,\,|X_t-x_0|\leq 2\delta\big)\,dr\\&\leq\int_0^{C_\delta}\mathbb{P}^{x_0}\big(|X_t-x_0|^\alpha\geq r/C\big)\,dr\end{aligned}$$

for any $\delta\in(0,1/2)$. By Lemma 3.1

$$\begin{aligned}\limsup_{t\to 0}\frac{1}{t}\mathbb{P}^{x_0}\big(|X_t-x_0|^\alpha\geq r/C\big)&=\limsup_{t\to 0}\frac{1}{t}\mathbb{P}^{x_0}\big(|X_t-x_0|\geq C^{-1/\alpha}r^{1/\alpha}\big)\\&\leq c\sup_{|\xi|\leq r^{-1/\alpha}C^{1/\alpha}}|q(x_0,\xi)|\leq C'r^{-\beta/\alpha}\end{aligned}\tag{13}$$

for any $\beta\in(\beta_\infty^x,\alpha)$ and suitable constants $c,C'>0$; thus, by Fatou's lemma,

$$\limsup_{t\to 0}\left|\frac{1}{t}\mathbb{E}^{x_0}([f\chi_\delta](X_t))\right|\leq C'\int_0^{C_\delta}r^{-\beta/\alpha}\,dr\xrightarrow{\delta\to 0}0.$$

Writing

$$\frac{1}{t}\mathbb{E}^{x_0}f(X_t)=\frac{1}{t}\mathbb{E}^{x_0}([f\chi_\delta](X_t))+\frac{1}{t}\mathbb{E}^{x_0}([f(1-\chi_\delta)](X_t))$$

and letting first $t \to 0$ and then $\delta \to 0$, proves the claim.

(A2) For fixed $R > 0$ let $\tau_R^{x_0}$ denote the exit time from the ball $B(x_0, R)$. The function

$$x \mapsto g(x) := f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0)\chi(x)$$

satisfies (A1) and, therefore, by the first part of this proof,

$$\begin{aligned}\lim_{t\to 0} \frac{1}{t} \mathbb{E}^{x_0} g(X_t) &= \fint (g(x_0 + y) - g(x_0))\, \nu(x_0, dy) \\ &= \fint (f(x_0 + y) - f(x_0) - \chi(y + x_0)\nabla f(x_0) \cdot y)\, \nu(x_0, dy).\end{aligned}$$

As $(\bullet - x_0)\chi(\bullet) \in C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ an application of Dynkin's formula shows

$$\frac{1}{t} \mathbb{E}^{x_0} \left((X_{t\wedge\tau_R^{x_0}} - x_0)\chi(X_{t\wedge\tau_R^{x_0}}) \right) \xrightarrow{t\to 0} b(x_0) + \fint y \left(\chi(y + x_0) - \mathbb{1}_{(0,1)}(|y|) \right) \nu(x_0, dy)$$

for any $R > 0$. Using the fact that $\operatorname{supp}\chi \subseteq B[x_0, 2]$ and applying Lemma 3.1, we find for some constant $c = c(x_0)$

$$\begin{aligned}&\left| \frac{1}{t} \mathbb{E}^{x_0} \left((X_{t\wedge\tau_R^{x_0}} - x_0)\chi(X_{t\wedge\tau_R^{x_0}}) \right) - \frac{1}{t} \mathbb{E}^{x_0} \left((X_t - x_0)\chi(X_t) \right) \right| \\ &\qquad \le \frac{4}{t} \mathbb{P}^{x_0} \left(\tau_R^{x_0} \le t \right) \le 4c \sup_{|\xi|\le R^{-1}} |q(x_0, \xi)| \xrightarrow{R\to\infty} 0,\end{aligned}$$

and therefore we conclude

$$\frac{1}{t} \mathbb{E}^{x_0} \left((X_t - x_0)\chi(X_t) \right) \xrightarrow{t\to 0} b(x_0) + \fint y \left(\chi(y + x_0) - \mathbb{1}_{(0,1)}(|y|) \right) \nu(x_0, dy).$$

Consequently,

$$\begin{aligned}\frac{1}{t} \left(\mathbb{E}^{x_0} f(X_t) - f(x_0) \right) &= \frac{1}{t} \mathbb{E}^{x_0} g(X_t) + \frac{1}{t} \nabla f(x_0) \cdot \mathbb{E}^{x_0} \left((X_t - x_0)\chi(X_t) \right) \\ &\xrightarrow{t\to 0} b(x_0) \cdot \nabla f(x_0) + \fint \left(f(x_0 + y) - f(x_0) - \nabla f(x_0) \cdot y \mathbb{1}_{(0,1)}(|y|) \right) \nu(x_0, dy),\end{aligned}$$

finishing the second part.

(A3) We begin with the particular case that $f(x_0) = 0$ and $\nabla f(x_0) = 0$. Since, by Theorem 3.2 and the dominated convergence theorem,

$$\begin{aligned}\frac{1}{t} \mathbb{E}^{x_0} ([f(1 - \chi_\delta)](X_t)) &\xrightarrow{t\to 0} \fint [f(1 - \chi_\delta)](x_0 + y)\, \nu(x_0, dy) \\ &\xrightarrow{\delta\to 0} \fint f(x_0 + y)\, \nu(x_0, dy),\end{aligned}$$

it is enough to show

$$\frac{1}{t} \mathbb{E}^{x_0} \left([f\chi_\delta](X_t) \right) \xrightarrow{t,\delta\to 0} \sum_{i,j=1}^{d} Q_{ij}(x_0)\partial_i\partial_j f(x_0). \tag{14}$$

In order to keep notation simple, we set $f_\delta(x) := f(x)\chi_\delta(x)$. Note that by Lemma 3.1

$$\begin{aligned}\left| \frac{1}{t} \mathbb{E}^{x_0} f_\delta(X_t) - \frac{1}{t} \mathbb{E}^{x_0} f_\delta(X_{t\wedge\tau_R^{x_0}}) \right| &\le 2\|f\|_\infty \frac{1}{t} \mathbb{P}^{x_0} \left(\tau_R^{x_0} \le t \right) \\ &\le 2c\|f\|_\infty \sup_{|\xi|\le R^{-1}} |q(x_0, \xi)| \xrightarrow{R\to\infty} 0,\end{aligned}$$

and therefore (14) follows if we can show that

$$\frac{1}{t} \mathbb{E}^{x_0} (f_\delta(X_{t\wedge\tau_R^x})) \xrightarrow{t,\delta\to 0} \sum_{i,j=1}^{d} Q_{ij}(x_0)\partial_i\partial_j f(x_0) \tag{15}$$

for every fixed $R > 0$. By Taylor's formula, there exists a continuous mapping $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\lim_{r\to 0} \varphi(r) = 0$ and

$$f(y) = \frac{1}{2} \sum_{i,j=1}^{d} (y^i - x_0^i)(y^j - x_0^j)\partial_i\partial_j f(x_0) + |y - x_0|^2 \varphi(|x_0 - y|)$$

for all $y = (y^1, \ldots, y^d) \in B(x_0, \delta)$. Thus,

$$\frac{1}{t}\mathbb{E}^{x_0}(f_\delta(X_{t\wedge\tau_R^x})) = I_1 + I_2$$

where

$$I_1 := \frac{1}{2t}\sum_{i,j=1}^d \partial_i\partial_j f(x_0)\mathbb{E}^{x_0}\big[(X^i_{t\wedge\tau_R^x} - x_0^i)(X^j_{t\wedge\tau_R^x} - x_0^j)\chi_\delta(X_{t\wedge\tau_R^x})\big]$$
$$I_2 := \frac{1}{t}\mathbb{E}^{x_0}\left[|X_{t\wedge\tau_R^x} - x_0|^2\varphi(|X_{t\wedge\tau_R^x} - x_0|)\chi_\delta(X_{t\wedge\tau_R^x})\right].$$

We estimate the terms separately. By the definition of χ_δ, we have

$$I_2 \le t^{-1}\sup_{r\le 2\delta}|\varphi(r)|\,\mathbb{E}^{x_0}(|X_{t\wedge\tau_R^x} - x_0|^2\chi(X_{t\wedge\tau_R^x})),$$

and so an application of Dynkin's formula yields

$$I_2 \le \sup_{r\le 2\delta}|\varphi(r)|\sup_{|y-x_0|\le R}|A(|\bullet - x_0|^2\cdot\chi(\bullet))(y)| \xrightarrow{\delta\to 0} 0.$$

Using that $\nabla\chi_\delta(x_0) = 0$ and $\nabla^2\chi_\delta(x_0) = 0$, it is not difficult to see from Dynkin's formula and the fundamental theorem of calculus that

$$I_1 \xrightarrow{t\to 0} \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j f(x_0)\left(Q_{ij}(x_0) + \int y_i y_j\chi_\delta(x_0+y)\,\nu(x_0,dy)\right)$$
$$\xrightarrow{\delta\to 0} \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j f(x_0)Q_{ij}(x_0).$$

Combining both convergence results proves (15) if $f(x_0) = 0$ and $\nabla f(x_0) = 0$. For the general case define

$$g(x) := f(x) - f(x_0) - \chi(x)\nabla f(x_0)\cdot(x - x_0), \qquad x \in \mathbb{R}^d,$$

and use exactly the same reasoning as in the proof of (A2). □

In Theorem 3.4 we have to assume that α is strictly larger than the Blumenthal–Getoor index $\beta_\infty^{x_0}$. It turns out that Theorem 3.4 also holds for $\alpha = \beta_\infty^{x_0}$ if $q(x_0,\cdot)$ satisfies the sector condition, has no diffusion part, and the fractional moment $\int_{|y|\le 1}|y|^{\beta_\infty^{x_0}}\,\nu(x_0,dy)$ is finite. This is a direct consequence of the following theorem.

3.5 Theorem *Let $(X_t)_{t\ge 0}$ be a rich Lévy-type process with symbol q and characteristics $(b,0,\nu)$. Suppose that $f \in C_\infty(\mathbb{R}^d)$ satisfies one of the following conditions for some fixed $x_0 \in \mathbb{R}^d$.*

(B1) *There exist $\alpha \in (0,1]$ and $C > 0$ such that $\int_{|y|\le 1}|y|^\alpha\,\nu(x_0,dy) < \infty$ and*

$$|f(x) - f(x_0)| \le C|x - x_0|^\alpha \qquad \text{for all } x \in B(x_0,1).$$

(B2) *f is differentiable at $x = x_0$ and there exist constants $\alpha \in (1,2)$ and $C > 0$ such that $\int_{|y|\le 1}|y|^\alpha\,\nu(x_0,dy) < \infty$ and*

$$|f(x) - f(x_0) - \nabla f(x_0)\cdot(x - x_0)| \le C|x - x_0|^\alpha \qquad \text{for all } x \in B(x_0,1).$$

If $q(x_0,\cdot)$ satisfies the sector condition, i. e. $|\operatorname{Im} q(x_0,\xi)| \le C'\operatorname{Re} q(x_0,\xi)$ for some constant $C' > 0$, then the limit

$$\lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}^{x_0}f(X_t) - f(x_0)\right)$$

exists and takes the value

(B1) $Lf(x_0) := \int (f(x_0+y) - f(x_0))\,\nu(x_0,dy)$;

(B2) $Lf(x_0) := b(x_0)\cdot\nabla f(x_0) + \int \left(f(x_0+y) - f(x_0) - \nabla f(x_0)\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(x_0,dy)$.

Proof. The proof is very similar to that of Theorem 3.4; the only modification is needed in (13) where we use the fact that $\int_{|y|\leq 1}|y|^\alpha\,\nu(x_0,dy)<\infty$ implies

$$\int_0^1 \sup_{|\xi|\leq r^{-1/\alpha}}|q(x_0,\xi)|\,dr=\alpha\int_1^\infty \frac{1}{s^{1+\alpha}}\sup_{|\xi|\leq s}|q(x_0,\xi)|\,ds<\infty$$

(cf. Lemma A.1 for details) to obtain an integrable majorant. □

In the remaining part of this section we extend Theorem 3.4 and Theorem 3.5 to functions f which are not necessarily bounded. In [21] it was shown that the implication

$$\sup_{x\in K}\int_{|y|\geq 1} g(y)\,\nu(x,dy)<\infty \implies \forall t>0:\ \sup_{x\in K}\sup_{s\leq t}\mathbb{E}^x g(X_{s\wedge\tau_K}-x)<\infty$$

holds for any twice differentiable submultiplicative function $g\geq 0$, any compact set $K\subseteq\mathbb{R}^d$, and any rich Lévy-type process; if $(X_t)_{t\geq 0}$ has bounded coefficients, then $K=\mathbb{R}^d$ is admissible. Here τ_K denotes as usual the first exit time from K. It is therefore a natural idea to replace

$$\frac{1}{t}\left(\mathbb{E}^x f(X_t)-f(x)\right)\quad\text{by}\quad \frac{1}{t}\left(\mathbb{E}^x f(X_{t\wedge\tau_K})-f(x)\right),$$

and to consider functions $f:\mathbb{R}^d\to\mathbb{R}$ which can be dominated by a submultiplicative function $g\geq 0$ with $\sup_{x\in K}\int_{|y|\geq 1} g(y)\,\nu(x,dy)<\infty$.

3.6 Definition Let $(b(x),Q(x),\nu(x,dy))$ be an x-dependent Lévy triplet and $K\subseteq\mathbb{R}^d$. We write $\Sigma(K)$ for the family of twice differentiable submultiplicative functions $g:\mathbb{R}^d\to(0,\infty)$ satisfying the following two integrability conditions.

(i). $M(K):=\sup_{x\in K}\int_{|y|\geq 1} g(y)\,\nu(x,dy)<\infty$ (integrability).

(ii). $M_R(K):=\sup_{x\in K}\int_{|y|\geq R} g(y)\,\nu(x,dy)\xrightarrow{R\to\infty}0$ (tightness).

3.7 Theorem (Behaviour at ∞) *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with symbol q and characteristics (b,Q,ν). Moreover, let $f:\mathbb{R}^d\to\mathbb{R}$ be a continuous mapping satisfying the following growth condition* (G).

(G) *There exist a compact set $K\subseteq\mathbb{R}^d$ and a function $g\in\Sigma(K)$ such that*

$$\lim_{|x|\to\infty}\left|\frac{f(x)}{g(x)}\right|<\infty.$$

If one of the conditions (A1)-(A3) *holds for some $x_0\in K$, then the limit*

$$\lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}^{x_0} f(X_{t\wedge\tau_K})-f(x_0)\right)$$

exists and equals $Lf(x_0)$ defined in Theorem 3.4; here $\tau_K:=\inf\{t\geq 0; X_t\notin K\}$ denotes the exit time from the set K. If $(X_t)_{t\geq 0}$ has bounded coefficients, then $K=\mathbb{R}^d$ is admissible.

Proof. We only consider the case that $(X_t)_{t\geq 0}$ has bounded coefficients and $g\in\Sigma(\mathbb{R}^d)$; the proof of the other assertion works analogously and just requires an additional stopping argument. For simplicity of notation we assume that $b(x)=0$ and $Q(x)=0$ for all $x\in\mathbb{R}^d$, see the remark at the end of the proof.

Let χ be a continuous function such that $1-\chi\in C_c^\infty(\mathbb{R}^d)$, $0\leq\chi\leq 1$, $\chi|_{B(0,1)}=0$ and $\chi|_{B^c(0,2)}=1$, and set $\chi_R(x):=\chi(R^{-1}x)$. Then $f(\bullet)\cdot(1-\chi_R(\bullet-x_0))$ satisfies the assumptions of Theorem 3.4 for each $R>0$ and therefore

$$\frac{1}{t}\left(\mathbb{E}^{x_0}(f(X_t)(1-\chi_R)(X_t-x_0))-f(x_0)\right)\xrightarrow{t\to 0}L(f(1-\chi_R)(\bullet-x_0))(x_0).$$

Since $\nabla\chi_R(x_0) = 0$, $\nabla^2\chi_R(x_0) = 0$ for each $R > 0$ and $\int_{|y|\geq 1} |f(y)|\,\nu(x_0, dy) < \infty$, it follows easily from the definition of $L(f(1-\chi_R)(\bullet - x_0))$ and the dominated convergence theorem that

$$\frac{1}{t}\left(\mathbb{E}^{x_0}(f(X_t)(1-\chi_R)(X_t - x_0)) - f(x_0)\right) \xrightarrow{t\to 0} L(f(1-\chi_R)(\bullet - x_0))(x_0) \xrightarrow{R\to\infty} Lf(x_0).$$

Consequently, it remains to show that

$$\limsup_{R\to\infty}\limsup_{t\to 0}\left|\frac{1}{t}\mathbb{E}^{x_0}\left(f(X_t)\chi_R(X_t - x_0)\right)\right| = 0.$$

Because of the growth condition (G) and the submultiplicativity of g, it suffices to prove

$$\limsup_{R\to\infty}\limsup_{t\to 0}\frac{1}{t}\mathbb{E}^{x_0}\left(g(X_t - x_0)\chi_R(X_t - x_0)\right) = 0. \tag{16}$$

By Theorem 2.2, $(X_t)_{t\geq 0}$ is a semimartingale with semimartingale characteristics $(0, 0, \mu)$ given by (8). Consequently, $(X_t)_{t\geq 0}$ has a canonical representation $X_t = x_0 + X_t^{(1)} + X_t^{(2)}$,

$$\begin{aligned} X_t^{(1)} &:= \int_0^t \int_{0<|y|<1} y\,(N(dy, ds) - \mu(dy, ds)) \\ X_t^{(2)} &:= \int_0^t \int_{|y|\geq 1} y\,N(dy, ds) \end{aligned}$$

where N denotes the jump measure of $(X_t)_{t\geq 0}$, cf. [19, Theorem II.2.34]. By the submultiplicativity of g, there exists a constant $c > 0$ such that

$$g(X_t - x_0) = g(X_t^{(1)} + X_t^{(2)}) \leq c g(X_t^{(1)}) g(X_t^{(2)}), \qquad t \geq 0.$$

Since any submultiplicative function grows at most exponentially, we can find constants $a, b > 0$ such that

$$g(X_t - x_0) \leq a \exp\left(b\sqrt{|X_t^{(1)}|^2 + 1} - 1\right) g(X_t^{(1)}), \qquad t \geq 0. \tag{17}$$

In order to keep our notation simple, we assume that $a = b = c = 1$. Moreover, we set

$$\varrho(x) := \exp\left(\sqrt{|x|^2 + 1} - 1\right)$$

and use the subscript to denote truncated functions, e. g.

$$\varrho_R(x) := \chi_R(x)\varrho(x) \quad \text{and} \quad g_R(x) := \chi_R(x) g(x).$$

From the definition of χ_R and the triangle inequality, it is not difficult to see that

$$\chi_R(x + y) \leq \chi_{R/4}(x) + \chi_{R/4}(y) \qquad \text{for all } \ x, y \in \mathbb{R}^d, \tag{18}$$

and therefore we obtain

$$\begin{aligned} g(X_t - x_0)\chi_R(X_t - x_0) &\leq \exp\left(\sqrt{|X_t^{(1)}|^2 + 1} - 1\right) g(X_t^{(2)})\chi_{R/4}(X_t^{(1)}) \\ &\quad + \exp\left(\sqrt{|X_t^{(1)}|^2 + 1} - 1\right) g(X_t^{(2)})\chi_{R/4}(X_t^{(2)}) \\ &= \varrho_{R/4}(X_t^{(1)}) g(X_t^{(2)}) + \varrho(X_t^{(1)}) g_{R/4}(X_t^{(2)}). \end{aligned}$$

Consequently, (16) follows if we can show

$$\lim_{R\to\infty}\lim_{t\to 0}\frac{1}{t}\mathbb{E}^{x_0}\left(\varrho_{R/4}(X_t^{(1)}) g(X_t^{(2)})\right) = 0 \tag{19}$$

$$\lim_{R\to\infty}\lim_{t\to 0}\frac{1}{t}\mathbb{E}^{x_0}\left(\varrho(X_t^{(1)}) g_{R/4}(X_t^{(2)})\right) = 0. \tag{20}$$

First we prove (19). Define a stopping time by

$$\tau := \tau_r := \inf\left\{t > 0; |X_t^{(1)}| + |X_t^{(2)}| \geq r\right\}$$

for fixed $r > 0$. Applying Itô's formula for semimartingales gives

$$
\begin{aligned}
&\mathbb{E}^{x_0}\big(\varrho_{R/4}(X^{(1)}_{t\wedge\tau})g(X^{(2)}_{t\wedge\tau})\big) \\
&= \mathbb{E}^{x_0}\left(\int_0^{t\wedge\tau}\int_{|y|\geq 1} \varrho_{R/4}(X^{(1)}_s)(g(X^{(2)}_s + y) - g(X^{(2)}_s))\,\nu(X_s, dy)\,ds\right) \qquad (21)\\
&\quad + \mathbb{E}^{x_0}\left(\int_0^{t\wedge\tau} \fint_{|y|<1} g(X^{(2)}_s)\left(\varrho_{R/4}(X^{(1)}_s + y) - \varrho_{R/4}(X^{(1)}_s) - \nabla\varrho_{R/4}(X^{(1)}_s)\cdot y\right)\nu(X_s, dy)\,ds\right).
\end{aligned}
$$

Since $g \geq 0$ is submultiplicative, the first term on the right-hand side of (21) is bounded above by

$$
\begin{aligned}
&\mathbb{E}^{x_0}\left(\int_0^t \fint \varrho_{R/4}(X^{(1)}_s)g(X^{(2)}_s)g(y)\,\nu(X_s, dy)\,ds\right) \\
&\quad\leq \left(\sup_{x\in\mathbb{R}^d}\int_{|y|\geq 1}|g(y)|\,\nu(x, dy)\right)\mathbb{E}^{x_0}\left(\int_0^t \varrho_{R/4}(X^{(1)}_s)g(X^{(2)}_s)\,ds\right).
\end{aligned}
$$

For the second term in (21) we apply Taylor's formula and use the fact that $\nabla^2\chi_{R/4}(z) = 0$ for all $z \in B(0, R/4) \cup B^c(0, R/2)$ to conclude that there exists a function $\psi \in C^2_b(\mathbb{R}^d)$ such that $\psi(z) = 0$ for all $z \in B(0, 1/16)$ and

$$
|\varrho_{R/4}(x+y) - \varrho_{R/4}(x) - \nabla\varrho_{R/4}(x)\cdot y| \leq |y|^2\varrho(x)\psi(x) \qquad \text{for all} \;\; x \in \mathbb{R}^d,\; |y| \leq 1
$$

for $R \geq 1$. Using this estimate for $x := X^{(1)}_s$, we find that the second term on the right-hand side of (21) is bounded above by

$$
\left(\sup_{x\in\mathbb{R}^d}\fint_{|y|\leq 1}|y|^2\,\nu(x, dy)\right)\mathbb{E}^{x_0}\left(\int_0^t \varrho(X^{(1)}_s)\psi(X^{(1)}_s)g(X^{(2)}_s)\,ds\right).
$$

Now it follows from Fatou's lemma, Definition 3.6 and Lemma 3.8 below that there exists an absolute constant $C > 0$ such that

$$
\frac{1}{t}\mathbb{E}^{x_0}\left(\varrho_{R/4}(X^{(1)}_t)g(X^{(2)}_t)\right) \leq \liminf_{r\to\infty}\frac{1}{t}\mathbb{E}^{x_0}\left(\varrho_{R/4}(X^{(1)}_{t\wedge\tau})g(X^{(2)}_{t\wedge\tau})\right) \leq \frac{C}{t}\int_0^t s\,ds
$$

(recall the definition of ϱ, $\varrho_{R/4}$ and note that $K = \mathbb{R}^d$), and this implies (19).

It remains to prove (20). Again an application of Itô's formula shows

$$
\begin{aligned}
&\mathbb{E}^{x_0}(\varrho(X^{(1)}_{t\wedge\tau})g_{R/4}(X^{(2)}_{t\wedge\tau})) \\
&= \mathbb{E}^{x_0}\left(\int_0^{t\wedge\tau}\int_{|y|\geq 1}\varrho(X^{(1)}_s)\left(g_{R/4}(X^{(2)}_s + y) - g_{R/4}(X^{(2)}_s)\right)\nu(X_{s-}, dy)\,ds\right) \qquad (22)\\
&\quad + \mathbb{E}^{x_0}\left(\int_0^{t\wedge\tau}\fint_{|y|<1} g_{R/4}(X^{(2)}_s)\left(\varrho(X^{(1)}_s + y) - \varrho(X^{(1)}_s) - \nabla\varrho(X^{(1)}_s)\cdot y\right)\nu(X_{s-}, dy)\,ds\right).
\end{aligned}
$$

Using the submultiplicativity of $g \geq 0$ and (18), we find that the first term on the right-hand side is bounded above by

$$
\begin{aligned}
&\mathbb{E}^{x_0}\left(\int_0^t\int_{|y|\geq 1}\varrho(X^{(1)}_s)\big[g_{R/16}(X^{(2)}_s)g(y) + g(X^{(2)}_s)g_{R/16}(y)\big]\,\nu(X_{s-}, dy)\,ds\right) \\
&\quad\leq M_{R/16}(\mathbb{R}^d)\int_0^t \mathbb{E}^{x_0}(\varrho(X^{(1)}_s)g(X^{(2)}_s))\,ds + M(\mathbb{R}^d)\int_0^t \mathbb{E}^{x_0}(\varrho(X^{(1)}_s)g_{R/16}(X^{(2)}_s)\,ds
\end{aligned}
$$

with $M(\mathbb{R}^d)$ and $M_{R/16}(\mathbb{R}^d)$ from Definition 3.6. On the other hand, a similar calculation as in the proof of (19) shows that the second term on the right-hand side of (22) is less or equal than

$$
C\mathbb{E}^{x_0}\left(\int_0^t g_{R/4}(X^{(2)}_s)\varrho(X^{(1)}_s)\psi(X^{(1)}_s)\,ds\right)
$$

where C is a suitable constant and $\psi \in C^2_b(\mathbb{R}^d)$ such that $\operatorname{supp}\psi \cap B(0, 1/16) = \varnothing$. If we combine both estimates, apply Lemma 3.8 and use that $\lim_{R\to\infty} M_{R/16}(\mathbb{R}^d) = 0$, we get (20).

In the general case, i. e. if $b(x) \neq 0$ or $Q(x) \neq 0$, we replace $X_t^{(1)}$ by

$$X_t^{(1)} := \int_0^t b(X_s)\,ds + X_t^C + \int_0^t \int_{0<|y|<1} y\,(N(dy,ds) - \mu(dy,ds))$$

where $(X_t^C)_{t\geq 0}$ denotes the continuous martingale part, cf. [19, Theorem II.2.34]; this gives additional terms when applying Itô's formula, but the reasoning works exactly as in the pure-jump case. □

3.8 Lemma *Let $(X_t)_{t\geq 0}$, K, g and $x_0 \in \mathbb{R}^d$ be as in Theorem 3.7. For any $T > 0$ and all functions $g, \theta \in C_b^2(\mathbb{R}^d)$ such that $\operatorname{supp}\theta \cap B(0,\varepsilon) = 0$ for some sufficiently small $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$\mathbb{E}^{x_0}\left(\exp\left[\sqrt{|X_{t\wedge\tau_K}^{(1)}|^2+1}-1\right] g(X_{t\wedge\tau_K}^{(2)})\right) \leq C$$
$$\mathbb{E}^{x_0}\left(\exp\left[\sqrt{|X_{t\wedge\tau_K}^{(1)}|^2+1}-1\right] g(X_{t\wedge\tau_K}^{(2)})\theta(X_t^{(1)})\right) \leq Ct$$
$$\mathbb{E}^{x_0}\left(\exp\left[\sqrt{|X_{t\wedge\tau_K}^{(1)}|^2+1}-1\right] g(X_{t\wedge\tau_K}^{(2)})\theta(X_t^{(2)})\right) \leq Ct$$

for all $t \leq T$; here τ_K denotes the exit time from the set K and $X_t - x_0 = X_t^{(1)} + X_t^{(2)}$ the decomposition from the proof of Theorem 3.7.

Proof. We know from the proof of [21, Theorem 4.1] that under the assumptions of Theorem 3.7

$$\sup_{t\leq T}\mathbb{E}^{x_0}\left(\exp\left[\sqrt{|X_{t\wedge\tau_K}^{(1)}|^2+1}-1\right] g(X_{t\wedge\tau_k}^{(2)})\right) < \infty,$$

and this proves the first assertion. The other two estimates now follow from a straightforward application of Itô's formula; mind that the initial term

$$\exp\left[\sqrt{|X_{t\wedge\tau_K}^{(1)}|^2+1}-1\right] g(X_{t\wedge\tau_K}^{(2)})\theta(X_t^{(i)})\Big|_{t=0} = 0$$

vanishes for $i \in \{1,2\}$ since $\theta(X_0^{(i)}) = 0$. □

Remark (i). The proof of Theorem 3.7 simplifies substantially if the submultiplicative function $g \in C^2(\mathbb{R}^d)$ satisfies the inequality

$$|\nabla^2 g(x)| \leq C|g(x)|, \qquad x \in \mathbb{R}^d, \tag{23}$$

for some absolute constant $C > 0$. In this case, we can apply Itô's formula directly to the mapping $x \mapsto g(x - x_0)\chi_R(x - x_0)$ to prove (16); there is no need to use the decomposition $X_t = x + X_t^{(1)} + X_t^{(2)}$ and estimate (17). Although there are many examples of submultiplicative functions satisfying (23), it does not hold true for all (twice differentiable) submultiplicative functions.

(ii). In Theorem 3.7 submultiplicativity of the dominating function g is required. This assumption can be weakened; it suffices to assume that there exist a subadditive function $a : \mathbb{R}^d \to \mathbb{R}$ and a submultiplicative function $m : \mathbb{R}^d \to (0,\infty)$ such that $g(x) = m(x) \cdot a(x)$ for all $x \in \mathbb{R}^d$, $a, m \in C^2(\mathbb{R}^d)$ and

$$\lim_{R\to\infty} \inf_{|x|\geq R} |a(x)| > 0.$$

The proof of Theorem 3.7 under this relaxed assumption is similar, but more technical.

Using exactly the same reasoning as in the proof of Theorem 3.7, we obtain a similar extension of Theorem 3.5 to unbounded functions.

3.9 Theorem *Let* $(X_t)_{t\geq 0}$ *be a rich Lévy-type process with characteristics* $(b,0,\nu)$ *and symbol* q*, and let* $f:\mathbb{R}^d\to\mathbb{R}$ *be a continuous function satisfying the growth condition* (G)*. Suppose that either* (B1) *or* (B2) *holds for some* $x_0\in K$ *and that* $q(x_0,\cdot)$ *satisfies the sector condition. Then the limit*

$$\lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}^{x_0}f(X_{t\wedge\tau_K})-f(x_0)\right)$$

exists and equals $Lf(x_0)$ *as defined in Theorem 3.5. If* $(X_t)_{t\geq 0}$ *has bounded coefficients, then* $K=\mathbb{R}^d$ *is admissible.*

We close this section with an application of Corollary 3.3, which has been announced (without proof) in the recent publication [21, remark following Theorem 5.2] on moments of Lévy-type processes.

3.10 Proposition *Let* $(X_t)_{t\geq 0}$ *be a rich Lévy-type process with symbol* q *and characteristics* (b,Q,ν)*. If there exist* $x\in\mathbb{R}^d$*,* $R\geq 0$ *and* $\alpha>0$ *such that*

$$\liminf_{t\to 0}\frac{1}{t}\mathbb{E}^x\left(|X_t-x|^\alpha \mathbb{1}_{\{|X_t-x|>R\}}\right)<\infty,$$

then

$$\int_{|y|>R}|y|^\alpha\,\nu(x,dy)\leq R^\alpha\nu(x,\{y\in\mathbb{R}^d;|y|>R\})\mathbb{1}_{R>0}+\liminf_{t\to 0}\frac{1}{t}\mathbb{E}^x(|X_t-x|^\alpha\mathbb{1}_{\{|X_t-x|>R\}});$$

in particular $\int_{|y|>R}|y|^\alpha\,\nu(x,dy)<\infty$.

For $R=0$ Proposition 3.10 shows

$$C:=\liminf_{t\to 0}\frac{1}{t}\mathbb{E}^x(|X_t-x|^\alpha)<\infty \implies \int\!\!\!\!\!\!\not\;\; |y|^\alpha\,\nu(x,dy)\leq C<\infty.$$

Proof of Proposition 3.10. Since the identity

$$\int|y|^\alpha\,\mu(dy)=\alpha\int_{(0,\infty)}\mu(|y|\geq r)r^{\alpha-1}\,dr \tag{$\star$}$$

holds for any $\alpha>0$ and any σ-finite measure μ, we have

$$\int_{|y|>R}|y|^\alpha\,\nu(x,dy)=\alpha\int_{(0,\infty)}\nu(x,\{y\in\mathbb{R}^d;|y|>R,\ |y|\geq r\})\,r^{\alpha-1}\,dr.$$

If $R=0$ then it follows from (10) and Fatou's lemma that

$$\int_{|y|>0}|y|^\alpha\,\nu(x,dy)\leq\alpha\liminf_{t\to 0}\frac{1}{t}\int_{(0,\infty)}\mathbb{P}^x(|X_t-x|\geq r)\,r^{\alpha-1}\,dr\overset{(\star)}{=}\liminf_{t\to 0}\frac{1}{t}\mathbb{E}^x(|X_t-x|^\alpha).$$

Here we use that the σ-finiteness of $\nu(x,dy)$ implies $\nu(x,\partial B(0,r))=0$ for Lebesgue-almost all $r>0$. If $R>0$, then we split the integral

$$\int_{|y|>R}|y|^\alpha\,\nu(x,dy)\leq R^\alpha\nu(x,\{y\in\mathbb{R}^d;|y|>R\})+\alpha\int_{(R,\infty)}\nu(x,\{y\in\mathbb{R}^d;|y|\geq r\})\,r^{\alpha-1}\,dr,$$

and use again (10) and Fatou's lemma to estimate the second term. □

4 Uniform limits

In the previous section we have seen that the pointwise limit $\lim_{t\to 0}t^{-1}(\mathbb{E}^{x_0}f(X_t)-f(x_0))$ exists for some fixed $x_0\in\mathbb{R}^d$ if $f\in C_\infty(\mathbb{R}^d)$ satisfies a Hölder condition at x_0. Now we turn to the question under which assumptions on the regularity of f the limit

$$\lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}^\bullet f(X_t)-f(\bullet)\right) \tag{24}$$

exists uniformly in $C_\infty(\mathbb{R}^d)$, i. e. under which assumptions f is contained in the domain $\mathcal{D}(A)$ of the generator of $(X_t)_{t\geq 0}$. It is well known that the limit exists (uniformly) for any function $f \in C^2_\infty(\mathbb{R}^d)$ and any Lévy-type process $(X_t)_{t\geq 0}$ with bounded coefficients, cf. [6, Theorem 2.37]. However, the results from the previous section suggest that the uniform limit may also exist for functions whose regularity varies from point to point, e. g. functions which satisfy

$$|f(x+y)-f(x)| \leq C|y|^{\alpha(x)} \qquad \text{for all } \; x,y \in \mathbb{R}^d, \; |y| \leq 1$$

for some absolute constant $C > 0$ and a suitable mapping $\alpha : \mathbb{R}^d \to [0,1]$. In this section, we will show that this is indeed true; more precisely we will establish that certain Hölder spaces of variable order are contained in the domain of the generator, cf. Corollary 4.7 and Corollary 4.8. The idea is to use the fact that for a Lévy-type process $(X_t)_{t\geq 0}$ the limit (24) exists uniformly if, and only if, the pointwise limit exists for each $x \in \mathbb{R}^d$ and the limit defines a function in $C_\infty(\mathbb{R}^d)$, cf. [28, Theorem 7.22]. At the end of this section we will present some examples, including stable-like and relativistic stable-like processes.

Let us begin with Lévy processes before we discuss the more general case of Lévy-type processes.

4.1 Theorem *Let $(L_t)_{t\geq 0}$ be a Lévy process with Lévy triplet (b,Q,ν). Denote by $(A,\mathcal{D}(A))$ its generator and fix $\alpha \in [0,2]$ such that $\int_{|y|\leq 1} |y|^\alpha \, \nu(dy) < \infty$.*

(i). *$C^2_\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ and $Af = b\cdot\nabla f + \frac{1}{2}\operatorname{tr}(Q\nabla^2 f) + \fint \left(f(\bullet+y) - f - \nabla f \cdot y \mathbb{1}_{(0,1)}(|y|)\right) \nu(dy)$ for $f \in C^2_\infty(\mathbb{R}^d)$.*

(ii). *If $Q=0$, $\alpha \in [0,1]$ and $b = \int_{|y|<1} y \, \nu(dy)$, then the Hölder space*

$$\mathcal{C}^\alpha_\infty := \left\{ f \in C_\infty(\mathbb{R}^d); \; \|f\|_\alpha := \sup_{x,y\in\mathbb{R}^d} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty \right\}$$

is contained in $\mathcal{D}(A)$ and $Af(x) = \fint (f(x+y)-f(x))\, \nu(dy)$ for any $f \in \mathcal{C}^\alpha_\infty$.

(iii). *If $Q=0$ and $\alpha \in [1,2]$, then*

$$\mathcal{C}^{1,\alpha-1}_\infty := \left\{ f \in C^1_\infty(\mathbb{R}^d); \; \nabla f \in \mathcal{C}^{\alpha-1}_\infty \right\} \subseteq \mathcal{D}(A)$$

and $Af(x) = b\cdot\nabla f(x) + \fint \left(f(x+y)-f(x)-\nabla f(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right) \nu(dy)$ for $f \in \mathcal{C}^{1,\alpha-1}_\infty$.

Part (ii) of Theorem 4.1 was recently proved by Cygan & Grzywny [14] for the particular case $\alpha = 1$.

4.2 Remark There are various concepts of Hölder (or Lipschitz) spaces in the literature. On the one hand, there are the "classical" Hölder spaces C^α equipped with the norm

$$\sum_{j=0}^{\lfloor\alpha\rfloor} \sum_{\substack{\beta\in\mathbb{N}_0^d \\ |\beta|=j}} \|\partial^\beta f\|_\infty + \max_{\substack{\beta\in\mathbb{N}_0^d \\ |\beta|=\lfloor\alpha\rfloor}} \sup_{x\neq y} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^{\alpha-\lfloor\alpha\rfloor}} \tag{$\star$}$$

where $\lfloor\alpha\rfloor$ denotes the biggest natural number less or equal than α. On the other hand, there are the Zygmund–Hölder spaces $\mathfrak{C}^\alpha$ consisting of all functions $f \in C^k$ such that the norm

$$\sum_{j=0}^{k} \sum_{\substack{\beta\in\mathbb{N}_0^d \\ |\beta|=j}} \|\partial^\beta f\|_\infty + \max_{\substack{\beta\in\mathbb{N}_0^d \\ |\beta|=k}} \sup_{\substack{x,h\in\mathbb{R}^d \\ h\neq 0}} \frac{|\partial^\beta f(x+h) + \partial^\beta f(x-h) - 2\partial^\beta f(x)|}{|h|^s}$$

is finite where $s \in (0,1]$ and $k \in \mathbb{N}$ are chosen such that $\alpha = k+s$, see Triebel [34, pp. 34]. If $\alpha \in (0,\infty)\setminus\mathbb{N}$ then $\mathfrak{C}^\alpha = C^\alpha$, cf. [33, Theorem 1(b), p. 201]; however for $\alpha\in\mathbb{N}$ we have a strict inclusion: $\mathfrak{C}^\alpha \supsetneq C^\alpha$. For $\alpha = 1$ it is possible to show that $\mathfrak{C}^1$ is strictly larger than the space of Lipschitz continuous functions Lip (cf. [31, p. 148]) which is, in turn, strictly larger than C^1.

Our spaces $\mathcal{C}^{1,\alpha-1}$, $\alpha \in [1,2)$, coincide with C^α with norm $(\star)$. There are the following relations between the Hölder spaces introduced in Theorem 4.1 and the just mentioned function spaces:

$$\begin{aligned} \mathcal{C}^\alpha_\infty &= C^\alpha \cap C_\infty(\mathbb{R}^d) = \mathfrak{C}^\alpha \cap C_\infty(\mathbb{R}^d), \qquad && \alpha \in (0,1), \\ \mathcal{C}^{1,\alpha-1}_\infty &= C^\alpha \cap C^1_\infty(\mathbb{R}^d) = \mathfrak{C}^\alpha \cap C^1_\infty(\mathbb{R}^d), \qquad && \alpha \in (1,2) \end{aligned}$$

and

$$\mathcal{C}^1_\infty = \mathrm{Lip} \cap C_\infty(\mathbb{R}^d), \qquad \mathcal{C}^{1,0}_\infty = C^1 \cap C_\infty(\mathbb{R}^d).$$

Proof of Theorem 4.1. (i) is well known, see e. g. [26, Theorem 31.5] or [6, Theorem 2.37]. The proofs of (ii) and (iii) are very similar, and therefore we only prove (ii). Pick a cut-off function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi \geq 0$, $\operatorname{supp}\chi \subseteq B(0,1)$ and $\int_{\mathbb{R}^d} \chi(x)\,dx = 1$. If we set $\chi_\varepsilon(x) := \varepsilon^{-1}\chi(\varepsilon^{-1}x)$, then the convolution $f_n := \chi_{1/n} * f$ is in $C^2_\infty(\mathbb{R}^d)$, hence in $\mathcal{D}(A)$, and $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$. As

$$\begin{aligned} |(f_n - f)(x+y) - (f_n - f)(x)| &\leq \left| \int \chi(z)(f(x+y+n^{-1}z) - f(x+n^{-1}z))\,dz \right| + |f(x+y) - f(x)| \\ &\leq 2\|f\|_\alpha |y|^\alpha \end{aligned}$$

for all $|y| \leq 1$ and

$$|(f_n - f)(x+y) - (f_n - f)(x)| \leq 2 \sup_{|r-s|\leq n^{-1}} |f(r) - f(s)| \xrightarrow{n\to\infty} 0,$$

we find

$$\sup_{x\in\mathbb{R}^d} \sup_{0<|y|\leq 1} \frac{|(f_n - f)(x+y) - (f_n - f)(x)|}{|y|^\alpha} \xrightarrow{n\to\infty} 0$$

which implies that

$$Af_n(x) = \fint (f_n(x+y) - f_n(x))\,\nu(dy) \xrightarrow{n\to\infty} \fint (f(x+y) - f(x))\,\nu(dy)$$

uniformly in $x \in \mathbb{R}^d$. Since the generator $(A, \mathcal{D}(A))$ is a closed operator, this finishes the proof. □

4.3 Example (Isotropic α-stable Lévy processes) Let $(L_t)_{t\geq 0}$ be an isotropic α-stable process for some $\alpha \in (0,2)$, i. e. a Lévy process with characteristic exponent $\psi(\xi) = |\xi|^\alpha$, $\xi \in \mathbb{R}^d$, and set $c_\alpha := \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma\left(\frac{\alpha+d}{2}\right)/\Gamma\left(1-\frac{\alpha}{2}\right)$. Then, by Theorem 4.1:

- If $\alpha \in (0,1)$, then Theorem 4.1 shows that the Hölder space $\mathcal{C}^\beta_\infty$ is contained in the domain of the generator A for any $\beta \in (\alpha, 1]$ and
$$Af(x) = c_\alpha \fint (f(x+y) - f(x))\,\frac{dy}{|y|^{d+\alpha}}, \qquad f \in \mathcal{C}^\beta_\infty,\ x \in \mathbb{R}^d.$$
- If $\alpha \in [1,2)$, then $\mathcal{C}^{1,\beta-1}_\infty \subseteq \mathcal{D}(A)$ for all $\beta \in (\alpha, 2]$ and
$$Af(x) = c_\alpha \fint \left(f(x+y) - f(x) - \nabla f(x)\cdot y \mathbb{1}_{(0,1)}(|y|)\right)\frac{dy}{|y|^{d+\alpha}}, \qquad f \in \mathcal{C}^{1,\beta-1}_\infty,\ x \in \mathbb{R}^d.$$

Let us mention that the domain $\mathcal{D}(A)$ of the generator of $(L_t)_{t\geq 0}$ is contained in the Zygmund–Hölder space $\mathfrak{C}^\alpha_\infty := \mathfrak{C}^\alpha \cap C_\infty$, see Remark 4.2 for the definition. In dimension $d = 1$ this follows by combining two results from interpolation theory [33, Theorem 1(a), p. 201; Theorem (d), p. 101] with the fact that the domain of the generator of one-dimensional Brownian motion equals $C^2_\infty(\mathbb{R})$ [28, Example 7.15]. For $d \geq 1$ it is possible to show that the resolvent R_λ, $\lambda > 0$, satisfies $R_\lambda(C_\infty(\mathbb{R}^d)) \subseteq \mathfrak{C}^\alpha_\infty$ using well-known heat kernel estimates for the transition density of $(L_t)_{t\geq 0}$; since $\mathcal{D}(A) = R_\lambda(C_\infty(\mathbb{R}^d))$ this gives the assertion.

In summary,

$$\mathfrak{C}^{\alpha+}_\infty := \bigcup_{\varepsilon>0} \mathfrak{C}^\alpha_\infty \subseteq \mathcal{D}(A) \subseteq \mathfrak{C}^\alpha_\infty. \tag{25}$$

4.4 Example (Compound Poisson processes) Let $(L_t)_{t\geq 0}$ be a Lévy process with Lévy triplet $(b, 0, \nu)$. Suppose that ν is a finite measure and $b = \int_{|y|<1} y\,\nu(dy)$ (e.g. $b = 0$ and $\nu|_{B(0,1)}$ symmetric). Then the domain $\mathcal{D}(A)$ of the generator of $(L_t)_{t\geq 0}$ equals $C_\infty(\mathbb{R}^d)$ and

$$Af(x) = \int (f(x+y) - f(x))\,\nu(dy), \qquad f \in C_\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

Next we extend Theorem 4.1 to Lévy-type processes.

4.5 Theorem *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with symbol q and characteristics (b, Q, ν). Assume that $(X_t)_{t\geq 0}$ has bounded coefficients and that $x \mapsto Q(x)$ is continuous. For fixed $x \in \mathbb{R}^d$ denote by*

$$\beta_\infty(x) := \inf\left\{\gamma > 0;\ \lim_{r\to\infty} \frac{1}{r^\gamma} \sup_{|\xi|\leq r} |q(x,\xi)| < \infty\right\} \in [0, 2]$$

the generalized Blumenthal–Getoor index at ∞. Let $\alpha : \mathbb{R}^d \to (0, 2]$ be a uniformly continuous mapping such that $\alpha(x) \geq \min\{\beta_\infty(x) + \varepsilon, 2\}$ and

$$\sup_{x\in\mathbb{R}^d} \int_{|y|\leq 1} |y|^{\alpha(x)-\varepsilon}\,\nu(x, dy) < \infty$$

for some absolute constant $\varepsilon > 0$. Suppose that $f \in C_\infty(\mathbb{R}^d)$ satisfies the following conditions.

(C1) *For any $x \in \{0 < \alpha \leq 1\}$ it holds that*

$$\sup_{0<|y|\leq 1} \frac{|f(x+y) - f(x)|}{|y|^{\alpha(x)}} < \infty.$$

(C2) *f is differentiable at every point $x \in \{1 < \alpha < 2\}$ and $g_j(x) := \partial_{x_j} f(x)$, $x \in \{1 < \alpha < 2\}$, has a C_∞-extension to $\mathbb{R}^d$ for each $j \in \{1, \ldots, d\}$. Moreover,*

$$\sup_{0<|y|\leq 1} \frac{|f(x+y) - f(x) - \nabla f(x)\cdot y|}{|y|^{\alpha(x)}} < \infty \qquad \text{for all}\ \ x \in \{1 < \alpha < 2\}.$$

(C3) *For any $x \in \{\alpha = 2\}$, f is twice differentiable in a neighbourhood of x and the function $h_{ij}(x) := \partial_{x_i}\partial_{x_j} f(x)$, $x \in \{\alpha = 2\}$, has a C_∞-extension to $\mathbb{R}^d$ for all $i, j \in \{1, \ldots, d\}$.*

Then f is in the domain $\mathcal{D}(A)$ of the generator A of $(X_t)_{t\geq 0}$ and

$$Af(x) = b(x)\cdot g(x) + \frac{1}{2}\operatorname{tr}(Q(x)h(x)) + \int \left(f(x+y) - f(x) - g(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(x, dy)$$

for all $x \in \mathbb{R}^d$ where $g := (g_1, \ldots, g_d)^\top$ and $h := (h_{ij})_{i,j=1,\ldots,d}$.

Before we prove Theorem 4.5, let us make some remarks and state two immediate corollaries of Theorem 4.5.

4.6 Remark (i). Depending on the local Hölder index $\alpha(x)$, the generator $Af(x)$, $f \in \mathcal{D}(A)$, has the following equivalent representations:

- $Af(x) = \int (f(x+y) - f(x))\,\nu(x, dy)$ for any $x \in \{0 < \alpha \leq 1\}$
- $Af(x) = b(x)\cdot\nabla f(x) + \int \left(f(x+y) - f(x) - \nabla f(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(x, dy)$ for any $x \in \{1 < \alpha < 2\}$
- $Af(x) = b(x)\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}\left(Q(x)\cdot\nabla^2 f(x)\right)$
  $+ \int \left(f(x+y) - f(x) - \nabla f(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(x, dy)$
  for any $x \in \{\alpha = 2\}$.

(ii). Since the regularity of the function f may vary from point to point and the triplet is x-dependent, Theorem 4.5 requires stronger assumptions than in the Lévy case.

(iii). Let q be a negative definite symbol with characteristics $(b,0,\nu)$ and suppose that q satisfies the sector condition, i. e. there exists a constant $C>0$ such that

$$|\operatorname{Im} q(x,\xi)| \le C \operatorname{Re} q(x,\xi) \qquad \text{for all } \ x,\xi \in \mathbb{R}^d. \tag{26}$$

Then $\fint_{|y|\le 1} |y|^{\alpha(x)-\varepsilon}\,\nu(x,dy) < \infty$ entails $\beta_\infty(x) \le \alpha(x)-\varepsilon$, cf. (12). Consequently, it suffices in this case to check the integrability condition

$$\sup_{x\in\mathbb{R}^d} \fint_{|y|\le 1} |y|^{\alpha(x)-\varepsilon}\,\nu(x,dy) < \infty.$$

On the other hand, if there exist constants $C>0$ and $\delta>0$ such that

$$|\operatorname{Re} q(x,\xi)| \le C|\xi|^{\beta_\infty^x+\delta} \qquad \text{for all } \ x,\xi \in \mathbb{R}^d,\ |\xi|\ge 1, \tag{27}$$

then any uniformly continuous function $\alpha:\mathbb{R}^d \to (0,2)$ with $\inf_{x\in\mathbb{R}^d}(\alpha(x)-\beta_\infty^x-\delta)>0$ satisfies the assumptions of Theorem 4.5; this follows from the identity

$$\int_{|y|\le 1} |y|^\kappa\,\nu(dy) \le c_\kappa \int \frac{\operatorname{Re}\psi(\xi)}{|\xi|^{d+\kappa}}\,d\xi, \qquad \kappa\in(0,2)$$

which holds for any continuous negative definite function $\psi:\mathbb{R}^d\to\mathbb{C}$ with triplet $(b,0,\nu)$, see (29) in the proof of Lemma A.1.

The sector condition (26) is, in particular, satisfied if $q(x,\cdot)$ is real-valued. This is equivalent to saying that $q(x,\cdot)$ symmetric for all $x\in\mathbb{R}^d$ (i. e. $q(x,\xi)=q(x,-\xi)$ for all $x,\xi\in\mathbb{R}^d$) or $b(x)=0$ and $\nu(x,dy)=\nu(x,-dy)$ for all $x\in\mathbb{R}^d$.

(iv). It is well known, cf. [6, Theorem 2.30], that the mapping $x\mapsto q(x,\xi)$ is continuous for all $\xi\in\mathbb{R}^d$ for any symbol q with $q(x,0)=0$. However, continuity of $q(\cdot,\xi)$ does, in general, not imply continuity of $x\mapsto Q(x)$; consider, for instance,

$$q(x,\xi) := \frac{1}{2}\xi^2 \mathbb{1}_{\{0\}}(x) + \frac{1-\cos(x\xi)}{x^2}\mathbb{1}_{\mathbb{R}^d\setminus\{0\}}(x), \qquad x,\xi\in\mathbb{R},$$

see [13, p. 11].

4.7 Corollary *Let $(X_t)_{t\ge 0}$ be a rich Lévy-type process with symbol q, $q(x,0)=0$ and characteristics $(b,0,\nu)$. Suppose that $(X_t)_{t\ge0}$ has bounded coefficients and $b(x)=\fint_{|y|<1} y\,\nu(x,dy)$ for all $x\in\mathbb{R}^d$. Let $\varepsilon>0$ and $\alpha:\mathbb{R}^d\to[\varepsilon,1]$ be uniformly continuous such that*

$$\sup_{x\in\mathbb{R}^d} \fint_{|y|\le 1} |y|^{\alpha(x)-\varepsilon}\,\nu(x,dy) < \infty.$$

If either the sector condition (26) holds or $\beta_\infty^x \le \alpha(x)-\varepsilon$ for all $x\in\mathbb{R}^d$, then the Hölder space of variable order

$$\mathcal{C}_\infty^{\alpha(\cdot)} := \left\{ f\in C_\infty(\mathbb{R}^d);\ \sup_{x\in\mathbb{R}^d}\sup_{0<|y|\le 1} \frac{|f(x+y)-f(x)|}{|y|^{\alpha(x)}} < \infty \right\}$$

is contained in the domain of the generator A and

$$Af(x) = \fint (f(x+y)-f(x))\,\nu(x,dy) \qquad \textit{for all } \ x\in\mathbb{R}^d,\ f\in\mathcal{C}_\infty^{\alpha(\cdot)}.$$

Proof. Under the assumptions of Corollary 4.7, we know from the remark following Theorem 4.5 that $\beta_\infty(x)\le\alpha(x)-\varepsilon$ for all $x\in\mathbb{R}^d$. Moreover, $\alpha(x)\in[0,1]$ for all $x\in\mathbb{R}^d$ and, by assumption, condition (C1) is satisfied for all $x\in\mathbb{R}^d$. Consequently, the assumptions of Theorem 4.5 are satisfied, and so Theorem 4.5 proves the assertion. □

Let us mention that among the first to consider Hölder spaces of variable order were Ross & Samko [25] who study fractional integrals of variable order. In [1] Hölder spaces of variable order are shown to be particular cases of Besov spaces with variable smoothness and integrability; see Andersson [2] for further characterizations.

4.8 Corollary *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with bounded coefficients and with symbol q and characteristics $(b,0,\nu)$. Let $\varepsilon>0$ be a constant and $\alpha:\mathbb{R}^d\to(\varepsilon,2]$ be a uniformly continuous mapping. Suppose that either the sector condition (26) is satisfied or $\alpha(x)-\varepsilon\geq\beta_\infty^x$ for all $x\in\mathbb{R}^d$. If*

$$\sup_{x\in\mathbb{R}^d}\int_{|y|\leq 1}|y|^{\alpha(x)-\varepsilon}\,\nu(x,dy)<\infty,$$

then the space

$$\mathcal{C}_\infty^{1,\alpha(\cdot)-1}:=\left\{f\in C_\infty^1(\mathbb{R}^d);\forall j=1,\dots,d:\partial_j f\in\mathcal{C}_\infty^{\alpha(\cdot)-1}\right\}$$

is contained in the domain of the generator A, and for all $x\in\mathbb{R}^d$ and $f\in\mathcal{C}_\infty^{1,\alpha(\cdot)-1}$

$$Af(x)=b(x)\cdot\nabla f(x)+\int\left(f(x+y)-f(x)-\nabla f(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(x,dy).$$

Proof. Since $\mathcal{C}_\infty^{1,\alpha(\cdot)-1}\subseteq C_\infty^1(\mathbb{R}^d)$, we may assume without loss of generality that $\alpha(x)\geq 1$ for all $x\in\mathbb{R}^d$; otherwise we could replace α by $\max\{\alpha,1\}$. As in the proof of Corollary 4.7, we find $\beta_\infty(x)+\varepsilon\leq\alpha(x)$ for all $x\in\mathbb{R}^d$. It remains to check that $f\in\mathcal{C}_\infty^{1,\alpha(\cdot)-1}$ satisfies the assumptions of Theorem 4.5. If $x\in\mathbb{R}^d$ is such that $\alpha(x)=1$ it is obvious from the mean value theorem that (C1) is satisfied. Now let $x\in\{1<\alpha<2\}$. Applying the mean value theorem to the auxiliary function $h(y):=f(x+y)-f(x)-\nabla f(x)\cdot y$ shows

$$|f(x+y)-f(x)-\nabla f(x)\cdot y|\leq|y|\sup_{\zeta\in B(0,|y|)}|\nabla f(x+\zeta)-\nabla f(x)|\leq C|y|\cdot|y|^{\alpha(x)-1}=C|y|^{\alpha(x)}$$

for all $x,y\in\mathbb{R}^d$, $|y|\leq 1$ and some absolute constant $C>0$; here we use that $\partial_j f\in\mathcal{C}_\infty^{\alpha(\cdot)-1}$ for all $j\in\{1,\dots,d\}$. This shows that condition (C2) holds true. □

Proof of Theorem 4.5. It follows from Theorem 3.4 that the pointwise limit

$$Lf(x)=\lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}^x f(X_t)-f(x)\right)$$

exists for all $x\in\mathbb{R}^d$ and is given by

- $Lf(x)=\int(f(x+y)-f(x))\,\nu(x,dy)$ for any $x\in\{0<\alpha\leq 1\}$;
- $Lf(x)=b(x)\cdot\nabla f(x)+\int\left(f(x+y)-f(x)-\nabla f(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(x,dy)$ for any $x\in\{1<\alpha<2\}$;
- $Lf(x)=b(x)\cdot\nabla f(x)+\frac{1}{2}\operatorname{tr}\left(Q(x)\nabla^2 f(x)\right)+\int\left(f(x+y)-f(x)-\nabla f(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(x,dy)$ for any $x\in\{\alpha=2\}$.

As $Q(x)=0$ for all $x\in\{0<\alpha<2\}$ and $\int_{|y|<1}y\,\nu(x,dy)=b(x)$ for all $x\in\{0<\alpha\leq 1\}$ (see Lemma A.2 in the appendix), we can write Lf in a closed form as

$$Lf(x)=b(x)\cdot g(x)+\frac{1}{2}\operatorname{tr}\left(Q(x)h(x)\right)+\int\left(f(x+y)-f(x)-g(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(x,dy).$$

In order to prove that f is contained in the domain of the generator A and $Af=Lf$, it suffices to show that $Lf\in C_\infty(\mathbb{R}^d)$, see e.g. [28, Theorem 7.22]. The triangle inequality, Taylor's formula and conditions (C1)-(C3) imply that there exists a constant $C>0$ such that

$$|f(x+y)-f(x)-g(x)\cdot y|\leq C|y|^{\alpha(x)}\qquad\text{for all } x,y\in\mathbb{R}^d,\ |y|\leq 1. \tag{28}$$

Fix a cut-off function $\chi\in C_c^\infty(\mathbb{R}^d)$ such that $\chi\geq 0$, $\operatorname{supp}\chi\subseteq B(0,1)$ and $\int_{\mathbb{R}^d}\chi(x)\,dx=1$. If we set $\chi_\varepsilon(x):=\varepsilon^{-1}\chi(\varepsilon^{-1}x)$, then the convolutions $f_n:=\chi_{1/n}*f$, $g_n:=\chi_{1/n}*g$ and $h_n:=\chi_{1/n}*h$ are $C_\infty^2(\mathbb{R}^d)$-functions and

$$\|f_n-f\|_\infty+\|g_n-g\|_\infty+\|h_n-h\|_\infty\xrightarrow{n\to\infty}0.$$

We are going to show that

$$\Delta_n(x,y):=(f_n-f)(x+y)-(f_n-f)(x)-(g_n-g)(x)\cdot y$$

satisfies an estimate similar to (28). By the very definition of the convolution, we have

$$\begin{aligned}\Delta_n(x,y) &= \int (f(x+y+z)-f(x+y))\chi_{1/n}(z)\,dz - \int (f(x+z)-f(x))\chi_{1/n}(z)\,dz\\ &\quad - \int (g(x+z)-g(x))\cdot y\chi_{1/n}(z)\,dz.\end{aligned}$$

Since $\operatorname{supp}\chi_{1/n}\subseteq B[0,1/n]$ and $0\le\chi\le 1$,

$$|\Delta_n(x,y)| \le 2\sup_{|r-s|\le n^{-1}}|f(r)-f(s)| + \sup_{|r-s|\le n^{-1}}|g(r)-g(s)|.$$

On the other hand, we have by (28)

$$|\Delta_n(x,y)| \le 2C\sup_{|x-z|\le n^{-1}}|y|^{\alpha(z)}\int_{B(0,1)}\chi(z)\,dz = 2C|y|^{\alpha(x)}\sup_{|x-z|\le n^{-1}}|y|^{\alpha(z)-\alpha(x)}.$$

As α is uniformly continuous, we can choose $N\in\mathbb{N}$ sufficiently large such that

$$|\alpha(x)-\alpha(z)|\le\varepsilon/2 \qquad \text{for all } \ x\in\mathbb{R}, z\in B(x,N^{-1}).$$

Combining both estimates, we find

$$\begin{aligned}&\frac{|\Delta_n(x,y)|}{|y|^{\alpha(x)-\varepsilon}}\\ &\le \min\left\{\frac{2\sup_{|r-s|\le n^{-1}}|f(r)-f(s)|+\sup_{|r-s|\le n^{-1}}|g(r)-g(s)|}{|y|^{\alpha(x)-\varepsilon}},\ 2C\sup_{|x-z|\le n^{-1}}|y|^{\varepsilon+(\alpha(z)-\alpha(x))}\right\}\\ &\le \min\left\{\frac{2\sup_{|r-s|\le n^{-1}}|f(r)-f(s)|+\sup_{|r-s|\le n^{-1}}|g(r)-g(s)|}{|y|^{2}},\ 2C|y|^{\varepsilon/2}\right\}\end{aligned}$$

for all $x\in\mathbb{R}^d$, $0<|y|\le 1$ and $n\ge N$. As $f\in C_\infty(\mathbb{R}^d)$ and $g\in C_\infty(\mathbb{R}^d)$ are uniformly continuous, this proves

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^d}\sup_{0<|y|\le 1}\frac{|\Delta_n(x,y)|}{|y|^{\alpha(x)-\varepsilon}} = 0.$$

In particular, there exist constants $C_n>0$ such that $C_n\to 0$ as $n\to\infty$ and

$$|(f_n-f)(x+y)-(f_n-f)(x)-(g_n-g)(x)\cdot y| \le C_n|y|^{\alpha(x)-\varepsilon}$$

for all $x,y\in\mathbb{R}^d$, $|y|\le 1$. If we set

$$Lf_n(x) := b(x)g_n(x)+\frac{1}{2}\operatorname{tr}(Q(x)h_n(x)) + \fint \left(f_n(x+y)-f_n(x)-g_n(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(x,dy),$$

then

$$\begin{aligned}|Lf_n(x)-Lf(x)| &\le \|b\|_\infty\|g_n-g\|_\infty + \|Q\|_\infty\|h_n-h\|_\infty + C_n\fint_{|y|\le 1}|y|^{\alpha(x)-\varepsilon}\,\nu(x,dy)\\ &\quad + 2\|f_n-f\|_\infty\sup_{x\in\mathbb{R}^d}\int_{|y|>1}\nu(x,dy).\end{aligned}$$

This expression converges to zero uniformly in x since $(X_t)_{t\ge 0}$ has bounded coefficients. As $Lf_n\in C_\infty(\mathbb{R}^d)$ for large $n\in\mathbb{N}$, see Lemma 4.9 below, we conclude that $Lf\in C_\infty(\mathbb{R}^d)$. □

For the proof of Theorem 4.5 we need the following auxiliary statement.

4.9 Lemma *Lf_n defined in the proof of Theorem 4.5 is a $C_\infty(\mathbb{R}^d)$-function for sufficiently large $n\in\mathbb{N}$.*

Proof. The mapping $x\mapsto Q(x)$ is, by assumption, continuous and bounded. As $h_n\in C_\infty^2(\mathbb{R}^d)$, this implies that $\operatorname{tr}(Q(\bullet)^\top h_n(\bullet))\in C_\infty(\mathbb{R}^d)$. Consequently, it is enough to show that

$$\tilde{L}f_n(x) := b(x)\cdot g_n(x) + \fint \left(f_n(x+y)-f_n(x)-g_n(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(x,dy)\in C_\infty(\mathbb{R}^d).$$

Since $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ and $(X_t)_{t\geq 0}$ has bounded coefficients, we have $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(A)$, and therefore

$$\begin{aligned} Af_n(x) &= b(x)\cdot\nabla f_n(x) + \frac{1}{2}\operatorname{tr}\big(Q(x)\nabla^2 f_n(x)\big) \\ &\quad + \fint \big(f_n(x+y) - f_n(x) - \nabla f_n(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\big)\,\nu(x,dy) \end{aligned}$$

is in $C_\infty(\mathbb{R}^d)$. Using again the fact that $Q \in C_b(\mathbb{R}^d)$ and $\nabla^2 f_n \in C_\infty(\mathbb{R}^d)$, we get

$$\tilde{A}f_n(x) := b(x)\cdot\nabla f_n(x) + \fint \big(f_n(x+y) - f_n(x) - \nabla f_n(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\big)\,\nu(x,dy) \in C_\infty(\mathbb{R}^d).$$

Let $x \in \mathbb{R}^d$. We distinguish between two cases.

$0 < \alpha(x) \leq 1+\varepsilon/2$: Using our assumption $\beta_\infty(x)+\varepsilon \leq \alpha(x)$, we find $\beta_\infty(x) < 1$ which implies, by Lemma A.2, $b(x) - \int_{|y|<1} y\,\nu(x,dy) = 0$. Thus, $\tilde{A}f_n(x) = \tilde{L}f_n(x)$.

$1+\varepsilon/2 < \alpha(x)$: Since α is uniformly continuous, we can choose $n \in \mathbb{N}$ (not depending on x) so large that $|\alpha(x)-\alpha(z)| \leq \varepsilon/4$ for all $z \in B[x,n^{-1}]$. Then $\alpha(z) > 1+\varepsilon/4$ for all $z \in B(x,n^{-1})$ and, therefore, $f|_{B(x,n^{-1})}$ is differentiable. Since $\operatorname{supp}\chi_{1/n} \subseteq B[0,1/n]$, this implies $\nabla f_n(x) = g_n(x)$. Hence, $\tilde{L}f_n(x) = \tilde{A}f_n(x)$.

Consequently, we have $\tilde{L}f_n = \tilde{A}f_n \in C_\infty(\mathbb{R}^d)$ for $n \in \mathbb{N}$ sufficiently large. □

We close this section with some examples. Recall the definition of the Hölder spaces of variable order $\mathcal{C}_\infty^{\alpha(\cdot)}$ and $\mathcal{C}_\infty^{1,\alpha(\cdot)-1}$ introduced in Corollary 4.7 and Corollary 4.8, respectively.

4.10 Example (Stable-like dominated process) Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with symbol q and characteristics $(b,0,\nu)$. Denote by $(A,\mathcal{D}(A))$ the generator of $(X_t)_{t\geq 0}$. Suppose that $(X_t)_{t\geq 0}$ has bounded coefficients and that there exist a constant $c>0$ and a mapping $\gamma:\mathbb{R}^d \to (0,2)$ such that $\inf_{x\in\mathbb{R}^d}\gamma(x) > 0$ and

$$\nu(x, A\cap B(0,1)) \leq c\int_{A\cap B(0,1)} \frac{dy}{|y|^{d+\gamma(x)}} \qquad \text{for all } \ A \in \mathcal{B}(\mathbb{R}^d\setminus\{0\}),\ x\in\mathbb{R}^d.$$

Let $\alpha:\mathbb{R}^d \to (0,2)$ be a uniformly continuous mapping such that $\inf_{x\in\mathbb{R}^d}(\alpha(x)-\gamma(x)) > 0$, and suppose that either the sector condition (26) is satisfied or $\inf_{x\in\mathbb{R}^d}(\alpha(x)-\beta_\infty^x) > 0$.

(i). If $\alpha(\mathbb{R}^d) \subseteq [0,1]$ and $b(x) = \fint_{|y|<1} y\,\nu(x,dy)$ for all $x\in\mathbb{R}^d$, then $\mathcal{C}_\infty^{\alpha(\cdot)} \subseteq \mathcal{D}(A)$ and

$$Af(x) = \fint (f(x+y)-f(x))\,\nu(x,dy), \qquad x\in\mathbb{R}^d,\ f\in\mathcal{C}_\infty^{\alpha(\cdot)}.$$

(ii). $\mathcal{C}_\infty^{1,\alpha(\cdot)-1} \subseteq \mathcal{D}(A)$ and

$$Af(x) = b(x)\cdot\nabla f(x) + \fint \big(f(x+y)-f(x)-\nabla f(x)\cdot y\mathbb{1}_{(0,1)}(|y|)\big)\,\nu(x,dy),\ x\in\mathbb{R}^d$$

for all $f \in \mathcal{C}_\infty^{1,\alpha(\cdot)-1}$.

4.11 Example Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with one of the following symbols.

- stable-like: $q(x,\xi) = |\xi|^{\gamma(x)}$ where $\gamma:\mathbb{R}^d\to(0,2)$ is a Hölder continuous mapping such that $\inf_{x\in\mathbb{R}^d}\gamma(x) > 0$.
- relativistic stable-like: $q(x,\xi) = (|\xi|^2+m(x)^2)^{\gamma(x)/2} - m(x)^{\gamma(x)}$ for Hölder continuous mappings $\gamma:\mathbb{R}^d\to(0,2)$ and $m:\mathbb{R}^d\to(0,\infty)$ such that

$$\inf_{x\in\mathbb{R}^d}\gamma(x) > 0 \quad \text{and} \quad 0 < \inf_{x\in\mathbb{R}^d} m(x) \leq \sup_{x\in\mathbb{R}^d} m(x) < \infty.$$

- TLP-like:[1] $q(x,\xi) = (|\xi|^2+m(x)^2)^{\gamma(x)/2}\cos\big[\gamma(x)\arctan\frac{|\xi|}{m(x)}\big] - m(x)^{\gamma(x)}$ for Hölder continuous mappings $\gamma:\mathbb{R}^d\to(0,1)$ and $m:\mathbb{R}^d\to(0,\infty)$ such that

$$0 < \inf_{x\in\mathbb{R}^d}\gamma(x) \leq \sup_{x\in\mathbb{R}^d}\gamma(x) < 1 \quad \text{and} \quad 0 < \inf_{x\in\mathbb{R}^d} m(x) \leq \sup_{x\in\mathbb{R}^d} m(x) < \infty.$$

[1] TLP is short for "truncated Lévy process".

- Lamperti stable-like: $q(x,\xi) = (|\xi|^2 + m(x))_{\gamma(x)} - (m(x))_{\gamma(x)}$ – $(z)_\gamma := \Gamma(z+\gamma)/\Gamma(z)$ denotes the Pochhammer symbol – for Hölder continuous mappings $\gamma : \mathbb{R}^d \to (0,1)$ and $m : \mathbb{R}^d \to (0,\infty)$ such that

$$0 < \inf_{x\in\mathbb{R}^d} \gamma(x) \le \sup_{x\in\mathbb{R}^d} \gamma(x) < 1 \quad \text{and} \quad 0 < \inf_{x\in\mathbb{R}^d} m(x) \le \sup_{x\in\mathbb{R}^d} m(x) < \infty.$$

Let $\alpha : \mathbb{R}^d \to [0,2]$ be a uniformly continuous map such that $\inf_{x\in\mathbb{R}^d}(\alpha(x) - \gamma(x)) > 0$. Then:

(i). $\mathcal{C}_\infty^{1,\alpha(\cdot)-1} \subseteq \mathcal{D}(A)$ and $Af(x) = \fint \left(f(x+y) - f(x) - \nabla f(x)\cdot y \mathbb{1}_{(0,1)}(|y|)\right) \nu(x,dy)$ for any $f \in \mathcal{C}_\infty^{1,\alpha(\cdot)-1}$ where $(0,0,\nu)$ denotes the characteristics of the symbol q.

(ii). If $\alpha(\mathbb{R}^d) \subseteq [0,1]$, then $\mathcal{C}_\infty^{\alpha(\cdot)} \subseteq \mathcal{D}(A)$ and $Af(x) = \fint (f(x+y) - f(x))\, \nu(x,dy)$ for any $f \in \mathcal{C}_\infty^{\alpha(\cdot)}$.

Example 4.11 is a direct consequence of Theorem 4.5 and Remark 4.6(ii) since all symbols satisfy the sector condition (26) and growth condition (27) with $\delta = 0$. Note that the existence of (rich) Lévy-type processes with the symbols mentioned in Example 4.11 has been established in [22] recently. Obviously, Example 4.11 applies, in particular, in the Lévy case, i. e. if the maps $\gamma(\bullet)$ and $m(\bullet)$ are constants.

We close this section with the following example.

4.12 Example (Lévy-driven SDE) Let $(L_t)_{t\ge0}$ be a k-dimensional Lévy process with Lévy triplet $(0,0,\nu)$ and characteristic exponent ψ. Suppose that the Lévy measure ν is symmetric and that there exists an $\alpha \in (0,2)$ such that $\fint_{|y|\le1} |y|^\alpha\, \nu(dy) < \infty$. For any bounded (globally) Lipschitz continuous function $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ the solution to the SDE

$$dX_t = \sigma(X_{t-})\, dL_t, \qquad X_0 = x,$$

is a rich Lévy-type process with symbol $q(x,\xi) = \psi(\sigma(x)^\top \xi)$, $x,\xi \in \mathbb{R}^d$. Moreover:,

(i). If $\alpha \in (0,1)$, then the Hölder space $\mathcal{C}_\infty^\beta$ is contained in the domain of the generator A of $(X_t)_{t\ge0}$ for any $\beta \in (\alpha,1]$ and

$$Af(x) = \fint (f(x+\sigma(x)y) - f(x))\, \nu(dy), \qquad f \in \mathcal{C}_\infty^\beta,\ x \in \mathbb{R}^d.$$

(ii). If $\alpha \in [1,2)$, then the Hölder space $\mathcal{C}_\infty^{1,\beta-1}$ is contained in the domain of the generator A of $(X_t)_{t\ge0}$ for any $\beta \in (\alpha,2]$ and

$$Af(x) = \fint \left(f(x+\sigma(x)y) - f(x) - \nabla f(x)\cdot\sigma(x)y\mathbb{1}_{(0,1)}(|y|)\right) \nu(dy), \qquad f \in \mathcal{C}_\infty^{1,\beta-1},\ x \in \mathbb{R}^d.$$

Proof. It is well known that the solution $(X_t)_{t\ge0}$ to the SDE is a rich Lévy-type process with symbol $q(x,\xi) = \psi(\sigma(x)^\top\xi)$, cf. Schilling & Schnurr [29, Corollary 3.7] or Kühn [23, Example 4.1]. Since the Lévy measure ν is symmetric, both ψ and q are real-valued; in particular, q satisfies the sector condition. Moreover, the characteristics of q are given by $(0,0,\nu(x,dy))$ where

$$\nu(x,B) := \int \mathbb{1}_B(\sigma(x)y)\, \nu(dy), \qquad x \in \mathbb{R}^d,\ B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\});$$

therefore, the boundedness of σ gives

$$\sup_{x\in\mathbb{R}^d} \fint_{|y|\le1} |y|^\alpha\, \nu(x,dy) \le \|\sigma\|_\infty^\alpha \int_{|y|\le1} |y|^\alpha\, \nu(dy) < \infty.$$

Now the assertion follows from Corollary 4.7 and Corollary 4.8. □

A Appendix

Recall that a function $\psi : \mathbb{R}^d \to \mathbb{C}$ with $\psi(0) = 0$ is *continuous negative definite*, if it admits a Lévy–Khintchine representation of the form (6). A continuous negative definite function ψ satisfies the *sector condition* if there exists a constant $C > 0$ such that

$$|\operatorname{Im}\psi(\xi)| \leq C \operatorname{Re}\psi(\xi) \qquad \text{for all } \ \xi \in \mathbb{R}^d.$$

The following lemma is used in the proof of Theorem 3.5.

A.1 Lemma *Let ψ be a continuous negative definite function with triplet $(b, 0, \nu)$, and let $\alpha \in (0, 2)$. The following statements are equivalent:*

(i). $\not\int_{|y|\leq 1} |y|^\alpha \, \nu(dy) < \infty$;

(ii). $\int_1^\infty \sup_{|\xi|\leq r} \operatorname{Re}\psi(\xi) \, \frac{dr}{r^{1+\alpha}} < \infty$;

(iii). $\int_{|\xi|\geq 1} \operatorname{Re}\psi(\xi) \, \frac{d\xi}{|\xi|^{d+\alpha}} < \infty$.

If ψ satisfies the sector condition, then we may replace $\operatorname{Re}\psi$ *by* $|\psi|$.

Proof. Obviously, it suffices to prove the first assertion. We prove (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

(i) ⇒ (ii): Since $1 - \cos(y \cdot \xi) \leq \frac{1}{2}|y\xi|^2$ for all $y, \xi \in \mathbb{R}^d$, we have

$$\psi^*(r) := \sup_{|\xi|\leq r} \operatorname{Re}\psi(\xi) \leq 2 \not\int \min\left\{1, |y|^2 r^2\right\} \nu(dy)$$

implying

$$\int_1^\infty \psi^*(r) \, \frac{dr}{r^{1+\alpha}} \leq 2 \int_1^\infty r^2 \not\int_{|y|<r^{-1}} |y|^2 \, \nu(dy) \, \frac{dr}{r^{1+\alpha}} + 2 \int_1^\infty \int_{|y|\geq r^{-1}} \nu(dy) \, \frac{dr}{r^{1+\alpha}} =: 2I_1 + 2I_2.$$

An application of Tonelli's theorem shows

$$\begin{aligned} I_1 = \int_1^\infty \not\int_{|y|<r^{-1}} r^{1-\alpha} |y|^2 \, \nu(dy) \, dr &= \not\int_{|y|\leq 1} \left(\int_{1\leq r<|y|^{-1}} r^{1-\alpha} \, dr \right) |y|^2 \, \nu(dy) \\ &= \frac{1}{2-\alpha} \not\int_{|y|\leq 1} |y|^2 \left(|y|^{\alpha-2} - 1 \right) \nu(dy) < \infty \end{aligned}$$

and

$$\begin{aligned} I_2 = \int_1^\infty \nu(\{y; \, |y| \geq r^{-1}\}) \, \frac{dr}{r^{1+\alpha}} &= \int_0^1 \nu(\{y; \, |y| \geq u\}) \, \frac{du}{u^{1-\alpha}} \\ &= \frac{1}{\alpha} \not\int_{|y|\leq 1} |y|^\alpha \, \nu(dy) < \infty. \end{aligned}$$

In the last step we use the identity

$$\int f(x) \, d\mu(x) = \int_0^\infty \mu(\{x; \, |f(x)| \geq r\}) \, dr$$

which holds for any σ-finite measure μ on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$ and any non-negative measurable function f. This proves (ii).

The implication (ii) ⇒ (iii) follows easily by introducing spherical coordinates and using the obvious estimate

$$\operatorname{Re}\psi(r\eta) \leq \psi^*(r) \qquad \text{for all } \ r \geq 0, \ \eta \in \mathbb{R}^d, \ |\eta| = 1.$$

It remains to prove that (iii) implies (i). To this end, we note that

$$|y|^\alpha = c \int (1 - \cos(y \cdot \xi)) \frac{1}{|\xi|^{d+\alpha}} \, d\xi$$

for the constant $c = \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma\left(\frac{\alpha+d}{2}\right)/\Gamma\left(1-\frac{\alpha}{2}\right)$, and therefore

$$\begin{aligned}\fint_{|y|\leq 1} |y|^\alpha \,\nu(dy) &= c\int \left(\fint_{|y|\leq 1} (1-\cos(y\cdot\xi))\,\nu(dy)\right)\frac{d\xi}{|\xi|^{d+\alpha}} \\ &\leq c\int \operatorname{Re}\psi(\xi)\,\frac{d\xi}{|\xi|^{d+\alpha}}\,d\xi < \infty,\end{aligned} \tag{29}$$

which completes the proof. □

For a continuous negative definite function ψ the Blumenthal–Getoor index at ∞ can be defined by

$$\beta_\infty := \inf\left\{\gamma > 0; \lim_{r\to\infty}\frac{1}{r^\gamma}\sup_{|\xi|\leq r}|\psi(\xi)| < \infty\right\},$$

cf. Schilling [27] or Blumenthal & Getoor [4]. The following auxiliary statement is needed in the proof of Theorem 4.5.

A.2 Lemma *Let ψ be a continuous negative definite function,*

$$\psi(\xi) = ib\cdot\xi + \frac{1}{2}\xi\cdot Q\xi + \fint \left(1-e^{i\xi\cdot y} + i\xi\cdot y\mathbb{1}_{(0,1)}(|y|)\right)\nu(dy), \qquad \xi\in\mathbb{R}^d,$$

and denote by $\beta_\infty \in [0,2]$ the Blumenthal–Getoor index at ∞.

(i). *If $\beta_\infty < 2$, then $Q = 0$.*

(ii). *If $\beta_\infty < 1$, then $b = \int_{|y|<1} y\,\nu(dy)$.*

Proof. (i) Since $|\xi|^{-2}|1-\cos(y\cdot\xi)| \leq \min\{2,|y|^2\}$ for all $|\xi|\geq 1$, an application of the dominated convergence theorem shows

$$\lim_{|\xi|\to\infty}\frac{1}{|\xi|^2}\fint (1-\cos(y\cdot\xi))\,\nu(dy) = 0.$$

Thus,

$$\lim_{|\xi|\to\infty}\frac{|\xi\cdot Q\xi|}{2|\xi|^2} \leq \lim_{|\xi|\to\infty}\frac{\operatorname{Re}\psi(\xi)}{|\xi|^2} + \lim_{|\xi|\to\infty}\frac{1}{|\xi|^2}\fint (1-\cos(y\cdot\xi))\,\nu(dy) = 0$$

which implies $Q = 0$.

(ii) We know from (i) that $Q = 0$. Since $\left|\int_{|y|\geq 1}\left(1-e^{iy\cdot\xi}\right)\nu(dy)\right| \leq 2\nu(\mathbb{R}^d\setminus B(0,1))$, we may assume, without loss of generality, that $\operatorname{supp}\nu\subseteq B[0,1]$. For any $\gamma\in(0,1)$ there exists some $c_\gamma > 0$ such that

$$\int |y|^\gamma\,\nu(dy) = c_\gamma \iint (1-\cos(y\cdot z))\,\frac{dz}{|z|^{1+\gamma}}\,\nu(dy) = c_\gamma\int \operatorname{Re}\psi(z)\,\frac{dz}{|z|^{1+\gamma}}.$$

As $\operatorname{supp}\nu\subseteq B[0,1]$, it follows easily from Taylor's formula that $|\operatorname{Re}\psi(z)| \leq C'|z|^2$ for some absolute constant $C' > 0$. On the other hand, by assumption, $|\operatorname{Re}\psi(z)| \leq C|z|^\beta$ for some $\beta\in(\beta_\infty,1)$. Consequently, we find $\fint |y|^\gamma\,\nu(dy) < \infty$ for all $\gamma > \beta$. This implies, in particular, that

$$\psi_0(\xi) := \fint \left(1-e^{iy\cdot\xi}\right)\nu(dy), \qquad \xi\in\mathbb{R}^d,$$

is well-defined. Using Markov's inequality and the elementary estimate $|\sin x| \leq |x|$, we find for all $\gamma\in(\beta,1)$

$$\begin{aligned}|\operatorname{Im}\psi_0(\xi)| &\leq \fint_{|y\xi|<1}|\sin(y\cdot\xi)|\,\nu(dy) + \int_{|y\cdot\xi|\geq 1} 1\,\nu(dy) \\ &\leq \int_{|y\cdot\xi|<1}|y\cdot\xi|\,\nu(dy) + \int_{|y\cdot\xi|\geq 1}|y\cdot\xi|^\gamma\,\nu(dy) \leq |\xi|^\gamma\fint |y|^\gamma\,\nu(dy).\end{aligned}$$

Thus,

$$2|\xi|^\gamma\fint |y|^\gamma\,\nu(dy) \geq |\operatorname{Im}\psi_0(\xi)| \geq \left|b + \fint_{|y|\leq 1} y\,\nu(dy)\right| |\xi| - |\operatorname{Im}\psi(\xi)|.$$

Dividing both sides by $|\xi|^\gamma$ and letting $|\xi|\to\infty$ proves the assertion. □

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